

On the geometric nature of characteristic classes of surface bundles

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Abstract

Each Morita–Mumford–Miller (MMM) class e_n assigns to each genus $g \geq 2$ surface bundle $\Sigma_g \rightarrow E^{2n+2} \rightarrow M^{2n}$ an integer $e_n^\#(E \rightarrow M) := \langle e_n, [M] \rangle \in \mathbb{Z}$. We prove that when n is odd the number $e_n^\#(E \rightarrow M)$ depends only on the diffeomorphism type of E , not on g , M or the map $E \rightarrow M$. Applying deep results of Pontryagin and Novikov, we generalize this to prove that $e_n^\#(E \rightarrow M)$ depends only on the topological cobordism type of E . Recent work of Hatcher implies that this stronger statement is false when n is even.

We give a general procedure to construct manifolds fibering as surface bundles in multiple ways, providing infinitely many examples to which our theorems apply. As an application of our results we give a new proof of the rational case of a recent theorem of Giansiracusa–Tillmann [GT, Theorem A] that the odd MMM classes e_{2i-1} vanish for any surface bundle which bounds a handlebody bundle. We show how the MMM classes can be seen as obstructions to low-genus fiberings. Finally, we discuss a number of open questions that arise from this work.

1 Introduction

The purpose of this paper is to explain our discovery of certain coincidences between certain characteristic numbers associated to surface bundles. This can be considered a step towards understanding the geometric meaning of the Morita–Mumford–Miller classes.

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Characteristic classes of surface bundles. Let Σ_g be a closed oriented surface of genus $g \geq 2$. A Σ_g -bundle over a base space B is a smooth oriented fiber bundle

$$\Sigma_g \rightarrow E \rightarrow B \tag{1}$$

with structure group $\text{Diff}^+(\Sigma_g)$.

An i -dimensional *characteristic class* c for Σ_g -bundles is a natural association of an element $c(E \rightarrow B) \in H^i(B; \mathbb{Q})$ to every bundle as in (1). To be *natural* means that if $\Sigma_g \rightarrow E' \rightarrow B'$ is any bundle and if $f: B \rightarrow B'$ is any continuous map covered by a bundle map $E \rightarrow E'$ then

$$c(E \rightarrow B) = f^*c(E' \rightarrow B').$$

If the association c is defined for all $g \geq 2$ then c is called a *characteristic class of surface bundles*. Characteristic classes are fundamental isomorphism invariants for surface bundles. When B is a closed, oriented, i -dimensional manifold, one obtains a numerical invariant $c^\#(E \rightarrow B) \in \mathbb{Q}$, called a *characteristic number*, by pairing $c \in H^i(B; \mathbb{Q})$ with the fundamental class $[B] \in H_i(B; \mathbb{Q})$:

$$c^\#(E \rightarrow B) := \langle c(E \rightarrow B), [B] \rangle \in \mathbb{Q}.$$

Notational convention. Unless otherwise specified, all manifolds are assumed to be smooth and oriented, all bundles in this paper are assumed to be bundles of closed manifolds, and all maps are assumed to be orientation-preserving.

Morita–Mumford–Miller classes. The main examples of characteristic classes of Σ_g -bundles were given by Miller, Morita and Mumford (see, e.g. [Mo]), and can be constructed as follows. Let $\Sigma \rightarrow E \xrightarrow{\pi} B$ be any surface bundle. The kernel of the differential $d\pi$ is a 2-dimensional oriented real vector bundle $T\pi \rightarrow E$. Associated to an oriented 2-plane bundle is its *Euler class* $e(T\pi) \in H^2(E)$; we will denote it simply by $e \in H^2(E)$. Let $\int_\Sigma: H^*(E) \rightarrow H^{*-2}(B)$ be the Gysin map, given by integration along the fiber Σ . The i -th *Morita–Mumford–Miller (or MMM) class* $e_i(E \rightarrow B) \in H^{2i}(B)$ is defined by the formula

$$e_i(E \rightarrow B) = \int_\Sigma e^{i+1}.$$

The naturality of the Euler class implies that each e_i is a characteristic class for Σ_g -bundles. Moreover, each e_i is *stable*; this means roughly that when it is possible to take the fiber-wise connected sum of $\Sigma_g \rightarrow E \rightarrow B$ with $\Sigma_1 \rightarrow \Sigma_1 \times B \rightarrow B$, the resulting

bundle $\Sigma_{g+1} \rightarrow E' \rightarrow B$ satisfies $e_i(E' \rightarrow B) = e_i(E \rightarrow B)$. More strongly, each e_i is *primitive*, meaning roughly that when the fiber-wise connected sum of $\Sigma_g \rightarrow E_1 \rightarrow B$ with $\Sigma_h \rightarrow E_2 \rightarrow B$ exists, the resulting bundle $\Sigma_{g+h} \rightarrow E' \rightarrow B$ satisfies

$$e_i(E' \rightarrow B) = e_i(E_1 \rightarrow B) + e_i(E_2 \rightarrow B).$$

Generalizing a construction due to Kodaira and Atiyah, Morita and Miller constructed explicit Σ_g -bundles to show that each polynomial $P(e_1, e_2, \dots)$ yields a characteristic class which is nonzero for g large enough (depending on P).

Madsen–Weiss [MW] proved the remarkable theorem that the MMM classes yield all the stable characteristic classes: any characteristic class of surface bundles which is stable must be a polynomial $P(e_1, e_2, \dots)$ in the e_i , and any characteristic class of surface bundles which is primitive must be one of the MMM classes e_i themselves. In spite of this progress, many basic geometric questions concerning the MMM classes e_i remain open.

Fibered 4-manifolds. The main result of this paper begins with an observation about e_1 . The Miller–Morita examples proving nontriviality of e_1 reduce to the original 4-dimensional surface bundles $\Sigma_g \rightarrow E^4 \rightarrow \Sigma_h$ constructed by Kodaira and Atiyah for certain specific g, h (see e.g. [Mo, §4.3.3]). It turns out that these 4-manifolds actually fiber as a surface bundle over a surface in multiple distinct ways. For example the Atiyah–Kodaira manifold

$$\Sigma_4 \rightarrow N^4 \rightarrow \Sigma_{17}$$

also fibers as

$$\Sigma_{49} \rightarrow N^4 \rightarrow \Sigma_2.$$

The associated characteristic classes

$$e_1(N^4 \rightarrow \Sigma_{17}) \in H^2(\Sigma_{17}; \mathbb{Z}) \quad \text{and} \quad e_1(N^4 \rightarrow \Sigma_2) \in H^2(\Sigma_2; \mathbb{Z})$$

are *a priori* unrelated. (Indeed, they come from cohomology classes living in two different groups, $H^2(\text{BDiff}^+(\Sigma_4); \mathbb{Z})$ and $H^2(\text{BDiff}^+(\Sigma_{49}); \mathbb{Z})$ respectively; see Section 2.5.) The base spaces of the two fiberings of N^4 are also different manifolds. However, pairing each of these characteristic classes with the fundamental class of the base space of the bundle gives (via a computation, e.g. following Hirzebruch [Hi]):

$$\langle e_1(N^4 \rightarrow \Sigma_{17}), [\Sigma_{17}] \rangle = 96 \quad \langle e_1(N^4 \rightarrow \Sigma_2), [\Sigma_2] \rangle = 96$$

and so the characteristic numbers of the two different bundles coincide.

This coincidence is explained by the Hirzebruch Signature Formula, which implies for any fibering $\Sigma_g \rightarrow N^4 \rightarrow \Sigma_h$ the formula

$$\langle e_1(N^4 \rightarrow \Sigma_h), [\Sigma_h] \rangle = 3 \cdot \sigma(N^4) \quad (2)$$

where $\sigma(N^4)$ is the *signature* of N^4 , i.e. the signature of the intersection pairing on $H^2(N^4; \mathbb{Q})$. In particular, since the right hand side of (2) depends only on N^4 and not on the choice of fibering $N^4 \rightarrow \Sigma_h$, the same holds for the left hand side.

