Non-technical research summary

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My research focuses on random curves and random surfaces which arise in statistical mechanics. In what follows, I give a broad overview of my research area which is intended to be accessible to a reader with some scientific background, not necessarily in mathematics.

1 Statistical mechanics

Statistical mechanics is the study of random systems which model various physical phenomena. For example, one might be interested in how randomly moving particles interact with each other, or how randomly assigned electrical charges organize themselves. Often such systems can be represented by random curves and/or random surfaces. For example, a curve could represent the the trajectory of a single particle or an interface between a region of positive charge and a region of negative charge. Surfaces can describe the density of particles in each region in the plane or an interface in three-dimensional space.

Many such systems are *discrete*, meaning that they depend on a finite number of random inputs (representing, e.g., the total number of particles). A central goal in statistical mechanics is to describe the large-scale behavior of these discrete systems as the number of random inputs tends to infinity. Mathematically, this behavior corresponds to the concept of a *scaling limit*, a continuous random object which the discrete objects converge to, in some sense, when we send the number of random inputs to infinity and "zoom out" (re-scale). The limiting objects are often much easier to understand than the original discrete random objects since one can ignore potentially complicated small-scale behavior. Intuitively, it is easier to think of a single physical object than it is to keep track of a huge number of individual atoms.

The most simple example of a scaling limit is the classical central limit theorem, which says that if $X_1, \ldots, X_n$ are independent, identically distributed random variables with zero mean and unit variance, then $\frac{1}{n^{1/2}}(X_1 + \cdots + X_n)$ converges in distribution to a standard Gaussian (normal) random variable. I will discuss several additional examples of scaling limits below.

2 Random curves

If A is a finite set, with N total elements, then one can sample a uniform random[1] element of A by choosing each element of A with probability $1/N$. We would like to consider a "uniform random curve" in the plane. The set of all possible curves is infinite, so there is not an obvious way to assign the same probability to every curve. However, one can get around this difficulty by first considering discrete random curves.

Suppose we start at $(0,0)$ and at each integer time we travel one unit of distance either up, down, left, or right, with equal probability. This gives us a random path on the square grid, called

[1] Here "uniform random" means that each possible outcome is assigned equal probability. For example, a roll of a fair die is a uniform random sample from $\{1, \ldots, 6\}$. A roll of a biased die is random but not uniform random.

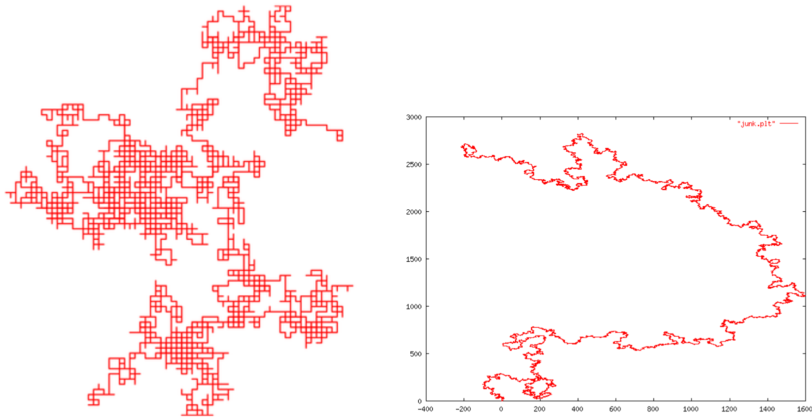

Figure 1: **Left:** A simulation of a random walk, taken from Wikipedia. **Right:** A simulation of a self-avoiding walk made by Tom Kennedy. The pictured curve is conjectured (but not yet proven) to converge to $\mathrm{SLE}_{8/3}$.

a *random walk* (see Figure 1, left). The first n steps of the walk constitute a uniform sample from the set of all n-step paths in the square grid started from 0. It was proven by Donsker [5] that the large-scale behavior of random walk after a large number of steps is described by planar *Brownian motion*, a certain type of continuous random curve in the plane which describes, e.g., the motion of a single particle in a dust cloud or stock market fluctuations. In a sense, one can think of Brownian motion as a uniform random continuous curve.

