

Counting closed geodesics in Moduli space

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1 Introduction

Let \mathcal{M}_g denote the moduli space of closed Riemann surfaces of genus g . We may write $\mathcal{M}_g = \mathcal{T}_g/\Gamma_g$, where \mathcal{T}_g is the Teichmüller space of genus g surfaces, and Γ_g is the mapping class group. In this paper, we investigate properties of Teichmüller geodesics on \mathcal{M}_g . Let $N(R)$ denote the number of closed Teichmüller geodesics in \mathcal{M}_g of length at most R . Then $N(R)$ is also the number of conjugacy classes of pseudo-Anosov elements of the mapping class group of translation length at most R . Our main result is the following:

Theorem 1.1 *As $R \rightarrow \infty$, we have*

$$N(R) \sim \frac{e^{hR}}{hR},$$

where $h = 6g - 6$.

In the above theorem and below, the notation $A \sim B$ means that the ratio A/B tends to 1. Even though we assume that the surface has no punctures here, most of the results also hold on $\mathcal{M}_{g,n}$, provided h is replaced by $6g - 6 + 2n$.

In the proof of Theorem 1.1, the key is to estimate the number of closed geodesics which stay outside of compact sets. Let $N_j(\delta, R)$ denote the number of closed geodesics γ of length at most R in \mathcal{M}_g such that for each point $X \in \gamma$, X has at least j simple closed curves of hyperbolic length less than δ .

Theorem 1.2 *Given $\epsilon > 0$ there exist $\delta > 0$ and $C = C(\epsilon)$ such that for all j , $1 \leq j \leq 3g - 3$, and all $R > 0$,*

$$N_j(\delta, R) \leq Ce^{(h-j+\epsilon)R}.$$

Let K be a compact subset of \mathcal{M}_g . We let $N^K(R)$ denote the number of geodesics in \mathcal{M}_g of length less than R which never intersect K . Letting $j = 1$ in Theorem 1.2, we obtain the following:

Corollary 1.3 *For every $\epsilon > 0$ there exist a compact set $K \subset \mathcal{M}_g$ and $C = C(\epsilon) > 0$ such that for all $R > 0$,*

$$N^K(R) \leq Ce^{(h-1+\epsilon)R}.$$

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Corollary 1.3 is complimentary to the following result:

Theorem 1.4 (Rafi, Hammenstadt) *For any compact $K \subset \mathcal{M}_g$, and sufficiently large R ,*

$$N^K(R) \geq e^{(h-1)R}.$$

1.1 Previous Results

The first results on this problem are due to Veech [Ve]. He proved that there exists a constant c_2 such that

$$h \leq \liminf_{R \rightarrow \infty} \frac{\log N(R)}{R} \leq \limsup_{R \rightarrow \infty} \frac{\log N(R)}{R} \leq c_2$$

and conjectured that $c_2 = h$. In a remark in a paper by Ursula Hamenstadt [H1] (see also [H3]), in which the main focus is different, she proves that $c_2 \leq (6g - 6 + 2n)(6g - 5 + 2n)$.

Sasha Bufetov [Bu] proved the formula

$$\lim_{R \rightarrow \infty} \frac{\log \tilde{N}(R)}{R} = h, \tag{1}$$

where $\tilde{N}(R)$ is the number of periodic orbits of the Rauzy-Veech induction such that the log of the norm of the renormalization matrix is at most R . This is a closely related problem; essentially $\tilde{N}(R)$ counts closed geodesics on a certain finite cover of \mathcal{M}_g . However the equation (1) does not easily imply

$$\lim_{R \rightarrow \infty} \frac{\log N(R)}{R} = h. \tag{2}$$

Very recently, Kasra Rafi [Ra1] proved Corollary 1.3 (which implies (2)) for the case of the five-punctured sphere.

We note that (2) is an immediate consequence of Theorem 1.1, which is a bit more precise. In order to prove Theorem 1.1 one needs Corollary 1.3 and certain recurrence results for geodesics, which are based on [Ath].

Remarks.

- The problem of understanding the asymptotics of the number $N_M(R)$ of primitive closed geodesics of length $\leq R$ on a given manifold M has been investigated intensively. An asymptotic formula for $N_M(R)$ on a compact hyperbolic surfaces was first proved by Huber. See §9 in [Bu] and references within for related work of Hejhal, Randol and Sarnak.

More generally, Margulis proved that on a compact n -manifold M of negative curvature

$$N_M(R) \sim \frac{e^{hR}}{hR},$$

where h is the topological entropy of the geodesic flow. In this case the techniques from uniformly hyperbolic dynamics can be applied to study the geodesic flow on the unit tangent bundle of M . See [Mar] for the proof of Margulis' theorem, and related results on Anosov and hyperbolic flows.