Geometric characteristic classes. With the above example in mind we make the following definition.

Definition 1.1 (Geometric characteristic class). An i -dimensional characteristic class c for surface bundles is *geometric* if the associated characteristic number $c^\#$ depends only on the diffeomorphism class of the total space of the bundle. More precisely, whenever

$$\Sigma_g \rightarrow E^{i+2} \rightarrow M^i$$

and

$$\Sigma_h \rightarrow X^{i+2} \rightarrow N^i$$

are surface bundles with diffeomorphic total spaces $E^{i+2} \approx X^{i+2}$, the associated characteristic numbers are equal:

$$c^\#(E^{i+2} \rightarrow M^i) = c^\#(X^{i+2} \rightarrow N^i) \in \mathbb{Q}.$$

We also have the corresponding notions of *geometric with respect to homeomorphisms, homotopy equivalences, cobordisms, etc.*, where each characteristic number $c^\#(E^{i+2} \rightarrow M^i)$ depends only on the homeomorphism type (resp. homotopy type, cobordism class, etc.) of the total space E^{i+2} , not on the choice of fibering of E^{i+2} . As we will see below, being geometric with respect to these more general equivalences can be a strictly more difficult condition to satisfy than being geometric with respect to diffeomorphisms.

The first main theorem of this paper is the following.

Theorem 1.2 (Odd MMM classes are geometric). *For each $n \geq 1$ the Morita–Mumford–Miller class e_{2n-1} is geometric.*

We prove Theorem 1.2 in Section 2. Theorem 1.2 leads to an obvious question:

Question 1.3. *Are the even MMM classes e_{2n} geometric?*

In all examples that we have considered of surface bundles fibering in multiple ways, the even MMM classes e_{2n} do not depend on the fibering, supporting an affirmative answer to Question 1.3. In particular, this holds for the surface bundles constructed in Section 3. On the other hand, we will see later in the introduction that a slightly weaker version of Question 1.3 has a negative answer.

Geometric with respect to cobordism. Theorem 1.2 actually follows from the following more general result.

Theorem 1.4 (Odd MMM classes are cobordism invariants). *For each $n \geq 1$ the MMM class e_{2n-1} is geometric with respect to topological cobordisms. In other words, let $\Sigma_g \rightarrow E^{4n} \rightarrow M^{4n-2}$ and $\Sigma_h \rightarrow \tilde{E}^{4n} \rightarrow \tilde{M}^{4n-2}$ be surface bundles. If E and \tilde{E} are topologically cobordant then the corresponding characteristic numbers coincide:*

$$e_{2n-1}^\#(E \rightarrow M) = e_{2n-1}^\#(\tilde{E} \rightarrow \tilde{M}).$$

We prove Theorem 1.4 by proving that for any surface bundle $\Sigma \rightarrow E^{4n} \rightarrow M^{4n-2}$, the characteristic number $e_{2n-1}^\#(E \rightarrow M)$ can be expressed as the characteristic number associated to a certain explicit polynomial in the Pontryagin classes of E . The polynomials involved are rather complicated. For example, for e_{11} and $\Sigma \rightarrow E^{24} \rightarrow M^{22}$ we will prove that (suppressing the $\#$ superscripts on the right side for readability):

$$\begin{aligned} e_{11}^\#(E \rightarrow M) = & p_1(E)^6 - 6p_1(E)^4p_2(E) + 6p_1(E)^3p_3(E) + 9p_1(E)^2p_2(E)^2 \\ & - 6p_1(E)^2p_4(E) + 6p_1(E)p_5(E) - 12p_1(E)p_2(E)p_3(E) \\ & - 2p_2(E)^3 + 6p_2(E)p_4(E) + 3p_3(E)^2 - 6p_6(E) \end{aligned}$$

It follows that $e_{2n-1}^\#$ does not depend on either M or on the fibering $E \rightarrow M$, but only on the diffeomorphism type of E . Moreover, recall that Pontryagin proved that the Pontryagin numbers are cobordism invariants, and Novikov proved that the rational Pontryagin classes are topological invariants. This implies that $e_{2n-1}^\#$ in fact only depends on the topological cobordism class of E .

In contrast to Theorem 1.4, we will see below that the *even* classes e_{2n} are *not* geometric with respect to cobordism; indeed no polynomial in the $\{e_{2i}\}$ is geometric with respect to cobordism. In Section 3 we give a general method of constructing closed oriented manifolds which fiber as surface bundles in more than one way. This gives many explicit examples to which Theorem 1.4 applies.

Other characteristic classes of surface bundles. We can ask a much more general question than Question 1.3. Suppose $P(t_1, t_2, \dots)$ is a homogeneous polynomial of total degree j , where t_i has degree i . For any surface bundle $\Sigma_g \rightarrow E^{2j+2} \rightarrow M^{2j}$ over a closed, oriented manifold M^{2j} , we have a well-defined characteristic number

$$P^\#(E \rightarrow M) := \langle P(e_1(E \rightarrow M), e_2(E \rightarrow M), \dots), [M] \rangle \in \mathbb{Z}$$

and so we can formulate the following question.

Question 1.5. *Which homogeneous polynomials P in the MMM classes e_i are geometric?*

Note that the property of being geometric is not *a priori* preserved under cup product. Thus an affirmative answer to Question 1.3 combined with Theorem 1.2 would still not imply that other polynomials in the MMM classes are geometric. For example, we do not know whether or not e_1^2 is geometric.

Handlebody bundles. The *genus g handlebody* V_g is the 3-manifold with $\partial V_g = \Sigma_g$ obtained as the regular neighborhood of a graph in \mathbb{R}^3 . Giansiracusa–Tillmann [GT, Theorem A] recently proved that the odd MMM classes e_{2n-1} vanish for any surface bundle which bounds a handlebody bundle. As a corollary to Theorem 1.4 we obtain another proof of this result in rational cohomology.

Corollary 1.6. *For any surface bundle $\Sigma_g \rightarrow E \rightarrow B$ over any base space B , if there exists a bundle $V_g \rightarrow W \rightarrow B$ whose fiberwise boundary is $\Sigma_g \rightarrow E \rightarrow B$, then $e_{2n-1}(E \rightarrow B) = 0 \in H^{4n-2}(B; \mathbb{Q})$ for all $n \geq 1$.*

The MMM classes can be defined in integral cohomology. In this case the theorem of Giansiracusa–Tillmann still holds, but this is not implied by Theorem 1.4.

As we remarked above, being geometric with respect to cobordism is a significantly more stringent condition than just being geometric. For example, we posed above the open question of whether the *even* MMM classes are geometric. But as pointed out

to us by Jeffrey Giansiracusa, handlebody bundles can be used to show that the even MMM classes are *not* geometric with respect to cobordism, as follows.

Hatcher [Hat] recently announced an analogue of the theorem of Madsen–Weiss for handlebody bundles. His theorem implies that the stable characteristic classes of handlebody bundles (equivalently, of surface bundles which bound handlebody bundles) are exactly the polynomials in the even MMM classes, and that each such polynomial gives a nontrivial characteristic class. Appealing to a theorem of Thom (see the proof of Corollary 1.6 for details), this implies that for every such polynomial $P(e_2, e_4, \dots)$, there is a bundle $\Sigma_g \rightarrow E^{4n+2} \rightarrow B^{4n}$ which bounds $V_g \rightarrow W^{4n+3} \rightarrow B^{4n}$ but has $P(e_2(E \rightarrow B), e_4(E \rightarrow B), \dots) \neq 0$. On the other hand, such a surface bundle is cobordant to the trivial bundle, and so its characteristic numbers must vanish. We thus conclude the following.

No polynomial in the even MMM classes is geometric with respect to cobordism.

MMM classes as obstructions to fibering. Any characteristic class which is geometric gives an obstruction to the existence of a fibering with fiber a surface of small genus. For example, we will use Theorem 1.2 below to prove the following corollary.

