

# Coarse differentiation of quasi-isometries I: spaces not quasi-isometric to Cayley graphs

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## Abstract

In this paper, we prove that certain spaces are not quasi-isometric to Cayley graphs of finitely generated groups. In particular, we answer a question of Woess and prove a conjecture of Diestel and Leader by showing that certain homogeneous graphs are not quasi-isometric to a Cayley graph of a finitely generated group.

This paper is the first in a sequence of papers proving results announced in [EFW1]. In particular, this paper contains many steps in the proofs of quasi-isometric rigidity of lattices in Sol and of the quasi-isometry classification of lamplighter groups. The proofs of those results are completed in [EFW2].

The method used here is based on the idea of *coarse differentiation* introduced in [EFW1].

## 1 Introduction and statements of rigidity results

For any group  $\Gamma$  generated by a subset  $S$  one has the associated Cayley graph,  $C_\Gamma(S)$ . This is the graph with vertex set  $\Gamma$  and edges connecting any pair of elements which differ by right multiplication by a generator. There is a natural  $\Gamma$  action on  $C_\Gamma(S)$  by left translation. By giving every edge length one, the Cayley graph can be made into a (geodesic) metric space. The distance on  $\Gamma$  viewed as the vertices of the Cayley graph is the *word metric*, defined via the norm:

$$\|\gamma\| = \inf\{\text{length of a word in the generators } S \text{ representing } \gamma \text{ in } \Gamma.\}$$

Different sets of generators give rise to different metrics and Cayley graphs for a group but one wants these to be equivalent. The natural notion of equivalence in this category is *quasi-isometry*:

**Definition 1.1.** *Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Given real numbers  $\kappa \geq 1$  and  $C \geq 0$ , a map  $f : X \rightarrow Y$  is called a  $(\kappa, C)$ -quasi-isometry if*

1.  $\frac{1}{\kappa}d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq \kappa d_X(x_1, x_2) + C$  for all  $x_1$  and  $x_2$  in  $X$ , and,
2. the  $C$  neighborhood of  $f(X)$  is all of  $Y$ .

This paper begins the proofs of results announced in [EFW1] by developing the technique of *coarse differentiation* first described there. Proofs of some of the results in [EFW1] are continued in [EFW2]. Even though quasi-isometries have no local structure and conventional derivatives do not make sense, we essentially construct a “coarse derivative” that models the large scale behavior of the quasi-isometry.

A natural question which has arisen in several contexts is whether there exist spaces not quasi-isometric to Cayley graphs. This is uninteresting without some assumption on homogeneity on the space, since Cayley graphs clearly have transitive isometry group. In this paper we prove that two types of spaces are not quasi-isometric to Cayley graphs. The first are non-unimodular three dimensional solvable groups which do not admit left invariant metrics of nonpositive curvature. The second are the Diestel-Leader graphs, homogeneous graphs first constructed in [DL] where it was conjectured that they were not quasi-isometric to any Cayley graph. We prove this conjecture, thereby answering a question raised by Woess in [SW, W].

Our work is also motivated by the program initiated by Gromov to study finitely generated groups up to quasi-isometry [Gr1, Gr2, Gr3]. Much interesting work has been done in this direction, see e.g. [E, EF, FM1, FM2, FM3, FS, KL, MSW, P1, S1, S2, Sh, W]. For a more detailed discussion of history and motivation, see [EFW1].

We state our results for solvable Lie groups first as it requires less discussion:

**Theorem 1.2.** *Let  $\text{Sol}(m, n) = \mathbb{R} \ltimes \mathbb{R}^2$  be a solvable Lie group where the  $\mathbb{R}$  action on  $\mathbb{R}^2$  is defined by  $z \cdot (x, y) = (e^{mz}x, e^{-nz}y)$  for  $m, n \in \mathbb{R}^+$  with  $m > n$ . Then there is no finitely generated group  $\Gamma$  quasi-isometric to  $G$ .*

If  $m > 0$  and  $n < 0$ , then  $G$  admits a left invariant metric of negative curvature. The fact that there is no finitely generated group quasi-isometric to  $G$  in this case, provided  $m \neq n$ , is a result of Kleiner [K], see also [P2]. When  $m = n$ , the group  $\text{Sol}(n, n)$  contains cocompact lattices which are (obviously) quasi-isometric to  $\text{Sol}(n, n)$ . In the sequel to this paper we prove that any group quasi-isometric to  $\text{Sol}(n, n)$  is virtually

a lattice in  $\text{Sol}(n, n)$  [EFW2]. Many of the partial results in this paper hold for  $m \geq n$  and are used in that paper as well. Note that the assumption  $m \geq n$  is only to fix orientation and that the case  $m < n$  can be reduced to this one by changing coordinates.

We also obtain the following (which is an immediate corollary of [FM3, Theorem 5.1] and Theorem 2.1 below:)

**Theorem 1.3.** *Sol( $m, n$ ) is quasi-isometric to Sol( $m', n'$ ) if and only if  $m'/m = n'/n$ .*

Before stating the next results, we recall a definition of the Diestel-Leader graphs,  $\text{DL}(m, n)$ . In this setting,  $m, n \in \mathbb{Z}^+$  and we assume  $m \geq n$ . Let  $T_1$  and  $T_2$  be regular trees of valence  $m + 1$  and  $n + 1$  respectively. Choose orientations on the edges of  $T_1$  and  $T_2$  so each vertex has  $n$  (resp.  $m$ ) edges pointing away from it. This is equivalent to choosing ends on these trees. We can view these orientations as defining height functions  $f_1$  and  $f_2$  on the trees (the Busemann functions for the chosen ends). If one places the point at infinity determining  $f_1$  at the bottom of the page and the point at infinity determining  $f_2$  at the top of the page, then the trees can be drawn as:

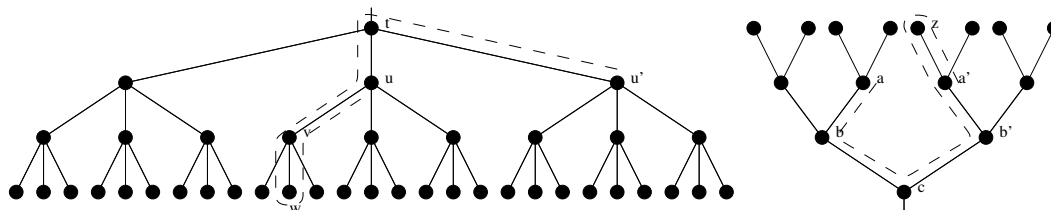


Figure 1: The trees for  $\text{DL}(3, 2)$ . Figure borrowed from [PPS].

The graph  $\text{DL}(m, n)$  is the subset of the product  $T_1 \times T_2$  defined by  $f_1 + f_2 = 0$ . There is strong analogy with the geometry of solvable groups which is made clear in section 3.

**Theorem 1.4.** *There is no finitely generated group quasi-isometric to the graph  $\text{DL}(m, n)$  for  $m \neq n$ .*

For  $n = m$  the Diestel-Leader graphs arise as Cayley graphs of lamplighter groups  $\mathbb{Z} \wr F$  for  $|F| = n$ . This observation was apparently first made by R.Moeller and P.Neumann [MN] and is described explicitly, from two slightly different points of view, in [Wo2] and [W]. In [EFW2] we classify lamplighter groups up to quasi-isometry and prove that any group quasi-isometric to a lamplighter group is a lattice

in  $\text{Isom}(DL(n, n))$  for some  $n$ . As discussed above, many of the technical results in this paper are used in those proofs.

We also obtain the following analogue of Theorem 1.3:

**Theorem 1.5.**  *$DL(m, n)$  is quasi-isometric to  $DL(m', n')$  if and only if  $m$  and  $m'$  are powers of a common integer,  $n$  and  $n'$  are powers of a common integer, and  $\log m' / \log m = \log n' / \log n$ .*

Unlike Theorem 1.3, the proof of this theorem is not completed in this paper. The case when  $m = n$  here requires additional arguments. For solvable groups,  $\text{Sol}(n, n)$  is always quasi-isometric to  $\text{Sol}(n', n')$  for all  $n$  and  $n'$ . As indicated by the statement of the theorem, this is not true for  $DL(n, n)$  and  $DL(n', n')$  which are only quasi-isometric when  $n$  and  $n'$  are powers of a common integer. This last statement is only proven in [EFW2].

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## 2 Quasi-isometries are height respecting

A typical step in the study of quasi-isometric rigidity of groups is the identification of all quasi-isometries of some space  $X$  quasi-isometric to the group, see §7 for more details. For us, the space  $X$  is either a solvable Lie group  $\text{Sol}(m, n)$  or  $DL(m, n)$ . In all of these examples there is a special function  $h : X \rightarrow \mathbb{R}$  which we call the height function and a foliation of  $X$  by level sets of the height function. We will call a quasi-isometry of any of these spaces *height respecting* if it permutes the height level sets to within bounded distance (In [FM4], the term used is horizontal respecting). For technical reasons, it is convenient to consider the more general question of quasi-isometries  $\text{Sol}(m, n) \rightarrow \text{Sol}(m', n')$ .

For  $\text{Sol}(m, n)$ , the height function is  $h(x, y, z) = z$ .

**Theorem 2.1.** *For any  $m \geq n > 0$ , any  $(\kappa, C)$ -quasi-isometry  $\phi : \text{Sol}(m, n) \rightarrow \text{Sol}(m', n')$  is within bounded distance of a height respecting quasi-isometry  $\hat{\phi}$ . Furthermore, this distance can be taken uniform in  $(\kappa, C)$  and therefore, in particular,  $\hat{\phi}$  is a  $(\kappa', C')$ -quasi-isometry where  $\kappa', C'$  depend only on  $\kappa$  and  $C$  and on  $m, n, m', n'$ .*

The proof of Theorem 2.1 is completed in this paper in the case  $m > n$ . The remaining case,  $m = n$ , is more difficult and is treated in [EFW2]. Most of the argument here applies in both cases and the only difference occurs at what is labelled “Step II” below.

In fact, Theorem 2.1 can be used to identify the self quasi-isometries of  $\text{Sol}(m, n)$  completely. We will need the following definition:

**Definition 2.2** (Product Map, Standard Map). A map  $\hat{\phi} : \text{Sol}(m, n) \rightarrow \text{Sol}(m', n')$  is called a *product map* if it is of the form  $(x, y, z) \rightarrow (f(x), g(y), q(z))$  or  $(x, y, z) \rightarrow (g(y), f(x), q(z))$ , where  $f, g$  and  $q$  are functions from  $\mathbb{R} \rightarrow \mathbb{R}$ . A product map  $\hat{\phi}$  is called *b-standard* if it is the composition of an isometry with a map of the form  $(x, y, z) \rightarrow (f(x), g(y), z)$ , where  $f$  and  $g$  is Bilipshitz with the Bilipshitz constant bounded by  $b$ .

It is easy to see that any height-respecting quasi-isometry is at a bounded distance from a standard map, and the standard maps from  $\text{Sol}(m, n)$  to  $\text{Sol}(m, n)$  form a group which is isomorphic to  $(\text{Bilip}(\mathbb{R}) \times \text{Bilip}(\mathbb{R})) \rtimes \mathbb{Z}/2\mathbb{Z}$  when  $m = n$  and  $(\text{Bilip}(\mathbb{R}) \times \text{Bilip}(\mathbb{R}))$  otherwise. Given a metric space  $X$ , one defines  $\text{QI}(X)$  to be the group of quasi-isometries of  $X$  modulo the subgroup of those at finite distance from the identity. Theorem 2.1 then implies that  $\text{QI}(\text{Sol}(m, n)) = (\text{Bilip}(\mathbb{R}) \times \text{Bilip}(\mathbb{R})) \rtimes \mathbb{Z}/2\mathbb{Z}$  when  $m = n$  and  $(\text{Bilip}(\mathbb{R}) \times \text{Bilip}(\mathbb{R}))$  otherwise. This explicit description was conjectured by Farb and Mosher in the case  $m = n$ .

Recall that  $\text{DL}(m, n)$  is defined as the subset of  $T_{m+1} \times T_{n+1}$  where  $f_1(x) + f_2(y) = 0$  where  $f_1$  and  $f_2$  are Busemann functions on  $T_{m+1}$  and  $T_{n+1}$  respectively. We fix the convention that Busemann functions decrease as one moves toward the end from which they are defined. We set  $h((x, y)) = f_m(x) = -f_n(y)$  which makes sense exactly on  $\text{DL}(m, n) \subset T_{m+1} \times T_{n+1}$ . Note that in this choice  $T_{m+1}$  branches downwards and  $T_{n+1}$  branches upwards. The reader can verify that the level sets of the height function are orbits for a subgroup of  $\text{Isom}(\text{DL}(m, n))$ .

**Theorem 2.3.** *For any  $m \geq n$ , any  $(\kappa, C)$ -quasi-isometry  $\varphi$  from  $\text{DL}(m, n)$  to  $\text{DL}(m', n')$  is within bounded distance of a height respecting quasi-isometry  $\hat{\phi}$ . Furthermore, the bound is uniform in  $\kappa$  and  $C$ .*

**Remark:** As above, this is proven here when  $m > n$  and in [EFW2] when  $m = n$ .

The discussion of standard and product maps in the setting of  $\text{DL}(m, n)$  is slightly more complicated. We let  $\mathbb{Q}_l$  be the  $l$ -adic rationals. The complement of a point in the boundary at infinity of  $T_{l+1}$  is easily seen to be  $\mathbb{Q}_l$ . Let  $x$  be a point in  $\mathbb{Q}_m$  and  $y$  a point in  $\mathbb{Q}_n$ . There is a unique vertical geodesic in  $\text{DL}(m, n)$  connecting  $x$

to  $y$ . To specify a point in  $\text{DL}(m, n)$  it suffices to specify  $x, y$  and a height  $z$ . We will frequently abuse notation by referring to the  $(x, y, z)$  coordinate of a point in  $\text{DL}(m, n)$  even though this representation is highly non-unique.

Theorem 2.3 can be used to identify the quasi-isometries of  $\text{DL}(m, n)$  completely. We need to define product and standard maps as in the case of solvable groups, but there is an additional difficulty introduced by the non-uniqueness of our coordinates. This is that maps of the form  $(x, y, z) \rightarrow (f(x), g(y), q(z))$ , even when one assumes they are quasi-isometries, are not well-defined, different coordinates for the same points will give rise to different images. We will say a quasi-isometry  $\psi$  is *at bounded distance* from a map of the form  $(x, y, z) \rightarrow (f(x), g(y), q(z))$  if  $d(\psi(p), (f(x), g(y), q(z)))$  is uniformly bounded for all points and all choices  $p = (x, y, z)$  of coordinates representing each point. It is easy to check that  $(x, y, z) \rightarrow (f(x), g(y), q(z))$  is defined up to bounded distance if we assume that the resulting map of  $\text{DL}(m, n)$  is a quasi-isometry. The bound depends on  $\kappa, C, m, n, m'$  and  $n'$ .