- The main difficulty for proving Theorem 1.1 is the fact that the Teichmüller flow is not hyperbolic. In order to overcome this difficulty, first in §4 we use Minsky’s product region theorem [Mi] to prove that the geodesic flow is biased toward the *thick part* of the moduli space. This result implies Theorem 1.2. Then we use the basic properties of the Hodge norm [ABEM] to prove a closing lemma for the Teichmüller geodesic flow in §6. But the Hodge norm behaves badly near smaller strata, i.e. near points with degenerating zeros of the quadratic differential. However, in §5 we show that the number of closed geodesics γ of length at most R such that γ spends at least θ -fraction of the time outside of a compact subset of the principal stratum is exponentially smaller than $N(R)$ (see Theorem 5.2). Finally, we obtain the counting result using the fact that in any compact subset of the principal stratum, the geodesic flow is uniformly hyperbolic ([Ve], [Fo], and [ABEM]).
- By results in [H2] the normalized geodesic flow invariant measure supported on the set of closed geodesics of length $\leq R$ in $\mathcal{Q}^1\mathcal{M}_g$ become equidistributed with respect to the Lebesgue measure μ (see §2.2) as $R \rightarrow \infty$.
- In a forthcoming joint work with Kasra Rafi, we generalize the results in this paper to the case of other strata of moduli spaces of Abelian and quadratic differentials.

Notation. In this paper, $A \approx B$ means that $A/C < B < AC$ for some universal constant C which only depends genus g . Also, $A = O(B)$ means that $A < BC$, for some universal constant C , which again could depend on g .

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2 Background and notation

In this section, we recall definitions and known results about the Teichmüller geometry of \mathcal{M}_g . For more details see [Hu].

2.1 Teichmüller space

A point in the *Teichmüller space* \mathcal{T}_g is a complex curve X of genus g equipped with a diffeomorphism $f : S_g \rightarrow X$. The map f provides a *marking* on X by S_g . Two marked surfaces $f_1 : S_g \rightarrow X$ and $f_2 : S_g \rightarrow Y$ define the same point in \mathcal{T}_g if and only if $f_1 \circ f_2^{-1} : Y \rightarrow X$ is isotopic to a holomorphic map. By the uniformization theorem, each point X in \mathcal{T}_g has a metric of constant curvature -1 . The space \mathcal{T}_g is a complex manifold of dimension $3g - 3$, diffeomorphic to a cell. Let Γ_g denote the mapping class group of S_g , or in other words the group of isotopy classes of orientation preserving self homeomorphisms of S_g . The mapping class group Γ_g acts on \mathcal{T}_g by changing the marking. The quotient space

$$\mathcal{M}_g = \mathcal{T}_g / \Gamma_g$$

is the moduli space of Riemann surfaces homeomorphic to S_g . The space \mathcal{M}_g is an orbifold. However, it is finitely covered by a manifold, and

$$\pi_1^{orb}(\mathcal{M}_g) = \Gamma_g.$$

Let

$$\mathfrak{p} : \mathcal{T}_g \rightarrow \mathcal{M}_g$$

denote the natural map from \mathcal{T}_g to $\mathcal{M}_g = \mathcal{T}_g/\Gamma_g$. The Teichmüller metric on marked surfaces is defined by

$$d_{\mathcal{T}}((f_1 : S_g \rightarrow X_1), (f_2 : S_g \rightarrow X_2)) = \frac{1}{2} \inf(\log K(h)),$$

where $h : X_1 \rightarrow X_2$ ranges over all quasiconformal maps isotopic to $f_1 \circ f_2^{-1}$. Here $K(h) \geq 1$ is the dilatation of h .

Dilatation of pseudo-Anosov elements. According to the Nielsen-Thurston classification, every irreducible mapping class element $\mathfrak{g} \in \Gamma_g$ of infinite order has a representative which is a pseudo-Anosov homeomorphism [Th]. By a theorem of Bers, every closed geodesic in \mathcal{M}_g is the unique loop of minimal length in its homotopy class. Given a pseudo-Anosov $\mathfrak{g} \in \Gamma_g$ the dilatation of \mathfrak{g} is defined by $K(\mathfrak{g})$. Then $\log(K(\mathfrak{g}))$ is the translation length of \mathfrak{g} as an isometry of \mathcal{T}_g [Be]. In other words,

$$\mathcal{L}(S_g) = \{ \log(K(\mathfrak{g})) \mid \mathfrak{g} \in \Gamma_g \text{ pseudo-Anosov} \}$$

is the length spectrum of \mathcal{M}_g equipped with the Teichmüller metric.