Corollary 1.7. *Let $n \geq 1$. Let $E^{4n} \rightarrow M^{4n-2}$ be any surface bundle for which $e_{2n-1}^\#(E \rightarrow M) \neq 0$. Then any fibering*

$$\Sigma_g \rightarrow E^{4n} \rightarrow N^{4n-2}$$

of E as a surface bundle must have $g \geq n + 1$. In fact this holds for any fibering of any manifold topologically cobordant to E^{4n} .

Outline of paper. In Section 2 we prove Theorem 1.2 and Theorem 1.4 and deduce various corollaries. In Section 2.3 we prove Corollary 1.6. In Section 2.5 we give another perspective on Theorem 1.4 in terms of bordism groups. In Section 3 we construct examples of closed manifolds fibering as surface bundles in more than one way. In Section 4 we describe different ways to generalize the notion of geometric characteristic class, including bundles with higher-dimensional fiber and vector bundles.

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2 Geometric characteristic classes of surface bundles

In the rest of the paper, all cohomology groups are with coefficients in \mathbb{Q} unless otherwise specified, and all manifolds are smooth unless otherwise specified.

2.1 Odd MMM classes are geometric

In this section we prove Theorem 1.2, which we now recall.

Theorem 1.2. *For each $n \geq 1$ the MMM class e_{2n-1} is geometric.*

Let $\Sigma \rightarrow E \rightarrow M$ with M be a surface bundle over a closed $2n$ -manifold M . We prove Theorem 1.2 by writing $e^{2n} \in H^{4n}(E)$ as a certain explicit polynomial in the Pontryagin classes $p_i(E) := p_i(TE)$. It follows immediately that $e_{2n-1}^\#$ only depends on the diffeomorphism type of E . Moreover, as mentioned in the introduction, Pontryagin proved that Pontryagin numbers are cobordism invariants, and Novikov proved that the rational Pontryagin classes are topological invariants. Thus from our proof we obtain the following strengthening of Theorem 1.2.

Theorem 1.4. *For each $n \geq 1$ the MMM class e_{2n-1} is geometric with respect to topological cobordisms. In other words, let $\Sigma_g \rightarrow E^{4n} \rightarrow M^{4n-2}$ and $\Sigma_h \rightarrow \tilde{E}^{4n} \rightarrow \tilde{M}^{4n-2}$ be surface bundles. If E and \tilde{E} are topologically cobordant then the corresponding characteristic numbers coincide:*

$$e_{2n-1}^\#(E \rightarrow M) = e_{2n-1}^\#(\tilde{E} \rightarrow \tilde{M}).$$

Proof. Fix $n \geq 1$ for the rest of the proof, and consider a surface bundle $\Sigma \rightarrow E^{4n} \xrightarrow{\pi} M^{4n-2}$ whose base and total space are closed, smooth, oriented manifolds. We will show that the characteristic number $e_{2n-1}^\#(E^{4n} \rightarrow M^{4n-2})$ can be written in terms of the Pontryagin numbers of $E = E^{4n}$, whose definition we recall below.

Recall that for a closed smooth manifold $E = E^{4n}$, the Pontryagin numbers of M are the characteristic numbers of the tangent bundle TE of E , defined as follows. Let $p_i = p_i(TE)$ denote the i th Pontryagin class of (the tangent bundle of) E . Let J be a sequence (j_1, \dots, j_n) with $j_i \geq 0$ and $\sum_i i \cdot j_i = n$. Then the characteristic class $p_J := p_1^{j_1} p_2^{j_2} \cdots p_n^{j_n}$ yields, for any real vector bundle $V \rightarrow B$, a cohomology class $p_J(V \rightarrow B) \in H^{4n}(B)$. The *Pontryagin number* $p_J^\#(E)$ associated to J is the integer defined by

$$p_J^\#(E) := \langle p_J(TE \rightarrow E), [E] \rangle \in \mathbb{Z}.$$

We now return to the MMM class e_{2n-1} . By definition

$$e_{2n-1}^\#(E \rightarrow M) = \langle e_{2n-1}(E \rightarrow M), [M] \rangle = \int_M e_{2n-1}(E \rightarrow M).$$

The definition of e_{2n-1} and properties of the Gysin map give

$$\int_M e_{2n-1}(E \rightarrow M) = \int_M \int_\Sigma e^{2n} = \int_E e^{2n}$$

where e is the Euler class of the 2-plane bundle $T\pi$ given by the kernel of the differential $d\pi: TE \rightarrow TM$. Note that TE splits as a direct sum

$$TE = T\pi \oplus \pi^*TM,$$

so the total Pontryagin class $p = 1 + p_1 + p_2 + \dots$ satisfies:

$$\begin{aligned} p(TE) &= p(T\pi) \cdot p(\pi^*(TM)) \\ &= (1 + p_1(T\pi)) \cdot (\pi^*p(TM)) \end{aligned} \tag{3}$$

Since $e \in H^2(E)$ is the Euler class of the 2-dimensional real vector bundle $T\pi \rightarrow E$, we have $e^2 = p_1(T\pi)$. Collecting terms in (3) thus gives

$$p_i(TE) = e^2 \pi^* p_{i-1}(TM) + \pi^* p_i(TM). \tag{4}$$

We will deduce the theorem from the following proposition.

Proposition 2.1. *Let $\mathbb{Z}[x_1, x_2, \dots]$ denote the ring of polynomials in variables x_i , graded so that x_i has degree i . For each $n \geq 1$ there exists a unique polynomial $f_n \in \mathbb{Z}[x_1, \dots, x_n]$, homogeneous of degree n , with the property that*

$$f_n(1 + x_1, x_1 + x_2, \dots, x_{n-1} + x_n) = 1 + f_n(x_1, \dots, x_n). \tag{5}$$

We prove Proposition 2.1 in §2.2 below by exhibiting the polynomials f_n explicitly. Assuming Proposition 2.1 for the moment, we complete the proof of the theorem. Note that

$$g_n(x_1, \dots, x_n) := f_n(1 + x_1, x_1 + x_2, \dots, x_{n-1} + x_n)$$

is not homogeneous; the homogenization \tilde{g}_n of the polynomial g_n is given by

$$\tilde{g}_n(t, x_1, \dots, x_n) := f_n(t + x_1, tx_1 + x_2, \dots, tx_{n-1} + x_n),$$

where t has degree 1. Since $g_n(x_1, \dots, x_n) = 1 + f_n(x_1, \dots, x_n)$ it follows that \tilde{g}_n coincides with the homogenization of $1 + f_n(x_1, \dots, x_n)$. This gives the following identity:

$$f_n(t + x_1, tx_1 + x_2, \dots, tx_{n-1} + x_n) = t^n + f_n(x_1, \dots, x_n) \quad (6)$$

For the sake of readability let $b_i \in H^{4i}(E)$ denote $\pi^*p_i(TM)$. With this notation equation (4) above becomes

$$p_i(TE) = e^2 b_{i-1} + b_i.$$

Now consider the cohomology class in $H^{4n}(E)$ defined by

$$f_n(p_1(TE), p_2(TE), \dots, p_n(TE)) \in H^{4n}(E).$$

We then have

$$\begin{aligned} f_n(p_1(TE), p_2(TE), \dots, p_n(TE)) &= f_n(e^2 + b_1, e^2 b_1 + b_2, \dots, e^2 b_{n-1} + b_n) \\ &= \tilde{g}_n(e^2, b_1, b_2, \dots, b_n) \\ &= e^{2n} + f_n(b_1, b_2, \dots, b_n) \quad \text{by (6)} \\ &= e^{2n} + f_n(\pi^*p_1(TM), \pi^*p_2(TM), \dots, \pi^*p_n(TM)) \\ &= e^{2n} + \pi^* f_n(p_1(TM), \dots, p_n(TM)) \end{aligned}$$

