Integrality in the Steinberg module and the top-dimensional cohomology of $\text{GL}_n\mathcal{O}_K$

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Abstract

We prove a new structural result for the spherical Tits building attached to $\text{GL}_n\mathcal{O}_K$ for many number fields $K$, and more generally for the fraction fields of many Dedekind domains $\mathcal{O}$: the Steinberg module $\text{St}_n(K)$ is generated by integral apartments if and only if the ideal class group $\text{cl}(\mathcal{O})$ is trivial. We deduce this integrality by proving that the complex of partial bases of $\mathcal{O}^n$ is Cohen–Macaulay. We apply this to prove new vanishing and nonvanishing results for $H^{\nu_n}(\text{GL}_n\mathcal{O}_K;\mathbb{Q})$, where $\mathcal{O}_K$ is the ring of integers in a number field and $\nu_n$ is the virtual cohomological dimension of $\text{GL}_n\mathcal{O}_K$. The (non)vanishing depends on the (non)triviality of the class group of $\mathcal{O}_K$. We also obtain a vanishing theorem for the cohomology $H^{\nu_n}(\text{GL}_n\mathcal{O}_K;V)$ with twisted coefficients $V$. The same results hold for $\text{SL}_n\mathcal{O}_K$ as well.

1 Introduction

1.1 The Tits building and the Steinberg module

One of the most fundamental geometric objects attached to the general linear group $\text{GL}_n K$ over a field $K$ is its associated Tits building, denoted $\mathcal{T}_n(K)$. The space $\mathcal{T}_n(K)$ is the $(n-2)$-dimensional simplicial complex whose $p$-simplices are flags of subspaces $0 \subset V_0 \subset \cdots \subset V_p \subset K^n$.

The group $\text{GL}_n K$ acts on $\mathcal{T}_n(K)$ by simplicial automorphisms. By the Solomon–Tits theorem, the space $\mathcal{T}_n(K)$ is homotopy equivalent to a wedge of $(n-2)$-dimensional spheres, so it has only one interesting homology group.

Definition 1.1 (The Steinberg module). The Steinberg module $\text{St}_n(K)$ is the $\text{GL}_n K$-module given by the top homology group $\tilde{H}_{n-2}(\mathcal{T}_n(K);\mathbb{Z})$.

The Solomon–Tits theorem also gives a generating set for $\text{St}_n(K)$. A frame for $K^n$ is a set $L = \{L_1, \ldots, L_n\}$ of lines in $K^n$ such that $K^n = L_1 \oplus \cdots \oplus L_n$. The apartment $A_L$ corresponding to $L$ is the full subcomplex of $\mathcal{T}_n(K)$ on the $2^n-2$ subspaces $\text{span}(L_i \mid i \in I)$ for $\emptyset \subset I \subset \{1, \ldots, n\}$. The apartment $A_L$ is homeomorphic to an $(n-2)$-sphere, so its fundamental class determines an apartment class

$$[A_L] \in \tilde{H}_{n-2}(\mathcal{T}_n(K);\mathbb{Z}) = \text{St}_n(K)$$

which is defined up to $\pm 1$. The Solomon–Tits theorem states that $\text{St}_n(K)$ is generated by the set of apartment classes.

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1.2 Integrality and non-integrality

Consider a Dedekind domain $\mathcal{O}$ with field of fractions $K$. While the action of $\text{GL}_n K$ on $\mathcal{T}_n(K)$ is transitive, the action of $\text{GL}_n \mathcal{O}$ is usually not, and indeed this action encodes arithmetic information about $\mathcal{O}$. For example, the number of orbits of the $\text{GL}_2 \mathcal{O}$-action on $\mathcal{T}_2(K)$ is the class number $|\text{cl}(\mathcal{O})|$ of $\mathcal{O}$. In this context, we have the following natural notion.

**Definition 1.2 (Integral apartment).** Let $\mathcal{O}$ be a Dedekind domain with field of fractions $K$. A frame $L = \{L_1, \ldots, L_n\}$ of $K^n$ is integral, and $A_L$ is an integral apartment, if

$$\mathcal{O}^n = (L_1 \cap \mathcal{O}^n) \oplus \cdots \oplus (L_n \cap \mathcal{O}^n).$$

In the case $\mathcal{O} = \mathbb{Z}$, and more generally when $\mathcal{O}$ is Euclidean, Ash–Rudolph [AR, Theorem 4.1] proved that $\text{St}_n(K)$ is generated by the fundamental classes of integral apartments. Our first theorem extends this result of Ash–Rudolph to a wide class of Dedekind domains of arithmetic type, whose definition we recall.

**Definition 1.3 (Dedekind domain of arithmetic type).** Let $K$ be a global field, i.e.

- either a number field ($[K : \mathbb{Q}] < \infty$) or a function field in one variable over a finite field ($[K : \mathbb{F}_q(T)] < \infty$). Let $S$ be a finite nonempty set of places of $K$; if $K$ is a number field, assume that $S$ contains all infinite places. The ring of $S$-integers

$$\mathcal{O}_S := \{x \in K \mid \text{ord}_p(x) \geq 0 \text{ for all } p \notin S\}$$

is a Dedekind domain, and we say that $\mathcal{O}_S$ is a Dedekind domain of arithmetic type.

**Theorem A (Integrality Theorem).** Let $\mathcal{O}$ be a Dedekind domain with $|\text{cl}(\mathcal{O})| = 1$ and field of fractions $K$. Assume either that $\mathcal{O}$ is Euclidean, or that $\mathcal{O} = \mathcal{O}_S$ is a Dedekind domain of arithmetic type such that $|S| > 1$ and $S$ contains a non-imaginary place. Then $\text{St}_n(K)$ is spanned by integral apartment classes.

Under the conditions of Theorem A, the set of integral apartments in $\mathcal{T}_n(K)$ is precisely the $\text{GL}_n \mathcal{O}$-orbit of a single standard apartment. Thus Theorem A implies that $\text{St}_n(K)$, which is obviously cyclic as a $\text{GL}_n K$-module, is in fact cyclic as a $\text{GL}_n \mathcal{O}$-module.

**Remark 1.4.** Ash–Rudolph’s proof [AR] of Theorem A when $\mathcal{O}$ is Euclidean is based on a beautiful generalization of the method of continued fractions to higher dimensions. Using the Euclidean function on $\mathcal{O}$ as a measure of “complexity”, they give an algorithm to write a non-integral apartment class as a sum of integral apartment classes. Our proof is quite different, even in the special case that $\mathcal{O}$ is Euclidean: non-integral apartments never actually show up in our proof, and we do not even make use of the fact that $\text{St}_n(K)$ is generated by apartment classes.

By Dirichlet’s theorem, the group of units $\mathcal{O}_S^\times$ has rank $|S| - 1$, so the assumption $|S| > 1$ is equivalent to $|\mathcal{O}_S^\times| = \infty$. The assumptions on $S$ rule out only two families of Dedekind domains of arithmetic type, for which Theorem A need not hold (see Remark 4.1 below):

- the ring of integers $\mathcal{O}_K$ in a totally imaginary number field $K$;
- those $\mathcal{O}_S$ in positive characteristic with finitely many units, such as $\mathbb{F}_q[T]$.

The assumption in Theorem A that $|\text{cl}(\mathcal{O}_S)| = 1$ obviously excludes many more examples. However, this assumption is necessary in a strong sense, as the following theorem shows.

**Theorem B (Non-Integrality Theorem).** Let $\mathcal{O}$ be a Dedekind domain with field of fractions $K$. If $1 < |\text{cl}(\mathcal{O})| < \infty$ and $n \geq 2$ then $\text{St}_n(K)$ is not generated by integral apartment classes.

Theorem B applies in particular to all number rings with nontrivial class group.
1.3 The top-degree cohomology of $\GL_n \mathcal{O}_K$ and $\SL_n \mathcal{O}_K$

When $\mathcal{O}_K$ is the ring of integers in a number field $K$, the Steinberg module $\text{St}_n(K)$ is directly connected to the cohomology of $\GL_n \mathcal{O}_K$ and $\SL_n \mathcal{O}_K$, as we now explain.

Let $r_1$ (resp. $2r_2$) be the number of real (resp. complex) embeddings of $\mathcal{O}_K$. Then $\mathcal{O}_K \otimes \mathbb{R} \cong \mathbb{R}^{r_1} \oplus \mathbb{C}^{r_2}$ and $\GL_n(\mathcal{O}_K \otimes \mathbb{R}) \cong (\GL_n \mathbb{R})^{r_1} \times (\GL_n \mathbb{C})^{r_2}$. The quotient $M_K$ of the associated symmetric space by the discrete group $\GL_n \mathcal{O}_K$ is a Riemannian orbifold with $H^*(M_K; \mathbb{Q}) \cong H^*(\GL_n \mathcal{O}_K; \mathbb{Q})$.

The computation of these cohomology groups is a fundamental problem in group theory, topology and number theory. The existence of $M_K$ implies that $H^i(\GL_n \mathcal{O}_K; \mathbb{Q}) = 0$ for $i > \dim M_K$. But since $M_K$ is not compact, it is not even clear what the largest $i$ for which $H^i(\GL_n \mathcal{O}_K; \mathbb{Q}) \neq 0$ is. The first progress on this basic problem came in 1973 by Borel and Serre [BS]. Recall that the virtual cohomological dimension of a virtually torsion-free group $\Gamma$ is

$$\text{vcd}(\Gamma) := \max \{ k \mid H^k(\Gamma; V \otimes \mathbb{Q}) \neq 0 \text{ for some } \Gamma\text{-module } V \}.$$ 

By constructing and analyzing a compactification of the orbifold $M_K$, Borel–Serre proved that $\text{vcd}(\GL_n \mathcal{O}_K)$ is equal to $\nu_n := r_1 \left(\frac{n+1}{2}\right) + r_2 \cdot n^2 - n$. In particular $H^k(\GL_n \mathcal{O}_K; \mathbb{Q}) = 0$ for $k > \nu_n$. The coefficient module $V$ used by Borel–Serre to certify that $H^n(\GL_n \mathcal{O}_K; V) \neq 0$ is infinite-dimensional. This left open the question as to whether or not $\GL_n \mathcal{O}_K$ has untwisted rational cohomology in the top dimension $\nu_n$.

In 1976 Lee–Szczarba [LS] Theorem 1.3 answered this question in the special case that $\mathcal{O}_K$ is Euclidean, proving that $H^n(\GL_n \mathcal{O}_K; \mathbb{Q}) = 0$ for these $\mathcal{O}_K$. Except for some explicit computations in low-dimensional cases (sometimes with the help of computers), there seems to have been no progress on this question since Lee–Szczarba.

Our next theorem extends the theorem of Lee–Szczarba from Euclidean number rings to many number rings with class number 1 (we remark that even in the Euclidean case our proof is different from that of Lee–Szczarba). In addition to untwisted coefficients $\mathbb{Q}$, it also gives a result for twisted coefficient systems arising from rational representations of the algebraic group $\GL_n$. Let $\nu'_n := \text{vcd} (\SL_n \mathcal{O}_K)$; an explicit formula for $\nu'_n$ is given in [4] below. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$, let $V_\lambda$ be the rational $\GL_n \mathcal{O}_K$-representation with highest weight $\lambda$, and define $\|\lambda\| = \sum_{i=1}^n (\lambda_i - \lambda_n)$.

**Theorem C (Vanishing Theorem).** Let $\mathcal{O}_K$ be the ring of integers in an algebraic number field $K$ with $|\text{cl}(\mathcal{O}_K)| = 1$. Suppose that $K$ has a real embedding or that $\mathcal{O}_K$ is Euclidean. Then

$$H^{n'}(\SL_n \mathcal{O}_K; V_\lambda) = H^n(\GL_n \mathcal{O}_K; V_\lambda) = 0$$

for $n \geq 2 + \|\lambda\|$. In particular, $H^{n'}(\SL_n \mathcal{O}_K; \mathbb{Q}) = H^n(\GL_n \mathcal{O}_K; \mathbb{Q}) = 0$ for $n \geq 2$.

For twisted coefficients, Theorem C seems to be new even in the case when $\mathcal{O}_K$ is Euclidean. As we discuss in Remark [4] below, the assumption that $K$ has a real embedding is necessary if $\mathcal{O}_K$ is not Euclidean.

In contrast to Theorem C, a different phenomenon occurs when $|\text{cl}(\mathcal{O}_K)| \neq 1$.

**Theorem D (Non-Vanishing Theorem).** Let $\mathcal{O}_K$ be the ring of integers in an algebraic number field $K$. Then for $n \geq 2$,

$$\dim H^{n'}(\SL_n \mathcal{O}_K; \mathbb{Q}) \geq \dim H^n(\GL_n \mathcal{O}_K; \mathbb{Q}) \geq (|\text{cl}(\mathcal{O}_K)| - 1)^{n-1}.$$ 

Theorem D is classical for $n = 2$; see Example 5.6. For $n \geq 3$ the result and method of proof are new. Unlike Theorem C there is no version of Theorem D for twisted coefficients. For example, if $K$ is any field of characteristic 0 then $H^i(\GL_n K; K^n) = 0$ for all $i \geq 0$. 


Remark 1.5. The cohomology of arithmetic groups such as $\text{SL}_n \mathcal{O}_K$ has been fruitfully studied via the theory of automorphic forms, see e.g. [FL, FS]. However, perhaps due to the power of these results, it can be difficult to extract concrete statements from them. For example, it is not clear how to see from [FL] even Borel–Serre’s result that the rational cohomology of $\text{SL}_n \mathcal{O}_K$ vanishes above its vcd, let alone the sharper vanishing of Theorem C.

Franke [F2] has also given a more practicable description of the Hecke-trivial subspace of $H^*(\text{SL}_n \mathcal{O}_K; \mathbb{Q})$. For example, for a quadratic imaginary field $K = \mathbb{Q}(\sqrt{-d})$, it can be deduced from [F2] Theorem 9 that $H^i(\text{SL}_n \mathcal{O}_K; \mathbb{Q})$ contains no Hecke-trivial cohomology for $i > \nu_n' - n + 1$. But as Theorem D shows, this cannot capture all the cohomology; indeed, the Hecke action on the classes we construct in Theorem D will be twisted by nontrivial ideal class characters.

1.4 Relationship between Integrality and Vanishing/Non-Vanishing

Theorems C (Vanishing) and D (Non-Vanishing) relate to the Integrality Theorem (Theorem A) as follows. While $\text{GL}_n \mathcal{O}_K$ does not satisfy Poincaré duality, Borel–Serre [BS] proved that it does satisfy Bieri–Eckmann duality with rational dualizing module $\text{St}_n(K)$. This means that for any $\mathbb{Q}\text{GL}_n \mathcal{O}_K$-module $V$ and any $i \geq 0$ we have

$$H^{\nu_n-i}(\text{GL}_n \mathcal{O}_K; V) \cong H_i(\text{GL}_n \mathcal{O}_K; \text{St}_n(K) \otimes V).$$

This implies in particular that

$$H^{\nu_n}(\text{GL}_n \mathcal{O}_K; \mathbb{Q}) \cong H_0(\text{GL}_n \mathcal{O}_K; \text{St}_n(K) \otimes \mathbb{Q}) \cong (\text{St}_n(K) \otimes \mathbb{Q})_{\text{GL}_n \mathcal{O}_K}. \quad (1.1)$$

Equation (1.1) converts the problem of computing $H^{\nu_n}(\text{GL}_n \mathcal{O}_K; \mathbb{Q})$ to the problem of understanding the $\text{GL}_n \mathcal{O}_K$-action on $\text{St}_n(K)$. This understanding is precisely what is given by the Integrality Theorem, which is applied directly to prove Theorem C.

As for Theorem D, the projection $\mathcal{T}_n(K) \to \mathcal{T}_n(K)/\text{GL}_n \mathcal{O}_K$ induces a map

$$\text{St}_n(K) \to \tilde{\mathcal{H}}_{n-2}(\mathcal{T}_n(K)/\text{GL}_n \mathcal{O}_K; \mathbb{Z}) \cong \mathbb{Z}^N \quad \text{where } N = (\text{cl}(\mathcal{O}_K) - 1)^{n-1}.$$ 

By Equation (1.1), this map rationally factors through $H^{\nu_n}(\text{GL}_n \mathcal{O}_K; \mathbb{Q})$. It turns out that $\mathcal{T}_n(K)/\text{GL}_n \mathcal{O}_K$ is also a spherical building (see Proposition 5.4 below). The hard part in proving Theorem D is showing that this map is surjective. This is difficult because the apartments of $\mathcal{T}_n(K)$ that generate $\text{St}_n(K)$ are made up of $n!$ simplices, while the apartments of $\mathcal{T}_n(K)/\text{GL}_n \mathcal{O}_K$ have only $2^{n-1}$ simplices. Since $n! \gg 2^{n-1}$, most apartments of $\mathcal{T}_n(K)$ will be draped over a huge number of apartments in the quotient, and it is difficult to describe the resulting homology class. However, we show that by making careful choices, each apartment of $\mathcal{T}_n(K)/\text{GL}_n \mathcal{O}_K$ can be lifted to a certain special apartment of $\mathcal{T}_n(K)$ for which all of the remaining $n! - 2^{n-1}$ simplices exactly cancel each other out, leaving the desired homology class. This requires a delicate combinatorial argument.

1.5 The complex $\mathcal{B}_n(\mathcal{O})$ of partial bases

The first step in our proof of Theorem A is to consider a certain “integral model” $\mathcal{B}_n(\mathcal{O})$ for the Tits building $\mathcal{T}_n(K)$. The complex of partial bases $\mathcal{B}_n(\mathcal{O})$ is the simplicial complex whose maximal simplices are bases of $\mathcal{O}^n$. Letting $C_{n-1}(\mathcal{B}_n(\mathcal{O}))$ be the $(n-1)$-chains of $\mathcal{B}_n(\mathcal{O})$, we define in [FL] an integral apartment class map

$$\phi: C_{n-1}(\mathcal{B}_n(\mathcal{O})) \to \text{St}_n(K).$$
Then the (Theorem A). This result is applied in abbreviated CM of dimension $d_n|_\text{type}$. Assume that Theorem E. But this cannot be so, as the story is more complicated: in Theorem 2.1 below Theorem E uses a stronger condition that can be thought of as "SR". The following theorem is the main technical result of this paper, and is of independent interest. Any Dedekind domain Remark 1.6. Note that, unlike our other results, in Theorem E we make no assumption about the class group $\text{cl}(\mathcal{O}_S)$. Theorem A is the assertion that $\mathcal{B}_n(\mathcal{O})$ is CM of dimension $n - 1$ for all $n \geq 1$.