By [AY] and [Iv] \mathcal{L}_g is discrete subset of \mathbb{R} . Hence the number $N(R)$ of conjugacy classes of pseudo-Anosov elements of the group Γ_g with dilatation factor $K(\mathfrak{g}) \leq R$ is finite. We remark that for any pseudo-Anosov $\mathfrak{g} \in \Gamma_g$ the number $K(\mathfrak{g})$ is an algebraic number. Moreover $\log(K(\mathfrak{g}))$ is equal to the minimal topological entropy of any element in the same homotopy class [FLP].

2.2 Moduli space of quadratic differentials

The cotangent space of \mathcal{T}_g at a point X can be identified with the vector space $Q(X)$ of holomorphic quadratic differentials on X . Recall that given $X \in \mathcal{T}_g$, a quadratic differential $q \in Q(X)$ is a tensor locally given by $\phi(z)dz^2$ where ϕ is holomorphic. Then the space $\mathcal{QT}_g = \{(q, X) \mid X \in \mathcal{T}_g, q \in Q(X)\}$ is the cotangent space of \mathcal{T}_g . In this setting, the Teichmüller metric corresponds to the norm

$$\|q\|_{\mathcal{T}} = \int_X |\phi(z)| |dz|^2$$

on \mathcal{QT}_g . Let $\mathcal{QM}_g \cong \mathcal{QT}_g/\Gamma_g$. Finally, let $\mathcal{Q}^1\mathcal{T}_g$ denote the Teichmüller space of unit area quadratic differentials on surfaces of genus g , and $\mathcal{Q}^1\mathcal{M}_g \cong \mathcal{Q}^1\mathcal{T}_g/\Gamma_g$. For simplicity, we let π denote the natural projection maps

$$\pi : \mathcal{Q}^1\mathcal{M}_g \rightarrow \mathcal{M}_g,$$

and

$$\pi : \mathcal{Q}^1\mathcal{T}_g \rightarrow \mathcal{T}_g.$$

Although the value of $q \in \mathcal{Q}^1(X)$ at a point $x \in X$ depends on the local coordinates, the zero set of q is well defined. As a result, there is a natural stratification of the space \mathcal{QM}_g by the

multiplicities of zeros of q . Define $\mathcal{QM}_g(a_1, \dots, a_k) \subset \mathcal{QM}_g$ to be the subset consisting of pairs (X, q) of holomorphic quadratic differentials on X with k zeros with multiplicities (a_1, \dots, a_k) . Then

$$\mathcal{QM}_g = \bigsqcup_{(a_1, \dots, a_k)} \mathcal{QM}_g(a_1, \dots, a_k).$$

It is known that each $\mathcal{QM}_g(a_1, \dots, a_k)$ is an orbifold of dimension $4g - 4 + 2k$. In particular $\dim(\mathcal{QM}_g(1, \dots, 1)) = \dim(\mathcal{QM}_g)$.

We recall that when $g > 1$, the Teichmüller metric is not even Riemannian. However, geodesics in this metric are well understood. A quadratic differential $q \in \mathcal{QT}_g$ with zeros at x_1, \dots, x_k is determined by an atlas of charts $\{\phi_i\}$ mapping open subsets of $S_g - \{x_1, \dots, x_k\}$ to \mathbb{R}^2 such that the change of coordinates are of the form $v \rightarrow \pm v + c$. Therefore the group $\mathrm{SL}_2(\mathbb{R})$ acts naturally on \mathcal{QM}_g by acting on the corresponding atlas; given $A \in \mathrm{SL}_2(\mathbb{R})$, $A \cdot q \in \mathcal{QM}_g$ is determined by the new atlas $\{A\phi_i\}$. The action of the diagonal subgroup $g_t = \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}$ is the Teichmüller geodesic flow for the Teichmüller metric. Moreover we have [Ve],[Mas]:

Theorem 2.1 (Veech-Masur) *The space $\mathcal{Q}^1\mathcal{M}_g$ carries a unique probability measure $\mu = \mu_g$ in the Lebesgue measure class such that :*

- *the action of $\mathrm{SL}_2(\mathbb{R})$ is volume preserving and ergodic;*
- *Teichmüller geodesic flow is mixing.*

By this theorem, $\mathcal{Q}^1\mathcal{T}_g$ carries a natural normalized smooth measure $\mu = \mu_g$, preserved by the action of Γ_g . We set

$$\mathbf{m} = \pi_*\mu. \tag{3}$$

Remark. In fact, the Teichmüller flow on $\mathcal{Q}^1\mathcal{M}_g$ is exponentially mixing with respect to μ [AR]. See also [AGY]. However, we will only use Theorem 2.1 in this paper.