But f_n has degree n and $p_i(TM)$ lies in $H^i(M)$, so the term $f_n(p_1(TM), \dots, p_n(TM))$ lies in $H^{4n}(M)$. Since M is a $(4n - 2)$ -dimensional manifold this term vanishes. We conclude that

$$f_n(p_1(TE), p_2(TE), \dots, p_n(TE)) = e^{2n}$$

so that

$$e_{2n-1}^\#(E \rightarrow M) = \int_E e^{2n} = \int_E f_n(p_1(TE), p_2(TE), \dots, p_n(TE)). \quad (7)$$

The righthand side of (7) is a linear combination of Pontryagin numbers. Indeed, write

$$f_n(x_1, \dots, x_n) = \sum_J a_J x^J$$

with $a_J \in \mathbb{Z}$, where x^J denotes the multinomial $x_1^{j_1} \cdots x_n^{j_n}$. We then have

$$e_{2n-1}^\#(E \rightarrow M) = \sum_J a_J \cdot p_J^\#(E).$$

Since the Pontryagin numbers $p_J^\#(E)$ only depend on E , we conclude that e_{2n-1} is geometric, as desired. \square

2.2 Constructing the polynomials f_n

Proof of Proposition 2.1. After giving a recursive definition of the polynomials f_n we verify that they have the desired property (5):

$$f_n(1 + x_1, \dots, x_{n-1} + x_n) = 1 + f_n(x_1, \dots, x_n).$$

We conclude by showing that any f_n satisfying this property is unique.

The first remark to make is that the property (5) is equivalent to the *a priori* weaker property that $f_n(1 + x_1, \dots, x_{n-1} + x_n)$ is congruent to 1 modulo terms of degree n , which is all we needed in the proof of Theorem 1.2. To see this, note that every term in $f_n(1 + x_1, \dots, x_{n-1} + x_n)$ has degree at most n , and expanding this polynomial by choosing one term from each input $x_{k-1} + x_k$, the terms of degree n are exactly those for which the x_k term is chosen every time (all other terms in the expansion will have lower degree); but this is exactly $f_n(x_1, \dots, x_n)$. We will need the stronger property (5) for the inductive step.

Recursively define $f_n \in \mathbb{Z}[x_1, \dots, x_n]$ by $f_1 := x_1$ and

$$f_n := \sum_{k=1}^{n-1} (-1)^{k-1} x_k f_{n-k} + (-1)^{n-1} n x_n \quad (8)$$

for $n > 1$. The first few polynomials f_n are:

$$\begin{aligned} f_1 &= x_1 \\ f_2 &= x_1^2 - 2x_2 \\ f_3 &= x_1^3 - 3x_1x_2 + 3x_3 \\ f_4 &= x_1^4 - 4x_1^2x_2 + 4x_1x_3 + 2x_2^2 - 4x_4 \\ f_5 &= x_1^5 - 5x_1^3x_2 + 5x_1^2x_3 + 5x_1x_2^2 - 5x_1x_4 - 5x_2x_3 + 5x_5 \\ f_6 &= x_1^6 - 6x_1^4x_2 + 6x_1^3x_3 + 9x_1^2x_2^2 - 6x_1^2x_4 + 6x_1x_5 \\ &\quad - 12x_1x_2x_3 - 2x_2^3 + 6x_2x_4 + 3x_3^2 - 6x_6 \end{aligned}$$

We prove by induction that the polynomials f_n have the property

$$f_n(1 + x_1, \dots, x_{n-1} + x_n) = 1 + f_n(x_1, \dots, x_n).$$

The claim is trivial for $n = 1$ since

$$f_1(1 + x_1) = 1 + x_1 = 1 + f_1(x_1).$$

The inductive step will only use the recursive definition (8) of f_n . As in the proof of Theorem 1.2 we define $g_n \in \mathbb{Z}[x_1, \dots, x_n]$ by

$$g_n := f_n(1 + x_1, \dots, x_{n-1} + x_n).$$

Assume that $g_k = 1 + f_k$ for $k < n$; our goal is to prove that $g_n = 1 + f_n$. For simplicity we set $x_0 = 1$. Then we compute:

$$\begin{aligned} g_n &= f_n(1 + x_1, \dots, x_{n-1} + x_n) \\ &= \sum_{k=1}^{n-1} (-1)^{k-1} (x_{k-1} + x_k) (1 + f_{n-k}) + (-1)^{n-1} n (x_{n-1} + x_n) \\ &= \sum_{k=1}^{n-1} (-1)^{k-1} (x_{k-1} + x_k) \quad + (-1)^{n-1} x_{n-1} \\ &\quad + \sum_{k=1}^{n-1} (-1)^{k-1} x_{k-1} f_{n-k} \quad + (-1)^{n-1} (n-1) x_{n-1} \\ &\quad + \sum_{k=1}^{n-1} (-1)^{k-1} x_k f_{n-k} \quad + (-1)^{n-1} n x_n \\ &= x_0 + \left(f_{n-1} - \sum_{j=1}^{n-2} (-1)^{j-1} x_j f_{n-1-j} - (-1)^{n-2} (n-1) x_{n-1} \right) + f_n \\ &= 1 + f_{n-1} - f_{n-1} + f_n = 1 + f_n. \end{aligned}$$

This completes the proof that the f_n constructed above satisfy property (5).

Since we will not need the uniqueness of the f_n in this paper, we only sketch its proof. If f'_n were another solution to (5) for some n , the difference $h_n := f_n - f'_n$ would be a homogeneous polynomial of degree n satisfying

$$h_n(1 + x_1, \dots, x_{n-1} + x_n) = (1 + f_n) - (1 + f'_n) = h_n.$$

But $h_n(1 + x_1, \dots, x_{n-1} + x_n)$ cannot be homogenous of degree n , as we now show. Let the *pseudo-degree* of a polynomial in $\mathbb{Z}[x_1, \dots, x_n]$ be its degree when the variables x_i are given degree 1, instead of degree i as usual. Let k be the pseudo-degree of h_n . We necessarily have $1 \leq k \leq n$. Then the smallest degree terms in $h_n(1 + x_1, \dots, x_{n-1} + x_n)$ are of degree $n - k < n$. Indeed each leading term of h_n (those of pseudo-degree k) contributes a linearly independent term of degree $n - k$. This contradiction shows that $h_n = 0$, and so f_n is unique. \square

We remark that the coefficients of f_n have the following closed form, which is not hard to prove by induction. For the monomial $x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ (we set $j = j_1 + \cdots + j_n$), the coefficient is $(-1)^j$ times the weighted number of sequences (a_1, \dots, a_j) with j_i entries equal to i , with the sequence (a_1, \dots, a_j) weighted by the integer a_1 . For example, for $x_1^2 x_2^2$ the sequences are

$$(1, 1, 2, 2), (1, 2, 1, 2), (1, 2, 2, 1), (2, 1, 1, 2), (2, 1, 2, 1), (2, 2, 1, 1)$$

so the coefficient of $x_1^2 x_2^2$ in f_6 is $1 + 1 + 1 + 2 + 2 + 2 = 9$. However, it seems to be more difficult to prove the key property (5) from this definition.

2.3 Odd MMM classes vanish on handlebody bundles

In this section we deduce Corollary 1.6, originally proved by Giansiracusa–Tillmann [GT, Theorem A], from Theorem 1.4.

Corollary 1.6. *If there exists a bundle $V_g \rightarrow W \rightarrow B$ whose fiberwise boundary is $\Sigma_g \rightarrow E \rightarrow B$, then $e_{2n-1}(E \rightarrow B) = 0 \in H^{4n-2}(B; \mathbb{Q})$ for all $n \geq 1$.*

Proof. When the base is a closed manifold B^{4n-2} of dimension $4n-2$, it follows immediately from Theorem 1.4 that $e_{2n-1}(E \rightarrow B) = 0$. Indeed, Theorem 1.4 implies that $e_{2n-1}(E \rightarrow B) = 0$ whenever $E^{4n} = \partial W^{4n+1}$, whether or not W is a handlebody bundle, or a fiber bundle at all. When the base space B is only a CW complex, consider an arbitrary homology class $x \in H_{4n-2}(B; \mathbb{Z})$; we will show that $\langle e_{2n-1}(E \rightarrow B), x \rangle = 0$.