Note that, unlike our other results, in Theorem E we make no assumption about the class group $\text{cl}(\mathcal{O}_S)$. Theorem E applies to any number ring $\mathcal{O}_S$ possessing a real embedding $\mathcal{O}_K \hookrightarrow \mathbb{R}$ (with the exception of $\mathbb{Z}$), as well as to every $\mathcal{O}_S$ with $|\mathcal{O}_S^\times| = \infty$ in which some prime $p$ is invertible. Maazen [Ma] proved earlier that for any Euclidean domain $\mathcal{O}$, the complex of partial bases $\mathcal{B}_n(\mathcal{O})$ is CM of dimension $n - 1$.

The proof of Theorem E is given in §2. It is a complicated inductive argument on the rank $n$. The base case $n = 2$ is where the specific arithmetic hypotheses on $\mathcal{O}_S$ are used. To make the induction work, we need to prove that a more general class of "complexes of I-bases" relative to an ideal $I \subset \mathcal{O}$ are CM.

Remark 1.6. Any Dedekind domain $\mathcal{O}$ satisfies Bass’s stable range condition $SR_3$, so by Van der Kallen [vdK] the complex $\mathcal{B}_n(\mathcal{O})$ is known to be $(n - 3)$-connected. Our proof of Theorem E uses a stronger condition that can be thought of as "$SR_{2,5}$" which was proved for Dedekind domains by Reiner. One might hope that an argument mimicking [vdK] would give Theorem E. But this cannot be so, as the story is more complicated: in Theorem 2.1 below we show that Theorem E is not true for all Dedekind domains, nor even those of arithmetic type. New ideas are needed.

Outline of paper. In §2 we prove that $\mathcal{B}_n(\mathcal{O})$ is CM for appropriate $\mathcal{O}$ (Theorem E). We apply this result in §3 to the integral apartment class map to prove the Integrality Theorem (Theorem A). This result is applied in §4 to prove the Vanishing Theorem (Theorem C) for the top degree cohomology of $\text{SL}_n \mathcal{O}_K$ and $\text{GL}_n \mathcal{O}_K$. The Non-Vanishing Theorem (Theorem D) and the Non-Integrality Theorem (Theorem B) are proved in §5.

2 $\mathcal{B}_n(\mathcal{O}_S)$ is CM of dimension $n - 1$

In this section we prove Theorem E. We also prove the following contrasting result.

Theorem 2.1. Let $\mathcal{O}_S$ belong to one of the families of Dedekind domains of arithmetic type:

1. $K = \mathbb{Q}(\sqrt{d})$ and $\mathcal{O}_S = \mathcal{O}_K$ for $d < 0$ squarefree with $d \notin \{-1, -2, -3, -7, -11\}$.
2. $[K : \mathbb{F}_q(T)] < \infty$ with $q$ odd, $S = \{f\}$, and either $|\text{cl}(\mathcal{O}_S)| \neq 1$ or $\deg f \geq 2$.

Then the $(n - 1)$-dimensional complex $\mathcal{B}_n(\mathcal{O}_S)$ is not Cohen–Macaulay for any $n \geq 2$.

In particular, Theorem 2.1 implies that $\mathcal{B}_2(\mathcal{O}_S)$ is not connected, so Van der Kallen’s result is sharp. For example, neither $\mathcal{B}_2(\mathbb{Z}[\sqrt{-5}])$ nor $\mathcal{B}_2(\mathbb{Z}[\frac{1 + \sqrt{-43}}{2}])$ is connected, even
though both are Dedekind domains and \( \mathbb{Z} \left[ \frac{1+\sqrt{-43}}{2} \right] \) is even a PID. Similarly, for \( K = \mathbb{F}_3(T) \) and \( f = \deg_{T^2+1} \) we have \( O_f \cong \mathbb{F}_3[X, \sqrt{X-X^2}] \), so \( B_2(\mathbb{F}_3[X, \sqrt{X-X^2}]) \) is not connected. Theorem 2.1 is proved in §2.3.

Outline. We begin with a short section of preliminaries, summarizing basic properties of Dedekind domains and Cohen–Macaulay complexes. We prove Theorem 2.1 by induction, but to make such an approach work we need a more general family of simplicial complexes; we introduce these in §2.1. The lengthy proof of Theorem 2.1 is outlined at the end of §2.2. The body of the proof then occupies §§2.3–2.6, and §2.7.

2.1 Dedekind domains and Cohen–Macaulay complexes

2.1.1 Dedekind domains

Let \( O \) be a Dedekind domain with field of fractions \( K \). We recall some basic facts about \( O \); for proofs see e.g. [Mi].

Projective modules. Every submodule of a projective \( O \)-module is projective. If \( M \) is a finitely generated projective \( O \)-module, the rank of \( M \) is defined by \( \text{rk}(M) := \dim_K(M \otimes_O K) \). The following standard lemma summarizes some properties of projective \( O \)-modules.

Lemma 2.2. Let \( O \) be a Dedekind domain with field of fractions \( K \) and let \( M \) be a finitely generated projective \( O \)-module of rank \( n \geq 1 \).

(a) A submodule \( U \subseteq M \) is a direct summand of \( M \) if and only if \( M/U \) is torsion-free.

(b) If \( U \) and \( U' \) are summands of \( M \) and \( U \subseteq U' \), then \( U \) is a summand of \( U' \).

(c) The assignment \( U \mapsto U \otimes_O K \) defines a bijection

\[ \{ \text{Direct summands of } M \} \longleftrightarrow \{ K \text{-subspaces of } M \otimes_O K \} \]

with inverse \( V \mapsto V \cap M \).

The class group. The class group \( \text{cl}(O) \) is the set of isomorphism classes of rank 1 projective \( O \)-modules. The tensor product endows \( \text{cl}(O) \) with the structure of an abelian group, which we write additively; the identity element is given by the free module \( O \). For every rank \( n \) projective \( O \)-module \( M \) there is a unique rank 1 projective \( O \)-module \( I \in \text{cl}(O) \) such that \( M \cong O^{n-1} \oplus I \). We write \( [M] = I \in \text{cl}(O) \) for this element. For two finite rank projective \( O \)-modules \( M \) and \( M' \), we have the identity \( [M \oplus M'] = [M] + [M'] \). Two such modules \( M \) and \( M' \) are isomorphic if and only if \( \text{rk}(M) = \text{rk}(M') \) and \( [M] = [M'] \).

The complex of partial bases. A partial basis for \( O^n \) is a set \( \{v_1, \ldots, v_k\} \) of elements of \( O^n \) that can be completed to a basis \( \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\} \) for \( O^n \). As we mentioned in the introduction, the complex of partial bases \( B_n(O) \) is the simplicial complex whose \((k-1)\)-simplices are partial bases \( \{v_1, \ldots, v_k\} \) for \( O^n \). Steinitz proved in 1911 that when \( O \) is a Dedekind domain, a vector \( v = (a_1, \ldots, a_n) \in O^n \) lies in a basis for \( O^n \) if and only if \( v \) is unimodular, that is, the coordinates \( \{a_1, \ldots, a_n\} \) generate the unit ideal of \( O \); see [Re]. The vertices of \( B_n(O) \) are therefore the unimodular vectors in \( O^n \).

Lemma 2.3. Let \( O \) be a Dedekind domain.

(a) A subset \( \{v_1, \ldots, v_k\} \subseteq O^n \) is a partial basis of \( O^n \) if and only if \( \text{span}(v_1, \ldots, v_k) \) is a rank \( k \) direct summand of \( O^n \).
(b) Let $M$ be a free summand of $\mathcal{O}^n$. A subset $\{v_1, \ldots, v_k\} \subset M$ is a partial basis of $M$ if and only if $\{v_1, \ldots, v_k\}$ is a partial basis of $\mathcal{O}^n$.

Proof. Set $U = \text{span}(v_1, \ldots, v_k)$. If $\{v_1, \ldots, v_k\}$ is a partial basis for $\mathcal{O}^n$ then $U$ is a rank $k$ direct summand of $\mathcal{O}^n$. We must prove the converse, so assume that $U$ is a rank $k$ direct summand of $\mathcal{O}^n$. Since $U$ is a rank $k$ projective module generated by $k$ elements, it must be free. Write $\mathcal{O}^n = U \oplus W$. Since $[W] = [\mathcal{O}^n] - [U] = 0 - 0 = 0$, we have $W \cong \mathcal{O}^{n-k}$. Adjoining a basis for $W$ then yields a basis for $\mathcal{O}^n$, so $\{v_1, \ldots, v_k\}$ is a partial basis and (a) follows. Part (b) follows from the characterization in part (a) together with Lemma 2.2(b). □

2.1.2 Cohen–Macaulay complexes

A space is $(-1)$-connected if and only if it is nonempty; every space is $\ell$-connected for $\ell \leq -2$. A simplicial complex $X$ is $d$-dimensional if it contains a $d$-simplex but no $(d + 1)$-simplices. It is $d$-spherical if it is $d$-dimensional and $(d - 1)$-connected. Finally, $X$ is Cohen–Macaulay (CM) of dimension $d$ if $X$ is $d$-spherical and for every $k$-simplex $\sigma^k$ of $X$, the link $\text{Link}_X(\sigma^k)$ is $(d - k - 1)$-spherical.

Low dimensions. A simplicial complex $X$ is CM of dimension 0 if $X$ is 0-dimensional. $X$ is CM of dimension 1 if $X$ is 1-dimensional and connected.

Links are CM. If $X$ is CM of dimension $d$, then $\text{Link}_X(\sigma^k)$ is CM of dimension $d - k - 1$ for all $\sigma^k \in X$ (where the definition requires only $(d - k - 1)$-spherical); see [Q, Proposition 8.6].

A sufficient condition to be CM. If a $d$-dimensional simplicial complex $X$ has the property that every simplex is contained in a $d$-simplex, then $\text{Link}_X(\sigma^k)$ is $(d - k - 1)$-dimensional for all $\sigma^k \in X$. Therefore, to prove that $X$ is CM of dimension $d$, it suffices to show that $X$ is $(d - 1)$-connected, and that for every $k$-simplex $\sigma^k \in X$ the link $\text{Link}_X(\sigma^k)$ is $(d - k - 2)$-connected.

2.2 The complex $B_n(I)$ of partial $I$-bases

One of the key insights in our proof of Theorem 2.2 is that to induct on $n$, one must strengthen the inductive hypothesis by working with a more general family of complexes.

Definition 2.4 (Partial $I$-basis). Fix a surjection $L: \mathcal{O}^n \rightarrow \mathcal{O}$. Given an ideal $I \subset \mathcal{O}$, a basis $\{v_1, \ldots, v_n\}$ for $\mathcal{O}^n$ is called an $I$-basis if

$$L(v_i) \equiv 0 \mod I \text{ or } L(v_i) \equiv 1 \mod I \quad \text{for all } 1 \leq i \leq n.$$ 

A partial basis is called a partial $I$-basis if it can be completed to an $I$-basis. The complex of partial $I$-bases, denoted $B_n(I)$, is the complex whose $(k - 1)$-simplices are partial $I$-bases $\{v_1, \ldots, v_k\}$.

Remark 2.5. The definition of $B_n(I)$ depends on the choice of $L$. However, if $L': \mathcal{O}^n \rightarrow \mathcal{O}$ is a different surjection, then both $\ker L$ and $\ker L'$ are projective summands with $\text{rk}(\ker L) = \text{rk}(\ker L') = n - 1$ and $[\ker L] = [\ker L'] = [\mathcal{O}^n] - [\mathcal{O}] = 0 \in \text{cl}(\mathcal{O})$. Therefore, there exists an isomorphism $\Psi: \mathcal{O}^n \rightarrow \mathcal{O}^n$ such that $L = L' \circ \Psi$. The resulting isomorphism $B_n(\mathcal{O}) \cong B_n(\mathcal{O})$ induces an isomorphism between the complexes $B_n(I)$ defined with respect to $L$ and $L'$. Because of this, we will not concern ourselves with the choice of $L$ except when it is necessary to clarify our proofs.
We record some observations regarding $I$-bases and partial $I$-bases.

(a) When $I = \mathcal{O}$, the condition of Definition 2.4 is vacuous, so $\mathcal{B}_n(I) = \mathcal{B}_n(\mathcal{O})$.

(b) If $I \subset J$ then any $I$-basis is a $J$-basis by definition. Therefore any partial $I$-basis is a partial $J$-basis, so $\mathcal{B}_n(I)$ is a subcomplex of $\mathcal{B}_n(J)$. In particular, $\mathcal{B}_n(I)$ is a subcomplex of $\mathcal{B}_n(\mathcal{O})$.

(c) The complex of partial $I$-bases $\mathcal{B}_n(I)$ is not necessarily a full subcomplex of $\mathcal{B}_n(\mathcal{O})$. In other words, a partial basis $\{v_1, \ldots, v_k\}$ may not be a partial $I$-basis even if $L(v_i)$ is congruent to 0 or 1 modulo $I$ for all $1 \leq i \leq k$. For example, let $\mathcal{O} = \mathbb{Z}$ and $I = 5\mathbb{Z}$, and take $L: \mathbb{Z}^2 \to \mathbb{Z}$ to be $L(a, b) = b$. The vector $v := (2, 5) \in \mathbb{Z}^2$ determines a partial basis $\{v\}$ since $\{(2, 5), (1, 3)\}$ is a basis for $\mathbb{Z}^2$. However, although $L(v) = 5 \equiv 0 \mod I$, the partial basis $\{v\}$ is not a partial $I$-basis. Indeed, any basis $\{(2, 5), (a, b)\}$ must have $2b - 5a = \pm 1$, which implies that $b \equiv 2$ or $3 \mod 5\mathbb{Z}$. Therefore $v$ is not contained in any $I$-basis for $\mathbb{Z}^2$.

(d) We allow the case $I = (0)$, in which case we simplify notation by writing $\mathcal{B}_n(0)$ for $\mathcal{B}_n(I)$. The subcomplex $\mathcal{B}_n(0)$ is contained in $\mathcal{B}_n(I)$ for any ideal $I$ since $(0) \subset I$. We will see in the proof of Proposition 2.14 below that $\mathcal{B}_n(0)$ is the full subcomplex of $\mathcal{B}_n(\mathcal{O})$ on the unimodular vectors $v \in \mathcal{O}^n$ which satisfy $L(v) = 0$ or $L(v) = 1$.

Our main result concerning the complexes $\mathcal{B}_n(I)$ is as follows.

**Theorem E’.** Let $\mathcal{O}_S$ be a Dedekind domain of arithmetic type. Assume that $|S| > 1$ and $S$ contains a non-imaginary place. Then $\mathcal{B}_n(I)$ is CM of dimension $n - 1$ for any ideal $I \subset \mathcal{O}_S$.

Theorem E is the special case $I = \mathcal{O}_S$ of Theorem E’. See Remark 2.21 for a discussion of why we are forced to introduce $\mathcal{B}_n(I)$.

**Outline of the proof of Theorem E’** The proof of Theorem E’ is by induction on $n$. In 2.3 we establish the base cases $n \leq 2$. In 2.4 we define a subcomplex $\mathcal{B}_n^{gd}(I) \subset \mathcal{B}_n(I)$ consisting of “good simplices” and prove that $(\mathcal{B}_n(I), \mathcal{B}_n^{gd}(I))$ is $(n - 2)$-connected. In 2.5 we prove that $(\mathcal{B}_n^{gd}(I), \mathcal{B}_n(0))$ is $(n - 2)$-connected. In 2.6 we prove that $\mathcal{B}_n(0)$ is $(n - 2)$-connected. In 2.7 we prove the connectivity of links, and assemble these pieces to prove Theorem E’.

### 2.3 Step 1: Base case, $n \leq 2$

Our inductive proof of Theorem E’ has two base cases. The first is easy.

**Proposition 2.6.** For any Dedekind domain $\mathcal{O}$ and any ideal $I \subset \mathcal{O}$, the complex $\mathcal{B}_1(I)$ is CM of dimension 0.

**Proof.** A surjection $L: \mathcal{O}^1 \to \mathcal{O}$ must be an isomorphism, so there exists a unique $v \in \mathcal{O}^1$ with $L(v) = 1$. Since $\{v\}$ is an $I$-basis of $\mathcal{O}^1$ for any $I$, the complex $\mathcal{B}_1(I)$ contains the 0-simplex $\{v\}$, and thus is 0-dimensional as desired. 

The second base case is more involved. It is the key place where the arithmetic hypotheses on $\mathcal{O}_S$ enter into the proof of Theorem E’.

**Proposition 2.7.** Let $\mathcal{O} = \mathcal{O}_S$ be a Dedekind domain of arithmetic type. Assume that $|S| > 1$ and that $S$ does not consist solely of imaginary places. Then for any ideal $I \subset \mathcal{O}$, the complex $\mathcal{B}_2(I)$ is CM of dimension 1.
Proof. The complex $B_2(I)$ is 1-dimensional, so it is enough to show that $B_2(I)$ is connected. Since $L$ is surjective we can fix a basis $\{e_1, e_2\}$ for $O^2$ such that $L(e_1) = 0$ and $L(e_2) = 1$. This is an $I$-basis, so it defines an edge $e = \{e_1, e_2\}$ of $B_2(I)$.

Let $\Gamma_1(O, I)$ be the group

$$\Gamma_1(O, I) := \{ (a b) \in SL_2 O \mid (a b) \equiv (0 1) \mod I \}.$$ 

Letting $SL_2 O$ act on $O^2$ via the basis $\{e_1, e_2\}$, the subgroup $\Gamma_1(O, I)$ consists of the automorphisms preserving the surjection $L$. In particular, the $\Gamma_1(O, I)$-orbit of $e$ consists of $I$-bases and thus is contained in $B_2(I)$. Let $Z \subset B_2(I)$ be the subgraph determined by this orbit.

If $I \neq O$ then the action of $\Gamma_1(O, I)$ on $Z$ is without inversions. Indeed, in this case vertices with $L(v) \equiv 0 \mod I$ cannot be exchanged with those with $L(v) \equiv 1 \mod I$. The edge $e$ is a fundamental domain for the action of $\Gamma_1(O, I)$ on $Z$. The $\Gamma_1(O, I)$-stabilizers of $e_1$ and $e_2$ are exactly the elementary matrices that lie in $\Gamma_1(O, I)$, so by a standard argument in geometric group theory (see, e.g., [Sc, Lemma I.4.1.2]) the complex $Z$ is connected if and only if the group $\Gamma_1(O, I)$ is generated by elementary matrices. This argument does not apply verbatim when $I = O$, but it is easy to verify nevertheless that $Z$ is connected if and only if $SL_2 O$ is generated by elementary matrices.