In some situations, one might want to consider a random curve which does not visit any point more than once (note that a random walk or Brownian motion will typically re-trace its past quite frequently). Extending the idea above, suppose we fix some large integer n, and look at the set of all paths of length n on the square grid started from $(0,0)$ which do not hit any point more than once. This is a large but finite set, so we can sample one of its elements uniformly at random. The discrete random curve thus obtained is called a *self-avoiding walk*.

As above, one might try to define a uniform random continuous self-avoiding curve by taking a scaling limit of the self-avoiding walk as $n \to \infty$. It was conjectured by Lawler, Schramm, and Werner [20] that the self-avoiding walk converges to a particular continuous random fractal curve called *Schramm-Loewner evolution* (SLE_κ) with parameter $\kappa = 8/3$. SLE_κ curves were first introduced by Schramm [29] and are in some sense the most natural types of random curves in the plane which do not cross themselves. The parameter κ controls the "windiness" of the curve: $\kappa = 0$ corresponds to a straight line, and the curve becomes more "rough" as κ increases. In particular, SLE_κ curves have no self-intersections for $\kappa \in (0,4]$; intersect, but do not cross, themselves for $\kappa \in (4,8)$; and fill space for $\kappa \geq 8$.

Proving that self-avoiding walk converges to $\mathrm{SLE}_{8/3}$ remains an open problem. Nevertheless, a number of other discrete random curves arising in statistical mechanics — for example, loop-erased random walk and interfaces between "clusters" in percolation or the Ising model — have been proven to converge to SLE_κ with other values of κ [1, 19, 30, 31, 33, 34].

Part of my research involves trying to prove that certain types of discrete random curves converge to SLE and to better understand the properties of SLE itself. Such properties include relationships between SLE_κ curves for different values of κ, various quantitative measures of the "fractality" of the curve, and the connections between SLE curves and other random objects (more on this below).

3 Random surfaces

What does it mean to choose a "uniform random surface"? As in the case of curves, one can start by considering a discrete random surface. A natural type of discrete random surface is a *random planar map*. A *planar map* is a graph—a set of "vertices" (points) and "edges" (lines between them) — which can be drawn in the plane in such a way that no two edges cross (see Figure 2, left). There are only finitely many random planar maps with a given number n of edges, so we can choose one uniformly at random. It is also interesting to consider non-uniform random planar maps, where some planar maps are more likely than others.

One can think of a random planar map as a discrete random surface by viewing the faces (regions enclosed by the edges) as polygons which are glued together along the edges to produce a surface.[2] When we send the number of edges to infinity, random planar maps are expected to converge in the scaling limit to random fractal surfaces called γ-*Liouville quantum gravity* (LQG) surfaces with parameter $\gamma \in [0,2]$. The parameter γ controls the "roughness" of the surface. The case $\gamma = 0$ corresponds to an ordinary smooth surface — like the plane or sphere — and arises as the scaling limit of non-random planar maps like the square grid. The case $\gamma = \sqrt{8/3}$ (sometimes called "pure gravity") describes the scaling limit of uniform random planar maps. LQG with other values of γ describes the scaling limit of certain types of non-uniform random planar maps, which physicists think of as "random surfaces decorated by matter". LQG surfaces were first introduced by Polyakov [26,27] in the context of bosonic string theory and were later defined in a mathematically rigorous way by various authors (see, e.g., [2,8,12,28]).

There are a few different ways to formulate the convergence of random planar maps to LQG. One way is to think of a planar map as a *metric space* (i.e., a set equipped with a notion of the distance between two points) where the distance between two vertices is the minimum number of edges in a path between them. Then, one can ask if there is a limiting metric space to which random planar maps converge when we re-scale distances appropriately.

In the case of uniform random planar maps, this problem has been solved by Le Gall and Miermont [21,22]: such maps converge to a limiting metric space called the *Brownian map* when the total number n of vertices tends to ∞ and distances are re-scaled by $n^{-1/4}$. Miller and Sheffield [23–25] showed that the Brownian map is equivalent to $\sqrt{8/3}$-LQG. The Brownian map can be thought of as a uniform random surface in the same sense that Brownian motion is a uniform random curve.