**Definition 2.4** (Product Map, Standard Map). A map  $\hat{\phi} : \text{DL}(m, n) \rightarrow \text{DL}(m', n')$  is called a *product map* if it is within bounded distance of the form  $(x, y, z) \rightarrow (f(x), g(y), q(z))$  or  $(x, y, z) \rightarrow (g(y), f(x), q(z))$ , where  $f : \mathbb{Q}_m \rightarrow \mathbb{Q}_{m'}$  (or  $\mathbb{Q}_{n'}$ ),  $g : \mathbb{Q}_n \rightarrow \mathbb{Q}_{n'}$  (or  $\mathbb{Q}_{m'}$ ) and  $q : \mathbb{R} \rightarrow \mathbb{R}$ . A product map  $\hat{\phi}$  is called *b-standard* if it is the composition of an isometry with a map within bounded distance of one of the form  $(x, y, z) \rightarrow (f(x), g(y), z)$ , where  $f$  and  $g$  is Bilipshitz with the Bilipshitz constant bounded by  $b$ .

Any height-respecting quasi-isometry is at a bounded distance from a standard map, and the standard self maps of  $\text{DL}(m, n)$  form a group which is isomorphic to  $(\text{Bilip}(\mathbb{Q}_m) \times \text{Bilip}(\mathbb{Q}_n)) \rtimes \mathbb{Z}/2\mathbb{Z}$  when  $m = n$  and  $(\text{Bilip}(\mathbb{Q}_m) \times \text{Bilip}(\mathbb{Q}_n))$  otherwise. Theorem 2.1 implies that  $\text{QI}(\text{DL}(m, n)) = (\text{Bilip}(\mathbb{Q}_m) \times \text{Bilip}(\mathbb{Q}_n))$  unless  $m = n$  when  $\text{QI}(\text{DL}(m, n)) = (\text{Bilip}(\mathbb{Q}_m) \times \text{Bilip}(\mathbb{Q}_n)) \rtimes \mathbb{Z}/2\mathbb{Z}$ .

### 3 Geometry of $\text{Sol}(m, n)$ and $\text{DL}(m, n)$

In this section we describe the geometry of  $\text{Sol}(m, n)$  and  $\text{DL}(m, n)$ , with emphasis on the geometric facts used in our proofs. In this section we allow the possibility that  $m = n$ .

#### 3.1 Geodesics, quasi-geodesics and quadrilaterals

The upper half plane model of the hyperbolic plane  $\mathbb{H}^2$  is the set  $\{(x, \xi) \mid \xi > 0\}$  with the length element  $ds^2 = \frac{1}{\xi^2}(dx^2 + d\xi^2)$ . If we make the change of variable  $z = \log \xi$ ,

we get  $\mathbb{R}^2$  with the length element  $ds^2 = dz^2 + e^{-z}dx^2$ . This is the *log model* of the hyperbolic plane  $\mathbb{H}^2$ . Note that changing  $ds^2$  to  $dz^2 + e^{-mz}dx^2$  we are choosing another metric of constant negative curvature, but changing the value of the curvature. This can be seen by checking that the substitution  $z \rightarrow \frac{z}{m}, x \rightarrow \frac{x}{m}$  is a homothety.

The length element of  $\text{Sol}(m, n)$  is:

$$ds^2 = dz^2 + e^{-2mz}dx^2 + e^{2nz}dy^2.$$

Thus planes parallel to the  $xz$  plane are hyperbolic planes in the log model. Planes parallel to the  $yz$  plane are *upside-down* hyperbolic planes in the log model. When  $m \neq n$ , these two families of hyperbolic planes have different normalization on the curvature. All of these copies of  $\mathbb{H}^2$  are isometrically embedded and totally geodesic.

- We use  $x, y, z$  coordinates on  $\text{Sol}(m, n)$ , with  $z$  called the *height*, and  $x$  called the *depth*. The planes parallel to the  $xz$  plane are right-side up hyperbolic planes (in the log model), and the planes parallel to the  $yz$  plane are upside-down hyperbolic planes (also in the log model).
- By “distance”, “area” and “volume” we mean these quantities in the  $\text{Sol}(m, n)$  metric.

We will refer to lines parallel to the  $x$ -axis as  $x$ -horocycles, and to lines parallel to the  $y$ -axis as  $y$ -horocycles. This terminology is justified by the fact that each ( $x$  or  $y$ )-horocycle is indeed a horocycle in the hyperbolic plane which contains it.

We now turn to a discussion of geodesics and quasi-geodesics in  $\text{Sol}(m, n)$ . Any geodesic in an  $\mathbb{H}^2$  leaf in  $\text{Sol}(m, n)$  is a geodesic. There is a special class of geodesics, which we call *vertical geodesics*. These are the geodesics which are of the form  $\gamma(t) = (x_0, y_0, t)$  or  $\gamma(t) = (x_0, y_0, -t)$ . We call the vertical geodesic *upward oriented* in the first case, and *downward oriented* in the second case. In both cases, this is a unit speed parametrization. Each vertical geodesic is a geodesic in two hyperbolic planes, the plane  $y = y_0$  and the plane  $x = x_0$ .

Certain quasi-geodesics in  $\text{Sol}(m, n)$  are easy to describe. Given two points  $(x_0, y_0, t_0)$  and  $(x_1, y_1, t_1)$ , there is a geodesic  $\gamma_1$  in the hyperbolic plane  $y = y_0$  that joins  $(x_0, y_0, t_0)$  to  $(x_1, y_0, t_1)$  and a geodesic  $\gamma_2$  in the plane  $x = x_1$  that joins  $(x_1, y_0, t_1)$  to a  $(x_1, y_1, t_1)$ . It is easy to check that the concatenation of  $\gamma_1$  and  $\gamma_2$  is a quasi-geodesic. In first matching the  $x$  coordinates and then matching the  $y$  coordinates, we made a choice. It is possible to construct a quasi-geodesic by first matching the  $y$  coordinates and then the  $x$  coordinates. This immediately shows that any pair of points not contained in a hyperbolic plane in  $\text{Sol}(m, n)$  can be joined by two distinct quasi-geodesics

which are not close together. This is an aspect of positive curvature. One way to prove that the objects just constructed are quasi-geodesics is to note the following: The pair of projections  $\pi_1, \pi_2 : \text{Sol}(m, n) \rightarrow \mathbb{H}^2$  onto the  $xz$  and  $yz$  coordinate planes can be combined into a quasi-isometric embedding  $\pi_1 \times \pi_2 : \text{Sol}(m, n) \rightarrow \mathbb{H}^2 \times \mathbb{H}^2$ .

This entire discussion is easily mimicked in  $\text{DL}(m, n)$  by replacing geodesics and horocycles in hyperbolic planes with geodesics and horocycles in the corresponding trees. When we want to state a fact that holds both for  $\text{Sol}(m, n)$  and  $\text{DL}(m, n)$ , we refer to the *model space* which we denote by  $X(m, n)$ .

We define the upper boundary  $\partial^+ X$  as the set of equivalence classes of vertical geodesic rays going up (where two rays are considered equivalent if they are bounded distance apart). The lower boundary  $\partial_- X$  is defined similarly. It is easy to see that if  $X = \text{Sol}(m, n)$  case,  $\partial^+ X \cong \mathbb{R}$  and  $\partial_- X \cong \mathbb{R}$ . If  $X = \text{DL}(m, n)$ , then  $\partial_- X \cong \mathbb{Q}_m$  and  $\partial^+ X \cong \mathbb{Q}_n$ . As discussed in Section 2, if  $x \in \partial_- X$ ,  $y \in \partial^+ X$  and  $z \in \mathbb{R}$ , we can define  $(x, y, z) \in X$  as the point at height  $z$  on the unique vertical geodesic connecting  $x$  and  $y$ .

**Notational Conventions.** In the following lemma and throughout the paper), we use the notation  $a = O(b)$  to mean that  $a < c_1 b$  where  $c_1$  is a constant depending only on the quasi-isometry constants  $(\kappa, C)$  of  $\phi$  and on the model space or spaces (i.e. on  $m, n, m', n'$ ). We use the notation  $a > \Omega(b)$  to mean that  $a > c_2 b$ , where  $c_2$  depends on the same quantities as  $c_1$ . We also use the notation  $a \gg b$  and  $a \ll b$  with the same meaning of the implied constants.

We state here a key geometric fact used at various steps in the proof.

**Lemma 3.1** (Quadrilaterals). *Let  $\epsilon > 0$  depending only on  $m', n'$ . Suppose  $p_1, p_2, q_1, q_2 \in X(m', n')$  and  $\gamma_{ij} : [0, \ell_{ij}] \rightarrow X(m', n')$  are vertical geodesic segments parametrized by arclength. Suppose  $C \gg 0$  and  $0 < D < \epsilon \ell_{ij}$ .*

*Assume that for  $i = 1, 2, j = 1, 2$ ,*

$$d(p_i, \gamma_{ij}(0)) \leq C \quad \text{and} \quad d(q_i, \gamma_{ij}(\ell_{ij})) \leq D,$$

*so that  $\gamma_{ij}$  connects the  $C$ -neighborhood of  $p_i$  to the  $D$ -neighborhood of  $q_j$ . Further assume that for  $i = 1, 2$  and all  $t$ ,  $d(\gamma_{i1}(t), \gamma_{i2}(t)) \geq (1/10)t - C$  (so that for each  $i$ , the two segments leaving the neighborhood of  $p_i$  diverge right away). Then there exists  $C_1 = O(C)$  such that exactly one of the following holds:*

- (a) *All four  $\gamma_{ij}$  are upward oriented, and  $p_2$  is within  $C_1$  of the  $x$ -horocycle passing through  $p_1$ .*
- (b) *All four  $\gamma_{ij}$  are downward oriented, and  $p_2$  is within  $C_1$  of the  $y$ -horocycle passing through  $p_1$ .*

We think of  $p_1, p_2, q_1$  and  $q_2$  as defining a quadrilateral. The content of the lemma is that any quadrilateral has its four "corners" in pairs that lie essentially along horocycles.

In particular, if we take a quadrilateral with geodesic segments  $\gamma_{ij}$  and with  $h(p_1) = h(p_2)$  and  $h(q_1) = h(q_2)$  and map it forward under a  $(\kappa, C)$ -quasi-isometry  $\phi : X(m, n) \rightarrow X(m', n')$ , and if we would somehow know that  $\phi$  sends each of the four  $\gamma_{ij}$  close to a vertical geodesic, then Lemma 3.1 would imply that  $\phi$  sends the  $p_i$  to a pair of points at roughly the same height.

To prove Lemma 3.1, we require a combinatorial lemma.

**Lemma 3.2** (Complete Bipartite Graphs). *Let  $\Gamma$  be a oriented graph with four vertices  $p_1, p_2, q_1, q_2$  and four edges, such that there is exactly one an edge connecting each  $p_i$  to each  $q_j$ . Then exactly one of the following is true:*

- (i) *All the edges of  $\Gamma$  are from some  $p_i$  to some  $q_j$ .*
- (ii) *All the edges of  $\Gamma$  are from some  $q_j$  to some  $p_i$ .*
- (iii) *There exist two vertices  $v_1$  and  $v_2$  which are connected by two distinct directed paths, and at least of the  $v_i$  is  $p_1$  or  $p_2$ .*

**Proof.** Since there are only 16 possibilities for  $\Gamma$ , one can check directly. One way to organize the check is to let  $k$  denote the sum of number of edges outgoing from  $p_1$  and the number of edges outgoing from  $p_2$ . If  $k = 0$ , (ii) holds, and if  $k = 4$ , then (i) holds. It is easy to check that for  $1 \leq k \leq 3$ , (iii) holds.  $\square$

**Proof of Lemma 3.1.** Let us assume for the moment that all the geodesics are downward oriented. Let  $x_{ij}, y_{ij}$  denote the  $x$  and  $y$  coordinates of the vertical geodesics  $\gamma_{ij}$ . By the assumptions near  $p_i$  we have for  $i = 1, 2$ ,  $C_2^{-1} \leq |x_{i1} - x_{i2}|e^{-m'h(p_i)} \leq C_2$ , where  $C_2 = O(C)$ . By the assumptions near  $q_j$  we have for  $j = 1, 2$ ,  $|x_{1j} - x_{2j}|e^{-m'h(q_j)} \leq D_1$  where  $D_1 = O(D)$ . Note that since for all  $i, j$ ,  $D < \epsilon \ell_{ij}$ , we have  $D_1 e^{m'h(q_j)} \ll C_2 e^{m'h(p_i)}$ . These inequalities imply that  $e^{-m'(h(p_1) - h(p_2))} = O(C_2)$ , and also that  $|x_{1j} - x_{2j}|e^{-m'h(p_1)} = O(C_2)$ . This proves the lemma under the assumption of downward orientation.

The case where all the vertical geodesics are upward oriented is identical (except that one considers differences in  $y$ -coordinates instead).

To reduce to the cases already considered, we apply Lemma 3.2 to the graph  $\Gamma$  consisting of the vertices  $p_1, p_2, q_1, q_2$  with edges the vertical geodesics "almost" connecting them. Suppose that possibility (iii) of Lemma 3.2 holds. Then, we would

then have two distinct oriented paths  $\eta_1$  and  $\eta_2$  connecting  $v_1$  and  $v_2$ . Each  $\eta_i$  is either a vertical geodesic, a concatenation two vertical geodesics, one of which ends near the beginning of the other, or a similar concatenation of three vertical geodesics. In each case it is easy to check that each  $\eta_i$  is close to a vertical geodesic  $\lambda'_i$ . (See Lemma 4.4 for a more general variant of this fact.) But this is a contradiction in view of the divergence assumptions, since any pair of vertical geodesics beginning and ending near the same point are close for their entire length. Thus either (i) or (ii) of Lemma 3.2 holds. □

## 3.2 Volume and measure

There is a large difference between the unimodular and nonunimodular examples we consider that has to do with the measures of sets, unimodularity and amenability. In the cases where  $m = n$  the spaces we consider are metrically amenable and have unimodular isometry group. When  $m \neq n$  the spaces are not metrically amenable and the isometry groups are not unimodular, though the isometry group remains amenable as a group. In particular it is immediately clear that  $\text{DL}(n, n)$  cannot be quasi-isometric to  $\text{DL}(m, n)$  with  $m \neq n$  (since one has metric Følner sets and the other does not. For the same reason,  $\text{Sol}(n, n)$  is not quasi-isometric to  $\text{Sol}(m', n')$  with  $m' \neq n'$ ).