2.3 Period coordinates on the strata

A *saddle connection* on $q \in \mathcal{Q}^1\mathcal{T}_g$ is a geodesic segment which joins a pair of singular points without passing through one in its interior. In general, a geodesic segment e joining two zeros of a quadratic differential $q = \phi dz^2$ determines a complex number $\mathrm{hol}_q(e)$ (after choosing a branch of $\phi^{1/2}$ and an orientation of e) by

$$\mathrm{hol}_q(e) = \mathrm{Re}(\mathrm{hol}_q(e)) + \mathrm{Im}(\mathrm{hol}_q(e)),$$

where

$$\mathrm{Re}(\mathrm{hol}_q(e)) = \int_e \mathrm{Re}(\phi^{1/2}),$$

and

$$\mathrm{Im}(\mathrm{hol}_q(e)) = \int_e \mathrm{Im}(\phi^{1/2}).$$

We recall that the period coordinates give $\mathcal{QT}_g(a_1, \dots, a_k)$ the structure of a piecewise linear manifold. For notational simplicity we discuss the case of $\mathcal{QT}_g(1, \dots, 1)$. Given $q_0 \in \mathcal{QT}_g(1, \dots, 1)$ there

is a triangulation E of the underlying surface by saddle connections, $h = 6g - 6$ directed edges $\delta_1, \dots, \delta_h$ of E , and an open neighborhood $U_{q_0} \subset \mathcal{QT}_g(1, \dots, 1)$ of q_0 such that the map

$$\psi_{E, q_0} : \mathcal{QT}_g(1, \dots, 1) \rightarrow \mathbb{C}^{6g-6}$$

by

$$\psi_{E, q_0}(q) = (\text{hol}_q(\delta_i))_{i=1}^h$$

is a local homeomorphism. Also for any other geodesic triangulation E' the map the map $\psi_{E', q_0} \circ \psi_{E, q_0}^{-1}$ is linear. For a discussion of these coordinates see [MasSm]. The measure μ_g in Theorem 2.1 (up to a constant) is given by the piecewise linear structure of \mathcal{QM}_g . This measure, up to normalization, also coincides with the measure defined by the Teichmüller norm on the unit cotangent bundle of \mathcal{T}_g . This measure is supported on $\mathcal{Q}^1\mathcal{M}_g(1, 1, \dots, 1)$; that is, $\mu_g(\mathcal{Q}^1\mathcal{M}_g - \mathcal{Q}^1\mathcal{M}_g(1, 1, \dots, 1)) = 0$.

2.4 Extremal and hyperbolic lengths of simple closed curves

Given a homotopy class of a simple closed curve α on a topological surface S_g and $X \in \mathcal{T}_g$, let $\ell_\alpha(X)$ be the length of the unique geodesic in the homotopy class of α with respect to the hyperbolic metric on X . The *extremal length* of a simple closed curve α on X is defined by

$$\text{Ext}_\alpha(X) = \sup_\rho \frac{\ell_\alpha(\rho)^2}{\text{Area}(X, \rho)}, \quad (4)$$

where the supremum is taken over all metrics ρ conformally equivalent to X , and $\ell_\alpha(\rho)$ denotes the length of α in the metric ρ .

Given simple closed curves α and β on S_g , the intersection number $i(\alpha, \beta)$ is the minimum number of points in which representatives of α and β must intersect. In general, we have:

$$i(\alpha, \beta) \leq \sqrt{\text{Ext}_\alpha(X)} \cdot \sqrt{\text{Ext}_\beta(X)}.$$

The following result [Ker] relates the ratios of extremal lengths to the Teichmüller distance:

Theorem 2.2 (Kerckhoff) *Given $X, Y \in \mathcal{T}_g$, the Teichmüller distance between X and Y is given by*

$$d_{\mathcal{T}}(X, Y) = \sup_\beta \log \left(\frac{\sqrt{\text{Ext}_\beta(X)}}{\sqrt{\text{Ext}_\beta(Y)}} \right),$$

where β ranges over all simple closed curves on S_g .