Vaserstein (see the Main Theorem of [Va], and Liehl [L] for a correction) proved that if $O = O_S$ is a Dedekind domain of arithmetic type with $|S| > 1$, then for any ideal $I \subset O$ the subgroup $E$ of $\Gamma_1(O, I)$ generated by elementary matrices is normal, and the quotient $\Gamma_1(O, I)/E$ is isomorphic to the relative $K$-group $SK_1(O, I)$. We remark that Vaserstein’s result does not apply to $I = (0)$, but in this case $\Gamma_1(O, I)$ consists solely of elementary matrices. In their resolution of the Congruence Subgroup Problem, Bass–Milnor–Serre [BMS] Theorem 3.6] proved that if $O$ is a Dedekind domain of arithmetic type that is not totally imaginary, then $SK_1(O, I) = 0$ for all ideals $I \subset O$. Our assumptions guarantee that both of these results apply, so $\Gamma_1(O, I)$ is generated by elementary matrices and $Z$ is connected.

It is thus enough to show that the pair $(B_2(I), Z)$ is 0-connected, i.e. that every vertex $v \in B_2(I)$ can be connected by a path to a vertex in $Z$. By definition, any vertex $v \in B_2(I)$ is contained in an $I$-basis $\{v, w\}$ for $O^2$. We first show that if $L(v) \equiv 1 \mod I$ then $v$ lies in $Z$.

By possibly replacing $w$ by $w - v$, we can assume that $L(w) \equiv 0 \mod I$. There exists a unique $g \in GL_2 O$ such that $g \cdot e_1 = w$ and $g \cdot e_2 = v$. Let $u = (\det g)^{-1}w$. The element $\gamma$ satisfying $\gamma \cdot e_1 = u$ and $\gamma \cdot e_2 = v$ lies in $SL_2 O$. Since $L(u) \equiv 0 \mod I$ and $L(v) \equiv 1 \mod I$, we see that $\gamma \in \Gamma_1(O, I)$. Therefore $v$ lies in the $\Gamma_1(O, I)$-orbit of $e_2$ and $v \in Z$.

If instead $L(v) \not\equiv 1 \mod I$, then since $\{v, w\}$ is an $I$-basis we must have $L(w) \equiv 1 \mod I$, so $w$ lies in $Z$. We conclude that every vertex of $B_2(I)$ is connected to $Z$ by a path of length at most 1, as desired.

We now pause to prove Theorem 2.1.

Proof of Theorem 2.1. We first prove the theorem in the case $n = 2$. We can break the hypotheses of Theorem 2.1 into the following three cases, and Cohn proved in each that $GL_2 O_S$ is not generated by elementary matrices together with diagonal matrices: for the quadratic imaginary case see [C] Theorem 6.1], for the function field case with $|cl(O_S)| \neq 1$ see [C], Corollary 5.6], and for the function field case with $\deg f \geq 2$ see [C], Theorem 6.2]. Thus certainly $SL_2 O_S$ cannot be generated by elementary matrices. We saw in the proof of Proposition 2.7 that the 1-dimensional complex $B_2(O_S)$ is connected if and only if $SL_2 O_S$ is generated by elementary matrices. Therefore $B_2(O_S)$ is disconnected and not Cohen–Macaulay.

Now assume that $n \geq 3$. Choose a basis $\{e_1, \ldots, e_n\}$ for $O^n$, let $\sigma^{n-3} = \{e_1, \ldots, e_{n-2}\}$ and $W = \langle e_{n-1}, e_n \rangle$. Let $B_W(O_S)$ denote the complex of partial bases of $W$; since $W \cong O^n_S$ there is an isomorphism $B_W(O_S) \cong B_2(O_S)$. By the previous paragraph, $B_W(O_S)$ is disconnected.
Consider the subcomplex \( L := \text{Link}_{\mathcal{B}_n(\mathcal{O}_S)}(\sigma^{n-3}) \). If \( \mathcal{B}_n(\mathcal{O}_S) \) were CM of dimension \( n - 1 \) then \( L \) would be \((n - 1) - (n - 3) - 1 = 1\)-spherical and therefore connected. Since \( \{w_1, w_2\} \) is a basis of \( W \) if and only if \( \{e_1, \ldots, e_{n-2}, w_1, w_2\} \) is a basis of \( \mathcal{O}^n \), there is a natural inclusion \( i : \mathcal{B}_W(\mathcal{O}_S) \hookrightarrow L \). Let \( \pi : \mathcal{O}^n \to W \) be the projection with kernel \( \langle e_1, \ldots, e_{n-2}\rangle \). Since \( \{e_1, \ldots, e_{n-2}, v_1, v_2\} \) is a basis of \( \mathcal{O}^n \) if and only if \( \{\pi(v_1), \pi(v_2)\} \) is a basis of \( W \), the projection \( \pi \) induces a simplicial map \( \pi : L \to \mathcal{B}_W(\mathcal{O}_S) \). By definition \( i : \mathcal{B}_W(\mathcal{O}_S) \hookrightarrow L \) is a section of this projection, so \( \pi : L \to \mathcal{B}_W(\mathcal{O}_S) \) is surjective. Since \( \mathcal{B}_W(\mathcal{O}_S) \) is disconnected, this shows that \( L = \text{Link}_{\mathcal{B}_n(\mathcal{O}_S)}(\sigma^{n-3}) \) is disconnected, so the \((n - 1)\)-dimensional complex \( \mathcal{B}_n(\mathcal{O}_S) \) is not Cohen–Macaulay.

**Remark 2.8.** Other examples can be deduced from Geller \[ G \]; for example \( \mathcal{B}_n(\mathcal{O}_f) \) is not Cohen–Macaulay for \( \mathcal{O}_f \cong \mathbb{F}_2[X, Y]/(X^2 + XY + Y^2 + X) \), which occurs as \( \mathcal{O}_f \subset \mathbb{F}_2(T) \) for \( f = T^2 + T + 1 \). In contrast, Vasertsein \[ V \] proved that when \( |S| \geq 2 \) the group \( \text{SL}_2 \mathcal{O}_S \) is generated by elementary matrices, so in that case \( \mathcal{B}_2(\mathcal{O}_S) \) is indeed connected.

### 2.4 Step 2: \((\mathcal{B}_n(I), \mathcal{B}_n^{gd}(I))\) is \((n - 2)\)-connected

We begin by defining certain bases to be “good bases”.

**Definition 2.9.** We say that a basis \( \{v_1, \ldots, v_n\} \) for \( \mathcal{O}^n \) is good if \( L(v_i) = 1 \) for at least one \( 1 \leq i \leq n \). We say that \( \{v_1, \ldots, v_k\} \) is a partial good I-basis if it is contained in a good I-basis. We denote by \( \mathcal{B}_n^{gd}(I) \) the complex of partial good I-bases.

The following proposition is the main result of this section.

**Proposition 2.10.** Let \( \mathcal{O} \) be a Dedekind domain and let \( I \subset \mathcal{O} \) an ideal. Assume that \( \mathcal{B}_2(I) \) is CM of dimension 1. Then for any \( n \geq 3 \) the pair \((\mathcal{B}_n(I), \mathcal{B}_n^{gd}(I))\) is \((n - 2)\)-connected.

#### 2.4.1 The link of an \((n - 2)\)-simplex

We need a few preliminary results before starting the proof of Proposition 2.10. The first ingredient in the proof of Proposition 2.10 is the following lemma.

**Definition 2.11.** Given a simplex \( \sigma = \{v_1, \ldots, v_k\} \) of \( \mathcal{B}_n(\mathcal{O}) \), define \( V_{\sigma} \) to be the direct summand of \( \mathcal{O}^n \) with basis \( \{v_1, \ldots, v_k\} \), and define \( I_{\sigma} \) to be the ideal \( L(V_{\sigma}) \subset \mathcal{O} \).

**Lemma 2.12.** Let \( n \geq 3 \). Consider an \((n - 2)\)-simplex \( \sigma^{n-2} \in \mathcal{B}_n(I) \), and let \( v \) be a vertex of \( \sigma \) with \( L(v) \neq 0 \). If \( I_{\sigma} \subset I \), assume additionally that \( L(v) \equiv 1 \mod I \). Then there exists a vertex \( w \) of \( \text{Link}_{\mathcal{B}_n(I)}(\sigma) \) such that \( I_{\{v, w\}} = \mathcal{O} \).

**Proof.** Since \( \sigma^{n-2} \in \mathcal{B}_n(I) \) is a partial I-basis, there exists some \( z \in \mathcal{O}^n \) such that \( \sigma \cup \{z\} \) is an I-basis. Let \( b_1 = L(v) \) and \( b_2 = L(z) \); by hypothesis \( b_1 \neq 0 \). Write \( \sigma^{n-2} = \{v, v_3, \ldots, v_n\} \) and let \( b_i = L(v_i) \) for \( 3 \leq i \leq n \). Since \( \sigma \cup \{z\} \) is a basis for \( \mathcal{O}^n \), the set \( \{b_1, \ldots, b_n\} \subset \mathcal{O} \) generates the unit ideal. Since \( n \geq 3 \) and \( b_1 \neq 0 \) and \( \{b_1, \ldots, b_n\} \) generates the unit ideal, we can apply a theorem of Reiner \[ R \] to conclude that there exists \( c_3, \ldots, c_n \in \mathcal{O} \) such that the two elements \( b_1 \) and \( \xi := b_2 + c_3b_3 + c_4b_4 + \cdots + c_nb_n \) generate the unit ideal (this theorem of Reiner is a strengthening for Dedekind domains of Bass’s stable range condition SR\(_3\)). Set \( y = z + c_3v_3 + \cdots + c_nv_n \). The ideal \( I_{\{v, y\}} \) is generated by \( L(v) = b_1 \) and \( L(y) = \xi \), so by construction \( I_{\{v, y\}} = \mathcal{O} \). Since \( \sigma \cup \{z\} \) is a basis for \( \mathcal{O}^n \), so is \( \sigma \cup \{y\} \).

If \( I_{\sigma} \subset I \) then \( L(v_3) \equiv \cdots \equiv L(v_n) \equiv 0 \mod I \), so \( L(y) \equiv L(z) \mod I \). Since \( z \in \mathcal{B}_n(I) \), we know that \( L(z) \) is congruent to 0 or 1 modulo \( I \). Therefore when \( I_{\sigma} \subset I \), the basis \( \sigma \cup \{y\} \) is an I-basis. We conclude that \( w := y \in \text{Link}_{\mathcal{B}_n(I)}(\sigma) \) satisfies the conditions of the lemma.
When \( I_\sigma \not\subset I \), the basis \( \sigma \cup \{y\} \) may not be an \( I \)-basis since \( L(y) = \xi \) may not be congruent to 0 or 1 modulo \( I \). In this case, by hypothesis \( L(v) \equiv 1 \mod I \), so the vector \( w := y - \xi v \) satisfies
\[
L(w) = L(y) - \xi \cdot L(v) \equiv \xi - \xi = 0 \mod I.
\]
Therefore the basis \( \sigma \cup \{w\} \) is an \( I \)-basis, so \( w \in \text{Link}_{B_n(I)}(\sigma) \). Since
\[
I_{\{v, w\}} = (L(v), L(w)) = (L(v), L(y) - \xi \cdot L(v)) = (L(v), L(y)) = (b_1, \xi) = O,
\]
the vector \( w \) satisfies the conditions of the lemma.

\[ \square \]

### 2.4.2 Characterizing good partial bases

The second ingredient in the proof of Proposition 2.13 is the following characterization of good partial bases.

**Proposition 2.13.** Let \( \sigma^k = \{v_1, \ldots, v_{k+1}\} \) be a \( k \)-simplex of \( B_n(O) \).

1. Assume that \( k < n - 1 \) and that \( L(v_i) \equiv 0 \) or 1 \( \mod I \) for \( i = 1, \ldots, k + 1 \). Then the following are equivalent.
   
   \begin{enumerate}
   
   
   \item \( \{v_1, \ldots, v_{k+1}\} \) is contained in a good basis.
   
   \item \( \{v_1, \ldots, v_{k+1}\} \) is contained in a good \( I \)-basis \( \{v_1, \ldots, v_n\} \) s.t. \( L(v_i) = 1 \) for \( i \geq k+2 \).

   \item There exists a complement \( W \) such that \( O^n = V_\sigma \oplus W \) and \( L(W) = O \).

   \end{enumerate}

2. If \( I_\sigma = (0) \) or \( I_\sigma = O \) or \( k < n - 2 \), then \( \sigma \) is contained in a good basis.

Before proving Proposition 2.13, we introduce some terminology. We say that \( \sigma \) is a *good simplex* if the three conditions of Proposition 2.13(1) are satisfied, and call the subspace \( W \) of condition (iii) a *good complement* for \( \sigma \). In this case we denote by \( B_W(I) \) the complex of partial \( I \)-bases for \( W \), whose simplices are subsets of \( W \) that can be extended to an \( I \)-basis for \( W \) (with respect to \( L \)). Since \( [W] = [O^n] - [V_\sigma] = 0 \), any good complement is a free \( O \)-module, and \( L|_W : W \to O \) is surjective by definition. Thus by Remark 2.5 there exists an isomorphism \( B_W(I) \cong B_\ell(I) \), where \( \ell = \text{rk}(W) \).

**Proof of Proposition 2.13.** To prove the first part of the proposition we will show that

\( (i) \implies (ii) \implies (iii) \implies (i) \).

Assume (i), so \( \sigma = \{v_1, \ldots, v_{k+1}\} \) is contained in a good basis

\[
\{v_1, \ldots, v_{k+1}, u_{k+2}, \ldots, u_n\}.
\]

By definition, some vector \( v \) lying in this good basis satisfies \( L(v) = 1 \). Define \( v_i = u_i + (1 - L(u_i))v \) for \( k + 2 \leq i \leq n \). The set \( \{v_1, \ldots, v_n\} \) is then a basis satisfying \( L(v_i) = 1 \) for \( k + 2 \leq i \leq n \). Since \( k + 1 < n \), this is a good basis, and since \( L(v_i) \equiv 0 \) or 1 \( \mod I \) for \( 1 \leq i \leq k + 1 \) by assumption, it is also an \( I \)-basis. Therefore (i) \( \implies (ii) \).

Assume (ii), and define \( W = \langle v_{k+2}, \ldots, v_n \rangle \). We certainly have \( O^n = V_\sigma \oplus W \), and since \( L(v_n) = 1 \) we have \( L(W) = O \). Therefore (ii) \( \implies (iii) \).

Finally, assume (iii), and choose any \( w \in W \) with \( L(w) = 1 \). We then have a splitting \( W = \langle w \rangle \oplus \text{ker} L|_W \). The submodule \( \text{ker} L|_W \) is free since \( \text{ker} L|_W = [W] - [O] = 0 \), so we can choose a basis \( \{u_{k+3}, \ldots, u_n\} \) for \( \text{ker} L|_W \). Then

\[
\{v_1, \ldots, v_{k+1}, w, u_{k+3}, \ldots, u_n\}
\]

is a good basis containing \( \sigma \). Therefore (iii) \( \implies (i) \).
We now prove the second part of the proposition. By Lemma 2.12, \( V_\sigma \) has a complement \( U \subset \mathcal{O}^n \) such that \( \mathcal{O}^n = V_\sigma \oplus U \). In particular \( \mathcal{O} = L(\mathcal{O}^n) = L(V_\sigma) + L(U) = I_\sigma + L(U) \). If \( I_\sigma = (0) \), this implies that \( L(U) = \mathcal{O} \), so \( U \) is a good complement for \( \sigma \). Otherwise, choose a basis \( \{u_{k+2}, \ldots, u_n\} \) for \( U \). If \( I_\sigma = \mathcal{O} \), there exists \( v \in V_\sigma \) with \( L(v) = 1 \). Then \( W := \langle u_{k+2} + (1 - L(u_{k+2})v, u_{k+3}, \ldots, u_n \rangle \) is a good complement for \( \sigma \).

It remains to handle the case when \((0) \neq I_\sigma \neq \mathcal{O}\) and \( k < n - 2 \), so \( \text{rk}(U) \geq 2 \). Since \( I_\sigma \neq \mathcal{O} \) we cannot have \( L(u_i) = 0 \) for all \( k + 2 \leq i \leq n \), so assume without loss of generality that \( L(u_n) \neq 0 \). Consider the \((n - 2)\)-simplex

\[
\sigma^{n-2} := \{v_1, \ldots, v_{k+1}, u_{k+3}, \ldots, u_n\} \in B_n(\mathcal{O}).
\]

Applying Lemma 2.12 with \( I = \mathcal{O} \) gives a vertex \( w \) in \( \text{Link}_{B_n(\mathcal{O})}(\sigma) \) such that \( I_{\{w, u_n\}} = \mathcal{O} \) and \( \{v_1, \ldots, v_{k+1}, w, u_{k+3}, \ldots, u_n\} \) is a basis for \( \mathcal{O}^n \). It follows that \( W := \langle w, u_{k+3}, \ldots, u_n \rangle \) is a good complement for \( \sigma \).

2.4.3 Completing the proof of Step 2

The pieces are now in place for the proof of Proposition 2.10

Proof of Proposition 2.10. Given a \( k \)-simplex \( \sigma^k \in B_n(I) \) with \( k \leq n - 3 \), Proposition 2.13(2) implies that \( \sigma \) is contained in \( B_n^{gd}(I) \). In other words, \( B_n^{gd}(I) \) contains the \((n - 3)\)-skeleton of \( B_n(I) \). Thus \( (B_n(I), B_n^{gd}(I)) \) is \((n - 3)\)-connected, and every \((n - 2)\)-simplex \( \sigma^{n-2} \in B_n(I) \) has \( \partial \sigma^{n-2} \subset B_n^{gd}(I) \). To prove that \( (B_n(I), B_n^{gd}(I)) \) is \((n - 2)\)-connected, it suffices to show that every \((n - 2)\)-simplex of \( B_n(I) \) that does not lie in \( B_n^{gd}(I) \) is homotopic relative to its boundary into \( B_n^{gd}(I) \).

Let \( \sigma^{n-2} \) be an \((n - 2)\)-simplex of \( B_n(I) \) that does not lie in \( B_n^{gd}(I) \). By Proposition 2.13(2), we must have \((0) \subset I_\sigma \subset \mathcal{O} \). Choose a vertex \( v \) of \( \sigma \) with \( L(v) \neq 0 \); if \( I_\sigma \not\subset I \) then choose \( v \) such that \( L(v) \neq 0 \) mod \( I \). Using Lemma 2.12, choose a vertex \( w \) of \( B_n(I) \) such that \( \sigma^{n-2} \cup \{w\} \in B_n(I) \) and \( I_{\{v, w\}} = \mathcal{O} \). Write \( \sigma^{n-2} = \sigma_0 \cup \{v\} \), let \( V = V_{\sigma_0} \) and let \( W = \langle v, w \rangle \).