The natural volume  $\text{vol}$  on  $\text{DL}(m, n)$  is the counting measure, the natural volume on  $\text{Sol}(m, n)$  is  $\text{vol} = e^{(n-m)z} dx dy dz$ . Note that for the unimodular case where  $m = n$ , the volume on  $\text{Sol}(m, n)$  is just the standard volume on  $\mathbb{R}^3$ . In the case when  $m \neq n$  we introduce a new measure. In the case of  $\text{Sol}(m, n)$  this is just  $\mu = dx dy dz$ . Note that on  $z$  level sets this is a rescaling of  $\text{vol}$  by a factor of  $e^{(n-m)z}$ . Analogously on  $\text{DL}(m, n)$ , we choose a height function  $h : \text{DL}(m, n) \rightarrow \mathbb{Z}$  and let  $\mu$  be counting measure times  $n^{h(x)} m^{-h(x)}$ . Recall that we are assuming that  $m \geq n$ .

We now define certain useful subsets of  $\text{Sol}(m, n)$ . Let  $B(L, \vec{0}) = [-\frac{e^{2mL}}{2}, \frac{e^{2mL}}{2}] \times [-\frac{e^{2nL}}{2}, \frac{e^{2nL}}{2}] \times [-\frac{L}{2}, \frac{L}{2}]$ . When  $m = n$ , then  $|B(L, \vec{0})| \approx Le^{2mL}$  and  $\text{Area}(\partial B(L, \vec{0})) \approx e^{2mL}$ , so  $B(L)$  is a Følner set.

To define the analogous object in  $\text{DL}(m, n)$ , we look at the set of points in  $\text{DL}(m, n)$  we fix a basepoint  $\vec{0}$  and a height function  $h$  with  $h(\vec{0}) = 0$ . Let  $L$  be an even integer and let  $\text{DL}(m, n)_L$  be the  $h^{-1}([-\frac{L+1}{2}, \frac{L+1}{2}])$ . Then  $B(L, \vec{0})$  is the connected component of  $\vec{0}$  in  $\text{DL}(m, n)_L$ . We are assuming that the top and bottom of the box are midpoints of edges, to guarantee that they have zero measure.

We call  $B(L, \vec{0})$  a box of size  $L$  centered at the identity. In  $\text{Sol}(m, n)$ , we define the box of size  $L$  centered at a point  $p$  by  $B(L, p) = T_p B(L, \vec{0})$  where  $T_p$  is left translation by  $p$ . We frequently omit the center of a box in our notation and write  $B(L)$ . For the case of  $\text{DL}(m, n)$  it is easiest to define the box  $B(L, p)$  directly. I.e. let  $\text{DL}(m, n)_{[h(p) - \frac{L+1}{2}, h(p) + \frac{L+1}{2}]}$  =  $h^{-1}([h(p) - \frac{L+1}{2}, h(p) + \frac{L+1}{2}])$  and let  $B(L, p)$  be the connected component of  $p$  in  $\text{DL}(m, n)_{[h(p) - \frac{L+1}{2}, h(p) + \frac{L+1}{2}]}$ . It is easy to see that isometries of  $\text{DL}(m, n)$  carry boxes to boxes.

We record the following lemma which holds for any model space  $X(m, n)$ .

**Lemma 3.3.** *When  $m = n$ , the fraction of the volume of  $B(L)$  which is within  $\epsilon L$  of the boundary of  $B(L)$  is  $O(\epsilon)$ . In all other cases, this is true for the  $\mu$ -measure but not the volume.*

We first describe  $B(L)$  in the case of  $\text{Sol}(m, n)$ . In this case, the top of  $B(L)$ , meaning the set  $[-\frac{e^{2mL}}{2}, \frac{e^{2mL}}{2}] \times [-\frac{e^{2nL}}{2}, \frac{e^{2nL}}{2}] \times \{\frac{L}{2}\}$ , is not at all square - the sides of this rectangle are horocyclic segments of lengths  $e^{2mL}$  and 1 - in other words it is just a small metric neighborhood of a horocycle. Similarly, the bottom is also essentially a horocycle but in the transverse direction. Further, we can connect the 1-neighborhood of any point of the top horocycle to the 1-neighborhood of any point of the bottom horocycle by a vertical geodesic segment, and these segments essentially sweep out the box  $B(L)$ . This picture is even easier to understand in the Diestel-Leader graphs  $\text{DL}(n, n)$ , where the boundary of the box is simply the union of the top and bottom "horocycles", and the vertical geodesics in the box form a complete bipartite graph between the two. Thus a box  $B(L)$  contains a very large number of quadrilaterals.

### 3.3 Tiling

The purpose of this subsection is to prove the following lemma.

**Lemma 3.4.** *Choose  $L/R \in \mathbb{Z}$ . We can write*

$$B(L) = \bigsqcup_{i \in I} B_i(R) \sqcup \Upsilon \tag{1}$$

where  $\mu(\Upsilon) = O(R/L)\mu(B(L))$  and the implied constant depends only on the model space.

*In the case when  $X(m, n) = \text{DL}(m, n)$ , then  $\Upsilon$  can be chosen to be empty. (This is also possible for  $\text{Sol}(m, n)$  if  $e^m$  and  $e^n$  are integers.)*

**Remark:** We will always refer to a decomposition as in equation (1) as a *tiling* of  $B(L)$ . We often omit specific reference to the set  $\Upsilon$  when discussing tilings.

**Proof.** For simplicity of notation, we assume  $B(L)$  is centered at the origin.

We give the proof first in the case of  $\text{DL}(m, n)$  where it is almost trivial. Since  $\frac{L}{R} \in \mathbb{Z}$ , we can partition  $[-\frac{L+1}{2}, \frac{L+1}{2}]$  into subsegments of length  $R$  which we label  $S_1, \dots, S_J$  where  $J = \frac{L}{R}$ . We can then look at the sets  $\text{DL}(m, n)_j = h^{-1}(S_j)$ . Each connected component of  $\text{DL}(m, n)_j$  is clearly a box  $B_{j,k}(R)$  of size  $R$ . Each  $B_{j,k}(R)$  is either entirely inside or entirely outside of  $B(L)$ . We choose only those  $k$  for which  $B_{j,k}(R) \subset B(L)$ . It is also clear that, after reindexing, we have chosen boxes such that  $B(L) = \bigsqcup B_i(R)$ .

In  $\text{Sol}(m, n)$  the proof is similar, though does not in general give an exact tiling. We simply take the box  $B(L)$  and cover it as best possible with boxes of size  $R$ . Since  $\frac{L}{R} \in \mathbb{Z}$ , if we take  $B(R, \vec{0})$  and look at translates by  $(0, 0, Rc)$  for  $c$  an integer between  $-\frac{L}{R}$  and  $\frac{L}{R}$ , the resulting boxes are all in  $B(L)$ . We then take the resulting box  $B(R, (0, 0, Rc))$  at height  $k$  and translate it by vectors of the form  $(ae^{mcR}, \vec{b}e^{ncR}, 0)$  where  $|a| \leq e^{m(L-(c+1)R)}$  and  $|b| \leq e^{n(L-(1-c)R)}$  are integers. This results in boxes  $B(R, (a, b, c))$  which we re-index as  $B_i(R)$ . It is clear that every point not in  $\bigsqcup_i B_i(R)$  is within  $R$  of the boundary of  $B(L)$ . Letting  $\Upsilon = B(L) - \bigsqcup_i B_i(R)$  we have that  $\mu(\Upsilon) < O(R/L)\mu(B(L))$  by Lemma 3.3. □

## 4 Step 1

All the results of this section hold for  $X(m, n)$  with  $m \geq n$ , so in particular for the case  $m = n$ . The case  $m = n$  will be used in the sequel [EFW2]. Also, all results in this section hold for quasi-isometric embeddings, i.e. maps satisfying (1) but not (2) of Definition 1.1. In the this part of the paper, our aim is to prove the following:

**Theorem 4.1.** *Suppose  $\theta > 0$ ,  $\epsilon > 0$ . Then there exist constants  $0 < \alpha < \beta < \Delta$  (depending on  $\theta, \epsilon, \kappa, C$  and the model spaces) such that the following holds: Let  $\phi : X(m, n) \rightarrow X(m', n')$  be a  $(\kappa, C)$  quasi-isometry and suppose  $r_0$  is sufficiently large (depending on  $\kappa, C, \theta, \epsilon$ ). Then for any  $L > \Delta r_0$  and any  $B(L)$ , there exists  $R$  with  $\alpha r_0 < R < \beta r_0$  such that if one tiles*

$$B(L) = \bigsqcup_{i \in I} B_i(R)$$

then there exists a subset  $I_g$  of  $I$  with  $\mu(\bigcup_{i \in I_g} B_i(R)) \geq (1 - \theta)\mu(B(L))$  so that for any  $i \in I_g$  there exists  $U_i \subset B_i(R)$  with  $\mu(U_i) \geq (1 - \theta)\mu(B_i(R))$  and a product map  $\hat{\phi}_i : B_i(R) \rightarrow X(m', n')$  such that

$$d(\phi|_{U_i}, \hat{\phi}_i) = O(\epsilon R). \quad (2)$$

**Remarks.** This theorem says that every sufficiently large box  $B(L)$  can be tiled by much smaller boxes  $B_i(R)$ , and for most (i.e.  $1 - \theta$  fraction) of the smaller boxes  $B_i(R)$  there exist a subset  $U_i$  containing  $1 - \theta$  fraction of the  $\mu$ -measure of  $B_i(R)$  on which the map is a product map, up to error  $O(\epsilon R) \ll R$ . In the case where  $n = m$ , the measure of  $B_i(R)$  is independent of  $i$ , and we have exactly  $|I_g| \geq (1 - \theta)|I|$ . When  $m \neq n$ , both the number of boxes of size  $R$  in a height level set tiling and  $\mu(B_i(R))$  are functions of height.

One should note that the number  $R$ , and the subset where control the map depends on  $\phi$ . Also in Theorem 4.1 there is no assertion that the product maps  $\hat{\phi}_i$  on the different boxes  $B_i(R)$  match up.

This theorem is in a sense an analogue of Rademacher's theorem that a lipschitz function (or map) is differentiable almost everywhere. The boxes  $B_i(R)$  with  $i \in I_g$  should be thought of as coarse analogues of points of differentiability.

## 4.1 Behavior of quasi-geodesics

We begin by discussing some quantitative estimates on the behavior of quasi-geodesic segments in  $X(m', n')$  (or equivalently in  $X(m, n)$ ). Throughout the discussion we assume  $\alpha : [0, r] \rightarrow X(m', n')$  is a  $(\kappa, C)$ -quasi-geodesic segment for a fixed choice of  $(\kappa, C)$ , i.e.  $\alpha$  is a quasi-isometric embedding of  $[0, r]$  into  $X(m', n')$ . A quasi-isometric embedding is a map that satisfies point (1) in Definition 1.1 but not point (2). All of our quasi-isometric embeddings are assumed to be continuous.

**Definition 4.2 ( $\epsilon$ -monotone).** A quasigeodesic segment  $\alpha : [0, r] \rightarrow X(m', n')$  is  $\epsilon$ -monotone if for all  $t_1, t_2 \in [0, r]$  with  $h(\alpha(t_1)) = h(\alpha(t_2))$  we have  $|t_1 - t_2| < \epsilon r$ .

In §5 and §6 we will also need a variant. The reader may safely ignore this variant on first reading this section.

**Definition 4.3 (Weakly  $(\eta, C_1)$ -monotone).** A quasigeodesic segment  $\alpha : [0, r] \rightarrow X(m', n')$  is weakly  $(\eta, C_1)$ -monotone if for any two points  $0 < t_1 < t_2 < r$  with  $h(\alpha(t_1)) = h(\alpha(t_2))$ , we have  $t_2 - t_1 < \eta t_2 + C_1$ .

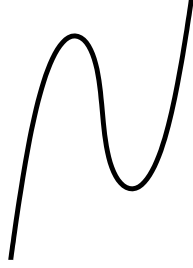


Figure 2: A quasigeodesic segment which is not  $\epsilon$ -monotone.

**Remark:** An  $\epsilon$ -monotone quasi-geodesic  $\alpha : [0, r] \rightarrow X(m', n')$  is a weakly  $(\epsilon, \epsilon r)$ -monotone quasi-geodesic.

The following fact about  $\epsilon$ -monotone geodesics is an easy exercise in hyperbolic geometry:

**Lemma 4.4.**

- (a) *Suppose  $\alpha : [0, r] \rightarrow X(m', n')$  is an  $\epsilon$ -monotone quasi-geodesic segment. Then, there exists a vertical geodesic segment  $\lambda$  in  $X(m', n')$  such that  $d(\alpha, \lambda) \leq \omega_1 \epsilon r$ , where  $\omega_1$  depends only on the model space  $X(m', n')$ .*
- (b) *Suppose  $\alpha : [0, r] \rightarrow X(m', n')$  is a weakly  $(\eta, C_1)$ -monotone quasi-geodesic segment. Then, there exists a vertical geodesic segment  $\lambda$  in  $X(m', n')$  such that  $d(\bar{\gamma}(t), \lambda(t)) \leq 2\kappa\eta t + \omega_2 C_1$ , where  $\omega_2$  depends only on  $X(m', n')$ .*

**Proof.** Both  $\epsilon$ -monotone and weakly  $(\eta, C)$ -monotone imply that the projections of  $\alpha$  onto both  $xz$  and  $yz$  hyperbolic planes are quasi-geodesics. The result is then a consequence of the Mostow-Morse lemma and the fact that the only geodesics shared by both families of hyperbolic planes are vertical geodesics. One can also prove the lemma by direct computation.  $\square$

**Remark:** The distance  $d(\alpha, \lambda)$  in (a) is the Hausdorff distance between the sets and does not depend on parametrizations. However, the parametrization on  $\lambda$  implied in (b) is not necessarily by arc length.

**Lemma 4.5** (Subdivision). *Suppose  $\alpha : [0, r] \rightarrow X(m', n')$  is a quasi-geodesic segment which is not  $\epsilon$ -monotone. Suppose  $N \gg 1$  (depending on  $\epsilon, \kappa, C$ ). Then*

$$\sum_{j=0}^{N-1} \left| h\left(\alpha\left(\frac{(j+1)r}{N}\right)\right) - h\left(\alpha\left(\frac{j r}{N}\right)\right) \right| \geq |h(\alpha(0)) - h(\alpha(r))| + \frac{\epsilon r}{8\kappa^2}.$$

**Proof.** Without loss of generality, we may assume that  $h(\alpha(0)) \geq h(\alpha(t_1)) = h(\alpha(t_3)) \geq h(\alpha(r))$ , where  $0 = t_0 < t_1 < t_3 < t_4 = r$  (if not, parametrize in the opposite direction). Since  $t_3 - t_1 > \epsilon r$ ,  $\alpha(t_3)$  and  $\alpha(t_1)$  are two points in  $X(m', n')$  which are at the same height and are at least  $\epsilon r/\kappa$  apart. Then, by  $X(m', n')$  geometry, any quasigeodesic path connecting  $\alpha(t_3)$  and  $\alpha(t_1)$  must contain a point  $q$  such that  $|h(q) - h(\alpha(t_1))| \geq (\epsilon r)/(4\kappa)$ . Hence, there exists a point  $t_2$  with  $t_1 < t_2 < t_3$  such that  $|h(\alpha(t_2)) - h(\alpha(t_1))| \geq (\epsilon r)/(4\kappa)$ . Hence,

$$\sum_{j=1}^4 |h(\alpha(t_j)) - h(\alpha(t_{j-1}))| \geq |h(\alpha(0)) - h(\alpha(r))| + \frac{\epsilon r}{4\kappa}$$

If  $N$  is large enough then the points  $t_1$ ,  $t_2$  and  $t_3$  have good approximations of the form  $jr/N$ , with  $j \in \mathbb{Z}$ . This implies the lemma.  $\square$

Informally, the proof amounts to the assertion that if  $N$  is sufficiently large, the total variation of the height increases after the subdivision by a term proportional to  $\epsilon$ . See Figure 3.