The relationship between the extremal length and hyperbolic length is complicated; in general, by the definition of extremal length

$$\frac{\ell_\alpha(X)^2}{4\pi(g-1)} \leq \text{Ext}_\alpha(X).$$

Also for any $X \in \mathcal{T}_g$ the extremal length can be extended continuously to the space of measured laminations [Ker] such that

$$\text{Ext}_{r, \lambda}(X) = r^2 \text{Ext}_\lambda(X).$$

As a result given X there exists a constant a_X such that

$$\frac{1}{a_X} \ell_\alpha(X) \leq \sqrt{\text{Ext}_\alpha(X)} \leq a_X \ell_\alpha(X).$$

However, by [M],

$$\frac{1}{\pi} \leq \frac{\text{Ext}_\alpha(X)}{\ell_\alpha(X)} \leq \frac{1}{2} e^{\ell_\alpha(X)/2}. \quad (5)$$

Hence as $\ell_\alpha(X) \rightarrow 0$,

$$\frac{\ell_\alpha(X)}{\text{Ext}_\alpha(X)} \approx 1.$$

Bounded pants decompositions. Recall that by a theorem of Bers, there exists L_g depending only on g such that for any surface $Y \in \mathcal{T}_g$ there exists a pants decomposition $\mathcal{P}_Y = \{\alpha_1, \dots, \alpha_{3g-3}\}$ of Y satisfying $\ell_{\alpha_i}(Y) \leq L_g$. By (5) the extremal length of each α_i on Y is bounded from above by C_g , where $C_g = e^{L_g} L_g$ is independent of Y . We call such a pants decomposition a *bounded* pants decomposition for Y .

Fix a small enough number $\epsilon_0 > 0$ such that:

- 1). Any two simple closed curves of extremal length $\leq \epsilon_0^2$ are disjoint;
- 2). any simple closed curve intersecting a simple closed curve of extremal length $\leq \epsilon_0^2$ on a surface of genus g has extremal length $\geq 2C_g$.

We say α is *short* on X if $\text{Ext}_\alpha(X) \leq \epsilon_0^2$. Let \mathcal{C}_X be the set of simple closed curves β on the surface X with $\text{Ext}_\beta(X) \leq \epsilon_0^2$. Note that by the definition any bounded pants decomposition of X contains all the elements of \mathcal{C}_X .

Now, define $G : \mathcal{T}_g \rightarrow \mathbb{R}_+$ by

$$G(Y) = 1 + \prod_{\beta \in \mathcal{C}_Y} \frac{1}{\sqrt{\text{Ext}_\beta(Y)}}. \quad (6)$$

Note that for any bounded pants decomposition $\mathcal{P}_X = \{\alpha_1, \dots, \alpha_{3g-3}\}$

$$G(X) \approx \prod_{\alpha \in \mathcal{P}_X} \frac{1}{\sqrt{\text{Ext}_\alpha(X)}}.$$

Also, if $d_{\mathcal{T}}(X, Y) = O(1)$ then $G(X) \approx G(Y)$. By the definition, G induces a proper function on \mathcal{M}_g .

2.5 Estimating extremal length and Minsky's product theorem

Let $\alpha_1, \dots, \alpha_j$ be a collection of disjoint simple closed curves on S_g , and $\epsilon \leq \epsilon_0^2$. Let

$$P_\epsilon(\alpha_1, \dots, \alpha_j) = \{X \in \mathcal{T}_g \mid \text{Ext}_X(\alpha_i) \leq \epsilon\}.$$

Let \mathbb{H}^2 denote the upper half plane model of the hyperbolic plane. Then using the Fenchel-Nielsen coordinates on \mathcal{T}_g , we can define $\phi_0 : P_\epsilon(\alpha_1, \dots, \alpha_j) \rightarrow (\mathbb{H}^2)^j$ by

$$\phi_0(X) = (\theta_1(X), \frac{1}{\ell_{\alpha_1}(X)}, \dots, \theta_j(X), \frac{1}{\ell_{\alpha_j}(X)}),$$

where θ_i is the twist coordinate around α_i . In fact, following Minsky, we get a map

$$\phi : P_\epsilon(\alpha_1, \dots, \alpha_j) \rightarrow (\mathbb{H}^2)^j \times \mathcal{T}',$$

where \mathcal{T}' is the quotient Teichmüller space obtained by collapsing all the α_i . The product region theorem [Mi] states that for sufficiently small ϵ the Teichmüller metric on $P_\epsilon(\alpha_1, \dots, \alpha_j)$ is within an additive constant of the supremum metric on $(\mathbb{H}^2)^j \times \mathcal{T}'$. More precisely, let $d'(\cdot, \cdot)$ denote the supremum metric on $(\mathbb{H}^2)^j \times \mathcal{T}'$. Then :

Theorem 2.3 *If $\epsilon > 0$ is small enough, then there exists a constant $b > 0$ depending only on the genus such that for all $X, Y \in P_\epsilon(\alpha_1, \dots, \alpha_j)$,*

$$|d_{\mathcal{T}}(X, Y) - d'(\phi(X), \phi(Y))| < b.$$

Note that by Theorem 2.2 this statement can be rewritten in terms of the ratios of extremal lengths of simple closed curves on X and Y .