Miller and I [12] recently showed that LQG surfaces for general $\gamma \in (0,2)$ can be defined as metric spaces, building on the works [3,6,13,17]. For $\gamma \neq \sqrt{8/3}$, these metric spaces should be the scaling limits of appropriate non-uniform random planar maps, but this has not yet been proven. Moreover, many important properties of distances in non-uniform random planar maps and γ-LQG surfaces for $\gamma \neq \sqrt{8/3}$ (e.g., the precise values of dimensions / scaling exponents) are not yet fully understood. One of the goals of my research is to obtain a better understanding of such distances.

Another way to talk about the limit of random planar maps is to think of vertices as points in the plane. One can then define the "discrete area" of a set to be $1/n$ times the number of vertices which it contains, where n is the total number of vertices, and ask if there is a random *measure* — i.e., a continuum area function — which describes the limit of these discrete areas as $n \to \infty$. For many types of random planar maps, it is expected that the limiting measure is the so-called γ-*LQG measure* defined in [8]. Note that the γ-LQG measure is very different in nature from the ordinary notion of "area" since the vertices of a random planar map will typically be much more concentrated in some regions of the plane than in others (see Figure 2, middle). In joint work with Miller and Sheffield [14], we proved this sort of convergence for a certain special type of random

[2] The surface obtained in this way always has the topology of the plane or the sphere. We are primarily interested in the metric/conformal structure of random surfaces rather than their topology.

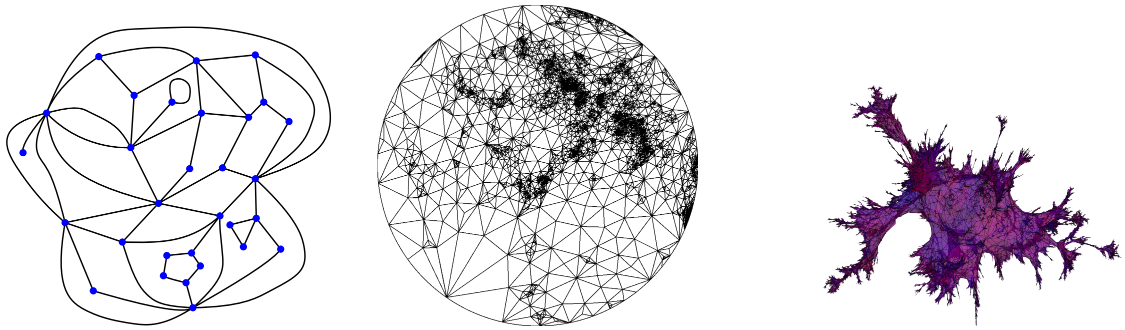

Figure 2: **Left:** A planar map. **Middle:** A simulation of a large random planar map drawn in the disk, made by Jason Miller. **Right:** A simulation of a large random planar map drawn in three-dimensional space, made by Jérémie Bettinelli.

planar maps. Another convergence result of this type (in the case of uniform triangulations) was proven by Holden and Sun [18]. In future research I would like to extend these results to other random planar maps.

There are a number of deep connections between γ-LQG surfaces and Schramm-Loewner evolution curves, which have important applications to both objects. For example, it was shown by Sheffield [32] that one can "cut" a certain type of LQG surface in two via an SLE curve to obtain two *independent* LQG surfaces. Many extensions of this result were proven by Duplantier, Miller, and Sheffield [7]. Such connections are a major tool in my research. For example, Miller and I [10,16] used the relationship between $\mathrm{SLE}_{8/3}$ and $\sqrt{8/3}$-LQG to prove that the self-avoiding walk on a random planar map converges to $\mathrm{SLE}_{8/3}$ (this is in contrast to the case of self-avoiding walk on the square grid discussed above, where the convergence has not been proven). As another example, in work with several different coauthors [4,9,11,15] we used the connections between SLE, LQG, and random planar maps to prove several facts about distances and random walk on certain types of random planar maps. We expect these connections to have many more applications in the future.

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