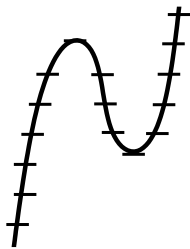


Figure 3: Proof of Lemma 4.5

**Choosing Scales:** Choose  $1 \ll r_0 \ll r_1 \ll \dots \ll r_S$ . In particular,  $C \ll r_0$  and for  $s \in [0, S-1] \cap \mathbb{Z}$ ,  $r_{s+1}/r_s > N$ .

**Lemma 4.6.** *Suppose  $L \gg r_S$ , and suppose  $\alpha : [0, L] \rightarrow X(m', n')$  is a quasi-geodesic segment. For each  $s \in [1, S]$ , subdivide  $[0, L]$  into  $L/r_s$  segments of length  $r_s$ . Let  $\delta_s(\alpha)$  denote the fraction of these segments whose images are not  $\epsilon$ -monotone. Then,*

$$\sum_{s=1}^S \delta_s(\alpha) \leq \frac{16\kappa^3}{\epsilon}.$$

*Proof.* By applying Lemma 4.5 to each non- $\epsilon$ -monotone segment on the scale  $r_S$ , we get

$$\begin{aligned} \sum_{j=1}^{L/r_{S-1}} |h(\alpha(jr_{S-1})) - h(\alpha((j-1)r_{S-1}))| &\geq \\ &\geq \sum_{j=1}^{L/r_S} |h(\alpha(jr_S)) - h(\alpha((j-1)r_S))| + \delta_S(\alpha) \frac{\epsilon L}{8\kappa^2}. \end{aligned}$$

Doing this again, we get after  $S$  iterations,

$$\begin{aligned} \sum_{j=1}^{L/r_0} |h(\alpha(jr_0)) - h(\alpha((j-1)r_0))| &\geq \\ &\geq \sum_{j=1}^{L/r_S} |h(\alpha(jr_S)) - h(\alpha((j-1)r_S))| + \frac{\epsilon L}{8\kappa^2} \sum_{s=1}^S \delta_s(\alpha). \end{aligned}$$

But the left-hand-side is bounded from above by the length and so bounded above by  $2\kappa L$ .  $\square$

## 4.2 Averaging

In this subsection we apply the estimates from above to images of geodesics under a quasi-isometry from  $X(m, n)$  to  $X(m', n')$ . The idea is to average the previous estimates over families of geodesics. In order to unify notation for the two possible model space types, we shift the parametrization of vertical geodesics in  $DL(m, n)$  so that they are parametrized by height minus  $\frac{1}{2}$ , i.e. by the interval  $[-\frac{L}{2}, \frac{L}{2}]$  rather than  $[-\frac{L+1}{2}, \frac{L+1}{2}]$ .

### Setup and Notation.

- Suppose  $\phi : X(m, n) \rightarrow X(m', n')$  is a  $(\kappa, C)$  quasi-isometry. Without loss of generality, we may assume that  $\phi$  is continuous.
- Let  $\gamma : [-\frac{L}{2}, \frac{L}{2}] \rightarrow X(m, n)$  be a vertical geodesic segment parametrized by arclength where  $L \gg C$ .
- Let  $\bar{\gamma} = \phi \circ \gamma$ . Then  $\bar{\gamma} : [-\frac{L}{2}, \frac{L}{2}] \rightarrow X(m', n')$  is a quasi-geodesic segment.

It follows from Lemma 4.6, that for every  $\theta > 0$  and every geodesic segment  $\gamma$ , assuming that  $S$  is sufficiently large, there exists  $s \in [1, S]$  such that  $\delta_s(\bar{\gamma}) < \theta$ . The difficulty is that  $s$  may depend on  $\gamma$ . In our situation, this is overcome as follows:

We will average the result of Lemma 4.6 over  $Y_L$ , the set of vertical geodesics in  $B(L)$ . Let  $|Y_L|$  denote the measure/cardinality of  $Y_L$ . We will always denote our average by  $\Sigma$ , despite the fact that when  $X(m, n) = \text{Sol}(m, n)$  this is actually an integral over  $Y_L$  and not a sum. When  $X(m, n) = \text{DL}(m, n)$  it is actually a sum. Changing order, we get:

$$\sum_{s=1}^S \left( \frac{1}{|Y_L|} \sum_{\gamma \in Y_L} \delta_s(\bar{\gamma}) \right) \leq \frac{32\kappa^3}{\epsilon}.$$

Let  $\delta > 0$  be a small parameter (In fact, we will choose  $\delta$  so that  $\delta^{1/4} = \min(\epsilon, \theta/256)$ , where  $\theta$  is as in Theorem 4.1). Then, if we choose  $S > \frac{32\kappa^3}{\epsilon\delta^4}$ , then there exists a scale  $s$  such that

$$\frac{1}{|Y|} \sum_{\gamma \in Y} \delta_s(\bar{\gamma}) \leq \delta^4. \quad (3)$$

**Conclusion.** On the scale  $R \equiv r_s$ , at least  $1 - \delta^4$  fraction of all vertical geodesic segments in  $B(L)$  have nearly vertical images under  $\phi$ .

From now on, we fix this scale, and drop the index  $s$ . We will refer to the scale  $R$  as the “big” scale, and the “edges” of length  $R$  (which are really unions of edges of smaller length) as “big” edges.

**Remark.** The difficulty is that, at this point, even though we know that most big edges have images under  $\phi$  which are nearly vertical, it is possible that some may have images which are going up, and some may have images which are going down.

### 4.3 Alignment

We assume that  $L/R \in \mathbb{Z}$ . As described in §3.3, we tile

$$B(L) = \bigsqcup_{i \in I} B_i(R).$$

Let  $Y_i$  denote the set of vertical geodesic segments in  $B_i(R)$ . We have

$$\frac{1}{|Y_L|} \sum_{\gamma \in Y_L} \delta_s(\bar{\gamma}) = \sum_{i \in I} \frac{\mu(B_i(R))}{\mu(B(L))} \left( \frac{1}{|Y_i|} \sum_{\lambda \in Y_i} \delta_s(\bar{\lambda}) \right) + O\left(\frac{R}{L}\right), \quad (4)$$

where  $\bar{\lambda} = \phi \circ \lambda$ , and  $\delta(\bar{\lambda}) = \delta_s(\bar{\lambda})$  is equal to 0 if  $\lambda$  is  $\epsilon$ -monotone, and equal to 1 otherwise. The error term of  $O(R/L)$  is due to the fact that the tiling may not be exact, see Lemma 3.4. To justify equation 4, one uses that  $\mu(B(L)) = |Y_L|L$  and  $\mu(B_i(R)) = |Y_i|R$ .

Since the left hand side is bounded by  $\delta^4$  and assuming  $R/L \ll \delta^2$ , we conclude the following:

**Lemma 4.7.** *Let us tile  $B(L)$  by boxes  $B_i(R)$  of size  $R$ , so that  $B(L) = \bigsqcup_{i \in I} B_i(R)$ . Then there exists a subset  $I_g$  of the indexing set  $I$  with  $\mu(\bigcup_{i \in I_g} B_i(R)) \geq (1 - \delta^2)\mu(B(L))$  such that if we let  $Y_i$  denote the set of vertical geodesics in  $B_i(R)$  then*

$$\frac{1}{|Y_i|} \sum_{\gamma \in Y_i} \delta_s(\bar{\gamma}) \leq 2\delta^2. \quad (5)$$

Note that  $R$  is the length of one big edge so that the set  $Y_i$  of vertical geodesics in  $B_i(R)$  consists of big edges connecting the top to the bottom. The equation (5) means that the fraction these edges which are not  $\epsilon$ -monotone is at most  $2\delta^2$ .

**Notation.** In the rest of §4.3 and in §4.4 we fix  $i \in I_g$  and drop the index  $i$ .

**Lemma 4.8** (Alignment). *Let  $e$  be an  $\epsilon$ -monotone big edge of  $B(R)$  going from the bottom to the top. We say that  $e$  is “upside-down” if  $\phi(e)$  is going down, and “right-side-up” if  $\phi(e)$  is going up. Then either the fraction of the big edges in  $B(R)$  which are upside down or the fraction of the big edges in  $B(R)$  which are right-side-up is at least  $1 - 4\delta$ .*

**Proof.** We have a natural notion of “top” vertices and “bottom” vertices so that each big edge connects a bottom vertex to a top vertex. (In this proof, a vertex must be on the top or bottom, since we are considering big edges only). Then  $B(R)$  is a complete bipartite graph. There must be a subset  $E$  of vertices of density  $1 - 4\delta$ , such such that for each vertex in  $v \in E$  the fraction of the edges incident to  $v$  which are not  $\epsilon$ -monotone is at most  $\delta/2$ . Let  $\Gamma_1$  be the subgraph of  $B(R)$  obtained by erasing any edge  $e$  such that  $\phi(e)$  is not  $\epsilon$ -monotone. We orient each edge  $e$  of  $\Gamma_1$  by requiring that  $\phi(e)$  is going down.

Let  $p_1, p_2 \in E$  be any two top vertices. Then we can find two bottom vertices  $q_1, q_2$  such that all four quasigeodesic segments  $\phi(\overline{p_1q_1})$ ,  $\phi(\overline{p_1q_2})$ ,  $\phi(\overline{p_2q_1})$  and  $\phi(\overline{p_2q_2})$  are all  $\epsilon$ -monotone,  $\overline{p_1q_1}$  and  $\overline{p_1q_2}$  diverge quickly at  $p_1$ , and  $\overline{p_2q_1}$  and  $\overline{p_2q_2}$  diverge quickly at  $p_2$ .

We now apply Lemma 3.1 to conclude that  $h(\phi(p_1)) = h(\phi(p_2)) + O(\epsilon R)$  and that all the segments  $\phi(\overline{p_i q_j})$  with  $i, j = 1, 2$  have the same orientation. Thus any two top vertices in  $E$  have images on essentially the same height, say  $h_1$ . Similarly, any two bottom vertices in  $E$  have images on the same height, say  $h_2$ . Since we must have  $h_1 > h_2$  or  $h_1 < h_2$ , the lemma holds.  $\square$

We define the *dominant orientation* to be right-side-up or upside-down so that the fraction of big edges which have the dominant orientation is at least  $1 - 4\delta$ .

## 4.4 Construction of a product map

Let  $Y'$  denote the space of pairs  $(\gamma, x)$  where  $\gamma \in Y$  is a vertical geodesic in  $B(R)$  and  $x \in \gamma$  is a point. Let  $|\cdot|$  denote uniform measure on  $Y'$ . (In the case of  $\text{DL}(m, n)$  this is just the counting measure.) The following lemma is a formal statement regarding subsets of  $Y'$  of large measure.

**Lemma 4.9.** *Suppose  $R \gg 1/\theta_1$  (where the implied constant depends only on the model space). Suppose  $E \subset Y'$ , with  $|E| \geq (1 - \theta_1)|Y'|$ . Then, there exists a subset  $U \subset B(R)$  such that:*

- (i)  $\mu(U) \geq (1 - 2\sqrt{\theta_1})\mu(B(R))$ , where  $\mu$  is defined in §3.2.
- (ii) *If  $x \in U$ , then for at least  $(1 - \sqrt{\theta_1})$  fraction of the vertical geodesics  $\gamma \in Y$  passing within distance  $1/2$  of  $x$ ,  $(\gamma, x) \in E$ .*

**Remark.** Note that for the case of  $\text{DL}(m, n)$ , any geodesic passing within distance  $(1/2)$  of  $x$  passes through  $x$ .

**Proof.** For  $x \in B(R)$ , let  $Y(x) \subset Y$  denote the set of geodesics which pass within  $1/2$  of  $x$ . For clarity, we first give the proof for the  $\text{DL}(m, n)$  case. Note that  $|Y(x)| = c\mu(\{x\})$ , where  $c$  depends on  $m, n$  and the location and size of  $B(R)$ . Note that

$$|Y|R = \sum_{x \in B(R)} \sum_{\gamma \in Y(x)} 1 = \sum_{x \in B(R)} |Y(x)| = \sum_{x \in B(R)} c\mu(\{x\}) = c\mu(B(R)) \quad (6)$$

Suppose  $f(\gamma, x)$  is any function of a geodesic  $\gamma$  and a point  $x \in \gamma$ : then,

$$\begin{aligned}
\frac{1}{|Y|R} \sum_{\gamma \in Y} \sum_{x \in \gamma} f(\gamma, x) &= \frac{1}{|Y|R} \sum_{x \in B(R)} \sum_{\gamma \in Y(x)} f(\gamma, x) \\
&= \frac{1}{|Y|R} \sum_{x \in B(R)} \frac{1}{|Y(x)|} \sum_{\gamma \in Y(x)} |Y(x)| f(\gamma, x) \\
&= \frac{1}{\mu(B(R))} \sum_{x \in B(R)} \frac{1}{|Y(x)|} \sum_{\gamma \in Y(x)} \mu(\{x\}) f(\gamma, x), \tag{7}
\end{aligned}$$

where in the last line we used (6).

We apply (7) with  $f$  the the characteristic function of the complement of  $E$ . We get,

$$\frac{1}{\mu(B(R))} \sum_{x \in B(R)} \mu(\{x\}) \left( \frac{1}{|Y(x)|} \sum_{\gamma \in Y(x)} f(\gamma, x) \right) < \theta_1 \tag{8}$$

Let  $F(x)$  denote the parenthesized quantity in the above expression. Let  $E_2 = \{x \in B(R) : F(x) > \sqrt{\theta_1}\}$ . Recall that Markov's inequality says that for any real-valued function  $f$ , and any real number  $a > 0$ , the measure of the set  $\{|f| > a\}$  is at most  $\frac{1}{a} \int |f|$ . Then, by this inequality,  $\mu(E_2)/\mu(B(R)) \leq \sqrt{\theta_1}/\theta_1 = \sqrt{\theta_1}$ , and for  $x \notin E_2$ , for at least  $(1 - \sqrt{\theta_1})$  fraction of the geodesics  $\gamma$  passing through  $x$ ,  $(\gamma, x) \in E$ .