Thom [Th, Theorem II.29] proved that every homology class in a closed orientable manifold has an integral multiple which can be represented by (the fundamental class of) a closed submanifold. This can be strengthened to show that every homology class in any CW complex has an odd integral multiple which can be represented by a closed submanifold, see e.g. Conner [Co, Corollary 15.3].

It follows that for some nonzero $k \in \mathbb{Z}$, the homology class $k \cdot x$ is represented by a map $M^{4n-2} \rightarrow B$. Let $\Sigma_g \rightarrow E' \rightarrow M$ be the pullback of $\Sigma_g \rightarrow E \rightarrow B$. By naturality:

$$\begin{aligned} \langle e_{2n-1}(E' \rightarrow M), [M] \rangle &= \langle e_{2n-1}(E \rightarrow B), [M] \rangle \\ &= \langle e_{2n-1}(E \rightarrow B), k \cdot x \rangle \\ &= k \cdot \langle e_{2n-1}(E \rightarrow B), x \rangle \end{aligned} \tag{9}$$

However, the pullback $V_g \rightarrow W' \rightarrow M$ of $V_g \rightarrow W \rightarrow B$ has fiberwise boundary $\Sigma_g \rightarrow E' \rightarrow M$. Thus by the argument at the beginning of the proof, Theorem 1.4 implies that $\langle e_{2n-1}(E' \rightarrow M), [M] \rangle = 0$. By (9), this implies that $\langle e_{2n-1}(E \rightarrow B), x \rangle = 0 \in \mathbb{Q}$. \square

As we mentioned in the introduction, Giansiracusa–Tillmann in fact proved the vanishing of e_{2n-1} in *integral* cohomology, which is not implied by Theorem 1.4.

2.4 Geometric classes as obstructions

As noted in the introduction, any characteristic class which is geometric gives an obstruction to the existence of a fibering for which the fiber is a surface of small genus. For example, we have the following corollary to the main theorems.

Corollary 1.7. *For $n \geq 1$, let $E^{4n} \rightarrow M^{4n-2}$ be a surface bundle with $e_{2n-1}^\#(E \rightarrow M) \neq 0$. Then any fibering*

$$\Sigma_g \rightarrow E^{4n} \rightarrow N^{4n-2}$$

of E as a surface bundle must have $g \geq n + 1$. In fact this holds for any fibering of any manifold topologically cobordant to E^{4n} .

Proof. The space $\text{BDiff}^+(\Sigma_g)$ mentioned in the introduction is the classifying space for bundles with structure group $\text{Diff}^+(\Sigma_g)$. From this it follows that characteristic classes of Σ_g -bundles correspond to cohomology classes in $H^*(\text{BDiff}^+(\Sigma_g))$.

For $g \geq 2$, Harer [Har] proved that the rational cohomological dimension of $\text{BDiff}^+(\Sigma_g)$ is $4g - 5$. Thus e_{2n-1} , which lies in $H^{4n-2}(\text{BDiff}^+(\Sigma_g); \mathbb{Q})$, vanishes for $g \leq n$. Any fibering $\Sigma_g \rightarrow E \rightarrow N$ with $g \leq n$ would have $e_{2n-1}^\#(E \rightarrow N) = 0$, but by Theorem 1.2 this contradicts our assumption that $e_{2n-1}^\#(E \rightarrow M) \neq 0$. Appealing to Theorem 1.4, we obtain the same bound $g \geq n + 1$ for any fibering $\Sigma_g \rightarrow \tilde{E} \rightarrow N'$ with \tilde{E} topologically cobordant to E . \square

2.5 Bordism groups

In this section we give a different perspective on Theorem 1.4 by bundling manifolds and surfaces together into cobordism classes. The oriented bordism group Ω_n^{SO} is the abelian group generated by oriented n -manifolds modulo cobordism. We can similarly consider the group generated by surface bundles over oriented n -manifolds modulo cobordism of bundles. This group is denoted $\Omega_n^{\text{SO}}(\coprod \text{BDiff}^+(\Sigma_g))$, as we now briefly explain.

Oriented bordism gives an extraordinary homology theory Ω_*^{SO} ; roughly, $\Omega_n^{\text{SO}}(X)$ is the abelian group generated by oriented n -manifolds M equipped with a map to X , modulo cobordism over X . The space $\text{BDiff}^+(\Sigma_g)$ is the classifying space

for bundles with structure group $\text{Diff}^+(\Sigma_g)$, which means that homotopy classes of maps from B to $\text{BDiff}^+(\Sigma_g)$ correspond to isomorphism classes of Σ_g -bundles over B . This property implies that $\Omega_n^{\text{SO}}(\text{BDiff}^+(\Sigma_g))$ is generated by Σ_g -bundles with base an oriented n -manifold, modulo cobordism as Σ_g -bundles. Taking all g together gives $\Omega_n^{\text{SO}}(\coprod \text{BDiff}^+(\Sigma_g))$, the oriented bordism group of surface bundles.

For each $n \geq 1$ there is a natural map

$$\Omega_n^{\text{SO}}(\coprod \text{BDiff}^+(\Sigma_g)) \rightarrow \Omega_{n+2}^{\text{SO}}$$

given by forgetting the bundle structure: a surface bundle $[\Sigma_g \rightarrow E^{n+2} \rightarrow B^n]$ is sent to the cobordism class of the total space $[E^{n+2}]$. This is one version of the Gysin map. Any i -dimensional characteristic class c of surface bundles yields a map

$$c^\# : \Omega_i^{\text{SO}}(\coprod \text{BDiff}^+(\Sigma_g)) \rightarrow \mathbb{Z}$$

by taking the corresponding characteristic number. We can then rephrase the definition of “geometric with respect to cobordism” as follows.

Definition 2.2 (Definition 1.1 for cobordism, restated). A characteristic class c of surface bundles is *geometric with respect to cobordism* if and only if the map $c^\# : \Omega_i^{\text{SO}}(\coprod \text{BDiff}^+(\Sigma_g)) \rightarrow \mathbb{Z}$ factors through the map $\Omega_n^{\text{SO}}(\text{BDiff}^+(\Sigma_g)) \rightarrow \Omega_{n+2}^{\text{SO}}$:

$$\begin{array}{ccc} \Omega_i^{\text{SO}}(\coprod \text{BDiff}^+(\Sigma_g)) & \longrightarrow & \Omega_{n+2}^{\text{SO}} \\ & \searrow c^\# & \downarrow \\ & & \mathbb{Z} \end{array}$$

For example, Theorem 1.4 states that the map $e_{2n-1}^\# : \Omega_{4n-2}^{\text{SO}}(\coprod \text{BDiff}^+(\Sigma_g)) \rightarrow \mathbb{Z}$ induced by the MMM class e_{2n-1} factors through Ω_{4n}^{SO} . Conversely, the result of Hatcher [Hat] described in the introduction implies that the map $e_{2n}^\# : \Omega_{4n}^{\text{SO}}(\coprod \text{BDiff}^+(\Sigma_g)) \rightarrow \mathbb{Z}$ does not factor through $\Omega_{4n+2}^{\text{SO}}$.

We emphasize that Definition 2.2 is strictly stronger than the notion of geometric characteristic class itself (Definition 1.1), which cannot be interpreted in this way. There is no reason that a characteristic number which only depends on the diffeomorphism class of the total space should necessarily depend only on the *cobordism class* of the total space.

3 Examples of bundles fibering in multiple ways

In this section we explain a general construction of closed, oriented manifolds that fiber as a surface bundle in more than one way. This construction thus provides many families of examples to which Theorem 1.4 applies.

Given a surface bundle $\Sigma \rightarrow N^n \rightarrow B^{n-2}$ with section $s: B \rightarrow N$,¹ we will construct a manifold E^{n+2} with distinct fiberings

$$\Sigma_g \rightarrow E^{n+2} \rightarrow P^n \quad \text{and} \quad \Sigma_h \rightarrow E^{n+2} \rightarrow Q^n.$$

In fact, by varying the parameters in the construction, each surface bundle $\Sigma \rightarrow N^n \rightarrow B^{n-2}$ yields an infinite family of such manifolds E^{n+2} .