The 1-dimensional complex \( B_W(I) \), whose edges consist of \( I \)-bases for \( W \), is contained in \( \text{Link}_{B_n(I)}(\sigma_0) \). As we discussed following Proposition 2.13, identifying \( W \) with \( \mathcal{O}^2 \) induces an isomorphism \( B_W(I) \cong B_2(I) \). We have assumed that \( B_2(I) \) is connected, so \( B_W(I) \) is connected.

Fix a vertex \( y \) of \( B_W(I) \) such that \( L(y) = 1 \), and let \( P \) be an embedded path in \( B_W(I) \) from \( v \) to \( y \). Let \( v = p_1, \ldots, p_m = y \) be an enumeration of the vertices of \( P \). For \( 1 \leq i < m \), the set \( \{p_i, p_{i+1}\} \) is a basis for \( W \), so \( L(p_i), L(p_{i+1}) = L(W) = \mathcal{O} \).

Regarding \( P \) as a triangulated closed interval that is mapped into \( \text{Link}_{B_n(I)}(\sigma_0) \), we obtain an embedding of the join \( \sigma_0 \ast P \) into \( B_n(I) \). Its boundary is

\[
\partial(\sigma_0 \ast P) = (\partial \sigma_0) \ast P \cup (\sigma_0 \ast \partial P).
\]

Since \( \partial P = \{v\} \cup \{y\} \), the second term \( \sigma_0 \ast \partial P \) consists of \( \sigma_0 \ast \{v\} = \sigma \) and \( \sigma_0 \ast \{y\} \). Since \( L(y) = 1 \), \( \sigma_0 \ast \{y\} \) is a partial good \( I \)-basis by Proposition 2.13, so \( \sigma_0 \ast \{y\} \in B_n^{gd}(I) \).

The first term \( \partial \sigma_0 \ast P \) can be written as

\[
\partial \sigma_0 \ast P = \bigcup_i \partial \sigma_0 \ast \{p_i, p_{i+1}\}.
\]

Since \( I_{\{p_i, p_{i+1}\}} = \mathcal{O} \), every \((n - 2)\)-simplex \( \tau \) occurring in the join \( \partial \sigma_0 \ast \{p_i, p_{i+1}\} \) has \( I_\tau = \mathcal{O} \), and thus lies in \( B_n^{gd}(I) \) by Proposition 2.13(2). We conclude that \( \sigma \) is the unique face of \( \sigma_0 \ast P \) that is not contained in \( B_n^{gd}(I) \). The subdivided \((n - 1)\)-simplex \( \sigma_0 \ast P \) thus provides the desired homotopy of \( \sigma \) into \( B_n^{gd}(I) \).
2.5 Step 3: \((B^{gd}_n(I), B_n(0))\) is \((n - 2)\)-connected

Since every vertex \(v \in B_n(0)\) has \(L(v) = 0\) or \(L(v) = 1\), every simplex \(\sigma \in B_n(0)\) has \(I_\sigma = (0)\) or \(I_\sigma = \mathcal{O}\). By Proposition 2.13(2), every such simplex is good, so \(B_n(0)\) is contained in \(B^{gd}_n(I)\). The main result of this section is the following proposition.

Proposition 2.14. Let \(\mathcal{O}\) be a Dedekind domain, and fix \(n \geq 3\). Assume that for all \(1 \leq m < n\) and all ideals \(I \subset \mathcal{O}\), the complex \(B_n(I)\) is CM of dimension \(m - 1\). Then for any ideal \(I \subset \mathcal{O}\), the pair \((B^{gd}_n(I), B_n(0))\) is \((n - 2)\)-connected.

Before proving Proposition 2.14 we establish three results that will be necessary in the proof.

2.5.1 Pushing into a subcomplex

The first ingredient is the following lemma on the connectivity of a pair.

Lemma 2.15. Let \(A\) be a full subcomplex of a simplicial complex \(X\), and fix \(m \geq 0\). Assume that for every \(k\)-simplex \(\sigma^k \in X\) which is disjoint from \(A\), the intersection \(\text{Link}_X(\sigma^k) \cap A\) is \((m - k - 1)\)-connected. Then the pair \((X, A)\) is \(m\)-connected.

The proof of Lemma 2.15 will use the language of posets. Recall that a poset \(P\) is said to be \(d\)-connected if its geometric realization \(|P|\) is \(d\)-connected. Similarly, the homology \(H_*(P; \mathbb{Z})\) is defined to be \(H_*(|P|; \mathbb{Z})\). Of course, a simplicial complex \(X\) can be regarded as a poset \(\mathcal{P}(X)\) whose elements are the simplices of \(X\) under inclusion. The canonical homeomorphism \(X \cong |\mathcal{P}(X)|\) shows that \(\mathcal{P}(X)\) is \(d\)-connected if and only if \(X\) is, and there is a canonical isomorphism \(H_*((X; \mathbb{Z}) \cong H_*(\mathcal{P}(X); \mathbb{Z})\).

Proof of Lemma 2.15. We can replace \(X\) by its \((m + 1)\)-skeleton without affecting the truth of the conclusion, so we can assume that \(X\) is finite-dimensional. Let \(A\) be the full complex on the set of vertices \(A(0) \subset X(0)\), so \(\mathcal{P}(A) = \{\tau \in \mathcal{P}(X) \mid \tau \subset A(0)\}\). Define \(B := X(0) \setminus A(0)\). A simplex \(\sigma \in \mathcal{P}(X)\) is disjoint from \(A\) if \(\sigma \cap A(0) = \emptyset\), i.e. if \(\sigma \subset B\); in this case, \(\text{Link}_X(\sigma) \cap A\) is \(\{\eta \in A \mid \sigma \cup \eta \in \mathcal{P}(X)\}\).

We will use PL Morse theory as described in [Be]. Consider the PL Morse function \(f: \mathcal{P}(X) \to \mathbb{N} \times -\mathbb{N}\) defined by

\[
f(\sigma) = (|\sigma \cap B|, -|\sigma \cap A(0)|),
\]

with the lexicographic order on \(\mathbb{N} \times -\mathbb{N}\). If \(\sigma \subsetneq \sigma'\), then we must have \(|\sigma \cap B| < |\sigma' \cap B|\) or \(|\sigma \cap A(0)| < |\sigma' \cap A(0)|\), so \(f(\sigma) \neq f(\sigma')\). This verifies that \(f\) has no “horizontal edges”. Our assumption that \(X\) is finite-dimensional implies that the image of \(f\) is finite.

Note that \(f^{-1}(\emptyset) \times -\mathbb{N}\) = \(\{\sigma \in \mathcal{P}(X) \mid |\sigma \cap B| = 0\} = \mathcal{P}(A)\). For any \(\sigma \in \mathcal{P}(X)\), define the downward link to be

\[
\text{Link}_\downarrow(\sigma) := \{\tau \in \mathcal{P}(X) \mid f(\tau) < f(\sigma)\}
\]

and either \(\tau \subsetneq \sigma\) or \(\tau \supsetneq \sigma\).

The fundamental theorem of PL Morse theory states that if \(\text{Link}_\downarrow(\sigma)\) is \((m - 1)\)-connected for all \(\sigma \in f^{-1}(\mathbb{N}_{>0} \times -\mathbb{N})\), then \((X, A)\) is \(m\)-connected. We wish to verify this condition, so consider \(\sigma \in f^{-1}(\mathbb{N}_{>0} \times -\mathbb{N})\).

We can uniquely write \(\sigma\) as the disjoint union of \(\sigma_B := \sigma \cap B\) and \(\sigma_A := \sigma \cap A(0)\), where \(|\sigma_B| > 0\) by assumption. Define subposets of \(\mathcal{P}(X)\) by

\[
T := \{\tau \in \mathcal{P}(X) \mid \tau \subsetneq \sigma, |\tau \cap B| < |\sigma \cap B|\},
\]

\[
R := \{\rho \in \mathcal{P}(X) \mid \rho \supsetneq \sigma, |\rho \cap B| = |\sigma \cap B|, |\rho \cap A(0)| > |\sigma \cap A(0)|\}.
\]
The downward link \( \text{Link}_i(\sigma) \) is the sub-poset \( T \cup R \) of \( P(X) \); since every \( \tau \in T \) and every \( \rho \in R \) satisfy \( \tau \subseteq \rho \), in fact the downward link is the join \( T \ast R \).

Now, either \( \sigma_A \neq \emptyset \) or \( \sigma_A = \emptyset \). In the first case, the poset \( T \) is contractible via the homotopy that takes \( \tau \) to \( \tau \cup \sigma_A \) and then to \( \sigma_A \). We conclude that in this case \( \text{Link}_i(\sigma) = T \ast R \) is contractible, and in particular is \((m - 1)\)-connected.

So suppose that \( \sigma_A = \emptyset \). In other words, \( \sigma \) is disjoint from \( A \). In this case, \( T \) is the poset of proper nonempty subsets of the \((k + 1)\)-element set \( \sigma^k \), so \( T \equiv P(\partial \Delta^k) \simeq S^{k-1} \); in particular, \( T \) is \((k - 1)\)-connected. The poset \( R \) consists of \( \rho \in P(X) \) that can be written as \( \rho = \sigma \cup \eta \) for some nonempty \( \eta \in P(A) \). The assignment \( \sigma \cup \eta \mapsto \eta \) gives an isomorphism of \( R \) with \( P(\text{Link}_X(\sigma) \cap A) \). Our assumption guarantees that \( R \) is \((m - k - 1)\)-connected, so \( \text{Link}_i(\sigma) = T \ast R \) is \((m - 1)\)-connected, as desired. \( \square \)

### 2.5.2 Topology of nicely fibered complexes

The second ingredient we will need concerns the topology of simplicial complexes that are “fibered” over a CM complex in the following sense.

**Definition 2.16.** Let \( F: X \to Y \) be a simplicial map between simplicial complexes. Let \( C \subset X^{(0)} \) be a subset of the vertices of \( X \) with \( F(C) = Y^{(0)} \).

We say that \( X \) is fibered over \( Y \) by \( F \) with core \( C \) if the following condition holds: a subset \( U = \{x_0, \ldots, x_k\} \subset X^{(0)} \) forms a \( k \)-simplex of \( X \) if and only if

1. \( F(U) = \{F(x_0), \ldots, F(x_k)\} \) is a \( k \)-simplex of \( Y \), and
2. if \( F(U) \) is a maximal simplex of \( Y \), then \( U \cap C \neq \emptyset \).

If \( C = X^{(0)} \), we say \( X \) is fully fibered over \( Y \) by \( F \); in this case condition (2) is vacuous.

The main result concerning fibered complexes is as follows.

**Lemma 2.17.** Assume that \( X \) is fibered over \( Y \) by \( F: X \to Y \) with core \( C \subset X^{(0)} \). If \( Y \) is CM of dimension \( d \) then \( X \) is CM of dimension \( d \).

To prove Lemma 2.17 we will make use of a theorem of Quillen that describes the topology of a poset map in terms of the topology of the fibers.

Given a poset \( A \), recall that the height \( \text{ht}(a) \) of an element \( a \) is the largest \( k \) for which there exists a chain \( a_0 \leq a_1 \leq \cdots \leq a_k = a \). When a \( k \)-simplex \( \sigma^k \in X \) is regarded as an element of the poset \( P(X) \), it has height \( k \). Given a map of posets \( F: A \to B \) and element \( b \in B \), the fiber \( F_{\leq b} \) is the subposet of \( A \) defined by \( F_{\leq b} := \{ a \in A \mid F(a) \leq b \} \).

A poset map \( F: A \to B \) is strictly increasing if \( a < a' \) implies \( F(a) < F(a') \). Finally, a poset \( P \) is said to be CM of dimension \( d \) if its geometric realization \( |P| \) is CM of dimension \( d \) as a simplicial complex. A simplicial complex \( X \) is CM of dimension \( d \) if and only if \( P(X) \) is CM of dimension \( d \) (this is not automatic since being CM is not homeomorphism-invariant; nevertheless it follows from \([Q \ (8.5)]\)).

**Theorem 2.18** (Quillen [Q Theorem 9.1, Corollary 9.7]). Let \( F: A \to B \) be a strictly increasing map of posets. Assume that \( B \) is CM of dimension \( d \) and that for all \( b \in B \), the fiber \( F_{\leq b} \) is CM of dimension \( \text{ht}(b) \). Then \( A \) is CM of dimension \( d \) and \( F_*: H_d(A; \mathbb{Z}) \to H_d(B; \mathbb{Z}) \) is surjective.

We also record the following observation, which follows directly from Definition 2.16.

**Lemma 2.19.** Let \( X \) be fibered over \( Y \) by \( F \) with core \( C \), and consider a simplex \( \sigma \in X \).

- If \( \sigma \cap C \neq \emptyset \), then the link \( \text{Link}_X(\sigma) \) is fully fibered over \( \text{Link}_Y(F(\sigma)) \) by \( F \).
• If $\sigma \cap C = \emptyset$, then the link $\text{Link}_X(\sigma)$ is fibered over $\text{Link}_Y(F(\sigma))$ by $F$ with core $C \cap \text{Link}_X(\sigma)^{(0)}$.

Proof of Lemma 2.17. We prove the lemma by induction on $d$. The base case $d = 0$ is trivial, so assume that $d > 0$ and that the lemma is true for all smaller values of $d$. We must prove two things about $X$: that for every $\sigma^k \in X$ the link $\text{Link}_X(\sigma^k)$ is CM of dimension $d - k - 1$, and that $X$ itself is $d$-spherical. By condition (1), the $k$-simplex $\sigma^k \in X$ projects to a $k$-simplex $F(\sigma^k) \in Y$. By Lemma 2.19 the complex $\text{Link}_X(\sigma^k)$ is fibered over $\text{Link}_Y(F(\sigma^k))$ by $F$ (with some core). Since $Y$ is CM of dimension $d$, the complex $\text{Link}_Y(F(\sigma^k))$ is CM of dimension $d - k - 1$. Applying the inductive hypothesis, we conclude that $\text{Link}_X(\sigma^k)$ is CM of dimension $d - k - 1$, as desired.

It remains to prove that $X$ is $d$-spherical. We begin by proving the lemma in the special case when $X$ is fibered over a single simplex $Y = \Delta^d$ with vertices $\{y_0, \ldots, y_d\}$. Define $Z$ to be the complex with $Z^{(0)} = X^{(0)}$ and where $U \subset Z^{(0)}$ forms a simplex if and only if $U$ contains at most one element from each $F^{-1}(y_i)$. The complex $Z$ is the join of the 0-dimensional complexes $F^{-1}(y_i)$. Since $F(C) = Y^{(0)}$, each $F^{-1}(y_i)$ is nonempty and thus 0-spherical. The join of an $r$-spherical and an $s$-spherical complex is $(r + s + 1)$-spherical [Q Example 8.1], so the $(d + 1)$-fold join $Z = F^{-1}(y_0) \ast \cdots \ast F^{-1}(y_d)$ is $d$-spherical.

Observe that $X \subset Z$. Since $Z$ is $d$-spherical, it is enough to prove that the pair $(Z, X)$ is $(d - 1)$-connected. Since $Y$ is itself a $d$-simplex, the only maximal simplex of $Y$ is $Y$ itself, and hence condition (2) applies only to $d$-simplices of $X$. Therefore $X$ and $Z$ coincide in dimensions up to $d - 1$ and $Z$ is obtained from $X$ by adding certain $d$-simplices $\sigma^d \in Z$ with $\sigma^d \notin X$. It thus suffices to show for each such $\sigma^d$ that the boundary $\partial \sigma^d \subset X$ is null-homotopic in $X$.

By condition (1), such a $d$-simplex can be written as $\sigma^d = \{x_0, \ldots, x_d\}$ with $F(x_i) = y_i$ for $0 \leq i \leq d$, and by condition (2) we must have $x_i \notin C$ for all $0 \leq i \leq d$. Since $F(C) = Y^{(0)}$, for each $0 \leq i \leq d$ we can choose $x'_i \in C$ with $F(x'_i) = y_i$. Consider the full subcomplex of $Z$ on $\{x_0, x'_0, \ldots, x_d, x'_d\}$. By construction, this is the $(d + 1)$-fold join of the 0-spheres $\{x_i, x'_i\}$, and thus is homeomorphic to a $d$-sphere. Every simplex of this $d$-sphere except $\sigma$ contains some $x'_i$ and thus lies in $X$. Therefore the full subcomplex of $X$ on $\{x_0, x'_0, \ldots, x_d, x'_d\}$ is a subdivided $d$-disk $D$ with $\partial D = \partial \sigma^d$, as desired.

This proves that $X$ is $d$-spherical if it is fibered over a $d$-simplex with any core; combined with the first paragraph, $X$ is CM of dimension $d$ in this case.

We now prove the lemma for a general CM complex $Y$ of dimension $d$. By condition (1), the map $F$ induces a height-preserving poset map $F: P(X) \to P(Y)$; we will verify the conditions of Theorem 2.18 for $F$.

Fix a simplex $\tau^k \in P(Y)$. Let $T \simeq \Delta^k$ be the subcomplex of $Y$ determined by $\tau^k$ and let $\bar{T}$ be the subcomplex of $X$ consisting of $\sigma^k \in X$ with $F(\sigma^k) \subset \tau^k$. The fiber $F_{\leq \tau^k}$ is the subposet of $P(X)$ consisting of simplices of $\bar{T}$, so $F_{\leq \tau^k} = P(\bar{T})$. Since $F$ is height-preserving, if $k < d$ then no $d$-simplex is contained in $\bar{T}$, so condition (2) is vacuous; in this case $\bar{T}$ is fully fibered over $T \simeq \Delta^k$ by $F$. If $\tau^d$ is a $d$-simplex, condition (2) is not vacuous, and in this case $\bar{T}$ is fibered over $T \simeq \Delta^d$ by $F$ with core $C \cap \bar{T}^{(0)}$. In either case the base $T$ is a single simplex, so the special case of the lemma implies that $\bar{T}$ is CM of dimension $\text{ht}(\tau^k) = k$, and thus so is $F_{\leq \tau^k} = P(\bar{T})$. Since $P(Y)$ is CM of dimension $d$ by assumption, this verifies the hypotheses of Theorem 2.18 for the map $F: P(X) \to P(Y)$. Theorem 2.18 implies that $P(X)$ is CM of dimension $d$, so $X$ is CM of dimension $d$ as well. \qed
2.5.3 A criterion for being fibered

The final ingredient we need is the following condition to verify that a complex is fibered over another.

