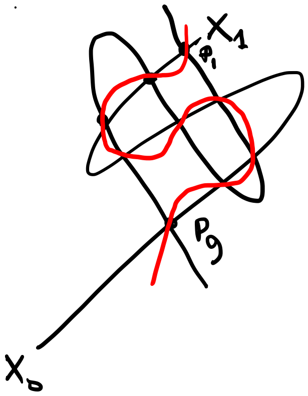
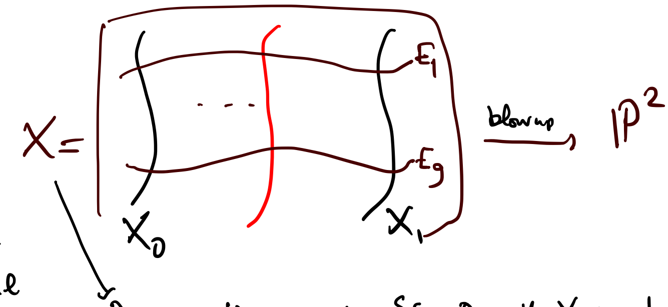
Rational elliptic surfaces (work with Heckman)

April 20 2020



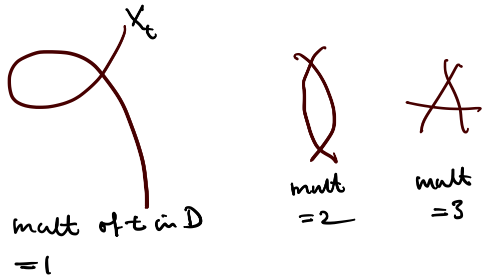
X_0, X_1 cubic curves in $\mathbb{P}^2$ meeting in $p_1, \dots, p_9$. Blow up these points: yields a fibered surface X over $P \subseteq \mathbb{P}^1$ (P = pencil generated by X_0 and X_1)

Exc. divisor E_i over p_i appears as section



$P \supset D$ discriminant $= \{t \in P$ with X_t singular$\}$ is a divisor of degree 12

Note that the projection $X \xrightarrow{\pi} P$ is intrinsic to X: the fibers are the anti-canonical divisors ≥ 0 on X



Def: A rational ell. surface is a rational surface for which the anti-canonical system defines a fibering and which admits a section

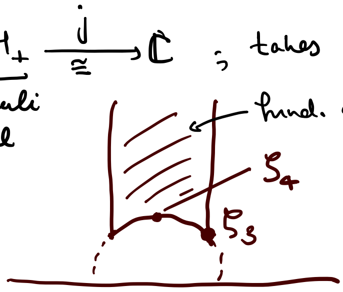
Fact: every rat. ell. surface arises via the above construction

Given a rat. ell. surface X, let D_X be the discriminant divisor on $P = |-K_X|$ (of deg 12)

Q: Can every 12-elt subset of a projective line occur in this manner up to proj. eq.? dim count tells us no: to get X it suffices to give $p_1, \dots, p_9$ in $\mathbb{P}^2$ up to proj. eq. This gives $8.2 - \dim PGL(3) = 8$ moduli. But 12 pts in $\mathbb{P}^1$ up to proj. eq. gives $12 - \dim PGL(2) = 9$ moduli. This suggest we get a hypersurface in the space $\mathrm{Sym}^{12} P$ of deg. 12 divisors (mod proj. equivalence)

Recall: $PSL_2\mathbb{Z} \backslash H_+ \underset{\cong}{\xrightarrow{j}} \mathbb{C}$; takes the special orbit of ρ_3 ($\leftrightarrow$ ell curve ▱) to 0, ρ_4 ($\leftrightarrow$ □) to 1

coarse moduli space of ell curves



fund. domain for $PSL_2\mathbb{Z}$-action

The moduli space is coarse, there is no way to let $PSL_2\mathbb{Z}\backslash H_+$ support an elliptic fibration (especially because of these special orbits)

We complete $PSL_2\mathbb{Z} \cup H_+$ like this

$\hat{H}_+ = H_+ \cup \mathbb{P}^1(\mathbb{Q}) \subset \mathbb{P}^1$ with a $PSL_2\mathbb{Q}$-inv. topology (not the induced one)

yields orbit space $PSL_2\mathbb{Z}\backslash\hat{H}_+ = \mathbb{C} \cup \{\infty\} = \mathbb{P}^1$ cusp. (use that $PSL_2\mathbb{Z}$ is transitive on $\mathbb{P}^1(\mathbb{Q})$)

Let X rational ell. surface
$\downarrow \pi$
$P \supset D$ discriminant

Then $J: P \to PSL_2\mathbb{Z}\backslash\hat{H} \cong \mathbb{P}^1$
$t \longmapsto j$-invariant of X_t.

<u>Assume D reduced</u>. Then $D = J^*(\infty)$. But $J^*(0)$ will be 3-divisible: $3D_0$ with D_0 of degree 4 and $J^*(1)$ will be 2-divisible: $2D_1$.

Since $(\infty) \in \mathbb{P}^1$ is in the "pencil" generated by (0) and (1), it follows that D is in the pencil generated by $3D_0$ and $2D_1$. This gives the correct no of moduli: $\dim \mathrm{Sym}^4 P + \dim \mathrm{Sym}^6 P + 1 - \dim PGL(2) = 4+6+1-3 = 8$.

we take a lin comb!