One caveat is that the total space E that we produce is only canonically defined as a topological manifold. We will find a smooth structure on E so that $\Sigma_g \rightarrow E \rightarrow P$ is a smooth fibering, and a smooth structure on E so that $\Sigma_h \rightarrow E \rightarrow Q$ is a smooth fibering, so that Theorem 1.4 applies to these fiberings. However, the argument does not imply that the resulting smooth structures on E are diffeomorphic (although we believe that they are).

The construction of the Atiyah–Kodaira manifold. To explain the various steps in the construction, we first sketch the construction of the Atiyah–Kodaira manifold M_Σ from a surface Σ . Fix an integer $k \geq 2$.

Choose a finite-sheeted normal covering $S \rightarrow \Sigma$, and let $f_1, \dots, f_k: S \rightarrow S$ be distinct deck transformations. Let $\widehat{S} \xrightarrow{\pi} S$ be the $k^{2g(S)}$ -sheeted normal cover determined by

$$1 \rightarrow \pi_1(\widehat{S}) \rightarrow \pi_1(S) \rightarrow H_1(S; \mathbb{Z}/k\mathbb{Z}) \rightarrow 1.$$

Consider the 4-manifold $\widehat{S} \times S$, and define the codimension-2 submanifolds $\Delta_1, \dots, \Delta_k$ to be the graphs of $f_i \circ \pi: \widehat{S} \rightarrow S$:

$$\Delta_i = \{(p, q) \in \widehat{S} \times S \mid f_i(\pi(p)) = q\}$$

Since the maps f_i are pointwise distinct, these graphs are disjoint. Using the Künneth formula, one can check that the cohomology class represented by $[\Delta_1] + \dots + [\Delta_k] \in H^2(\widehat{S} \times S; \mathbb{Z})$ is divisible by k . It follows (see [Hi, Proposition 6]) that $\widehat{S} \times S$ admits

¹The condition that $\Sigma \rightarrow N \rightarrow B$ has a section is not very restrictive, since any surface bundle $\Sigma \rightarrow B \rightarrow Y$ induces a surface bundle $\Sigma \rightarrow B \times_Y B \rightarrow B$ with section $s: B \rightarrow B \times_Y B$ given by the diagonal embedding.

a cyclic k -sheeted cover branched over the union $\Delta := \Delta_1 \cup \dots \cup \Delta_k$. (As we will see below, this description does not uniquely determine the branched cover, but all the different covers yield the same total space.) This branched cover $M_\Sigma \rightarrow \widehat{S} \times S$ is the desired 4-manifold M_Σ .

Note that the composition $M_\Sigma \rightarrow \widehat{S} \times S \rightarrow \widehat{S}$ is a surface bundle, whose fiber is the cyclic k -sheeted cover Σ_h of S branched over $\Delta \cap S$, which consists of k points. Similarly, the composition $M_\Sigma \rightarrow \widehat{S} \times S \rightarrow S$ is a surface bundle, whose fiber is the cyclic k -sheeted cover Σ_g of \widehat{S} branched over $\Delta \cap \widehat{S}$, which consists of $k^{2g(S)+1}$ points.

Atiyah-Kodaira for families. We now describe a procedure which begins with a surface bundle $\Sigma \rightarrow N \rightarrow B$ with section $s: B \rightarrow N$, and constructs a bundle $M_\Sigma \rightarrow E \rightarrow B'$, where B' is a finite cover of B . One artifact of this construction is that although M_Σ is a smooth manifold, the resulting M_Σ -bundles have structure group $\text{Homeo}^+(M_\Sigma)$ and thus are not necessarily smooth.

We begin by taking $\Gamma = \text{Diff}^+(\Sigma, *)$, which is the structure group for Σ -bundles endowed with a section. At various stages in the proof we will need to replace Γ by a finite index subgroup. In the end we will have a homomorphism $\phi: \Gamma \rightarrow \text{Homeo}^+(M_\Sigma)$ for some finite index subgroup $\Gamma < \text{Diff}^+(\Sigma, *)$. For any surface bundle $\Sigma \rightarrow N \rightarrow B$, by pulling back to a finite cover of the base $B' \rightarrow B$ we can reduce the structure group to Γ . We then apply the usual fiber-replacement determined by the homomorphism of structure groups $\phi: \Gamma \rightarrow \text{Homeo}^+(M_\Sigma)$ to the Σ -bundle $\Sigma \rightarrow N' \rightarrow B'$ to obtain an M_Σ -bundle $M_\Sigma \rightarrow E \rightarrow B'$. We will then use the two fiberings $M_\Sigma \rightarrow \widehat{S}$ and $M_\Sigma \rightarrow S$ to construct two fiberings of E .

Defining the homomorphism $\Gamma \rightarrow \text{Homeo}^+(M_\Sigma)$. We maintain the same choices as in the previous section, so $S \rightarrow \Sigma$ is a finite-sheeted normal cover and $f_1, \dots, f_k: S \rightarrow S$ are distinct deck transformations. Let G be the deck group of the cover $S \rightarrow \Sigma$; we additionally assume that each f_i is nontrivial. We fix a lift in S of $* \in \Sigma$, which by abuse of notation we also denote $* \in S$. A diffeomorphism $\varphi \in \text{Diff}(\Sigma, *)$ lifts to $\text{Diff}(S, *)$ if and only if it preserves the finite index subgroup $\pi_1(S, *) \triangleleft \pi_1(\Sigma, *)$. For such φ , we have an induced action on the quotient $\pi_1(\Sigma, *)/\pi_1(S, *) \approx G$. We replace Γ by the finite-index subgroup of $\text{Diff}^+(\Sigma, *)$ consisting of diffeomorphisms φ which preserve $\pi_1(S, *)$ and act trivially on the quotient G .

Our choice of a basepoint $* \in S$ gives an unambiguous lifting $\Gamma \hookrightarrow \text{Diff}(S, *)$. As before $\pi: \widehat{S} \rightarrow S$ is the characteristic cover with deck group $H_1(S; \mathbb{Z}/k\mathbb{Z})$. Since this cover is characteristic, every diffeomorphism of S lifts to \widehat{S} . Choosing a basepoint $* \in \widehat{S}$, we obtain a lifting $\text{Diff}(S, *) \hookrightarrow \text{Diff}(\widehat{S}, *)$. Since Γ acts trivially on G , the image of Γ in $\text{Diff}(S, *)$ commutes with the action of G by deck transformations; in particular, the image of Γ commutes with f_i . Thus the diagonal action of $\Gamma \subset \text{Diff}(S, *)$ on $\widehat{S} \times S$ preserves the subsets $\Delta_i = \{(p, f_i \circ \pi(p))\}$ and their union Δ .

At this point the action of Γ on $\widehat{S} \times S$ fixes the basepoint $(*, *) \in \widehat{S} \times S$ (which does not lie on Δ_i since f_i is nontrivial) and preserves Δ . Let γ_i be the small circle

$$\gamma_i := \{(\theta, f_i \circ \pi(*)) \mid \text{dist}(\theta, *) = \varepsilon\}$$

linking once with Δ_i . A cyclic k -sheeted cover of $\widehat{S} \times S$ branched over Δ is determined by a homomorphism

$$\pi_1(\widehat{S} \times S \setminus \Delta) \rightarrow \mathbb{Z}/k\mathbb{Z} \quad \text{satisfying} \quad \gamma_i \mapsto 1 \in \mathbb{Z}/k\mathbb{Z} \text{ for all } i.$$

As our final modification, we replace Γ by the finite-index subgroup which acts trivially on $H^1(\widehat{S} \times S \setminus \Delta; \mathbb{Z}/k\mathbb{Z})$. (If $k = 2$, for convenience later we consider instead the action on $H^1(\widehat{S} \times S \setminus \Delta; \mathbb{Z}/4\mathbb{Z})$.) Fix a homomorphism $\pi_1(\widehat{S} \times S \setminus \Delta) \rightarrow \mathbb{Z}/k\mathbb{Z}$. This determines a branched cover $M_\Sigma \rightarrow \widehat{S} \times S$. Fix a basepoint $* \in M_\Sigma$ lying above $(*, *)$. Our modification guarantees that Γ preserves the kernel of this homomorphism, so choosing the lift fixing $* \in M_\Sigma$, the action $\Gamma \hookrightarrow \text{Diff}(\widehat{S} \times S \setminus \Delta)$ lifts to $\text{Diff}(M_\Sigma \setminus \Delta)$.