This completes the proof for the DL( $m, n$ ) case. In Sol( $m, n$ ) the computation is essentially the same, except for the fact that  $|Y(x)|$  (i.e. the measure of set of geodesics passing within  $(1/2)$  of  $x$ ) can become smaller when  $x$  is within  $(1/2)$  of the boundary of  $B(R)$ . However, the relative  $\mu$  measure of such points is  $O(1/R)$  by Lemma 3.3. Therefore (6) and (7) hold up to error  $O(1/R) < \theta_1$ .  $\square$

**Corollary 4.10.** *There exists a subset  $U \subset B(R)$  with  $\mu(U) > (1 - 8\sqrt{\delta})\mu(B(R))$  such that for  $x \in U$ ,  $(1 - 2\sqrt{\delta})$ -fraction of the geodesics passing within  $(1/2)$  of  $x$  are  $\epsilon$ -monotone and have the dominant orientation.*

**Proof.** Let  $E$  denote the set of pairs  $(\gamma, x)$  where  $\gamma \in Y$  is a dominantly oriented  $\epsilon$ -monotone geodesic segment, and  $x$  is a point of  $\gamma$ . Let  $U \subset B(R)$  be the subset constructed by Lemma 4.9. Since  $|E| \geq (1 - 4\delta)|Y'|$ ,  $\mu(U) \geq (1 - 8\sqrt{\delta})\mu(B(R))$ .  $\square$

**Lemma 4.11.** *Suppose  $\phi$  and  $B(R)$  and  $U$  are as in Corollary 4.10. Then, there exist functions  $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ ,  $q : \mathbb{R} \rightarrow \mathbb{R}$ , and a subset  $U_1 \subset B(R)$  with  $\mu(U_1) > (1 - 128\delta^{1/4})\mu(B(R))$  such that for  $(x, y, z) \in U_1$ ,*

$$d(\phi(x, y, z), (\psi(x, y, z), q(z))) = O(\epsilon R) \tag{9}$$

**Proof.** We assume that the dominant orientation is right-side-up (the other case is identical). Now suppose  $p_1, p_2 \in U$  belong to the same  $x$ -horocycle. By the construction of  $U$  there exist  $q_1, q_2$  in  $B(R)$  (above  $p_1, p_2$ ) such that for each  $i = 1, 2$  the two geodesic segments  $\overline{p_i q_1}$  and  $\overline{p_i q_2}$  leaving  $p_i$  diverge quickly, and each of the quasi-geodesic segments  $\phi(\overline{p_i q_j})$  is  $\epsilon$ -monotone. Then by Lemma 4.4, each of the  $\phi(\overline{p_i q_j})$  is within  $O(\epsilon R)$  of a quasi-geodesic segment  $\lambda_{ij}$ . Now by applying Lemma 3.1 to the  $\lambda_{ij}$  we see that  $\phi(p_1)$  and  $\phi(p_2)$  are on the same  $x$ -horocycle, up to an error of  $O(\epsilon R)$ . Thus, the restriction of  $\phi$  to  $U$  preserves the  $x$ -horocycles. A similar argument (but now we will pick  $q_1, q_2$  below  $p_1, p_2$ ) shows that the restriction of  $\phi$  to  $U$  preserves the  $y$ -horocycles. We can now conclude that  $\phi$  is height respecting on a slightly smaller set  $U_1$ , i.e. there exist functions  $\psi : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  and  $q : \mathbb{R} \rightarrow \mathbb{R}$  such that for  $(x, y, z) \in U_1$ , (9) holds.  $\square$

**Proposition 4.12.** *Suppose  $\phi$  and  $B(R)$  and  $U$  are as in Corollary 4.10. Then, there exist functions  $f, g, q$ , a corresponding product map  $\hat{\phi}$ , and a subset  $U_2 \subset B(R)$  with  $\mu(U_2) > (1 - 256\delta^{1/4})\mu(B(R))$  such that for  $(x, y, z) \in U_2$ ,*

$$d(\phi(x, y, z), \hat{\phi}(x, y, z)) = O(\epsilon R).$$

**Proof.** To simplify language, we assume that the dominant orientation is right-side-up (the other case is identical). Let  $z_1$  (resp.  $z_2$ ) denote the height of the bottom (resp. top) of  $B(R)$ . If  $(x, y, z) \in B(R)$ , we let  $\gamma_{xy} : [z_1, z_2] \rightarrow B(R)$  denote the vertical geodesic segment  $\gamma_{xy}(t) = (x, y, t)$ . Let  $F_1$  (resp.  $F_2$ ) denote the subset of the bottom (resp. top) face of  $B(R)$  which is within  $8\delta^{1/4}$  of a point of  $U$ . Since  $\mu(U) \geq (1 - 8\sqrt{\delta})\mu(B(R))$ , each  $F_i$  has nearly full  $\mu$ -measure. In fact if we let  $U' \subset B(R)$  denote the set of points  $(x, y, z)$  such that  $(x, y, z_1) \in F_1$ ,  $(x, y, z_2) \in F_2$ , and  $\gamma_{xy}$  has  $\epsilon$ -monotone image under  $\phi$ , then  $\mu(U') \geq (1 - 8\delta^{1/4})\mu(B(R))$ .

Note that  $F_1$  is an  $O(1)$  neighborhood of a (subset of) a segment of a  $x$ -horocycle, say  $\{(x, y_1, z_1) : x \in A\}$ . Since the restriction of  $\phi$  to  $U$  preserves the  $x$ -horocycles,  $\delta^{1/4} < \epsilon$ , there exist numbers  $y'_1$  and  $z'_1$  and a function  $f : A \rightarrow \mathbb{R}$  or  $\mathbb{Q}_m$  such that for  $x \in A$ ,  $\phi(x, y_1, z_1)$  is at most  $O(\epsilon R)$  distance from  $(f(x), y'_1, z'_1)$ . Similarly,  $F_2$  is bounded distance from a set of the form  $\{(x_2, y, z_2) : y \in A'\}$ , and there exists a function  $g : A' \rightarrow \mathbb{R}$  or  $\mathbb{Q}_n$  such that the restriction of  $\phi$  to  $F_2$  is  $O(\epsilon R)$  distance from a map of the form  $(x_2, y, z_2) \rightarrow (x'_2, g(y), z'_2)$ .

Let  $U_1$  be as in Lemma 4.11. Now suppose  $p = (x, y, z) \in U' \cap U_1$ . Since  $p \in U'$ ,  $\phi(p)$  is  $O(\epsilon R)$  from a vertical geodesic connecting a point in the  $O(\epsilon R)$  neighborhood of  $\phi(x, y, z_1)$  to a point in the  $O(\epsilon R)$  neighborhood of  $\phi(x, y, z_2)$ . Hence,  $\phi(p)$  is within

$O(\epsilon R)$  distance of the vertical geodesic connecting  $(f(x), y'_1, z'_1)$  to  $(x'_2, g(y), z'_2)$ . This, combined with (9) implies the proposition, and hence Theorem 4.1.  $\square$

**Remark:** The product map  $\hat{\phi}$  produced in the proof of Proposition 4.12 is not defined on the entire box. Since we are not assuming anything about the regularity of the maps  $f, g$  and  $q$  which define  $\hat{\phi}$ , one can choose an arbitrary extension to a product map defined on the box. This is sufficient for our purposes here.

**Order in which constants are chosen.**

- We may assume that  $\epsilon$  is sufficiently small so that in Lemma 3.1, the  $O(\epsilon r)$  error term is smaller than  $(r/100)$ .
- We choose  $N = N(\epsilon, \kappa, C)$ , so that Lemma 4.5 works. We may assume  $N \in \mathbb{Z}$ .
- As described in §4.2, we choose  $\delta = \delta(\epsilon, \theta, \kappa, C)$ , so that  $\delta^{1/4} < \epsilon$  (see proof of Proposition 4.12),  $256\delta^{1/4} < \theta$  (see Proposition 4.12) and also  $\delta^2 < \theta$ , see Lemma 4.7.
- We choose  $S = S(\delta, \kappa, \epsilon)$  so that  $S > \frac{32\kappa^3}{\epsilon\delta^4}$ , as described in §4.2.
- For  $s = 1, \dots, S$ , write  $r_s = r_0 N^s$ .
- Write  $L = \Delta r_0$ . Choose  $\Delta = N^p$  for some  $p \in \mathbb{Z}$ , and so that for  $R = r_S$ , the  $O(R/L)$  error term in (4) is at most  $\delta^2$ . Then the same is true for any  $R = r_s$ ,  $1 \leq s \leq S$ .

Now assume  $r_0$  is sufficiently large so that Lemma 4.9 holds with  $R = r_0$  and  $\theta_1 = 2\delta$ . Theorem 4.1 holds with  $\alpha = 1$  and  $\beta = N^S$ .

## 5 Step II

In this section, we assume that  $m > n$ . We prove the following theorem:

**Theorem 5.1.** *For every  $\delta > 0$ ,  $\kappa > 1$  and  $C > 0$  there exists a constant  $L_0 > 0$  (depending on  $\delta, \kappa, C$ ) such that the following holds: Suppose  $\phi : X(m, n) \rightarrow X(m', n')$  is a  $(\kappa, C)$  quasi-isometry. Then for every  $L > L_0$  and every box  $B(L)$ , there exists a subset  $U \subset B(L)$  with  $|U| \geq (1 - \delta)|B(L)|$  and a height-preserving map  $\hat{\phi}(x, y, z) = (\psi(x, y, z), q(z))$  such that*

(i)

$$d(\phi|_U, \hat{\phi}) = O(\delta L).$$

(ii) For  $z_1, z_2$  heights of two points in  $B(L)$ , we have

$$\frac{1}{2\kappa}|z_1 - z_2| - O(\delta L) < |q(z_1) - q(z_2)| \leq 2\kappa|z_1 - z_2| + O(\delta L). \quad (10)$$

(iii) For all  $x \in U$ , at least  $(1 - \delta)$  fraction of the vertical geodesics passing within  $O(1)$  of  $x$  are  $(\eta, O(\delta L))$ -weakly monotone.

**Remark.** It is not difficult to conclude from Theorem 5.1 that  $\hat{\phi}$  is in fact a product map (not merely height-preserving). However, we will not need this.

**Note.** Theorem 5.1 is true also for the case  $m = n$ ; its proof for that case is the content of [EFW2]. The proof presented in this section is much simpler, but applies only to the case  $m > n$ . The main point of the proof is to show that if  $m > n$ , then in the notation of Theorem 4.1, for each  $i \in I_g$ , the maps  $\hat{\phi}_i$  must preserve the up direction. This is done in §5.3. The deduction of Theorem 5.1 from that fact is in §5.4.

## 5.1 Volume estimates

The following is a basic property shared for example by all homogeneous spaces and all spaces with a transitive isometry group (such as  $X(m, n)$ ).

**Lemma 5.2.** For  $p \in X(m, n)$ , let  $D(p, r)$  denote the metric ball of radius  $r$  centered at  $p$ . Then, for every  $b \gg a > 0$  there exists  $\omega = \omega(a, b) > 1$  with  $\log \omega = O(b - a)$  such that for all  $p, q \in X(m, n)$ ,

$$\omega(a, b)^{-1}|D(p, a)| \leq |D(q, b)| \leq \omega(a, b)|D(p, a)|,$$

where  $|\cdot|$  denotes volume (relative to the  $X(m, n)$  metric). Also  $\log \omega(a, b) = O(b - a)$ , where the implied constant depends on the model space  $X(m, n)$ .

**Proof.** The first statement is immediate since  $X(m, n)$  is a homogeneous space. The second is a simple calculation.  $\square$

In this section we prove some fairly elementary facts about quasi-isometries quasi-preserving volume. The main tool is the following basic covering lemma.

**Lemma 5.3.** *Let  $X$  be a metric space and let  $\mathcal{F}$  be a collection of points in  $X$ . Then for any  $a > 0$  there is a subset  $\mathcal{G}$  in  $\mathcal{F}$  such that:*

- (i) *The sets  $\{B(x, a) | x \in \mathcal{G}\}$  are pairwise disjoint.*
- (ii)  $\bigcup_{\mathcal{F}} B(x, a) \subset \bigcup_{\mathcal{G}} B(x, 5a)$ .

This lemma and its proof (which consists of picking  $\mathcal{G}$  by a greedy algorithm) can be found in [He, Chapter 1]. This argument is implicit in almost any reference which discusses covering lemmas.

Recall that we are assuming that  $\phi$  is a continuous  $(\kappa, C)$  quasi-isometry.

From this we can deduce the following fact about quasi-isometries of  $X(m, n)$ . This fact holds much more generally for metric measure spaces which satisfy Lemma 5.2.

**Proposition 5.4.** *Let  $\phi : X(m, n) \rightarrow X(m', n')$  be a continuous  $(\kappa, C)$  quasi-isometry. Then for any  $a \gg C$  there exists  $\omega_1 > 1$  with  $\log \omega_1 = O(a)$  such that for any  $U \subset X(m, n)$ ,*

$$\omega_1^{-1} |\phi(N_a(U))| \leq |N_a(U)| \leq \omega_1 |N_a(\phi(U))|$$

where  $N_a(U) = \{x \in X(m, n) : d(x, U) < a\}$ .

**Proof.** We assume that  $a > 4\kappa C$ . Note that we are assuming that every point is within distance  $C$  of the image of  $\phi$ . Let  $\mathcal{F}$  be the covering of  $N_a(U)$  consisting of all balls of radius  $a$  centered in  $U$ . By Lemma 5.3, we can find a (finite) subset  $\mathcal{G}$  of  $U$  such that  $\bigcup_{x \in \mathcal{G}} D(x, 5a)$  cover  $N_a(U)$  and such that the balls centered at  $\mathcal{G}$  are pairwise disjoint. Hence,

$$\sum_{x \in \mathcal{G}} |D(x, a)| \leq |N_a(U)| \leq \sum_{x \in \mathcal{G}} |D(x, 5a)|.$$

Now  $\phi(N_a(U))$  is covered by  $\bigcup_{x \in \mathcal{G}} \phi(D(x, 5a)) \subset \bigcup_{x \in \mathcal{G}} D(\phi(x), 5\kappa a + C)$ . Hence,

$$|\phi(N_a(U))| \leq \sum_{x \in \mathcal{G}} |D(\phi(x), 5\kappa a + C)| \leq \omega(a, 5\kappa a + C) \sum_{x \in \mathcal{G}} |D(x, a)| \leq \omega(a, 5\kappa a + C) |N_a(U)|.$$

For the other inequality,

$$\begin{aligned} |N_a(\phi(U))| &\geq |N_a(\phi(\mathcal{G}))| = \left| \bigcup_{x \in \mathcal{G}} D(\phi(x), a) \right| \geq \left| \bigcup_{x \in \mathcal{G}} D(\phi(x), a/\kappa - C) \right| \\ &= \sum_{x \in \mathcal{G}} |D(\phi(x), a/\kappa - C)| \geq \omega(a/\kappa - 2C, 5a)^{-1} \sum_{x \in \mathcal{G}} |D(x, 5a)| \geq \omega_1^{-1} |N_a(U)| \end{aligned}$$

□

**Terminology.** The “coarse volume” of a set  $E$  means the volume of  $N_a(E)$  for a suitable  $a$ . If the set  $E$  is essentially one dimensional (resp. two dimensional) we use the term “coarse length” (resp. “coarse area”) instead of coarse volume but the meaning is still the volume of  $N_a(E)$ . We also use  $\ell(\cdot)$  to denote coarse length.