Let \mathcal{C}_g denote the set of all multicurves on S_g , and $\{\alpha_1, \dots, \alpha_{3g-3}\}$ be a pants decomposition of S_g . Consider the Dehn-Thurston parameterization [HP]

$$DT : \mathcal{C}_g \rightarrow (\mathbb{Z}_+ \times \mathbb{Z})^{3g-3}$$

defined by

$$DT(\beta) = (i(\beta, \alpha_i), \text{tw}(\beta, \alpha_i))_{i=1}^{3g-3},$$

where $i(\cdot, \cdot)$ denotes the geometric intersection number and $\text{tw}(\beta, \alpha_i)$ is the twisting parameter of β around α_i . See [HP] for more details.

The proof of Theorem 2.3 relies on the following estimates for the extremal lengths of arbitrary simple closed curves on a surface, see [Mi] (Theorem 5.1, and equation (4.3)):

Theorem 2.4 (Minsky) *Suppose $Y \in \mathcal{T}_g$, and let $\mathcal{P} = \mathcal{P}_Y = \{\alpha_1, \dots, \alpha_{3g-3}\}$ be a bounded pants decomposition on Y . Then given a simple closed curve β on S_g , $\text{Ext}_\beta(Y)$ is bounded from above and below by*

$$\max_{1 \leq j \leq 3g-3} \left[\frac{i(\beta, \alpha_j)^2}{\text{Ext}_{\alpha_j}(Y)} + \text{tw}^2(\beta, \alpha_j) \text{Ext}_{\alpha_j}(Y) \right] \quad (7)$$

up to a multiplicative constant depending only on g .

This result gives an upper bound for the Dehn-Thurston coordinates of a simple closed curve in terms of its extremal length.

We remark that the definition of the twist used in equation (4.3) in [Mi] is different from the definition we are using here. We follow the definition used in [HP]. In terms of our notation,

$$\text{tw}(h_\alpha^r(\beta), \alpha) = \text{tw}(\beta, \alpha) + r \cdot i(\beta, \alpha_i), \quad (8)$$

where $h_\alpha \in \Gamma_g$ is the right Dehn twist around α .

Let $\mathcal{P}_X = \{\alpha_1, \dots, \alpha_n\}$ be a bounded pants decomposition of $X \in \mathcal{T}_g$ (see §2.4). Then:

- For any simple closed curve β on S_g ,

$$\text{tw}(\beta, \alpha_i) \leq c_1 \cdot \frac{\sqrt{\text{Ext}_\beta(X)}}{\sqrt{\text{Ext}_{\alpha_i}(X)}}, \quad i(\beta, \alpha_i) \leq \sqrt{\text{Ext}_\beta(X)} \cdot \sqrt{\text{Ext}_{\alpha_i}(X)}, \quad (9)$$

where $c_1 > 0$ only depend on g .

- Let $\alpha \in \mathcal{P}_X$ and $r_0 > 0$. We claim that if

$$d_{\mathcal{T}}(h_{\alpha}^k(X), X) \leq r_0,$$

then

$$|k| \leq c \frac{e^{r_0}}{\text{Ext}_{\alpha}(X)},$$

where $c > 0$ only depends on g and $h_{\alpha} \in \Gamma_g$ is the right Dehn twist around α . Note that by [Mi], if $\alpha \in \mathcal{P}_X$ then there exists a simple closed curve β such that

$$\text{Ext}_{\beta}(X) \approx \frac{1}{\text{Ext}_{\alpha}(X)}, \quad (10)$$

and $i(\alpha, \beta) = 1$. We can use Theorem 2.4 to estimate $\sqrt{\text{Ext}_{h_{\alpha}^{-k}(\beta)}(X)} = \sqrt{\text{Ext}_{\beta}(h_{\alpha}^k(X))}$. As a result

$$\frac{1}{\sqrt{\text{Ext}_{\alpha}(X)}} + |k| \sqrt{\text{Ext}_{\alpha}(X)} \leq c' \sqrt{\text{Ext}_{h_{\alpha}^{-k}(\beta)}(X)},$$

where c' only depends on g . By Theorem 2.2, we have:

$$\sqrt{\text{Ext}_{h_{\alpha}^{-k}(\beta)}(X)} \leq e^{r_0} \sqrt{\text{Ext}_{\beta}(X)}.$$