Lemma 2.20. Let $X$ and $Y$ be $d$-dimensional simplicial complexes with the property that every simplex is contained in a $d$-simplex. Let $F: X \to Y$ be a simplicial map and $C \subset X^{(0)}$ be a subset such that the following two conditions hold.

(i) A subset $U = \{x_0, \ldots, x_d\}$ forms a $d$-simplex of $X$ if and only if $F(U)$ is a $d$-simplex of $Y$ and $U \cap C \neq \emptyset$.

(ii) For each $d$-simplex $\tau^d \in Y$ there exists $U \subset C$ with $F(U) = \tau^d$.

Then $X$ is fibered over $Y$ by $F$ with core $C$.

Proof. The maximal simplices of $Y$ are precisely the $d$-simplices, so when $|U| = d + 1$ condition (i) is equivalent to conditions (1) and (2) of Definition 2.16. Since every vertex of $Y$ is contained in a $d$-simplex, condition (ii) implies that $F(C) = Y^{(0)}$. It remains to verify conditions (1) and (2) for subsets $U \subset X^{(0)}$ with $|U| < d + 1$.

Every simplex $\tau^k \in X$ is contained in a $d$-simplex $\sigma^d \in X$. By condition (i), its image $F(\sigma^d)$ is a $d$-simplex of $Y$, so $F(\tau^k)$ is a $k$-simplex of $Y$. Conversely, consider a subset $U = \{x_0, \ldots, x_k\}$ with $k < d$ such that $F(U)$ is a $k$-simplex of $Y$. Then $F(U)$ is contained in a $d$-simplex $\eta^d = F(U) \cup \{y_{k+1}, \ldots, y_d\} \in Y$. Using the assumption that $F(C) = Y^{(0)}$, choose lifts $x_i \in C$ with $F(x_i) = y_i$ for $k + 1 \leq i \leq d$. By construction, $U' := \{x_0, \ldots, x_d\}$ has $F(U') = \eta^d$ and $U' \cap C \neq \emptyset$, so by condition (i) $U'$ is a $d$-simplex of $X$. It follows that $U$ is a $k$-simplex of $X$, as desired. \qed

2.5.4 Completing the proof of Step 3

We are now ready to prove Proposition 2.14.

Proof of Proposition 2.14. We want to show that $(\mathcal{B}^d_n(I), \mathcal{B}_n(0))$ is $(n - 2)$-connected. We will prove this by applying Lemma 2.15 with $X = \mathcal{B}^d_n(I)$ and $A = \mathcal{B}_n(0)$ and $m = n - 2$.

To apply Lemma 2.15 we need to verify the two hypotheses: first, that $\mathcal{B}_n(0)$ is a full subcomplex of $\mathcal{B}^d_n(I)$, and second, that for any $k$-simplex $\sigma^k$ of $\mathcal{B}^d_n(I)$ that is disjoint from $\mathcal{B}_n(0)$, the intersection

$$\text{Link}_0(\sigma^k) := \mathcal{B}_n(0) \cap \text{Link}_{\mathcal{B}^d_n(I)}(\sigma^k)$$

is $(n - k - 3)$-connected.

We begin by proving that $\mathcal{B}_n(0)$ is the full subcomplex of $\mathcal{B}_n(\mathcal{O})$ on the vertex set

$$\mathcal{B}_n(0)^{(0)} := \{v \in \mathcal{B}_n(\mathcal{O}) \mid L(v) = 0 \text{ or } L(v) = 1\}.$$

Let $\sigma = \{v_1, \ldots, v_k\} \in \mathcal{B}_n(\mathcal{O})$ be a partial basis with $v_i \in \mathcal{B}_n(0)^{(0)}$ for $1 \leq i \leq k$. We have either $I_{\sigma} = \mathcal{O}$ (if some $v_i$ has $L(v_i) = 1$) or $I_{\sigma} = (0)$ (if all $v_i$ have $L(v_i) = 0$). By Proposition 2.13(2), $\sigma$ is contained in a good basis. By Proposition 2.13(1)(ii) this implies that $\sigma$ is contained in a good (0)-basis, so $\sigma \in \mathcal{B}_n(0)$. This proves that $\mathcal{B}_n(0)$ is a full subcomplex of $\mathcal{B}_n(\mathcal{O})$, and thus of $\mathcal{B}^d_n(I)$, verifying the first hypothesis.

For the second hypothesis, consider a $k$-simplex $\sigma^k \in \mathcal{B}^d_n(I)$ that is disjoint from $\mathcal{B}_n(0)^{(0)}$; we will prove that $\text{Link}_0(\sigma^k)$ is CM of dimension $n - k - 2$. Using Proposition 2.13(1)(iii), choose a good complement $W$. By definition, we have $\mathcal{O}^a = V_{\sigma} \oplus W$ and $L(W) = \mathcal{O}$. Let $\pi: \mathcal{O}^a \to W$ be the projection with kernel $V_{\sigma}$.

We will prove the following claims, using the terminology of Definition 2.16.
1. If \( I_\sigma \neq O \), then \( \text{Link}_0(\sigma) \) is fully fibered over \( \mathcal{B}_W(I_\sigma) \) by \( \pi \).
2. If \( I_\sigma = O \), then \( \text{Link}_0(\sigma) \) is fibered over \( \mathcal{B}_W(O) \) by \( \pi \) with core

\[
C := \{ v \in \text{Link}_0(\sigma) \mid L(v) = 1 \} \subset \text{Link}_0(\sigma)^{(0)}.
\]

The bulk of the argument does not depend on whether \( I_\sigma = O \) or not, so we begin without making any assumption on \( I_\sigma \).

By definition, every simplex of \( \mathcal{B}_W(I_\sigma) \) is contained in an \((n-k-2)\)-simplex, corresponding to an \( I_\sigma \)-basis of \( W \). Similarly, if \( \eta \in \text{Link}_0(\sigma^k) \) then \( \sigma \cup \eta \in \mathcal{B}_n^{\text{id}}(I) \) is a partial good \( I \)-basis. By Proposition 2.13(1)(ii) we can enlarge this to a good \( I \)-basis \( \sigma \cup \tau \) with \( \tau \in \mathcal{B}_n(0) \). Therefore every simplex \( \eta \in \text{Link}_0(\sigma^k) \) is contained in an \((n-k-2)\)-simplex \( \sigma^{n-k-2} \in \text{Link}_0(\sigma^k) \). This means that in verifying the claim, we can work only with \((n-k-2)\)-simplices, and appeal to Lemma 2.20 for the general case.

A subset \( U = \{u_{k+2}, \ldots, u_n\} \subset \mathcal{B}_n(0)^{(0)} \) forms an \((n-k-2)\)-simplex of \( \text{Link}_0(\sigma^k) \) if \( \sigma \cup U = \{v_1, \ldots, v_{k+1}, u_{k+2}, \ldots, u_n\} \) is a good \( I \)-basis, or equivalently if \( \sigma \cup U \) is a good basis.

Second, assume that \( I_\sigma \neq O \). In this case any basis \( \sigma \cup U \) with \( U \subset \mathcal{B}_n(0)^{(0)} \) must be good, because if \( L(u_i) = 0 \) for all \( k+2 \leq i \leq n \) we would have \( L(O^n) = I_\sigma \neq O \). Therefore \( \sigma \cup U \) is a good basis if and only if \( \pi(U) \) is an \( I_\sigma \)-basis of \( W \). This verifies condition (i) of Lemma 2.20.

Let \( \tau = \{w_{k+2}, \ldots, w_n\} \) be an \( I_\sigma \)-basis of \( W \). For \( i = k+2, \ldots, n \), if \( L(w_i) \equiv 0 \mod I_\sigma \), choose \( v'_i \in V_\sigma \) with \( L(v'_i) = -L(w_i) \); if \( L(w_i) \equiv 1 \mod I_\sigma \), choose \( v'_i \in V_\sigma \) with \( L(v'_i) = 1 - L(w_i) \). In either case, define \( u_i = w_i + v'_i \); by construction, we have \( L(u_i) = 0 \) or \( L(u_i) = 1 \) and \( \pi(u_i) = \pi(w_i) + \pi(v'_i) = w_i \). Setting \( U = \{u_{k+2}, \ldots, u_n\} \) we see that \( \pi(U) \) is the \( I_\sigma \)-basis \( \tau \) of \( W \). We can therefore apply Lemma 2.20 to conclude that \( \text{Link}_0(\sigma) \) is fibered over \( \mathcal{B}_W(I_\sigma) \) by \( \pi \) with core \( C \), verifying Claim 1.

Now that Claims 1 and 2 established, we finish the proof of the proposition. Since \( W \cong O^{n-k-1} \), Remark 2.5 yields an isomorphism \( \mathcal{B}_W(I_\sigma) \cong \mathcal{B}_{n-k-1}(I_\sigma) \). Our hypothesis guarantees that \( \mathcal{B}_{n-k-1}(I_\sigma) \) is CM of dimension \( n-k-2 \), so the same is true of \( \mathcal{B}_W(I_\sigma) \). We established in Claims 1 and 2 above that \( \text{Link}_0(\sigma) \) is fibered over \( \mathcal{B}_W(I_\sigma) \) by \( \pi \) (either with core \( C \) or with core \( \text{Link}_0(\sigma)^{(0)} \)). Lemma 2.17 thus implies that \( \text{Link}_0(\sigma) \) is CM of dimension \( n-k-2 \); in particular \( \text{Link}_0(\sigma) \) is \((n-k-3)\)-connected. This verifies the hypothesis of Lemma 2.15 for the pair \( (\mathcal{B}_n^{\text{id}}(I), \mathcal{B}_n(0)) \) with \( m = n-2 \); we conclude that \( (\mathcal{B}_n^{\text{id}}(I), \mathcal{B}_n(0)) \) is \((n-2)\)-connected, as desired. \( \square \)
Remark 2.21. Many of the ingredients of the proof that \( \text{Link}_0(\sigma^k) \) is \((n - k - 3)\)-connected were introduced only for this step; this step thereby accounts for much of the complexity of our proof of Theorem E.

- Even if we had only been interested in \( \mathcal{B}_n(\mathcal{O}) \), we were forced in the proof of Proposition 2.14 to consider the complex \( \mathcal{B}_{n-k-1}(I_\sigma) \). This is the entire reason that we introduced the complexes \( \mathcal{B}_n(I) \) relative to nontrivial ideals of \( \mathcal{O} \).
- In the proof of Proposition 2.14 we needed the assumption that \( \mathcal{B}_m(I_\sigma) \) was CM of dimension \( m - 1 \) for \( m < n \). This is the reason that we are forced to prove in Theorem E that \( \mathcal{B}_n(I) \) is Cohen–Macaulay of dimension \( n - 1 \), rather than simply that \( \mathcal{B}_n(I) \) is \((n - 1)\)-spherical. This weaker statement would not suffice as an inductive hypothesis; Lemma 2.17 is not true if one only assumes that \( Y \) is \( d\)-spherical.
- In the case \( I_\sigma = \mathcal{O} \) above, we applied Lemma 2.17 in a case where the core \( C \) only contained some of the vertices of the domain. This is the reason that we introduced the notion of “fibered with core \( C \)”; we will apply Lemma 2.17 in Step 4 and in the proof of Theorem E as well, but only in the fully fibered case.

2.6 Step 4: \( \mathcal{B}_n(0) \) is \((n - 2)\)-connected

The main result of this section is the following proposition.

Proposition 2.22. Let \( \mathcal{O} \) be a Dedekind domain, and assume that \( \mathcal{B}_{n-1}(\mathcal{O}) \) is CM of dimension \( n - 2 \). Then \( \mathcal{B}_n(0) \) is \((n - 2)\)-connected.

Proof. Define \( Z_n \) to be the full subcomplex of \( \mathcal{B}_n(0) \) spanned by those vertices \( v \) of \( \mathcal{B}_n(0) \) with \( L(v) = 0 \). The complex \( Z_n \) consists of partial bases contained in the summand \( \ker L \cong \mathcal{O}^{n-1} \).

In fact, by Lemma 2.3b) these are precisely the partial bases of the summand \( \ker L \), so we have an isomorphism \( Z_n \cong \mathcal{B}_{n-1}(\mathcal{O}) \). Therefore our assumption guarantees that \( Z_n \) is CM of dimension \( n - 2 \).

Consider a \( k \)-simplex \( \sigma^k = \{v_0, v_1, \ldots, v_k\} \in \mathcal{B}_n(0) \) disjoint from \( Z_n \), i.e. with \( L(v_0) = \cdots = L(v_k) = 1 \). Let \( \eta^{k-1} = \{v_1 - v_0, \ldots, v_k - v_0\} \) and \( \tau^k = \{v_0\} \cup \eta \in \mathcal{B}_n(0) \). Since \( V_\sigma = V_\tau \), the links \( \text{Link}_{\mathcal{B}_n(0)}(\sigma^k) \) and \( \text{Link}_{\mathcal{B}_n(0)}(\tau^k) \) coincide. Furthermore, given \( v^{n-k-2} \in Z_n \) the set \( \{v_0\} \cup \eta \cup v \) is a basis for \( \mathcal{O}^n \) if and only if \( \eta \cup v \) is a basis for \( \ker L \). In other words,

\[ \text{Link}_{\mathcal{B}_n(0)}(\tau^k) \cap Z_n = \text{Link}_{\mathcal{Z}_n}(\eta^{k-1}). \]

Since \( Z_n \) is CM of dimension \( n - 2 \), we know that \( \text{Link}_{\mathcal{Z}_n}(\eta^{k-1}) \) is CM of dimension \((n - 2) - (k - 1) - 1 = n - k - 2 \), and is thus \((n - k - 3)\)-connected. We now apply Lemma 2.15 with \( X = \mathcal{B}_n(0) \) and \( A = Z_n \) and \( m = n - 2 \); the conclusion is that \( (\mathcal{B}_n(0), Z_n) \) is \((n - 2)\)-connected.

Fix a vector \( v \in \mathcal{O}^n \) with \( L(v) = 1 \). Since \( \{u_2, \ldots, u_n\} \subset \ker L \) is a basis for \( \ker L \) if and only if \( \{v, u_2, \ldots, u_n\} \) is a basis for \( \mathcal{O}^n \), every simplex of \( Z_n \) is contained in \( \text{Link}_{\mathcal{B}_n(0)}(v) \). We conclude that the image of \( Z_n \hookrightarrow \mathcal{B}_n(0) \) can be contracted to \( v \). Together with the fact that \( (\mathcal{B}_n(0), Z_n) \) is \((n - 2)\)-connected, this implies that \( \mathcal{B}_n(0) \) is \((n - 2)\)-connected, as desired. \( \square \)

2.7 Assembling the proof: \( \mathcal{B}_n(I) \) is CM of dimension \( n - 1 \)

We are finally ready to combine the steps of the previous sections and prove Theorem E.

Proof of Theorem E. We prove by induction on \( n \) that \( \mathcal{B}_n(I) \) is CM of dimension \( n - 1 \) for all ideals \( I \subset \mathcal{O} \). The base cases are \( n = 1 \) and \( n = 2 \). Proposition 2.6, which holds for any
Dedekind domain $\mathcal{O}$, states that $B_1(I)$ is CM of dimension 0. Proposition 2.7 states that under precisely our hypotheses, $B_2(I)$ is CM of dimension 1.

Now fix $n \geq 3$ and an ideal $I \subset \mathcal{O}$, and assume that $B_m(J)$ is CM of dimension $m - 1$ for all $m < n$ and all $J \subset \mathcal{O}$. Under these assumptions, Proposition 2.10 states that $(B_n(I), B^{\text{gd}}_n(I))$ is $(n - 2)$-connected; Proposition 2.14 states that $(B^{\text{gd}}_n(I), B_n(0))$ is $(n - 2)$-connected; and Proposition 2.22 states that $B_n(0)$ itself is $(n - 2)$-connected. Together these imply that the $(n - 1)$-dimensional complex $B_n(I)$ is $(n - 2)$-connected, and thus $(n - 1)$-spherical.

It remains to prove that for every $k$-simplex $\sigma^k \in B_n(I)$, the space $\text{Link}_{B_n(I)}(\sigma^k)$ is CM of dimension $n - k - 2$. If $k = n - 1$ this is vacuous. If $k = n - 2$, we must prove that $\text{Link}_{B_n(I)}(\sigma^k)$ is nonempty. But the partial $I$-basis $\sigma^{n - 2}$ is by definition contained in an $I$-basis $\sigma^{n - 2} \cup \{v\}$, so $\text{Link}_{B_n(I)}(\sigma^k)$ contains the 0-simplex $\{v\}$.

Now assume that $k < n - 2$. By Lemma 2.13(1), $\sigma$ has a good complement $W$. Let $\pi: \mathcal{O}^n \to W$ be the projection with kernel $V_\sigma$. We will prove that $\text{Link}_{B_n(I)}(\sigma)$ is fully fibered over $B_W(I + I_\sigma)$ by $\pi$ using Lemma 2.20.

By definition, every simplex of $\text{Link}_{B_n(I)}(\sigma)$ or $B_W(I + I_\sigma)$ is contained in an $(n - k - 2)$-simplex. Given a subset $U = \{u_{k+2}, \ldots, u_n\}$ with $L(u_i) \equiv 0$ or $1 \mod I$ for each $i = k + 2, \ldots, n$, the set $\sigma \cup U$ is a basis of $\mathcal{O}^n$ if and only if $\pi(U)$ is a basis of $W$. Moreover $L(\pi(u)) \equiv L(u) \mod I_\sigma$, so in this case $\pi(U)$ is an $(I + I_\sigma)$-basis of $W$. Therefore such a subset $U$ is an $(n - k - 2)$-simplex of $\text{Link}_{B_n(I)}(\sigma)$ if and only if $\pi(U)$ is an $(n - k - 2)$-simplex of $B_W(I + I_\sigma)$, verifying condition (i) of Lemma 2.20.

It remains to check condition (ii) of Lemma 2.20, so consider an $(I + I_\sigma)$-basis $\tau = \{u_{k+2}, \ldots, u_n\}$ of $W$. For each $i$ we have $L(u_i) \equiv 0$ or $1 \mod I + I_\sigma$, so there exists $v_i \in V_\sigma$ such that $L(w_i + v_i) \equiv 0$ or $1 \mod I$. The set $U := \{w_{k+2} + v_{k+2}, \ldots, w_n + v_n\}$ projects under $\pi$ to $\tau$, so it is an $(n - k - 2)$-simplex of $\text{Link}_{B_n(I)}(\sigma)$.