## 5.2 The trapping lemma

For a path  $\gamma$ , let  $\ell(\gamma)$  denote the length of  $\gamma$  (measured in the  $X(m, n)$ -metric).

Recall that we are assuming  $m \geq n$ .

**Lemma 5.5.** *Suppose  $L$  is a constant  $z$  plane, and suppose  $U$  is a bounded set contained in  $L$ . Suppose  $k > r > 0$  and  $\gamma$  is a path which stays at least  $k$  units below  $L$ , i.e. that  $\max_{x \in \gamma}(h(x)) < h(L) - k$ . Suppose also that any vertical geodesic ray starting at  $U$  and going down intersects the  $r$ -neighborhood of  $\gamma$ . Then,*

$$\ell(\gamma) \geq e^{c_1 k - c_2 r} \text{Area}(U)$$

where  $c_1 > 0$  and  $c_2 > 0$  depend only on the model space, and both the length and the area are measured using the  $X(m, n)$  metric.

**Proof.** We give a proof for  $DL(m, n)$ , the proof for  $Sol(m, n)$  is similar. Let  $\Delta$  denote the  $r$ -neighborhood of  $\gamma$ , then  $|\Delta| \leq e^{c_2 r} \ell(\gamma)$ . Pick  $N$  so that  $\Delta$  stays above height  $h(L) - N$ . Let  $\mathcal{A}$  denote the set of vertical segments of length  $N$  which start at height  $h(L)$  and go down. Let  $\mathcal{A}_U$  denote the elements of  $\mathcal{A}$  which start at points of  $U$ . Then  $|\mathcal{A}_U| = |U|m^N$ . Now for  $0 < s < N$ , any point at height  $h(U) - s$  intersects exactly  $n^s m^{N-s}$  elements of  $\mathcal{A}$ . Thus, by the assumption on the height of  $\Delta$ , any point of  $\Delta$  can intersect at most  $n^k m^{N-k}$  elements of  $\mathcal{A}$ . But by assumption,  $\Delta$  intersects any element of  $\mathcal{A}_U$ . Thus,  $|\Delta| \geq (|U|m^N)/(n^k m^{N-k}) = |U|(m/n)^k$ , which implies the lemma. (Recall that in our notation, length = area = cardinality in  $DL(m, n)$ ).

□

**Remark:** When  $m = n$ , Lemma 5.5 still holds (but with  $c_1 = 0$ ), and also in addition with the word *below* replaced by the word *above*. When  $m \neq n$ , volume decreases on upwards projection.

### 5.3 Vertical Orientation preserved

Given a box  $B(R)$ , an  $y$ -horocycle  $H$  in  $B(R)$ , and a number  $\rho$ , we let the *shadow*,  $\text{Sh}(H, \rho)$ , of  $H$  in  $B(R)$  be the set of points that can be reached by a vertical geodesics going straight down from the  $\rho$  neighborhood of  $H$ . Note that if  $H$  is the top of the box,  $\text{Sh}(H, 1)$  is the entire box. Similar definitions hold for  $x$ -horocycles, but then the shadow will be above the horocycle.

The goal of this subsection is the following:

**Theorem 5.6.** *Suppose  $m > n$  and that  $\epsilon$  and  $\theta$  are sufficiently small (depending only on the model space). Let  $I, I_g, U_i$  and  $\hat{\phi}_i$  be as in Theorem 4.1. Suppose  $i \in I_g$ . Then the product map  $\hat{\phi}_i : B(R) \rightarrow X(m', n')$  can be written as  $\hat{\phi}_i(x, y, z) = (f_i(x), g_i(y), q_i(z))$ , with  $q_i : \mathbb{R} \rightarrow \mathbb{R}$  coarsely orientation preserving.*

**Remark.** The result of Theorem 5.6 is *false* in the case  $m = n$ , since there exist “flips”, i.e. isometries which reverse vertical orientation. This is the point where the proof in the case  $m = n$  diverges from the case  $m > n$ .

In the rest of §5.3 we prove Theorem 5.6. We pick  $i \in I_g$  and suppress the index  $i$  for the rest of this subsection.

Pick  $1 \gg \rho_2 \gg \rho_1 \gg \epsilon$  to be determined later (see the end of this subsection).

**Lemma 5.7.** *All but  $O(4\sqrt{\theta})$  proportion of the  $y$ -horocycles  $H$  that are above the middle of the box  $B(R)$  have all but  $O(\sqrt{\theta})$  fraction of the  $\mu$ -measure of both  $N_{\rho_1 R}(H)$  and  $\text{Sh}(H, \rho_1 R)$  in  $U$ .*

**Proof.** Let  $L$  be a constant  $z$  plane above the middle of the box (and not too close to the top). We choose horocycles  $H_i$  in  $L$  such that  $L = \coprod_i N_{\rho_1 R}(H_i)$ . The subset of  $B(R)$  below  $L$  is then the disjoint union of the shadows  $\coprod_i \text{Sh}(H_i, \rho_1 R)$ . Since at least half the measure of  $B(R)$  is below  $L$ , it follows that there is some  $i$  so that all but  $O(\sqrt{\theta})$  of the  $\mu$ -measure of  $\text{Sh}(H_i, \rho_1 R)$  is in  $U$ . To guarantee the same fact about  $N_{\rho_1 R}(H)$ , we pick  $L$  such that  $N_{\rho_1 R}(L)$  has all but  $O(\sqrt{\theta})$  fraction of its  $\mu$ -measure in  $U$ .  $\square$

**Lemma 5.8.** *For any  $H$  as in Lemma 5.7, there exists a constant  $z$  plane  $L$  such that  $N_{\rho_1 R}(L) \cap \text{Sh}(H, \rho_1 R) \cap U$  contains all but  $O(\theta^{1/4})$  fraction of the  $\mu$ -measure in  $N_{\rho_1 R}(L) \cap \text{Sh}(H, \rho_1 R)$ . Furthermore, we can choose  $L$  and  $H$  such that  $\rho_2 R < d(L, H) < 2\rho_2 R$ .*

**Proof.** Let

$$E = \text{Sh}(H, \rho_1 R) \cap h^{-1}(h(H) - 2\rho_2 R, h(H) - \rho_2 R).$$

By Lemma 5.7,  $\mu(E \cap U) \geq (1 - c\sqrt{\theta}/\rho_2)\mu(E) \geq (1 - \theta^{1/4})\mu(E)$ , where  $c$  is the implied constant in Lemma 5.7, and we have assumed that  $c\theta^{1/2}/\rho_2 \leq \theta^{1/4}$ . Now this is another application of Fubini's theorem, where we partition  $E$  into its intersections with neighborhood of constant  $z$  planes.  $\square$

**Lemma 5.9.** *Let  $L$  be as in the conclusion of Lemma 5.8. There are subsets  $S_1, S_2$  of  $L \cap B(R)$  such that*

1.  $N_{\rho_1 R}(S_i) \cap U$  contains all but  $O(\theta^{1/4})$  fraction of the  $\mu$ -measure of  $N_{\rho_1 R}(S_i)$ .
2. for  $s_i \in S_i$  any path joining  $s_1$  to  $s_2$  of length less than  $\kappa^3 \rho_2 R$  passes within  $O(\rho_1 R)$  of  $H$ .
3. For  $i = 1, 2$ ,  $\text{Area}(S_i) \gg \frac{1}{6} \text{Area}(\text{Sh}(H, \rho_1 R) \cap L) > e^{c\rho_2 R} \ell(H \cap B(R))$  where  $c$  depends on the model spaces.

**Proof.** We divide  $L \cap \text{Sh}(H, \rho_1 R)$  into equal thirds where each third has the entire  $y$ -extent and a third of the  $x$ -extent. We let  $\tilde{S}_1$  and  $\tilde{S}_2$  be the two non-middle thirds. Now let  $S_i$  be the portion of  $\tilde{S}_i$  which is at least  $\kappa^3 \rho_2 R$  away from the edges of  $B(R)$ . The area of each of these regions is much more than the coarse area  $\ell(H \cap B(R))$  since projecting upwards decreases area and each region projects upwards onto  $H$ .  $\square$

The proof of Theorem 5.6 involves deriving contradiction to the reversal of orientation of vertical geodesics under  $\hat{\phi}$  on  $B_i(R)$ . The goal is to show that if the orientation were to reverse, we could find a path in the target joining  $\hat{\phi}(s_1)$  to  $\hat{\phi}(s_2)$  that contradicts Lemma 5.9.

**Proof of Theorem 5.6.** We assume that vertical orientation is not preserved but reversed. This means the  $z$  component  $q(z)$  of the product map in Theorem 4.1 is orientation reversing. Let  $H_p$  denote the  $x$ -horocycle through  $p$ . For  $i = 1, 2$ , let

$$S'_i = \{p \in S_i \cap U \ : \ \text{for } j = 1, 2, \ell(H_p \cap U \cap S_j) > 0.5\ell(H_p \cap S_j) \}$$

Let  $W_i = \hat{\phi}(S'_i)$ . By Proposition 5.4, and Lemma 5.9 part 3, we have

$$\text{Area}(W_i) \geq e^{-D\rho_1 R} \text{Area}(S'_i) > e^{(c\rho_2 - D\rho_1)R} \ell(H) \tag{11}$$

where  $D$  depends only on the model spaces,  $\kappa$  and  $C$ . Assuming  $\epsilon$  is sufficiently small,  $\ell(H) \ll \text{Area}(W_i)$ . Then by Lemma 5.5, 99.9% of the geodesics going down from  $W_i$

do not enter the  $O(\rho_1 R)$  neighborhood of  $\phi(H)$ . Let  $W'_i$  denote the set of  $p \in W_i$  so that 99% of the geodesics going down from (the  $1/2$ -neighborhood of)  $p$  do not enter the  $O(\rho_1 R)$  neighborhood of  $\phi(H)$ . Then  $\text{Area}(W'_i) \geq 0.9 \text{Area}(W_i)$ .

Let  $\mathcal{H}$  denote the set of  $y$ -horocycles  $H'$  such that  $W_i \cap H'$  is non-empty for some (or equivalently for all)  $i \in \{1, 2\}$ . Let  $\mathcal{H}_i = \{H' \in \mathcal{H} : W'_i \cap H' \neq \emptyset\}$ . We claim that  $\mathcal{H}_1 \cap \mathcal{H}_2 \neq \emptyset$ . Indeed,  $W_i = \coprod_{H' \in \mathcal{H}} W_i \cap H'$ , and  $\text{Area}(W_i) = |\mathcal{H}|c_i$ , where  $c_i = |W_i \cap H'|$  is independent of  $H' \in \mathcal{H}$ . Now,

$$0.9c_i|\mathcal{H}| = 0.9 \text{Area}(W_i) \leq \text{Area}(W'_i) \leq c_i|\mathcal{H}_i|.$$

Thus, we have  $|\mathcal{H}_i| \geq 0.9|\mathcal{H}|$ , and hence there exists  $H' \in \mathcal{H}_1 \cap \mathcal{H}_2$ . By the definition of the  $\mathcal{H}_i$ , we can find  $p_1, p_2$  such that  $p_i \in H' \cap W'_i$ . Then for  $i = 1, 2$ , by the definition of  $W'_i$ , we can find geodesics  $\gamma_i$  going down from  $p_i$ , such that  $\gamma_i$  do not enter the  $O(\rho_1 R)$  neighborhood of  $\phi(H)$ , and such that  $\gamma_1$  and  $\gamma_2$  meet at some point  $p'$ . By construction  $d(p', W_i) \leq 2\kappa^2 \rho_2 R$ . Concatenating subsegments of these two geodesics yields a path connecting  $p_1$  to  $p_2$  of length  $d(p_1, p_2)$  which avoids the  $O(\rho_1 R)$  neighborhood of  $\phi(H)$ . Pulling back, we have a path of length at most  $16\kappa^3 \rho_2 R$  and avoiding the  $O(\rho_1 R)$  neighborhood of  $H$ , which connects a point within  $O(\epsilon R)$  of  $S_1$  to a point within  $O(\epsilon R)$  of  $S_2$ . This contradicts Lemma 5.9.  $\square$

**Remark:** The only place in this paper where we make essential use of the fact that  $\phi$  satisfies (2) of Definition 1.1 is in pulling back the path connecting  $p_1$  and  $p_2$  at the end of the proof of Theorem 5.6.

**Choice of constants.** Let  $A$  be the largest constant depending only on  $\kappa, C$  and the model spaces, which arises in the course of the argument in §5.3.

We choose  $\rho_2$  so that  $A\rho_2 < 1$ . Similarly we choose  $\rho_1$  so that  $A\rho_1 < \rho_2$  and  $\epsilon A\epsilon < \rho_1$ . We also choose  $\theta$  so that  $A\theta^{1/4} < \rho_1$ . We also make sure that  $\epsilon$  and  $\theta$  are sufficiently small so that Theorem 5.6 applies. In addition we choose  $r_0$  in the statement of Theorem 4.1 such that the constant  $e^{(c\rho_2 - D\rho_1)R}$  that appears in the proof of Theorem 5.6 is at least 1000. Our other choices guarantee that  $\rho_2 > \frac{D}{c}\rho_1$ , so this is just a lower bound on  $R$  and therefore  $r_0$ .

## 5.4 Proof of Theorem 5.1

**The uniform set and the exceptional set.** Let  $I_g$  and  $U_i, i \in I_g$  be as in Theorem 4.1. Let  $W = \bigcup_{i \in I_g} U_i$ .

Recall that  $Y_L$  is the set of vertical geodesics in  $B(L)$ . Here we will work with a fixed geodesic  $\gamma \in Y_L$ . Let  $W^c \subset B(L)$  denote the complement of  $W$  in  $B(L)$ . For a

point  $x \in \gamma$  and  $T > 0$  let

$$P(x, \gamma, T) = |W^c \cap \gamma \cap D(x, T)|,$$

where  $D(x, T)$  is the ball of radius  $T$  centered at  $x$  (so that  $\gamma \cap D(x, T)$  is an interval of length  $2T$  centered at  $x$ ).