By continuity we can extend the resulting diffeomorphisms across Δ to all of M_Σ (the action of Γ on each component Δ_i of the ramification locus can be identified with the action of Γ on \widehat{S}). It is easy to check that the resulting map of M_Σ is orientation-preserving. However, the extended map will not be smooth along the ramification locus $\Delta \subset M_\Sigma$. Thus we have defined the desired homomorphism

$$\phi: \Gamma \rightarrow \text{Homeo}^+(M_\Sigma).$$

We finish by emphasizing a key property: since Γ acts on $\widehat{S} \times S$ diagonally, the projection $\widehat{S} \times S \rightarrow S$ is Γ -equivariant, as is the projection $\widehat{S} \times S \rightarrow \widehat{S}$. Since the action on M_Σ lifts the action on $\widehat{S} \times S$, the fiberings $M_\Sigma \rightarrow S$ and $M_\Sigma \rightarrow \widehat{S}$ are also Γ -equivariant.

Fiberings as surface bundles. As mentioned before, for any surface bundle $\Sigma \rightarrow N \rightarrow B$ we can reduce the structure group to Γ by passing to a finite cover $B' \rightarrow B$. Then applying $\phi: \Gamma \rightarrow \text{Homeo}^+(M_\Sigma)$, we obtain a bundle $M_\Sigma \rightarrow E \rightarrow B'$.

Let P be the total space of the bundle $S \rightarrow P \rightarrow B'$, and Q the total space of the bundle $\widehat{S} \rightarrow Q \rightarrow B'$, determined by the maps $\Gamma \hookrightarrow \text{Diff}(S)$ and $\Gamma \hookrightarrow \text{Diff}(\widehat{S})$ respectively. Since the fibering $M_\Sigma \rightarrow S$ is Γ -equivariant, it induces a fibering $E \rightarrow P$; the fiber Σ_g of $\Sigma_g \rightarrow M_\Sigma \rightarrow S$ is also the fiber of $\Sigma_g \rightarrow E \rightarrow P$. Similarly, the Γ -equivariant fibering $\Sigma_h \rightarrow M_\Sigma \rightarrow \widehat{S}$ induces a fibering $\Sigma_h \rightarrow E \rightarrow Q$.

Smoothing the resulting bundles. Since the structure group of $M_\Sigma \rightarrow E \rightarrow B'$ is $\text{Homeo}^+(M_\Sigma)$, we only know that E is a topological manifold. But to apply the results of this paper, we need $E \rightarrow P$ and $E \rightarrow Q$ to be smooth fiberings of smooth manifolds. Fortunately this is possible, as we illustrate for $\Sigma_g \rightarrow E \rightarrow P$.

We know that the map $E \rightarrow P$ is a topological fiber bundle, with fiber Σ_g and structure group $\text{Homeo}^+(\Sigma_g)$. But for $g \geq 2$ the identity component $\text{Homeo}_0(\Sigma_g)$ is contractible [FM, Theorem 1.14], so the structure group can be taken to be the discrete group $\text{Mod}(\Sigma_g) := \pi_0 \text{Homeo}^+(\Sigma_g)$. (We remark that $\text{Diff}_0(\Sigma_g)$ is also contractible, so $\text{Mod}(\Sigma_g) \approx \pi_0 \text{Diff}^+(\Sigma_g)$ as well.) Since $\text{Mod}(\Sigma_g)$ is discrete, it follows that a surface bundle $\Sigma_g \rightarrow E \rightarrow P$ is determined by its *monodromy representation* $\rho: \pi_1(P) \rightarrow \text{Mod}(\Sigma_g)$.

We will need the fact that for our construction, the image of ρ is torsion-free. This can be seen as follows. Let $R = P \times_{B'} Q$ be the total space of the bundle $\widehat{S} \times S \rightarrow R \rightarrow B'$, so that we have a bundle $\widehat{S} \rightarrow R \rightarrow P$. From the bundle $M_\Sigma \rightarrow E \rightarrow B'$ we see that E is a branched cover of R ; writing $\Sigma_g \rightarrow E \rightarrow P$, we see that E is obtained from $\widehat{S} \rightarrow R \rightarrow P$ by a fiberwise branched cover. It follows that the monodromy $\rho_E: \pi_1(P) \rightarrow \text{Mod}(\Sigma_g)$ factors through the monodromy $\rho_R: \pi_1(P) \rightarrow \text{Mod}(\widehat{S})$.

Recall that P is the total space of the bundle $S \rightarrow P \rightarrow B'$. The restriction of the bundle $\widehat{S} \rightarrow R \rightarrow P$ to $S \subset P$ is just the product $\widehat{S} \rightarrow \widehat{S} \times S \rightarrow S$. Thus ρ_R vanishes on the fundamental group $\pi_1(S)$ of the fiber, so we may think of ρ_R as a map $\pi_1(B') \rightarrow \text{Mod}(\widehat{S})$, induced by the action $\Gamma \rightarrow \text{Diff}(\widehat{S})$ described earlier. Our assumptions on Γ imply that the image of the monodromy $\pi_1(B') \rightarrow \text{Mod}(\widehat{S})$ is contained in the subgroup $\text{Mod}(\widehat{S})[k]$ which acts trivially on $H^1(\widehat{S}; \mathbb{Z}/k\mathbb{Z})$. Indeed we assumed that that Γ acts trivially on $H^1(\widehat{S} \times S \setminus \Delta; \mathbb{Z}/k\mathbb{Z})$. Since $H^1(\widehat{S}; \mathbb{Z}/k\mathbb{Z})$ injects into $H^1(\widehat{S} \times S \setminus \Delta; \mathbb{Z}/k\mathbb{Z})$, this implies the claim. For $k = 2$ we made a further assumption which implies the image is contained in $\text{Mod}(\widehat{S})[4]$.

The subgroup $\text{Mod}(\widehat{S})[k]$ is known to be torsion-free for $k \geq 3$ [FM, Theorem 6.9]. Note that P is a finite cover of N , so it is naturally a smooth manifold. Now the desired property follows from the following proposition.

Proposition 3.1. *For any surface bundle $\Sigma \rightarrow E \rightarrow B$, if B is a smooth manifold and the image of the monodromy representation $\pi_1(B) \rightarrow \text{Mod}(\Sigma)$ is torsion-free, there is a smooth structure on E making $\Sigma \rightarrow E \rightarrow B$ into a smooth bundle of smooth manifolds.*

Note that the hypothesis of the proposition is always satisfied after passing to some finite cover $B' \rightarrow B$ (corresponding for $\text{Mod}(\Sigma)[k]$ for some $k \geq 3$, for example).

Proof of Proposition 3.1. If Γ is a torsion-free subgroup of $\text{Mod}(\Sigma_g)$, a finite-dimensional model for the classifying space $B\Gamma$ is given by the appropriate cover \mathcal{M}_g^Γ of \mathcal{M}_g , the *moduli space* of genus g Riemann surfaces [FM, §12.6]. The classifying space \mathcal{M}_g^Γ is a smooth manifold of dimension $6g - 6$, and there is a universal surface bundle \mathcal{C}_g^Γ fitting into a smooth bundle of smooth manifolds $\Sigma_g \rightarrow \mathcal{C}_g^\Gamma \rightarrow \mathcal{M}_g^\Gamma$. Since \mathcal{M}_g^Γ is the classifying space $B\Gamma$, we know that Σ_g -bundles over a base X with structure group Γ correspond to maps $f: X \rightarrow \mathcal{M}_g^\Gamma$ up to homotopy. This correspondence is made explicit by sending the map f to the pullback bundle $\Sigma_g \rightarrow f^*\mathcal{C}_g^\Gamma \rightarrow X$; homotopic maps yield isomorphic bundles.