Lemma 2.20 now implies that $\text{Link}_{B_n(I)}(\sigma)$ is fully fibered over $B_W(I + I_\sigma)$ by $\pi$. Since $W \cong \mathcal{O}^{n-k-1}$, the complex $B_W(I + I_\sigma)$ is isomorphic to $B_{n-k-1}(I + I_\sigma)$, which we have assumed by induction is CM of dimension $n - 2$. Applying Lemma 2.17 shows that $\text{Link}_{B_n(I)}(\sigma^k)$ is CM of dimension $n - k - 2$, as desired.

We have proved that $B_n(I)$ is $(n - 1)$-spherical and that $\text{Link}_{B_n(I)}(\sigma^k)$ is CM of dimension $n - k - 2$ for every $\sigma^k \in B_n(I)$, so $B_n(I)$ is CM of dimension $n - 1$. Since the ideal $I$ was arbitrary, this concludes the inductive step, and thus concludes the proof of Theorem E. □

### 3 Integral apartments

In this section we use Theorem E to prove Theorem A.

**Proof of Theorem A.** Let $C_*$ be the reduced chain complex of the simplicial complex $B_n(\mathcal{O})$, so $C_{-1} \cong \mathbb{Z}$. We define the integral apartment class map

$$\phi: C_{n-1} \to \text{St}_n(K)$$

as follows. An ordered basis $v = (v_1, \ldots, v_n)$ of $\mathcal{O}^n$ determines an $(n - 1)$-simplex of $B_n(\mathcal{O})$, and thus a generator $[v] = [v_1, \ldots, v_n]$ of $C_{n-1}$. Reordering the basis simply changes the orientation of this simplex, so we have $[v_{\tau(1)}, \ldots, v_{\tau(n)}] = (-1)^\tau[v_1, \ldots, v_n]$ for $\tau \in S_n$; there are no other relations between the elements $[v] \in C_{n-1}$. Given $v$, define lines $L_1, \ldots, L_n$ in $K^n$ via the formula $L_i := \text{span}_K(v_i)$. Since $\{v_1, \ldots, v_n\}$ is a basis of $\mathcal{O}^n$, Lemma 2.2(c) implies that $L_i \cap \mathcal{O}^n = \text{span}_\mathcal{O}(v_i)$ and $\mathcal{O}^n = \text{span}_\mathcal{O}(v_1) \oplus \cdots \oplus \text{span}_\mathcal{O}(v_n)$, so the frame $L_v := \{L_1, \ldots, L_n\}$ is integral.
Let \( A_V := A_{L_V} \) be the integral apartment determined by this integral frame, and define \( \phi([v]) = [A_V] \in \text{St}_n(K) \). Reordering the basis vectors gives the same frame, and thus the same apartment, but possibly with reversed orientation: \([A_{r(v)}] = (-1)^r[A_v] \). This shows that the map \( \phi \) is well-defined. Our goal is to prove that \( \phi \) is surjective, i.e. that \( \text{St}_n(K) \) is generated by integral apartment classes. We will do this by factoring the map \( \phi \) as follows.

For a simplicial complex \( X \), the isomorphism \( b: H_i(X; \mathbb{Z}) \cong H_i(\mathcal{P}(X); \mathbb{Z}) \) determined by the homeomorphism between \( X \) and its barycentric subdivision is induced by the chain map

\[
\{(x_1, \ldots, x_k)\} \mapsto \sum_{\tau \in S_k} (-1)^{\tau} \left[ \{x_{\tau(1)}\} \subset \{x_{\tau(1)}, x_{\tau(2)}\} \subset \cdots \subset \{x_{\tau(1)}, \ldots, x_{\tau(k)}\} \right].
\]

Let \( \mathcal{B}_n(\mathcal{O})^{(n-2)} \) be the \((n-2)\)-skeleton of \( \mathcal{B}_n(\mathcal{O}) \) and let \( \mathcal{P}(n-2) := \mathcal{P}(\mathcal{B}_n(\mathcal{O})^{(n-2)}) \) be the poset of simplices of \( \mathcal{B}_n(\mathcal{O})^{(n-2)} \). In the introduction we considered \( \mathcal{T}_n(K) \) to be a simplicial complex, but let us define \( \mathcal{T}_n(K) \) to be the poset of proper nonzero \( K \)-subspaces of \( K^n \); the simplicial complex of flags from the introduction is then its geometric realization. Define the poset map \( F: \mathcal{P}(n-2) \to \mathcal{T}_n(K) \) by

\[
F(\{v_1, \ldots, v_r\}) = \text{span}_K(v_1, \ldots, v_r).
\]

Since \( \text{span}_K(v_1, \ldots, v_r) \) is an \( r \)-dimensional subspace of \( K^n \), the restriction to simplices of \( \mathcal{B}_n(\mathcal{O})^{(n-2)} \) (i.e. to simplices satisfying \( 0 < r < n \)) guarantees that this is a proper nonzero subspace, so \( F \) is well-defined and also height-preserving.

Our claim is that the integral apartment class map \( \phi: \mathcal{C}_{n-1} \to \text{St}_n(K) \) factors as the composition

\[
\phi: C_{n-1} \xrightarrow{\partial} \ker(\partial: C_{n-2} \to C_{n-3}) = \tilde{H}_{n-2}(\mathcal{B}_n(\mathcal{O})^{(n-2)}; \mathbb{Z}) \xrightarrow{b} \tilde{H}_{n-2}(\mathcal{P}(n-2); \mathbb{Z}) \xrightarrow{F_*} \tilde{H}_{n-2}(\mathcal{T}_n(K); \mathbb{Z}) = \text{St}_n(K).
\]

Under \( F_* \), the term corresponding to \( \sigma \in S_n \) is taken to \([L_{\sigma(1)} \subset L_{\sigma(1)} + L_{\sigma(2)} \subset \cdots \subset L_{\sigma(1)} + \cdots + L_{\sigma(n-1)}]\). These are precisely the \((n-2)\)-simplices making up the apartment \( A_v \) (compare with [5.1] and Example [5.1] below), and so \( F_*([b([v])] = [A_v] \), as claimed.

To prove that \( \phi \) is surjective, we prove that the maps \( \partial \) and \( F_* \) occurring in (3.1) are surjective. We first establish that under our assumptions, \( \mathcal{B}_n(\mathcal{O}) \) is CM of dimension \( n - 1 \). In the case when \( |S| > 1 \) and \( S \) contains a non-imaginary place, this is the statement of Theorem E. In the remaining case when \( \mathcal{O} \) is Euclidean, this was proved by Maazen [Ma] (see [valKL, Appendix] and [DP, Proof of Theorem B] for published proofs). By definition, the cokernel of \( \partial: C_{n-1} \to \ker(\partial: C_{n-2} \to C_{n-3}) = \tilde{H}_{n-2}(\mathcal{B}_n(\mathcal{O}); \mathbb{Z}) \). We have just established that \( \mathcal{B}_n(\mathcal{O}) \) is CM of dimension \( n - 1 \), and thus certainly \((n-2)\)-acyclic. Therefore \( \tilde{H}_{n-2}(\mathcal{B}_n(\mathcal{O}); \mathbb{Z}) = 0 \) and \( \partial: C_{n-1} \to \ker(\partial: C_{n-2} \to C_{n-3}) \) is surjective.

To show that \( F_* \) is surjective, we will apply Theorem 2.18 to the height-preserving poset map \( F \), so we must verify that its hypotheses are satisfied. The Solomon–Tits theorem says that \( \mathcal{T}_n(K) \) is CM of dimension \((n-2)\) (see e.g. [Q, Example 8.2]). Given \( V \in \mathcal{T}_n(K) \), let \( \ell = \dim(V) \), so \( \text{ht}(V) = \ell - 1 \). Lemma 2.18(c) implies that \( V \cap \mathcal{O}^n \) is a rank-\( \ell \) direct summand of \( \mathcal{O}^n \). Since \( \text{cl}(\mathcal{O}) = 1 \), every f.g. projective \( \mathcal{O} \)-module is free, so \( V \cap \mathcal{O}^n \) is isomorphic to
Lemma 2.3(b) states that \( \{v_1, \ldots, v_k\} \subset F_{\leq V} \) exactly if \( \{v_1, \ldots, v_k\} \) is a partial basis of \( V \cap O^n \), so we have an isomorphism \( F_{\leq V} \cong \mathcal{P}(B_o(O)) \). We established in the previous paragraph that \( \mathcal{P}(B_o(O)) \) is CM of dimension \( \ell - 1 = \ell(V) \), so this verifies the remaining hypothesis of Theorem 2.18. Applying Theorem 2.18 we deduce that the map

\[
F_* : \Pi_{n-2}(\mathcal{P}(n-2); \mathbb{Z}) \to \Pi_{n-2}(\mathcal{T}_n(K); \mathbb{Z}) = St_n(K)
\]

is surjective. Since all other maps in (3.1) are isomorphisms, this shows that the integral apartment class map \( \phi : C_{n-1} \to St_n(K) \) is surjective. \( \Box \)

4 Vanishing Theorem

In this section we deduce Theorem C from Theorem A. We first remark that by Dirichlet’s unit theorem, \( O_K^\times \cong \mathbb{Z}^{r_1 + r_2 - 1} \times \mu_K \). This is a virtual Poincaré duality group with virtual cohomological dimension \( r_1 + r_2 - 1 \), so Bieri–Eckmann [BE, Theorem 3.5] implies that \( vcd(SL_n O_K) = vcd(GL_n O_K) - vcd(O_K^\times) \), i.e. that

\[
vcd(SL_n O_K) = v'_n := r_1 \left( \frac{n+1}{2} \right) + r_2 (n^2 - 1) - (n - 1).
\]

Proof of Theorem C. In the special case \( \lambda = 0 \), Theorem C states that \( H^n(\text{GL}_n O_K; K) = 0 \) for all \( n \geq 2 \) (and similarly for \( \text{SL}_n O_K \)). Since \( H^*(\text{GL}_n O_K; K) \cong H^*(\text{GL}_n O_K; \mathbb{Q}) \otimes_{\mathbb{Q}} K \), the last claim of the theorem follows from the first ones, so those are all we need to prove.

Fix \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \) with \( \lambda_1 \geq \cdots \geq \lambda_n \). As we explained in §1.3 Borel–Serre duality implies that \( St_n(K) \) is the rational dualizing module for both \( \text{SL}_n O_K \) and \( \text{GL}_n O_K \), which implies that

\[
H^n(\text{GL}_n O_K; V_\lambda) = (St_n(K) \otimes V_\lambda)_{\text{GL}_n O_K},
\]

\[
H^n(\text{SL}_n O_K; V_\lambda) = (St_n(K) \otimes V_\lambda)_{\text{SL}_n O_K}.
\]

Since \( \text{SL}_n O_K \subset \text{GL}_n O_K \), the coinvariants \( (St_n(K) \otimes V_\lambda)_{\text{GL}_n O_K} \) are a quotient of \( (St_n(K) \otimes V_\lambda)_{\text{SL}_n O_K} \). The upshot is that to prove Theorem C it is enough to prove that \( (St_n(K) \otimes V_\lambda)_{\text{SL}_n O_K} = 0 \) when \( n \geq 2 + \|\lambda\| \).

Define \( \lambda' = (\lambda'_1, \ldots, \lambda'_{n-1}, 0) \) by \( \lambda'_i = \lambda_i - \lambda_n \); observe that \( \|\lambda'\| = \|\lambda\| \). As representations of \( \text{SL}_n O_K \), the representations \( V_\lambda \) and \( V_{\lambda'} \) are isomorphic; they only differ as representations of \( \text{GL}_n O_K \). Using Schur–Weyl duality, we can embed the \( \text{SL}_n K \)-representation \( V_{\lambda'} \) as a direct summand of \( (K^n)^{\otimes \ell} \) with \( \ell := \|\lambda\| \). It follows that it is enough to prove that \( (St_n(K) \otimes (K^n)^{\otimes \ell})_{\text{SL}_n O_K} = 0 \) when \( n \geq 2 + \ell \).

Fix an integral frame \( L = \{L_1, \ldots, L_n\} \) with integral apartment class \( [A_L] \subset St_n(K) \). We next show that \( [A_L] \otimes (K^n)^{\otimes \ell} \) vanishes in the quotient \( (St_n(K) \otimes (K^n)^{\otimes \ell})_{\text{SL}_n O_K} \) if \( n \geq 2 + \ell \). For each \( 1 \leq i \leq n \), the intersection \( L_i \cap O_K^\times \) is a rank 1 projective \( O_K \)-module. Since \( |\text{cl}(O_K)| = 1 \) this module is free, so we can choose a generator \( v_i \) for each \( L_i \cap O_K^\times \). The assumption that \( L \) is an integral frame means precisely that \( \{v_1, \ldots, v_n\} \) is an \( O_K \)-basis for \( O_K^\times \). In particular \( \{v_1, \ldots, v_n\} \) is a \( K \)-basis for \( K^n \), so \( S = \{v_{i_1} \otimes \cdots \otimes v_{i_\ell} \mid 1 \leq i_j \leq n \} \) is a basis for \( (K^n)^{\otimes \ell} \).

Consider an arbitrary basis vector \( s = v_{i_1} \otimes \cdots \otimes v_{i_\ell} \) \( S \). Under the assumption \( n \geq 2 + \ell \), we can find \( 1 \leq j < j' \leq n \) which are distinct from all of the \( i_1, \ldots, i_\ell \). There exists a unique \( g \in \text{SL}_n O_K \) such that

\[
g(v_i) = \begin{cases} v_{j'} & \text{if } i = j, \\ -v_j & \text{if } i = j', \\ v_i & \text{otherwise.} \end{cases}
\]

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We have $g(s) = s$ and $g(L) = L$ by construction. However $g$ exchanges the two lines $L_j$ and $L_{j'}$, so it reverses the orientation of the apartment $A_L$, i.e. $g([A_L]) = -[A_L]$. This means that $[A_L] \otimes s$ and $-[A_L] \otimes s$ become equal, and thus vanish, in the coinvariant quotient $(\text{St}_n(K) \otimes (K^n)^{\otimes \ell})_{\text{St}_n \circ K}$. Since $s$ was arbitrary, $[A_L] \otimes (K^n)^{\otimes \ell}$ vanishes in this quotient for every integral apartment $A_L$. Theorem A states that $\text{St}_n(K)$ is generated by integral apartment classes, and so we conclude that $(\text{St}_n(K) \otimes (K^n)^{\otimes \ell})_{\text{St}_n \circ K} = 0$, as desired. \hfill \qed

**Remark 4.1.** Some condition on $K$ beyond $|\text{cl}(O_K)| = 1$ is necessary in Theorem C. Indeed, for $d < 0$ squarefree let $O_d$ denote the ring of integers in the quadratic imaginary field $K_d = \mathbb{Q}(\sqrt{d})$. Those $d < 0$ for which $O_d$ is non-Euclidean but satisfies $|\text{cl}(O_d)| = 1$ are exactly $d \in \{-19, -43, -67, -163\}$. Although $H^2(\text{SL}_2(O_{-43}); \mathbb{Q}) = 0$, we have

$$H^2(\text{SL}_2(O_{-43}); \mathbb{Q}) = \mathbb{Q}, \quad H^2(\text{SL}_2(O_{-67}); \mathbb{Q}) = \mathbb{Q}^2, \quad H^2(\text{SL}_2(O_{-163}); \mathbb{Q}) = \mathbb{Q}^6.$$  

For these calculations see Vogtmann [Völ] and Rahm [Ra]. Presumably, similar things occur for $\text{SL}_n(O_d)$ for $n \geq 3$, but we could not find such calculations in the literature. It is likely that these four fields provide the only such counterexamples to the conclusion of Theorem C; indeed, Weinberger [W] proved that the generalized Riemann hypothesis implies that if $O_K$ has class number 1 and infinitely many units, then $O_K$ is Euclidean.

## 5 Non-Vanishing and Non-Integrality Theorems

In this section we prove Theorem $\mathcal{B}$ and Theorem $\mathcal{D}$; in fact we prove stronger versions of these results in Theorems $\mathcal{B}′$ and $\mathcal{D}′$ which apply to modules $M$ that need not be free. After summarizing in §5.1 some basic properties of spherical buildings, in §5.2 we prove Proposition 5.5, the key technical theorem of this section. We apply this proposition in §5.3 to prove Theorems $\mathcal{B}′$ and $\mathcal{D}′$.

### 5.1 Spherical buildings

In this section we briefly review the classical theory of spherical buildings. See [BR] for a general reference.

A $d$-dimensional spherical building $Y$ is a simplicial complex satisfying certain axioms. Associated to $Y$ is a finite Coxeter group $W$ of rank $d + 1$ called the Weyl group and a canonical collection of subcomplexes $A \subset Y$ called apartments. Each apartment is isomorphic to the Coxeter complex of $W$ and is thus acted upon by $W$. In particular, each apartment $A$ is homeomorphic to a sphere $S^d$ and defines a fundamental class $[A] \in \tilde{H}_d(Y)$. The class $[A]$ is only defined up to $\pm 1$, but for our purposes this ambiguity will never pose a problem, so we do not mention it again. A $d$-simplex $C$ in an apartment $A$ is called a chamber of $A$. The apartment $A$ is the union of the translates $wC$ for $w \in W$, and $[A] = \sum_{w \in W} (-1)^w [wC]$.

Any $d$-dimensional spherical building $Y$ is CM of dimension $d$. The top homology $\tilde{H}_d(Y)$ is spanned by the fundamental classes $[A]$ of apartments; indeed, for any fixed chamber $C$, a basis for $\tilde{H}_d(Y)$ is given by the apartment classes $[A]$ of all apartments $A$ containing $C$.

The following three examples will be of the most interest to us. A poset $P$ is said to be a $d$-dimensional spherical building if its realization $|P|$ is.