**Lemma 5.10.** *For every  $\eta_1 > 0$  there exists  $\eta > 0$  (with  $\eta \rightarrow 0$  as  $\eta_1 \rightarrow 0$ ) such that the following holds: Suppose  $\gamma$  is a geodesic ray leaving  $x$ , and for any  $T > 1$ ,  $P(x, \gamma, T) < \eta_1 T$ . Then,  $\bar{\gamma} = \phi \circ \gamma$  is  $(\eta, C_1)$ -weakly-monotone, where  $C_1 = O(\eta_1 R)$ .*

**Proof.** Parametrize  $\gamma$  so that  $\gamma(0) = x$ . Without loss of generality, we may assume that  $\gamma$  is going up. Let  $\bar{\gamma} = \phi \circ \gamma$ . Suppose  $0 < t_1 < t_2$  are such that  $h(\bar{\gamma}(t_1)) = h(\bar{\gamma}(t_2))$ . Write  $q(t) = h(\bar{\gamma}(t))$ . Subdivide  $[t_1, t_2]$  into intervals  $I_1, \dots, I_N$  of length  $\leq \eta_1 R$  and so that the length of all but the first and last is exactly  $\eta_1 R$ . We may assume  $N \geq 3$ . Let  $J \subset [1, \dots, N]$  be the set of  $j \in \mathbb{Z}$  such that  $\bar{\gamma}(I_j) \cap W \neq \emptyset$ . For  $j \in J$ , pick  $s_j$  such that  $\bar{\gamma}(s_j) \in W$ , and pick  $s \in I_j$  arbitrarily otherwise. Now

$$0 = q(t_2) - q(t_1) = q(t_2) - q(s_{N'}) + \sum_{\substack{j=3 \\ j \text{ odd}}}^{N'} (q(s_j) - q(s_{j-2})) + q(s_1) - q(t_1), \quad (12)$$

where  $N'$  is either  $N$  or  $N - 1$  depending on whether  $N$  is odd or even.

Let  $Q_0 = \{\text{odd } j \in [3, N'] : \gamma(s_j) \in U_i, \gamma(s_{j-2}) \in U_i\}$  (same  $U_i$ ). Let  $Q_1$  denote the set of odd  $j \in [3, N']$  such that  $\gamma(s_j)$  and  $\gamma(s_{j+1})$  are in different boxes  $B_i(R)$ . Finally, let  $Q_2$  denote the set of odd  $j \in [3, N']$  such that  $\gamma(I_j) \subset W^c$  or  $\gamma(I_{j-2}) \subset W^c$ . By assumption,  $|Q_2| \leq t_2/R$  and also  $|Q_1| \leq t_2/R$ . Then,  $|Q| \geq (1/3)(t_2 - t_1)/(\eta_1 R)$ . Note that if  $j \in Q$ , then  $|q(s_j) - q(s_{j-2})| \geq \eta_1 R/(2\kappa)$ , and for any  $j$ ,  $|q(s_{j+2}) - q(s_j)| \leq 4\kappa\eta_1 R$ . Hence,

$$\sum_{j \in Q} q(s_j) - q(s_{j-2}) \geq |Q| \frac{\eta_1 R}{\kappa} \geq \frac{t_2 - t_1}{6\kappa}.$$

Also,

$$\left| \sum_{j \in Q_1 \cup Q_2} q(s_j) - q(s_{j-2}) \right| \leq |Q_1 \cup Q_2| 2\kappa\eta_1 R \leq 2\kappa\eta_1 t_2$$

Plugging into (12) we see that

$$0 \geq \frac{t_2 - t_1}{6\kappa} - 2\kappa\eta_1 t_2 - O(\eta_1 R),$$

or

$$\frac{t_2 - t_1}{6\kappa} \leq 2\kappa\eta_1 t_2 + O(\eta_1 R)$$

which implies the lemma.  $\square$

Pick  $A \gg 1$  (in fact we will eventually choose  $A = (4(128/\delta)^4)$ ). Suppose  $\gamma \in Y_L$ . We define a point  $x \in \gamma$  to be  $A$ -uniform along  $\gamma$ , if for all  $T > 1$ ,

$$\frac{P(x, \gamma, T)}{T} < A \frac{|\gamma \cap W^c|}{L}$$

**Lemma 5.11.** *Let  $\theta(\gamma)$  denote the proportion of non- $A$ -uniform points along  $\gamma$ . Then,  $\theta(\gamma) \leq 2/A$ .*

**Proof.** This is a standard application of the Vitali covering lemma. Let  $\nu = \frac{|\gamma \cap W^c|}{L}$ . Suppose  $x$  is non-uniform, then there is an interval  $I_x$  centered at  $x$  such that

$$|I_x \cap W^c| \geq A\nu|I_x|.$$

The intervals  $I_x$  obviously cover the non-uniform set of  $\gamma$ , and, by Vitali, we can choose a disjoint subset  $I_j$  which cover at least half the measure of the non-uniform set. Then,

$$|\bigcup I_j| \leq \sum |I_j| \leq (A\nu)^{-1}|I_j \cap W^c| \leq (A\nu)^{-1}|\gamma \cap W^c|$$

Dividing both sides by  $L$  (the length  $\gamma$ ), and recalling that  $|\gamma \cap W^c|/L = \nu$ , we obtain the estimate.  $\square$

$$\text{Let } \theta_1 = \frac{\theta}{\eta_1} + \frac{2}{A}.$$

**Corollary 5.12.** *There exists a subset  $U \subset B(L)$  with  $\mu(U) > (1 - 2\sqrt{\theta_1})\mu(B(L))$  such that for  $x \in U$ ,  $(1 - \sqrt{\theta_1})$ -fraction of the geodesics passing within  $(1/2)$  of  $x$  are right-side-up  $(\eta, \eta_1 R)$ -weakly-monotone.*

**Proof.** Let  $Y'$  denote the space of pairs  $(\gamma, x)$  where  $\gamma \in Y_L$  is a vertical geodesic in  $B(L)$  and  $x \in \gamma$  is a point. Let  $E \subset Y'$  denote the set of pairs  $(\gamma, x)$  such that  $|\gamma \cap W| \geq (1 - \eta_1/A)L$ , and  $x$  is  $A$ -uniform along  $\gamma$ . Then, by Lemma 5.11 we have  $|E| \geq (1 - \theta_1)|Y'|$ . Let  $U$  be the subset constructed by applying Lemma 4.9. Then  $\mu(U) \geq (1 - 2\sqrt{\theta_1})\mu(B(L))$ , and for  $x \in U$  by Lemma 5.10 at least  $(1 - \sqrt{\theta_1})$  fraction of the geodesic rays leaving  $x$  are  $(\eta, O(\eta_1 R))$ -weakly-monotone.  $\square$

**Lemma 5.13.** *Suppose  $\phi$  and  $B(L)$  and  $U$  are as in Corollary 5.12, and  $\eta$  is sufficiently small (depending only on the model space). Then, there exist functions  $\psi$ ,  $q$ , and a subset  $U_1 \subset B(L)$  with  $\mu(U_1) > (1 - 128\theta_1^{1/4})\mu(B(L))$  such that for  $(x, y, z) \in U_1$ ,*

$$d(\phi(x, y, z), (\psi(x, y, z), q(z))) = O(\delta L) \quad (13)$$

**Proof.** This proof is identical to that of Lemma 4.11. □

**Proof of Theorem 5.1.** Choose  $\eta$  so that Lemma 5.13 holds. Choose  $\eta_1$  so that Lemma 5.10 holds, and also that the  $O(\eta_1 R)$  term in Lemma 5.10 is at most  $\delta L$ . Choose  $A^{-1} = (\delta/128)^4/4$  and choose  $\theta = (\delta/128)^4/\eta_1$  so that  $128\theta_1^4 < \delta$ . Now the theorem follows from combining Corollary 5.12 and Lemma 5.13. □

## 6 Step III

In this section, we complete the proof of Theorem 2.1 and Theorem 2.3. We assume that  $\phi$  is a  $\kappa, C$  quasi-isometry from  $X(m, n)$  to  $X(m', n')$  satisfying the conclusion of Theorem 5.1. All the arguments in this section are valid also in the case  $m = n$  (and are used in [EFW2]).

### 6.1 A weak version of height preservation

In this subsection, our main goal is to prove the following:

**Theorem 6.1.** *Let  $\phi : X(m, n) \rightarrow X(m', n')$  be a  $(\kappa, C)$  quasi-isometry satisfying the conclusions of Theorem 5.1. Then for any  $\theta \ll 1$  there exists  $M > 0$  (depending on  $\theta, \kappa, C$ ) such that for any  $x$  and  $y$  in  $X(m, n)$  with  $h(x) = h(y)$ ,*

$$|h(\phi(x)) - h(\phi(y))| \leq \theta d(x, y) + M. \quad (14)$$

**Note.** This is a step forward, since the theorem asserts that (14) holds for all pairs  $x, y$  of equal height (and not just on a set of large measure).

We would like to restrict Theorem 5.1 to the neighborhood of a constant  $z$  plane. Let  $\nu = \sqrt{\delta}$ . Fix a constant  $z$  plane  $P$ . For notational convenience, assume that  $P$  is at height 0. Let  $R(L) \subset P$  denote the intersection of  $P$  with a box  $B(2L)$  whose top face is at height  $L$  and bottom face at  $-L$ . Then  $R(L)$  is a rectangle (in fact, when  $m = n$ , with this choice of  $P$ ,  $R(L)$  is a square in the euclidean metric). We

will call  $L$  the *size* of  $R(L)$ . Let  $R^+(L)$  denote the “thickening” of  $R(L)$  in the  $z$ -direction by the amount  $\nu L$ , i.e.  $R^+(L)$  is the intersection of  $B(2L)$  with the region  $\{p \in X(m, n) : -(\nu/2)L \leq h(p) \leq (\nu/2)L\}$ , where as above  $h(\cdot)$  denotes the height function.

We now have the following corollary of Theorem 5.1:

**Corollary 6.2.** *Suppose  $L > L_0$ . Then for every rectangle  $R(L) \subset P$  there exists  $U \subset R^+(L)$  with  $\mu(U) \geq (1-\nu)\mu(R^+(L))$  and a standard map  $\hat{\phi} : U \rightarrow X(m', n')$  such that  $d(\phi|_U, \hat{\phi}) \leq \nu L$ . Furthermore, for any  $p \in U$ , for 99% of the geodesics  $\gamma$  leaving  $p$ ,  $\phi(\gamma \cap B(2L))$  is within  $\delta L$  of a vertical geodesic segment (in the right direction).*

**The tilings.** Choose  $\beta \ll 1$  depending only on  $\kappa, C, m$  and  $n$ . When  $m = n$ ,  $\beta \approx \frac{1}{\kappa^4}$ . Let  $L_j = (1 + \beta)^j L_0$ . For each  $j > 0$  we tile  $P$  by rectangles  $R$  of size  $L_j$ ; we denote the rectangles by  $R_{j,k}$ ,  $k \in \mathbb{N}$ . For  $x \in X(m, n)$ , let  $R_j[x]$  denote the unique rectangle in the  $j$ 'th tiling to which the orthogonal projection of  $x$  to  $P$  belongs.

**Warning.** Despite the fact that  $L_{j+1} = (1 + \beta)L_j$ , the number of rectangles of the form  $R_j[x]$  needed to cover a rectangle of the form  $R_{j+1}[y]$  is very large (on the order of  $e^{\beta L_j}$ ). This is because the Euclidean size of  $R(L_j)$  is approximately  $e^{L_j}$ .

**The sets  $U_j$ .** For each rectangle  $R_{j,k}$ , Corollary 6.2 gives us a subset of  $R_{j,k}^+$  which we will denote by  $U_{j,k}$ . Let

$$U_j = \bigcup_{k=1}^{\infty} U_{j,k}.$$

In view of Corollary 6.2, for any  $x \in U_j$ ,

$$\sup_{y \in R_j^+[x] \cap U_j} |h(\phi(y)) - h(\phi(x))| \leq 2\nu L_j \quad (15)$$

We also have the following generalization:

**Lemma 6.3.** *For any  $x \in U_j$  and any  $y \in R_{j+1}^+[x] \cap U_j$ ,*

$$|h(\phi(y)) - h(\phi(x))| \leq 12\nu L_j.$$

**Proof.** Let  $R_j[p]$  be a rectangle on the same “row” as  $R_j[x]$  and the same “column” as  $R_j[y]$ . Then, since  $\nu \ll 1$ , there exists an  $x$ -horocycle  $H$  which intersects both  $R_j^+[x] \cap U_j$  and  $R_j^+[p] \cap U_j$ ; let us denote the points of intersection by  $x_1$  and  $p_1$  respectively.

Now for  $i = 1, 2$  choose (sufficiently different) vertical geodesics  $\gamma_i$  coming down from (near)  $x_1$  and  $\gamma'_i$  coming down from (near)  $p_1$  such that for  $i = 1, 2$ ,  $\gamma_i(L_{j+1})$  and  $\gamma'_i(L_{j+1})$  are close. (here all the geodesics are parametrized by arclength). In view of Corollary 6.2, since  $x_1$  and  $p_1$  are in  $U_j$ , we may assume that there exist vertical geodesics  $\lambda_i$  and  $\lambda'_i$  such that for  $0 \leq t \leq L_j$ ,  $d(\gamma_i(t), \lambda_i) \leq \nu + \eta t$  where  $\eta \ll 1$ . Similarly,  $d(\gamma'_i(t), \lambda'_i) \leq \nu + \eta t$ .