Now from (10) we get

$$\frac{1}{\sqrt{\text{Ext}_{\alpha}(X)}} + |k| \sqrt{\text{Ext}_{\alpha}(X)} = O(e^{r_0} \sqrt{\text{Ext}_{\beta}(X)}) \implies |k| = O\left(\frac{e^{r_0}}{\text{Ext}_{\alpha}(X)}\right). \quad (11)$$

In other words, the number of twists around $\alpha_i \in \mathcal{P}_X$ which one can take and still stay in $B_{\mathcal{T}}(X, r)$ is $O(e^r/a_i)$, where $a_i = \text{Ext}_{\alpha_i}(X)$. A generalization of this argument is used in the proof of Lemma 7.3 [ABEM].

Remark. As a result of Theorem 7.2, for any $X \in \mathcal{T}_g$ the number of $\mathfrak{g} \in \Gamma_g$ such that $d_{\mathcal{T}}(\mathfrak{g} \cdot X, X) \leq r$ is $O(\prod_{i=1}^{3g-3} r e^r/a_i)$. When $r > 0$ is fixed, one can obtain this bound by showing that up to uniformly bounded index, the set of $\{\mathfrak{g} \in \Gamma_g \mid d_{\mathcal{T}}(\mathfrak{g} \cdot X, X) \leq r\}$ only consists of twists about $\alpha_i \in \mathcal{P}_X$.

3 Net points in Teichmüller space

Recall that a set \mathcal{N} is a (c_1, c_2) separated net on a metric space \mathcal{X} if $\mathcal{N} \subset \mathcal{X}$, every point of \mathcal{X} is within c_2 of a net point, and the minimal distance between net points is at least c_1 . In §4 we will use nets in order to discretize \mathcal{T}_g .

3.1 Volumes of balls of fixed radius

Let $B_{\mathcal{T}}(X, L) \subset \mathcal{T}_g$ denote the ball of radius L with respect to the Teichmüller metric, and let \mathbf{m} be the smooth measure defined by (3) on \mathcal{T}_g . Then we have:

Lemma 3.1 *There exists $L_0 > 0$ (depending only on g) and for every $L > L_0$ there exist constants $0 < c_1 < c_2$ such that for all $X \in \mathcal{T}_g$,*

$$c_1 \leq \mathbf{m}(B_{\mathcal{T}}(X, L)) \leq c_2$$

The constants c_1 and c_2 depend on L and g (but not on X).

Proof. Suppose $X \in \mathcal{T}_g$. Let $\alpha_1, \dots, \alpha_k$ be the simple closed curves on X with extremal length less than C_g^2 . Let a_i denote the extremal length of α_i . Note that by (5) the hyperbolic length $\ell_{\alpha_i}(X) \approx a_i$. By Theorem 2.2 for all $Y \in B_{\mathcal{T}}(X, L)$,

$$\text{Ext}_Y(\alpha_i) \leq L_1^2 \text{Ext}_X(\alpha_i) \leq C L_1^2 a_i,$$

where $L_1 = e^L$, and C depends only on the genus. Then, in view of the definition of extremal length (equation (4)), for any area 1 holomorphic quadratic differential q on $Y \in B_{\mathcal{T}}(X, L)$,

$$\ell_q(\alpha_i)^2 \leq \text{Ext}_Y(\alpha_i) \leq C L_1^2 a_i, \quad (12)$$

where $\ell_q(\cdot)$ denotes length in the flat metric defined by q , and C depends only on the genus. Thus any flat metric in the conformal class of a surface in $B_{\mathcal{T}}(X, L)$ has closed curves of flat length at most $L_1 \sqrt{C a_i}$.

Let \mathcal{F} be a fundamental domain for the action of Γ_g on \mathcal{T}_g . For $k \leq 3g - 3$ and $0 \leq c_1 \leq c_2 \leq \dots \leq c_k \leq 1$, let $Q(c_1, \dots, c_k)$ denote the subset of $X \in \mathcal{T}_g$ for which there exist disjoint curves β_1, \dots, β_k with $\text{Ext}_X(\beta_i) \leq c_i$. Then, for any area 1 holomorphic quadratic differential $q \in \pi^{-1}(X)$, the length of α_i in the flat metric induced by q is at most $\sqrt{c_i}$. Then, by the definition of the measure $\mathbf{m}(\cdot)$,

$$\mathbf{m}(Q(c_1, \dots, c_k) \cap \mathcal{F}) \leq C \prod_{i=1}^k c_i. \quad (13)$$

In view of (12),

$$B_{\mathcal{T}}(X, L) \subset Q(CL_1^2 a_1, \dots, CL_1^2 a_k).$$

Then, in view of (13), for any $\mathfrak{g} \in \Gamma_g$,

$$\mathbf{m}(B_{\mathcal{T}}(X, L) \cap \mathfrak{g}\mathcal{F}) \leq C \prod_{i=1}^k CL_1^2 a_i. \quad (14)$$

Let $I_{X,L}$ denote the set of elements $\mathfrak{g} \in \Gamma_g$ such that $B_{\mathcal{T}}(X, L) \cap \mathfrak{g}\mathcal{F}$ is non-empty. We will now estimate the size of $I_{X,L}$.