**Example 5.1 (Poset of subspaces).** Given an $n$-dimensional vector space $W$ over a field $K$, let $\mathcal{T}(W)$ be the poset of proper, nonzero $K$-subspaces of $W$. Its realization is the spherical
Tits building associated to GL(W), an \((n - 2)\)-dimensional spherical building with Weyl group \(W = S_n\). The chambers of \(T(W)\) are the complete flags

\[
0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_{n-1} \subsetneq W.
\]

The Coxeter complex of \(S_n\) can be identified with the barycentric subdivision of \(\partial \Delta^{n-1}\). As discussed in the introduction, apartments of \(T(W)\) are in bijection with frames of \(W\), i.e. sets of lines \(L = \{L_1, \ldots, L_n\}\) such that \(W = L_1 \oplus \cdots \oplus L_n\). The apartment \(A_L\) corresponding to \(L\) satisfies \([A_L] = \sum_{\sigma \in S_n} (-1)^\sigma V_\sigma\), where \(V_\sigma\) is the chamber

\[
L_\sigma(1) \subsetneq L_\sigma(2) \subsetneq \cdots \subsetneq L_\sigma(1) + \cdots + L_\sigma(n-1).
\]

**Example 5.3 (Poset of summands).** Given a finite rank projective \(O\)-module \(M\) over a Dedekind domain \(O\), let \(T(M)\) denote the poset of proper nonzero \(O\)-submodules \(U \subsetneq M\) which are direct summands of \(M\). Recall from Lemma 2.2(c) that the assignment \(V \mapsto V \cap M\) gives a bijection between the \(K\)-subspaces \(V \subsetneq M \otimes K\) and the direct summands of the \(O\)-module \(M\). This assignment preserves containment, so it gives a canonical isomorphism \(T(M \otimes K) \cong T(M)\). In particular, if \(M\) has rank \(n\), then \(T(M)\) is an \((n - 2)\)-dimensional spherical building. The apartments of \(T(M)\) correspond to frames \(L = \{L_1, \ldots, L_n\}\) of \(M \otimes K\), or equivalently to sets \(I = \{I_1, \ldots, I_n\}\) of rank 1 summands of \(M\) such that \(I_1 + \cdots + I_n\) has rank \(n\). The formula (5.1) might suggest that the chambers of \(A_I\) are given by e.g. \(I_1 \subsetneq I_1 + I_2 \subsetneq \cdots\), but this is not the case: indeed \(I_1 + I_2\) need not be a summand of \(M\) at all! The corresponding summand is \(M \cap (I_1 + I_2) \otimes K\), the “saturation” of \(I_1 + I_2\). For \(n \in \mathbb{N}\), let \([n] = \{1, \ldots, n\}\), and given \(X \subset [n]\) set

\[
U_X := M \cap \left( \bigoplus_{x \in X} I_x \otimes K \right).
\]

The chambers of \(A_I\) are then

\[
V_\sigma : \quad U_{\sigma([1])} \subsetneq U_{\sigma([2])} \subsetneq \cdots \subsetneq U_{\sigma([n-1])} \quad \text{for} \quad \sigma \in S_n,
\]

and the apartment class \([A_I] \in \bar{\Pi}_{n-2}(T(M); \mathbb{Z})\) is given by \([A_I] = \sum_{\sigma \in S_n} (-1)^\sigma V_\sigma\). A frame \(I = \{I_1, \ldots, I_n\}\) is \(M\)-integral if \(I_1 + \cdots + I_n = M\). In this case we have \(M = I_1 \oplus \cdots \oplus I_n\) and \(U_X = \bigoplus_{x \in X} I_x\), and the chambers of the \(M\)-integral apartment \(A_I\) are indeed given by

\[
I_{\sigma(1)} \subsetneq I_{\sigma(2)} \oplus I_{\sigma(2)} \subsetneq \cdots \subsetneq I_{\sigma(1)} \oplus \cdots \oplus I_{\sigma(n-1)}.
\]

**Example 5.3 (Join of discrete posets).** Let \(T\) be a finite set. Define a poset \(X_m(T) = \{1, \ldots, m\} \times T\) by declaring \((p, t) < (p', t)\) exactly when \(p < p'\). The poset \(X_m(T)\) is an \((m - 1)\)-dimensional spherical building with Coxeter group \(W = (\mathbb{Z}/2)^m\). The chambers of \(X_m(T)\) are chains

\[
(1, t_1) < (2, t_2) < \ldots < (m, t_m)
\]

with \(t_i \in T\) for \(1 \leq i \leq m\). We will denote this chamber by \((t_1, \ldots, t_m)\). The apartments of \(X_m(T)\) are in bijection with ordered sequences \(S = (S_1, \ldots, S_m)\), where \(S_i = \{a_i, b_i\}\) is a 2-element subset of \(T\) with \(a_i \neq b_i\). For \(e = (\varepsilon_1, \ldots, \varepsilon_m) \in (\mathbb{Z}/2)^m\), let \(c^e_k = b_k\) if \(\varepsilon_k = 1\) (this convention will simplify some of our later formulas). Denote the corresponding chamber by \(C_e = (c^e_1, \ldots, c^e_m)\). The apartment \(A_S\) is the union of the \(2^m\) chambers \(C_e\), and its fundamental class is

\[
[A_S] = \sum_e (-1)^e [C_e] \quad \text{where} \quad (-1)^e = (-1)^{\sum_i \varepsilon_i}.
\]

\(X_m(T)\) is the \(m\)-fold join of the discrete set \(T\), so it is homotopy equivalent to a wedge of \((|T| - 1)^m\) copies of \(S^{m-1}\). In particular, \(\bar{\Pi}_{m-1}(X_m(T); \mathbb{Z}) \cong \mathbb{Z}^N\) with \(N = (|T| - 1)^m\).
5.2 Cycles in $\mathcal{T}(M)$ and the class group $\text{cl}(O)$

Let $M$ be a rank $n$ projective $O$-module. In this section we relate the Tits building $\mathcal{T}(M)$ to the much simpler complex $X_{n-1}(\text{cl}(O))$ that we introduced in Example 5.3. Define a map

$$\psi: \mathcal{T}(M) \to X_{n-1}(\text{cl}(O))$$

via the formula $\psi(U) = (\text{rk}(U), [U])$. Given $U, U' \in \mathcal{T}(M)$ with $U \subsetneq U'$, Lemma 2.2(b) says that $U$ is a summand of $U'$, so $\psi(U) < \psi(U')$. Thus $\psi$ is a poset map. Elements $g \in \text{GL}(M)$ and $U \in \mathcal{T}(M)$ determine an isomorphism $g: U \cong g(U)$, so $\psi(g(U)) = \psi(U)$. Therefore $\psi$ induces a map $\overline{\psi}: |\mathcal{T}(M)| / \text{GL}(M) \to |X_{n-1}(\text{cl}(O))|$. 

**Proposition 5.4.** Let $M$ be a rank $n$ projective $O$-module. The map $\overline{\psi}: |\mathcal{T}(M)| / \text{GL}(M) \to |X_{n-1}(\text{cl}(O))|$ defined above is then an isomorphism of cell complexes.

**Proof.** The proposition is equivalent to the following two claims; the first implies that $\overline{\psi}$ is surjective and the second implies that $\overline{\psi}$ is injective.

1. Let $(j_1, c_1) \leq \cdots \leq (j_p, c_p)$ be a chain in $X_{n-1}(\text{cl}(O))$. Then there exists a chain $U_1 \subsetneq \cdots \subsetneq U_p$ in $\mathcal{T}(M)$ such that $\psi(U_i) = (j_i, c_i)$ for all $1 \leq i \leq p$.

2. For $\ell = 1, 2$ let $V_1^\ell \subsetneq \cdots \subsetneq V_q^\ell$ be a chain in $\mathcal{T}(M)$. Assume that $\psi(V_1^1) = \psi(V_2^1)$ for all $1 \leq i \leq q$. Then there exists some $g \in \text{GL}(M)$ such that $g(V_1^1) = V_2^1$ for all $1 \leq i \leq q$.

We begin with the first claim. Set $(j_0, c_0) = (0, 0)$ and $(j_{p+1}, c_{p+1}) = (n - j_p, [M])$. For $1 \leq i \leq p + 1$, let $\hat{U}_i$ be a projective $O$-module of rank $j_i$ with $\hat{U}_i = c_i - c_{i-1}$. Then $\hat{U}_1 \oplus \cdots \oplus \hat{U}_{p+1}$ is a projective $O$-module of rank $n$ satisfying

$$[\hat{U}_1 \oplus \cdots \oplus \hat{U}_{p+1}] = c_1 + (c_2 - c_1) + \cdots + (c_p - c_{p-1}) + ([M] - c_p) = [M],$$

so there exists an isomorphism $\eta: \hat{U}_1 \oplus \cdots \oplus \hat{U}_{p+1} \to M$. The desired summands $U_i$ are then $U_i := \eta(\hat{U}_1) \oplus \cdots \oplus \eta(\hat{U}_i)$.

We now turn to the second claim. For both $\ell = 0$ and $\ell = 1$, set $V_0^\ell = 0$ and $V_{q+1}^\ell = M$. For $1 \leq i \leq q + 1$, Lemma 2.2(b) implies that $V_i^\ell$ is a summand of $V_i^\ell$, so there exists some $\hat{V}_i^\ell \subsetneq V_i^\ell$ such that $V_i^\ell = \hat{V}_i^\ell \oplus \hat{V}_i^\ell$. We thus have $V_i^\ell = \hat{V}_1^\ell \oplus \cdots \oplus \hat{V}_q^\ell$. Observe that

$$[\hat{V}_1^1] = [V_1^1] - [V_1^0] = c_1 - c_0 = [V_2^2] - [V_2^1] = [\hat{V}_2^1].$$

and similarly $\text{rk}(\hat{V}_1^1) = \text{rk}(\hat{V}_2^2)$. Therefore there exists an isomorphism $\zeta_i: \hat{V}_i^1 \to \hat{V}_i^2$. The desired $g \in \text{GL}(M)$ is then the composition

$$M = \hat{V}_1^1 \oplus \cdots \oplus \hat{V}_{q+1}^1 \xrightarrow{\zeta_1 \oplus \cdots \oplus \zeta_{q+1}} \hat{V}_1^2 \oplus \cdots \oplus \hat{V}_{q+1}^2 = M. \quad \square$$

We now turn to the main technical theorem of §5 which asserts that the map

$$\psi_*: \tilde{H}_{n-2}(\mathcal{T}(M); \mathbb{Z}) \to \tilde{H}_{n-2}(X_{n-1}(\text{cl}(O)); \mathbb{Z})$$

is surjective. We will use the notation for apartments and chambers in these buildings that was introduced in Examples 5.2 and 5.3. The reason that this result is so hard to prove is that for a general apartment $A_I$ in $\mathcal{T}(M \otimes O K)$, it seems quite difficult to describe the image $\psi_*([A_I])$ of its fundamental class in the homology of $X_{n-1}(\text{cl}(O))$. For one thing, we saw in Example 5.2 that the summands $U_X$ that make up the chambers of $A_1$ are not easily understood; for example, knowing the images $\psi(I_k)$ of the rank 1 summands does not allow us to determine $\psi(U_X)$. For another, the apartment $A_1$ consists of $n!$ chambers, while the apartments in $X_{n-1}(\text{cl}(O))$ are made up of only $2^{n-1}$ chambers. Since $n! \gg 2^{n-1}$, in general we expect that $\psi(A_1)$ will be draped over many apartments in $X_{n-1}(\text{cl}(O))$. However, the following proposition guarantees that we can find certain special apartments $A_1$ whose image is just a single apartment $A_1$, with the excess $n! - 2^{n-1}$ chambers all “folded away” and canceling each other out precisely, leaving only the desired homology class.
Proposition 5.5 (Detecting spaces of cycles). Let $\mathcal{O}$ be a Dedekind domain, let $M$ be a rank $n$ projective $\mathcal{O}$-module, and let $A_s$ be an apartment of $X_{n-1}(\mathrm{cl}(\mathcal{O}))$. Then there exists a frame $I$ of $M$ such that $\psi_s([A_I]) = [A_s] \in \mathbb{H}_{n-2}(X_{n-1}(\mathrm{cl}(\mathcal{O})); \mathbb{Z})$. In particular, $\psi_s : \mathbb{H}_{n-2}(\mathcal{T}(M); \mathbb{Z}) \to \mathbb{H}_{n-2}(X_{n-1}(\mathrm{cl}(\mathcal{O})); \mathbb{Z})$ is surjective.

Proof. Write $S = (S_1, \ldots, S_{n-1})$ with $S_i = \{a_i, b_i\}$ for some $a_i, b_i \in \mathrm{cl}(\mathcal{O})$. For $e = (\varepsilon_1, \ldots, \varepsilon_{n-1}) \in (\mathbb{Z}/2)^{n-1}$, let $c^e_k = b_k$ if $\varepsilon_k = 0$, and $c^e_k = a_k$ if $\varepsilon_k = 1$ (as we noted in Example [5.3], this convention will simplify some of our formulas). The apartment $A_s$ is the union of the $2^{n-1}$ chambers $C_e = (c^e_1, \ldots, c^e_{n-1})$, and its fundamental class is

$$[A_s] = \sum_e (-1)^e [C_e] \quad \text{with } (-1)^e = (-1)^{\sum_i \varepsilon_i}.$$

Using Proposition 5.4 we may choose summands $B_k$ for $1 \leq k \leq n-1$ and $A^{(i)}_k$ for $1 \leq i \leq k \leq n-1$ with the following properties:

- $0 = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_{n-1} \subseteq B_n = M$ and $\text{rk}(B_k) = k, [B_k] = b_k$.
- $B_{i-1} \subseteq A^{(i)}_i \subseteq A^{(i)}_{i+1} \subseteq \cdots \subseteq A^{(i)}_{n-1} \subseteq M$ and $\text{rk}(A^{(i)}_k) = k, [A^{(i)}_k] = a_k, A^{(i)}_k \subseteq B_{k+1}$.

(First, choose all the $B_k$. Then for each $i$, choose the $A^{(i)}_k$ inductively using $B_{i-1} \subseteq A^{(i)}_i \subseteq B_{i+1}$ and $A^{(i)}_{k-1} \subseteq A^{(i)}_k \subseteq B_{k+1}$.) We now define

$$I_1 := B_1 \quad \text{and} \quad I_k := \bigcap_{i=1}^{k-1} A^{(i)}_{k-1} \quad \text{for } 1 < k \leq n.$$

Set $I = \{I_1, \ldots, I_n\}$. We will prove that $I$ is a frame of $M$ and that $\psi_s([A_I]) = [A_s]$.

Claim 1. For each $1 \leq k \leq n$ the intersection $I_k$ is a rank 1 summand of $M$. Also, $I_1 + \cdots + I_k$ is a rank $k$ $\mathcal{O}$-submodule of $M$ (not necessarily a summand). In particular, the rank of $I_1 + \cdots + I_n$ is $n$, so $I$ is a frame of $M$.

Proof of Claim 1. The bijection $V \mapsto M \cap V$ of Lemma 2.2(c) preserves intersections, so the intersection of summands of $M$ is also a summand of $M$. In particular, each $I_k$ is a summand of $M$ and $I_k \otimes \mathcal{O} K = \bigcap_{i=1}^{k-1} A^{(i)}_{k-1} \otimes \mathcal{O} K$. The intersection of the $k$-1 hyperplanes $A^{(i)}_{k-1} \otimes \mathcal{O} K, \ldots, A^{(k-1)}_{k-1} \otimes \mathcal{O} K$ inside the $k$-dimensional vector space $B_k \otimes \mathcal{O} K$ cannot vanish, so $\text{rk}(I_k) \geq 1$. We will prove equality holds, but not until the end of the proof.

Since each $A^{(i)}_{k-1}$ is contained in $B_k$, we have $I_k \subseteq B_k$. Since $B_j \subseteq B_k$ for $j < k$, we see that $I_1 + \cdots + I_k \subseteq B_k$ for all $k$.

We next claim that $I_k \cap B_{k-1} = 0$ for all $k$. The proof will be by induction on $k$. The base case $k = 1$ is trivial since $I_1 = B_1$, so assume that $k > 1$. For $1 \leq i \leq k-2$, we have $[A^{(i)}_{k-1}] = a_{k-1} \neq b_{k-1} = [B_{k-1}]$, so $A^{(i)}_{k-1} \neq B_{k-1}$. It follows that $A^{(i)}_{k-1} \cap B_{k-1}$ is a summand of rank at most $k-1$. But by construction, the rank $k-2$ summand $A^{(i)}_{k-2}$ is contained in both $A^{(i)}_{k-1}$ and in $B_{k-1}$, so it must be their intersection: $A^{(i)}_{k-1} \cap B_{k-1} = A^{(i)}_{k-2}$ for $1 \leq i \leq k-2$. In a similar way, we have $A^{(i)}_{k-1} \cap B_{k-1} = B_{k-2}$. Combining these, we find that

$$I_k \cap B_{k-1} = \left( \bigcap_{i=1}^{k-1} A^{(i)}_{k-1} \right) \cap B_{k-1} = \bigcap_{i=1}^{k-1} (A^{(i)}_{k-1} \cap B_{k-1})$$

$$= ( \bigcap_{i=1}^{k-2} A^{(i)}_{k-1} ) \cap B_{k-2} = I_{k-1} \cap B_{k-2}.$$
By induction, this is zero, as claimed.

Finally, we prove $I_k$ has rank 1 and that $I_1 + \cdots + I_k$ has rank $k$. The proof of this is by induction on $k$. The base case $k = 1$ is trivial, so assume that $k > 1$, that $I_{k-1}$ has rank 1, and that $I_1 + \cdots + I_{k-1}$ has rank $k-1$. From the containments

$$I_1 + \cdots + I_{k-1} \subset I_1 + \cdots + I_k \subset B_k$$

we see that $k-1 \leq \text{rk}(I_1 + \cdots + I_k) \leq k$. Assume for the sake of contradiction that $\text{rk}(I_1 + \cdots + I_k) < k$ or that $\text{rk}(I_k) > 1$. In either case, the subspaces $I_k \otimes \mathcal{O} K$ and $(I_1 + \cdots + I_{k-1}) \otimes \mathcal{O} K$ must intersect nontrivially inside the $k$-dimensional vector space $B_k \otimes \mathcal{O} K$. Multiplying by an appropriate denominator, it follows that $(I_1 + \cdots + I_{k-1}) \cap I_k \neq 0$. However we proved above that $B_{k-1} \cap I_k = 0$; since $I_1 + \cdots + I_{k-1} \subset B_{k-1}$, this is a contradiction. Therefore $\text{rk}(I_1 + \cdots + I_k) = k$ and $\text{rk}(I_k) = 1$, as desired. This concludes the proof of Claim 2. \hfill \Box

It remains to prove that $\psi_*([A_1]) = [A_S]$. As explained in Example 5.2, the chambers of the apartment $A_1$ are given by

$$V_\sigma := U_{\sigma([1])} \leq U_{\sigma([2])} \leq \cdots \leq U_{\sigma([n-1])}$$

for $\sigma \in S_n$.

The image $\psi(V_\sigma) \subset \psi(A_{L}) \subset X_{n-1}(\text{cl}(\mathcal{O}))$ is the chamber determined by the sequence

$$([U_{\sigma([1])}], [U_{\sigma([2])}], \ldots, [U_{\sigma([n-1])}]).$$

For most subsets $X \subset [n]$, we cannot say anything about the summand $U_X$ or its class $[U_X]$, but there are two exceptions.

**Claim 2.** We have the following trichotomy.

1. $U_{[k]} = B_k$, so $[U_{[k]}] = b_k$.
2. If $X = [k+1] \setminus \{i\}$ for some $1 \leq i \leq k$, then $U_X = A_k^{(i)}$, so $[U_X] = a_k$.
3. For other subsets $X \subset [n]$, we know nothing.