The bundle $\Sigma_g \rightarrow E \rightarrow B$ of the proposition is classified by some map $f: B \rightarrow \mathcal{M}_g^\Gamma$. Since every continuous map between smooth manifolds can be approximated by a smooth map, let $\tilde{f}: B \rightarrow \mathcal{M}_g^\Gamma$ be a smooth map homotopic to f . The pullback of a smooth bundle along a smooth map is smooth, making the pullback $\Sigma_g \rightarrow \tilde{f}^*\mathcal{C}_g^\Gamma \rightarrow B$ into a smooth bundle. Since this is isomorphic to the bundle $\Sigma_g \rightarrow E \rightarrow B$, we can think of this as giving a smooth structure on E with respect to which $\Sigma_g \rightarrow E \rightarrow B$ is a smooth bundle of smooth manifolds, as desired. \square

4 Variations on geometric classes

4.1 Geometric classes for bundles of the “wrong” dimension

Our definition of geometric for a d -dimensional characteristic class c required us to look at a closed, oriented d -manifold as base space, since we needed a canonical homology class on which to evaluate c , thus obtaining a number which could compare for various fibering. However, when the base manifold has dimension different from d , or is no longer closed or orientable, there is still a reasonable notion of “geometric”.

Definition 4.1 (Weakly geometric characteristic classes). A d -dimensional characteristic class c for surface bundles is *weakly geometric* if the vanishing or nonvanishing of c does not depend on the fibering. More precisely, whenever

$$\Sigma_g \rightarrow E \rightarrow B$$

and

$$\Sigma_h \rightarrow E' \rightarrow B'$$

are two surface bundles with homeomorphic total spaces $E \approx E'$, then $c(E \rightarrow B) = 0$ if and only if $c(E' \rightarrow B') = 0$.

Note that we do *not* assume that B is a closed d -manifold, only that it is a manifold. Thus despite the name, geometric does not imply weakly geometric. For example, we do not know if the odd MMM classes e_{2n-1} are weakly geometric.

Question 4.2. *Which MMM classes are weakly geometric for surface bundles with base a closed manifold of any dimension?*

For the first MMM class e_1 , Question 4.2 is equivalent to the following question. If $E \rightarrow M$ is a surface bundle, the preimage of a closed surface in M is a 4-manifold in E . Let $E \rightarrow M$ be a surface bundle so that some closed surface in M has preimage with nonzero signature. If $E \rightarrow M'$ is another fibering of E as a surface bundle, must there exist such a surface in M' ? There is also the following *a priori* stronger question: if N is a 4-dimensional submanifold of E and if $E \rightarrow M$ restricts to N as a submersion, must every fibering $E \rightarrow M'$ restrict to N as a submersion? An affirmative answer would imply that e_1 is weakly geometric, by showing that the nontriviality of e_1 is carried on submanifolds N which cannot be avoided by choosing another fibering.

4.2 Generalized MMM classes

There are higher-dimensional versions of the Morita–Mumford–Miller classes, which are characteristic classes of orientable bundles with fiber a d -dimensional manifold. For any bundle $M^d \rightarrow E \rightarrow B$ we have the d -dimensional vertical tangent bundle $T_M E$, and for any polynomial P in the Euler class and Pontryagin classes of $T_M E$, integrating along the fiber yields a characteristic class ε_P . We define such a characteristic class to be geometric if the associated characteristic number depends only on the total space, exactly as in Definition 1.1.

Ebert [Eb, Theorem B] has proved that if d is even, every polynomial in the characteristic classes ε_P is nonzero for some bundle $M^d \rightarrow E^{n+d} \rightarrow B^n$, and for d odd he described exactly which polynomials in the ε_P vanish. In light of the results of this paper, the following question seems quite natural.

Question 4.3. *Which polynomials in the higher MMM classes ε_P are geometric? In particular, is $\varepsilon_{e^2} \in H^d$ geometric?*

One possible conjecture is that ε_P should be geometric with respect to cobordism whenever P is a polynomial only in the Pontryagin classes. This is supported by Giansiracusa–Tillman, who prove for such P in [GT, Corollary C] that ε_P is invariant with respect to *fiberwise* cobordism.

4.3 Vector bundles

It is natural to ask whether the notion of “geometric characteristic classes” can be extended to vector bundles. We first point out that there are many examples of distinct vector bundles with diffeomorphic total space, so this question is far from vacuous.

Mazur [Ma] proved for compact manifolds that if there is a tangential homotopy equivalence $M^k \rightarrow N^k$, then $M \times \mathbb{R}^n$ and $N \times \mathbb{R}^n$ are diffeomorphic for sufficiently large n . In particular, although the 3–dimensional lens spaces $L(7, 1)$ and $L(7, 2)$ are not homeomorphic, the trivial bundle $L(7, 1) \times \mathbb{R}^4$ is diffeomorphic to $L(7, 2) \times \mathbb{R}^4$. In a similar vein, Siebenmann proved [Si, Theorem 2.2] that if $\mathbb{R}^n \rightarrow E \rightarrow M$ is a vector bundle of rank $n > 2$ over a compact manifold M , and N is any smooth compact submanifold of E so that $N \hookrightarrow E$ is a homotopy equivalence, then there is a vector bundle $\mathbb{R}^n \rightarrow V \rightarrow N$ so that V is diffeomorphic to E .

The soul theorem of Cheeger–Gromoll [CG] states that associated to any complete non-negatively curved metric on a Riemannian manifold E is a compact totally geodesic submanifold $M \subset E$ (called the *soul* of E) so that E is diffeomorphic to the total space of the normal bundle NM . By varying the metric on E , this provides numerous examples of distinct vector bundles with diffeomorphic total space; for example, Belegradek [Be, Theorem 1] shows there exist infinitely many compact 7–manifolds M_i with vector bundles $\mathbb{R}^5 \rightarrow E_i \rightarrow M_i$ whose base spaces M_i are pairwise non-homeomorphic, but whose total spaces E_i are all diffeomorphic to $S^3 \times S^4 \times \mathbb{R}^5$.

It is not difficult to show that the Euler class is geometric, in the sense as above that if $\mathbb{R}^n \rightarrow E \rightarrow M^n$ and $\mathbb{R}^n \rightarrow E \rightarrow N^n$ are vector bundles with the same total

space, the Euler numbers $e^\#(E \rightarrow M) = \langle e(E \rightarrow M), [M] \rangle$ and $e^\#(E \rightarrow N) = \langle e(E \rightarrow N), [N] \rangle$ coincide. However for vector bundles we can strengthen the notion of geometric characteristic class by dropping the restriction that our bundles have base space an n -dimensional manifold. For surface bundles, our focus on characteristic numbers was necessary because there is no way to directly compare the cohomology groups of different base spaces. But for any vector bundles $\mathbb{R}^n \rightarrow E \rightarrow M$ and $\mathbb{R}^n \rightarrow E \rightarrow N$, the canonical homotopy equivalences $E \rightarrow M$ and $E \rightarrow N$ induce an identification $H^*(M) \approx H^*(N)$ by which we can compare characteristic classes directly. For the Euler class, it is proved in Belegradek–Kwasik–Schultz [BKS, Proposition 5.1] that $e(E \rightarrow M) \in H^n(M)$ and $e(E \rightarrow N) \in H^n(N)$ coincide under this identification. It would be very interesting to know which other characteristic classes of vector bundles are geometric in this stronger sense; for the rational Pontryagin classes this is established in [BKS, Proposition 5.3] when $n \leq 3$.

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