Thus, in particular,  $h(\phi(\gamma_i(L_j))) \leq h(\phi(x_1)) - L_j/\kappa + \eta \leq h(H) - L_j/(2\kappa)$ , and similarly  $h(\phi(\gamma'_i(L_j))) \leq h(\phi(p_1)) - L_j/(2\kappa)$ . Now note that  $d(\gamma_i(L_j), \gamma'_i(L_j)) = \beta L_j + O(1)$ . Hence  $d(\phi(\gamma_i(L_j)), \phi(\gamma'_i(L_j))) \leq 2\kappa\beta L_j + O(1)$ , and by assumption  $\kappa^2\beta \ll 1$ . Then by Lemma 3.1,  $\phi(x_1)$  and  $\phi(p_1)$  are near the same horocycle, and thus, in particular,

$$|h(\phi(x_1)) - h(\phi(p_1))| \leq 4\nu L_j \quad (16)$$

Similarly, we can find  $p_2 \in R_j^+[p] \cap U_j$  and  $y_2 \in R_j^+[y] \cap U_j$  such that  $p_2$  and  $y_2$  are on the same  $y$ -horocycle. Then, by the same argument,

$$|h(\phi(p_2)) - h(\phi(y_2))| \leq 4\nu L_j \quad (17)$$

Hence, in view of (16), (17), and (15),

$$|h(\phi(x)) - h(\phi(y))| \leq 12\nu L_j,$$

as required. □

**Lemma 6.4.** *Suppose  $p \in R_j^+[x] \cap U_j$ ,  $q \in R_{j+1}^+[x] \cap U_{j+1}$ . Then,*

$$|h(\phi(p)) - h(\phi(q))| \leq 16\nu L_{j+1} \quad (18)$$

**Proof.** Note that the orthogonal projection of  $U_j \cap R_{j+1}^+[x]$  to  $R_{j+1}[x]$  has full  $\mu$ -measure (up to order  $\nu$ ). The same is true of  $U_{j+1} \cap R_{j+1}^+[x]$ . Thus, the projections intersect, and thus we can find  $p' \in U_j \cap R_{j+1}^+[x]$  and  $q' \in R_{j+1}^+[x] \cap U_{j+1}$  such that  $d(p', q') \leq 2\nu L_{j+1}$ . Now, in view of Lemma 6.3,

$$|h(\phi(p)) - h(\phi(p'))| \leq 12\nu L_j$$

and in view of (15),

$$|h(\phi(q')) - h(\phi(q))| \leq 2\nu L_{j+1}$$

This implies (18). □

**Proof of Theorem 6.1.** We have

$$R_0[x] \subset R_1[x] \subset R_2[x] \subset \dots$$

and

$$R_0[y] \subset R_1[y] \subset R_2[y] \subset \dots$$

There exists  $N$  with  $L_N$  comparable to  $d(x, y)$  such that (after possibly shifting the  $N$ 'th grid by a bit)  $R_N[x] = R_N[y]$ . Now for  $0 \leq j \leq N$ , pick  $x_j \in R_j^+[x] \cap U_j$ ,  $y_j \in R_j^+[y] \cap U_j$ . We may assume that  $x_N = y_N$ . Now, using Lemma 6.4,

$$\begin{aligned} |h(\phi(x_0)) - h(\phi(y_0))| &\leq \sum_{j=0}^{N-1} |h(\phi(x_{j+1})) - h(\phi(x_j))| + \sum_{j=0}^{N-1} |h(\phi(y_{j+1})) - h(\phi(y_j))| \\ &\leq 2 \sum_{j=0}^{N-1} 16\nu L_{j+1} \\ &\leq \frac{32\nu}{\beta} L_N, \end{aligned}$$

where in the last line we used that  $L_j = (1 + \beta)^j L_0$ . Now since  $x_0 \in R_0[x]$ ,  $d(x, x_0) \leq L_0$ , so  $|h(\phi(x) - \phi(x_0))| = O(L_0)$ . Similarly,  $|h(\phi(y) - h(\phi(y_0)))| = O(L_0)$ . Also note that  $L_{N+1}$  is within a factor of 2 of  $d(x, y)$ . Thus the theorem follows.  $\square$

## 6.2 Completion of the proof of height preservation

**Lemma 6.5.** *Let  $\phi : X(m, n) \rightarrow X(m', n')$  be a  $(\kappa, C)$  quasi-isometry. Then for any  $\eta \ll 1$  there exists  $C_1 > 0$  (depending on  $\eta, \kappa, C$ ) such that for any vertical geodesic ray  $\gamma$ ,  $\phi \circ \gamma$  is  $(\eta, C_1)$ -weakly monotone.*

**Proof.** This is a corollary of Theorem 6.1. Suppose  $\gamma$  is a vertical geodesic ray parametrized by arclength, and  $\bar{\gamma} = \phi \circ \gamma$ . Suppose  $0 < t_1 < t_2$  are such that  $h(\bar{\gamma}(t_1)) = h(\bar{\gamma}(t_2))$ . We now apply Theorem 6.1 to  $\phi^{-1}$  instead of  $\phi$  (with  $x = \bar{\gamma}(t_1)$  and  $y = \bar{\gamma}(t_2)$ ). We get  $|h(\gamma(t_1)) - h(\gamma(t_2))| \leq \theta d(\bar{\gamma}(t_1), \bar{\gamma}(t_2)) + O(M)$ , i.e.

$$|t_2 - t_1| \leq \theta \kappa^2 |t_2 - t_1| + O(M)$$

i.e.  $\bar{\gamma}$  is  $(\theta \kappa^2, O(M))$ -weakly monotone.  $\square$

**Proof of Theorem 2.1 and Theorem 2.3.** Suppose  $p_1$  and  $p_2$  are two points of  $X(m, n)$ , with  $h(p_1) = h(p_2)$ . We can find  $q_1, q_2$  in  $X(m, n)$  such that  $p_1, p_2, q_1, q_2$

form a quadrilateral. By Lemma 6.5, each of the segments  $\gamma_{ij}$  connecting a point in the  $O(1)$  neighborhood of  $p_i$  to a point in the  $O(1)$  neighborhood of  $q_j$  maps under  $\phi$  to an  $O(\eta, C_1)$ -weakly monotone quasi-geodesic segment. Then by Lemma 4.4, and Lemma 3.1, we see that  $h(\phi(p_1)) = h(\phi(p_2)) + O(C_1)$ .  $\square$

## 7 Deduction of rigidity results

The purpose of this section is to apply the previous results on self quasi-isometries of  $\text{Sol}(m, n)$  and the DL-graphs to understand all finitely generated groups quasi-isometric to either one. This follows a standard outline: if  $\Gamma$  is quasi-isometric to  $X$  then  $\Gamma$  quasi-acts on  $X$  (in this case that just means there is a homomorphism  $\Gamma \rightarrow \text{QI}(X)$  with uniformly bounded constants). We then need to show that such a quasi-action can be conjugated to an isometric action. The basic ingredients we need to do this are the following:

**Theorem 7.1.** *[FM2] Every uniform quasi-similarity action on  $\mathbb{R}$  is bilipschitz conjugate to a similarity action.*

The proof of this theorem makes substantial use of work of Hinkannen [H] who had shown that a uniform quasi-symmetric action was quasi-symmetrically conjugate to a symmetric action.

**Theorem 7.2.** *[MSW] Let  $\Gamma$  have a uniform quasi-similarity action on  $\mathbb{Q}_m$ . If the  $\Gamma$  action is cocompact on the space of pairs of distinct points in  $\mathbb{Q}_m$  then there is some  $n$  and a similarity action of  $\Gamma$  on  $\mathbb{Q}_n$  which is bilipschitz conjugate to the given quasi-similarity action.*

It is useful to think about these results in a quasi-action interpretation. One can view  $\mathbb{R}$  as  $S^1 - \{pt\}$ , and interpret a uniform quasi-similarity action on  $\mathbb{R}$  as the boundary of a quasi-action on  $\mathbb{H}^2$  fixing a point at infinity. The result of Farb and Mosher then says that this quasi-action is quasi-conjugate to an isometric action on  $\mathbb{H}^2$ . The interpretation of the second result is similar, with a tree of valence  $m + 1$  replacing  $\mathbb{H}^2$ . The hypothesis of cocompactness on pairs in that theorem then translates to cocompactness of the quasi-action on the tree.

We now state and prove a result that immediately implies Theorem 1.2. This result is also used in [EFW2].

**Theorem 7.3.** *Assume every  $(\kappa, C)$  self quasi-isometry of  $\text{Sol}(m, n)$  is at bounded distance from a  $b$ -standard map where  $b = b(\kappa, C)$ . Then any uniform group of quasi-isometries of  $\text{Sol}(m, n)$  is virtually a lattice in  $\text{Sol}(m, n)$ .*

**Proof.** Let  $f : \Gamma \rightarrow \text{Sol}(m, n)$  be a quasi-isometry. For each  $\gamma$  in  $\Gamma$  we have the self-quasi-isometry  $T_\gamma$  of  $\text{Sol}(m, n)$  given by

$$x \mapsto f(\gamma f^{-1}(x))$$

By theorem 2.1,  $T_\gamma$  is bounded distance from a standard map. On a subgroup  $\Gamma'$  of  $\Gamma$  of index at most two, this gives a homomorphism  $\Phi : \Gamma' \rightarrow QSim(\mathbb{R}) \times QSim(\mathbb{R})$ . By theorem 7.1, each of these quasi-similarity actions on  $\mathbb{R}$  can be bilipschitz conjugated to a similarity action. This gives  $\Psi : \Gamma' \rightarrow Sim(\mathbb{R}) \times Sim(\mathbb{R})$ .

Since the quasi-isometries  $T_\gamma$  have uniformly bounded constants, we know that the stretch factors of the two quasi-similarity actions  $\Phi$  are approximately on the curve  $(e^{mt}, e^{-nt})$  - meaning that the products weighted by these factors are uniformly close to 1. This therefore is true for  $\Psi$  as well. So, in the sequence:

$$\Gamma' \rightarrow Sim(\mathbb{R}) \times Sim(\mathbb{R}) \rightarrow \mathbb{R} \times \mathbb{R}$$

where the final map is just the log of the stretch factor, we know that the image lies within a bounded neighborhood of the line  $ny = -mx$ . Since the image is a subgroup, this implies it must lie on this line. Since the subgroup of  $Sim(\mathbb{R}) \times Sim(\mathbb{R})$  above this line is  $\text{Sol}(m, n)$ , we have produced a homomorphism

$$\Psi : \Gamma' \rightarrow \text{Sol}(m, n)$$

We now show that the kernel is finite and the image discrete and cocompact. This follows essentially from the fact that the map  $f$  is a quasi-isometry.

Consider a compact subset  $K \subset \text{Sol}(m, n)$ . The set  $F = \Psi^{-1}(K)$  consists of maps with uniformly bounded stretch factors, and which move the origin at most a bounded amount. Transporting this information back to the standard maps of  $\text{Sol}(m, n)$ , we see that for  $\gamma \in F$  the maps  $T_\gamma$  move the identity a uniformly bounded amount. However, the quasi-action  $T$  of  $\Gamma$  on  $\text{Sol}(m, n)$  is the  $f$ -conjugate of the left action of  $\Gamma$  on  $\Gamma$ . This action is proper, so we conclude that  $F$  is finite. This implies that  $\Psi$  has finite kernel and discrete image. In the same way, the fact that the  $\Gamma$  action on  $\Gamma$  is transitive implies that the image of  $\Psi$  is cocompact.

Thus the image of  $\Gamma'$  is a lattice in  $\text{Sol}(m, n)$ . □

This proves Theorem 1.2, since if  $m = n$ , the group  $\text{Sol}(m, n)$  is not unimodular and therefore does not contain lattices.

We next aim to prove Theorem 1.4. In fact, we show

**Theorem 7.4.** *Assume every  $(\kappa, C)$  self quasi-isometry of  $\text{DL}(m, n)$  is at bounded distance from a  $b$ -standard map where  $b = b(\kappa, C)$ . Then any uniform group of quasi-isometries of  $\text{DL}(m, n)$  is virtually a lattice in  $\text{Isom}(\text{DL}(n', n'))$  where  $n', m, n$  are all powers of a common integer.*

Some complications arise from the differences between Theorem 7.2 and Theorem 7.1. We need the following theorem of Cooper:

**Theorem 7.5.** *[Co] The metric spaces  $\mathbb{Q}_p$  and  $\mathbb{Q}_q$  are bilipschitz equivalent if and only if there are integers  $d, s, t$  so that  $p = d^s$  and  $q = d^t$ .*

This immediately implies a weaker version Theorem 1.5. We now turn to theorem 7.4.

**Proof.** We proceed as in the previous proof for  $\text{Sol}(m, n)$ . The first difference is that to apply theorem 7.2 we need to know that the quasi-similarity actions of  $\Gamma'$  on  $\mathbb{Q}_n$  and  $\mathbb{Q}_m$  are cocompact on pairs of points. As discussed above, this is equivalent to asking the corresponding quasi-action on the trees of valence  $n + 1$  and  $m + 1$  to be cocompact. This then follows immediately from the fact that  $\Gamma'$  is cocompact on  $\text{DL}(m, n)$ .

Thus we have  $\Psi : \Gamma' \rightarrow \text{Sim}(\mathbb{Q}_a) \times \text{Sim}(\mathbb{Q}_b)$  for some  $a$  and  $b$ . Thus we know that we have  $d_i, s_i, t_i$  for  $i = 1, 2$  with  $n = d_1^{s_1}$ ,  $m = d_2^{s_2}$  and:

$$\Psi : \Gamma' \rightarrow \text{Sim}(\mathbb{Q}_{d_1^{t_1}}) \times \text{Sim}(\mathbb{Q}_{d_2^{t_2}})$$

We know, as before, that the weighted stretch factors are approximate inverses. In this case the stretch factors are in  $\mathbb{Z}$  - in  $\text{Sim}(\mathbb{Q}_m)$  one can stretch only by powers of  $m$ . Thus the image is a subgroup lying on the line  $\{(a, b); a * \log d_1 * \frac{t_1}{s_1} + b * \log d_2 * \frac{t_2}{s_2} = 0\}$ . For this to be a non-empty subgroup of  $\mathbb{Z}^2$  we must have  $\frac{\log d_1}{\log d_2}$  rational, which implies that there is a  $d$  with  $d_1 = d^u$ ,  $d_2 = d^v$  for some  $u$  and  $v$ .

There is still some ambiguity in the choices, since many groups occur as subgroups of  $\text{Sim}(\mathbb{Q}_{p^k})$  for many different  $k$ . As in the construction of [MSW] we can make the choices unique by choosing the  $t_i$  the maximum possible, so that all powers of  $d_i^{t_i}$  occur as stretch factors. With these choices we are forced to have the line  $\{(a, b) : a + b = 0\}$  as this is the only line of negative slope in  $\mathbb{Z}^2$  surjecting to both factors. Thus we have  $\Psi : \Gamma' \rightarrow \text{Sim}(\mathbb{Q}_{d^{t_1}}) \times \text{Sim}(\mathbb{Q}_{d^{t_2}})$ , with the image contained in the subgroup having inverse stretch factors. This group is, up to finite index,  $\text{Isom}(\text{DL}(d^{t_1}, d^{t_2}))$ . So we have:

$$\Psi : \Gamma' \rightarrow \text{Isom}(\text{DL}(d^{t_1}, d^{t_2}))$$

Exactly as before, one can see that the kernel is finite and the image is a lattice, which implies that  $t_1 = t_2$ . This implies that  $\Gamma$  is amenable, and hence it and  $DL(m, n)$  have metric Følner sets. This is true only for  $m = n$ , which completes the proof.  $\square$

This immediately implies Theorem 1.4, since  $DL(m, n)$  is only amenable as a metric space when  $m = n$ .

**Proof of Theorem 1.3.** Since all  $Sol(n, n)$  are obviously quasi-isometric to one another, it suffices to consider the case  $m \neq n$ . This then follows immediately from Theorem 2.1 and [FM3, Theorem 5.1].  $\square$

**Proposition 7.6.** *Theorem 2.3 implies Theorem 1.5.*

**Proof.** In view of Theorem 2.3, the proof of this result is similar to the last one. The point is that (up to permuting  $m$  and  $n$ ) the quasi-isometry  $DL(m, n) \rightarrow DL(m', n')$  induces quasi-similarities  $\mathbb{Q}_n \rightarrow \mathbb{Q}_{n'}$  and  $\mathbb{Q}_m \rightarrow \mathbb{Q}_{m'}$ . Theorem 7.5 then implies that  $m$  and  $m'$  are both powers of some number  $d$  and that  $n$  and  $n'$  are both powers of some number  $s$ . However, since the quasi-similarities both come from the same map on vertical geodesics, the scale factors must agree. This immediately implies  $\log m' / \log m = \log n' / \log n$ .  $\square$

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