**Proof of Claim 2** Case 1: We saw above that $I_1 + \cdots + I_k \subset B_k$, and Claim 1 states that $\text{rk}(I_1 + \cdots + I_k) = k$, so $(I_1 + \cdots + I_k) \otimes \mathcal{O} K = B_k \otimes \mathcal{O} K$. Since $U_{[k]} = M \cap ((I_1 + \cdots + I_k) \otimes \mathcal{O} K)$ by definition, this means that $U_{[k]} = M \cap (B_k \otimes \mathcal{O} K)$. But $B_k$ is a summand of $M$, so $M \cap (B_k \otimes \mathcal{O} K) = B_k$.

Case 2: We begin by observing that $I_x \subset A_k^{(i)}$ whenever $x \leq k+1$ and $i \neq x$. For $i < x$ this follows from $I_x \subset A_{x-1}^{(i)}$ (by the definition of $I_x$) and $A_{x-1}^{(i)} \subset A_k^{(i)}$. For $i > x$, we use instead that $I_x \subset B_x \subset B_{i-1} \subset A_i^{(i)} \subset A_k^{(i)}$. Let $X = [k+1] \setminus \{i\}$ for some $1 \leq i \leq k$. It follows from Claim 1 that $I_1 + \cdots + \tilde{I}_i + \cdots + I_{k+1}$ has rank $k$, and the beginning of this paragraph shows that this submodule is contained in the rank $k$ summand $A_k^{(i)}$. Just as above, this implies that $U_X = M \cap ((I_1 + \cdots + \tilde{I}_i + \cdots + I_{k+1}) \otimes \mathcal{O} K)$ coincides with $A_k^{(i)}$. This concludes the proof of Claim 2. \hfill \Box

We can analyze this trichotomy in terms of $\sigma \in S_n$, as follows. Given $\sigma \in S_n$, think of $\sigma$ as a function $\sigma : [n] \to [n]$, and define $s_\sigma : [n] \to [n]$ by

$$s_\sigma(k) = \sup\{\sigma(i) \mid 1 \leq i \leq k\}.$$
The function \( s_\sigma \) is monotone increasing and satisfies \( k \leq s_\sigma(k) \leq n \). If \( s_\sigma(k) = k \) then it must be that \( \sigma([k]) = [k] \), so \( [U_{\sigma([k])}] = [B_k] = b_k \). If \( s_\sigma(k) = k + 1 \), it must be that \( \sigma([k]) = [k + 1] - \{i\} \) for some \( 1 \leq i \leq k \), so \( [U_{\sigma([k])}] = [A^{(i)}_k] = a_k \). To sum up, we have

\[
[U_{\sigma([k])}] = \begin{cases} 
  b_k & \text{if } s_\sigma(k) = k \\
  a_k & \text{if } s_\sigma(k) = k + 1 \\
  \text{unknown} & \text{otherwise}
\end{cases} \tag{5.3}
\]

Accordingly, we divide the permutations \( \sigma \in S_n \) into two subsets: we say that \( \sigma \in S_n \) is good if \( s_\sigma(k) \in \{k, k + 1\} \) for all \( k \), and say that \( \sigma \) is bad otherwise.

**Claim 3.** The good permutations are the \( 2^{n-1} \) permutations of the form

\[
\sigma_e := (1 \ 2)^{\varepsilon_1}(2 \ 3)^{\varepsilon_2} \cdots (n-1 \ n)^{\varepsilon_{n-1}} \in S_n
\]

for some \( e = (\varepsilon_1, \ldots, \varepsilon_{n-1}) \in (\mathbb{Z}/2)^{n-1} \). For such a good permutation \( \sigma = \sigma_e \),

\[
s_\sigma(k) = k \iff \varepsilon_k = 0 \quad s_\sigma(k) = k + 1 \iff \varepsilon_k = 1 \quad \text{for } k < n.
\]

**Proof of Claim 3.** The proof is by induction on \( n \). The base case is \( n = 1 \), where the claim is trivial. Assume now that \( n \geq 2 \) and that the claim is true for smaller \( n \). Let \( \sigma \in S_n \) be a good permutation. Since \( s_\sigma(k) \geq \sigma(k) \), the condition \( s_\sigma(k) \leq k + 1 \) for all \( k \) implies that \( \sigma(k) \neq n \) for any \( k < n - 1 \). Therefore we have either \( \sigma(n) = n \) or \( \sigma(n - 1) = n \). In the first case, the restriction \( \sigma|_{n-1} \) is a good permutation in \( S_{n-1} \), so the claim holds by induction with \( \varepsilon_{n-1} = 0 \); since \( \sigma(n) = n \) in this case, we must have \( s_\sigma(n - 1) = n - 1 \). In the second case, we have \( s_\sigma(n - 1) = n \). Moreover, the permutation \( \sigma' := \sigma(n - 1 \ n) \) satisfies \( \sigma'(n) = n \) and is good (since \( \sigma'|_{n-2} = \sigma|_{n-2} \)), so applying the first case to \( \sigma' \) proves Claim 3. \( \square \)

We can now divide the chambers of \( A_1 \) between the two chains defined by

\[
[A_1]^{\text{good}} := \sum_{\sigma \in S_n \text{ good}} (-1)^\sigma V_\sigma \quad \text{and} \quad [A_1]^{\text{bad}} := \sum_{\sigma \in S_n \text{ bad}} (-1)^\sigma V_\sigma,
\]

so \( [A_1] = [A_1]^{\text{good}} + [A_1]^{\text{bad}} \) as chains in \( C_{n-2}(T(M)) \).

**Claim 4.** We have \( \psi_*([A_1]^{\text{good}}) = [A_S] \).

**Proof of Claim 4.** Consider \( e = (\varepsilon_1, \ldots, \varepsilon_{n-1}) \in (\mathbb{Z}/2)^{n-1} \). By Claim 3, \( s_{\sigma_e}(k) = k + \varepsilon_k \) for all \( 1 \leq k < n \). By (5.3) this implies \( [U_{\sigma_e([k])}] = c_k^e \) for all \( k \) (recall that \( c_k^e \) is either \( b_k \) or \( a_k \) depending on whether \( \varepsilon_k \) is 0 or 1). Therefore the image \( \psi(V_\sigma_e) \) of the chamber \( V_\sigma_e \) is precisely the chamber \( C_e \). Since

\[
(-1)^{\sigma_e} = (-1)^{\sum \varepsilon_i} = (-1)^e,
\]

the orientations of \( V_\sigma_e \) and \( C_e \) agree, proving Claim 4. \( \square \)

To deal with \( [A_1]^{\text{bad}} \), we define an involution \( \sigma \mapsto \overline{\sigma} \) on the set of bad permutations \( \sigma \in S_n \) as follows. Given a bad permutation \( \sigma \), define

\[
x_\sigma = \sup\{x \in \{2, \ldots, n-1\} \mid s_\sigma(x - 1) > x\},
\]

\[
y_\sigma = \inf\{y \in \{x_\sigma + 1, \ldots, n\} \mid s_\sigma(y) = y\}.
\]

There exists some \( x \) with \( s_\sigma(x - 1) > x \) if and only if \( \sigma \) is bad, so \( x_\sigma \) is well-defined. The second condition is always satisfied by \( y = n \), so \( y_\sigma \) is well-defined as well. We define

\[
\overline{\sigma} := \sigma(x_\sigma \ y_\sigma) \in S_n.
\]
Claim 5. The map $\sigma \mapsto \overline{\sigma}$ is an involution on the set of bad permutations.

Proof of Claim 5. Let $\sigma \in S_n$ be a bad permutation. We will show that $s_\sigma = s_{\overline{\sigma}}$; since $x_\sigma$ and $y_\sigma$ only depend on $s_\sigma$, this implies that $x_\sigma = x_{\overline{\sigma}}$ and $y_\sigma = y_{\overline{\sigma}}$ and thus that $\overline{\sigma} = \sigma$.

For $k < x_\sigma$ or $k \geq y_\sigma$ we have $\sigma([k]) = \overline{\sigma}([k])$, so $s_\sigma(k) = s_{\overline{\sigma}}(k)$ automatically. Considering the remaining terms we find that

$$\text{for } x_\sigma \leq k < y_\sigma, \quad \text{we must have } s_\sigma(k) = k + 1. \quad (5.4)$$

Indeed, to have $s_\sigma(k) > k + 1$ would contradict the definition of $x_\sigma$, and to have $s_\sigma(k) = k$ would contradict the definition of $y_\sigma$. It follows that $\sigma(k) = k + 1$ for $x_\sigma < k < y_\sigma$. By the pigeonhole principle, $\sigma(y_\sigma) \leq x_\sigma + 1$. But if $\sigma(y_\sigma)$ were $x_\sigma + 1$ we could not have $s_\sigma(x_\sigma) = x_\sigma + 1 = s_\sigma(x_\sigma - 1)$, so in fact $\sigma(y_\sigma) \leq x_\sigma$. For the same reason, we must have $\sigma(x_\sigma) \leq x_\sigma$.

Since $\sigma(x_\sigma) \leq x_\sigma$ and $\sigma(y_\sigma) \leq x_\sigma$ while $\sup \sigma([k]) = s_\sigma(k) = k + 1$ for $x_\sigma \leq k < y_\sigma$, we see that neither $\sigma(x_\sigma)$ nor $\sigma(y_\sigma)$ ever realizes this supremum. Therefore exchanging $\sigma(x_\sigma)$ and $\sigma(y_\sigma)$ will not change $\sup \sigma([k])$ for any $k$. This shows that $s_\sigma = s_{\overline{\sigma}}$, proving Claim 5. \hfill \Box

Claim 6. We have $\psi_\ast([A_1]^{\text{bad}}) = 0$.

Proof of Claim 6. Let $\sigma \in S_n$ be a bad permutation. By definition, we have $(-1)^\sigma = (-1)^{\overline{\sigma}}$, so it is enough to prove that the chambers $\psi(\sigma) = \psi(\overline{\sigma})$ coincide. In other words, we must show that $[U_{\sigma([k])}] = [U_{\overline{\sigma}([k])}]$ for all $1 \leq k \leq n - 1$. For all $k < x_\sigma$ or $k \geq y_\sigma$, we have $(x_\sigma, y_\sigma)([k]) = [k]$, so $\sigma([k]) = \sigma([k])$ and the claim is automatic. For the remaining $x_\sigma \leq k < y_\sigma$, we showed in (5.4) that $s_\sigma(k) = s_{\overline{\sigma}}(k) = k + 1$. By (5.3), for such $k$ we have $[U_{\sigma([k])}] = a_k = [U_{\overline{\sigma}([k])}]$. This concludes the proof of Claim 6. \hfill \Box

By definition $[A_1] = [A_1]^{\text{good}} + [A_1]^{\text{bad}}$, and Claims 4 and 5 state that

$$\psi_\ast([A_1]^{\text{good}}) = [A_S] \quad \text{and} \quad \psi_\ast([A_1]^{\text{bad}}) = 0,$$

so $\psi_\ast([A_1]) = [A_S]$, as desired. This concludes the proof of Proposition 5.5. \hfill \Box

5.3 Proof of the Non-Vanishing and Non-Integrality Theorems

In this section we prove Theorems [D] and [E]. In fact, we prove the stronger Theorems [D'] and [E'] which apply to the automorphism groups of arbitrary (not necessarily free) finite rank projective modules. If $M$ is a rank $n$ projective $\mathcal{O}_K$-module, its automorphism group $GL(M)$ is also a lattice in $GL_n(\mathcal{O}_K \otimes \mathbb{R})$, commensurable with but not in general isomorphic to $GL_n \mathcal{O}_K$. As before, $\nu_n = \text{vcd}(GL_n \mathcal{O}_K)$ and $\nu'_n = \text{vcd}(\text{SL}_n \mathcal{O}_K)$.

Theorem D'. Let $K$ be a number field with ring of integers $\mathcal{O}_K$ and let $M$ be a rank $n$ projective $\mathcal{O}_K$-module. Then for all $n \geq 2$,

$$\dim H^n_{\mathcal{U}}(\text{SL}(M); \mathbb{Q}) \geq \dim H^n_{\mathcal{U}}(\text{GL}(M); \mathbb{Q}) \geq (|\text{cd}(\mathcal{O}_K)| - 1)^{n-1}.$$

Theorem [D] is the special case of Theorem [D'] with $M \cong \mathcal{O}_K^n$.

Proof. Define $\text{St}(M) := \text{H}_n(M; \mathbb{Z})$. The automorphism group $GL(M)$ acts naturally on $\mathcal{T}(M)$, and thus on $\text{St}(M)$. Since $\text{SL}(M)$ is commensurable with $\text{SL}_n \mathcal{O}_K$, it follows from [BE] Theorems 3.2, 3.3] that $GL(M)$ and $\text{SL}(M)$ are virtual duality groups with rational
Theorem B'. Let \( O \) be a Dedekind domain with \( 1 < \| \text{cl}(O) \| < \infty \), and let \( M \) be a rank \( n \) projective \( O \)-module for some \( n \geq 2 \). If \( \| \text{cl}(O) \| = 2 \), assume that \( M \) is not the unique non-free \( O \)-module of rank 2. Then \( \text{St}(M) \) is not generated by \( M \)-integral apartment classes.

**Proof.** Let \( \text{St}^{\text{Int}}(M) \) be the subspace of \( \text{St}(M) \) spanned by \( M \)-integral apartment classes. By Proposition 5.5, the map \( \psi_* : \text{St}(M) \rightarrow \tilde{H}_{n-2}(X_n(\text{cl}(O)); \mathbb{Z}) \) is surjective. Thus to prove that \( \text{St}^{\text{Int}}(M) \neq \text{St}(M) \) it is enough to prove that \( \psi_*|_{\text{St}^{\text{Int}}(M)} \) is a proper subspace of \( \tilde{H}_{n-2}(X_n(\text{cl}(O)); \mathbb{Z}) \).

Consider an \( M \)-integral frame \( I = \{ I_1, \ldots, I_n \} \) with \( [I_k] = c_k \) and \( M = I_1 \oplus \cdots \oplus I_n \); the latter property implies that \( c_1 + \cdots + c_n = [M] \in \text{cl}(O) \). We claim first that \( \psi_*([A_I]) \) only depends on \( \{ c_1, \ldots, c_n \} \), and not on the particular integral frame \( I \). To see this, recall from Example 5.2 that the chambers of an integral apartment \( A_I \) are of the form

\[
V_\sigma = I_{\sigma(1)} \subseteq I_{\sigma(1)} \oplus I_{\sigma(2)} \subseteq \cdots \subseteq I_{\sigma(1)} \oplus \cdots \oplus I_{\sigma(n-1)}
\]

for \( \sigma \in S_n \). Given \( X \subset [n] \), we have \( \bigoplus_{k \in X} I_k = \sum_{k \in X} c_k \), so the chamber \( V_\sigma \) is taken by \( \pi \) to the chamber

\[
C_\sigma = (c_{\sigma(1)}, c_{\sigma(1)} + c_{\sigma(2)}, \ldots, c_{\sigma(1)} + \cdots + c_{\sigma(n-1)}).
\]

\[\text{Equation (5.5)}\]
The claim follows.

Next, assume for the moment that \( c_i = c_j \) for some \( i \neq j \). Define the involution \( \sigma \mapsto \sigma \) on \( S_n \) by \( \sigma = (i \ j) \sigma \). From (5.5) we see that \( C_\sigma \) will coincide with \( C_\sigma \) for all \( \sigma \in S_n \). Since \((-1)^2 = (-1) \cdot (-1)^\sigma \), the chamber \( V_\sigma \) occurs with opposite orientation from \( V_\sigma \) in \([A_1]\). We conclude that \( \psi_*([A_1]) = 0 \) unless the classes \( c_1, \ldots, c_n \in cl(O) \) are distinct, in which case (up to sign) \( \psi_*([A_1]) \) only depends on the unordered set \( \{c_1, \ldots, c_n\} \subset cl(O) \). This already shows that when \( n > |cl(O)| \), the image \( \psi_*({\text{St}}^{\text{Int}}(M)) \) vanishes.

We complete the proof by an elementary counting argument. By Example 5.3 the rank of \( \tilde{H}_{n-2}(X_n(cl(O)); \mathbb{Z}) \) is \( (|cl(O)| - 1)^{n-1} \). The precise number of \( n \)-element subsets \( \{c_1, \ldots, c_n\} \subset cl(O) \) satisfying \( c_1 + \cdots + c_n = [M] \) is hard to calculate and depends on the group structure of \( cl(O) \). One case is easy, however: when \( cl(O) = \mathbb{Z}/2\mathbb{Z} \), the relation \( c_1 + \cdots + c_n = [M] \) can never be satisfied by distinct elements \( c_i \in cl(O) \) for any \( n \geq 2 \), excepting the single case \( n = 2 \) and \([M] \neq 0 \). Therefore, whenever \( |cl(O)| = 2 \) (excepting the excluded case when \( n = 2 \) and \([M] \neq 0 \)) we have \( \psi_*({\text{St}}^{\text{Int}}(M)) = 0 \). Since \( \tilde{H}_{n-2}(X_n(cl(O)); \mathbb{Z}) = \mathbb{Z} \) when \( |cl(O)| = 2 \), this concludes the proof in that case.

For all other nontrivial class groups, the relation \( c_1 + \cdots + c_n = [M] \) implies that \( c_n \) is determined by \( \{c_1, \ldots, c_{n-1}\} \). This lets us bound the rank of \( \psi_*({\text{St}}^{\text{Int}}(M)) \) by \( \binom{|cl(O)|}{n-1} \). Elementary estimates show that

\[
(|cl(O)| - 1)^{n-1} > \binom{|cl(O)|}{n-1}
\]

for all \( n \geq 2 \) whenever \( |cl(O)| > 2 \). Therefore \( \psi_*({\text{St}}^{\text{Int}}(M)) \) must be a proper subspace of \( \tilde{H}_{n-2}(X_n(cl(O)); \mathbb{Z}) \), as desired. \( \square \)

In the excluded case when \( cl(O) \cong \mathbb{Z}/2\mathbb{Z} \), let \( I \subset O \) be a non-principal ideal, so that \( M \cong O \oplus I \). It can be verified as in Proposition 2.7 that \( St(M) \) is spanned by \( M \)-integral apartment classes if and only if the group \( SL(M) \) is generated by diagonal matrices together with elementary matrices of the form \( \begin{pmatrix} 1 & I \\ 0 & 1 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 0 \\ I^{-1} & 1 \end{pmatrix} \). It would be interesting to know when this condition holds for \( SL(M) \); we are not aware of any results on this subject.

References