Note that as \mathcal{M}_g is finitely covered by a manifold, (11) implies that

$$|I_{X,L}| \leq C_L G(X)^2,$$

where

$$G(X) \approx \prod_{i=1}^k \frac{1}{\sqrt{a_i}},$$

and C_L only depends on L and g . Also, See Theorem 7.2 (for the case of $\mathcal{B} = \emptyset$).

On the other hand, by Theorem 2.4, the number of twists around α_i which one can take and still stay in $B_{\mathcal{T}}(X, L)$ is $\approx L_1^2/a_i$. Thus,

$$|I_{X,L}| \approx C'_L \prod_{i=1}^k \frac{1}{a_i}. \quad (15)$$

See §2.5. The upper bound of Lemma 3.1 follows from (14).

We now briefly outline the proof of the lower bound. Let $R \subset \mathcal{Q}^1\mathcal{M}_g/\Gamma_g$ be the set of flat structures such that each $S \in R$ has flat cylinders C_i with width (i.e. core curve) between $\sqrt{a_i}$ and $\sqrt{a_i}/2$ and height between $1/(g\sqrt{a_i})$ and $1/(2g\sqrt{a_i})$, and the rest of the arcs in a triangulation of $S \in R$ have length comparable to 1. The period coordinates give the structure of a piecewise linear integral manifold on $\mathcal{Q}^1\mathcal{M}_g$. By work of Masur [Mas], (up to normalization) the measure μ_g is the canonical measure defined by this piecewise linear integral structure (See §2.3). Hence, by the definition of the measures μ_g and \mathbf{m} we get

$$\mathbf{m}(\pi(R)) \geq \mu(R) > c \prod_{i=1}^k a_i, \quad (16)$$

where c depends only on the genus. Note that by [Ra2] for any $Y \in \pi(R)$, the only short simple closed curves on Y (in the hyperbolic or extremal metric) are the core curves of the cylinders C_i , and the extremal length of the core curve of C_i is within a constant multiple of a_i . Thus, there exists a constant L' depending only on the genus such that

$$\pi(R) \subset B_{\mathcal{T}}(X, L').$$

Note that the above equation takes place in \mathcal{T}_g/Γ_g . We may think of it as taking place in \mathcal{T}_g if we identify $\pi(R)$ with a subset of the fundamental domain \mathcal{F} . Then, for any $\mathbf{g} \in I_{X,L'}$,

$$\mathbf{g}\pi(R) \subset B_{\mathcal{T}}(X, 2L').$$

Thus, in view of (15) and (16),

$$\mathbf{m}(B_{\mathcal{T}}(X, 2L')) \geq |I_{X,L'}| \mathbf{m}(\pi(R)) \geq c,$$

where c depends only on the genus. □

3.2 Choosing a net in Teichmüller space

Let $c > 2L_0$ (the constant in Lemma 3.1), $\tilde{\mathcal{N}}$ be a $(c, 2c)$ -net in \mathcal{M}_g , and $\mathcal{N} \subset \mathfrak{p}^{-1}(\tilde{\mathcal{N}})$ be a net in \mathcal{T}_g such that $\mathfrak{p}(\mathcal{N}) = \tilde{\mathcal{N}}$. As before, $\mathfrak{p} : \mathcal{T}_g \rightarrow \mathcal{M}_g$ is the natural projection to the moduli space. In other words,

1. given $X \in \mathcal{T}_g$ there exists $Y \in \mathcal{N}$ such that $d_{\mathcal{T}}(X, Y) \leq 2c$, and
2. for any $Y_1 \neq Y_2 \in \mathcal{N}$, we have $d_{\mathcal{T}}(Y_1, Y_2) \geq c$.

In view of Lemma 3.1, for any $(c, 2c)$ separated net \mathcal{N} in any Teichmüller space (including \mathbb{H}^2), for $\tau \gg 1$ and any X ,

$$c_1 |B_{\mathcal{T}}(X, \tau) \cap \mathcal{N}| \leq \mathbf{m}(B_{\mathcal{T}}(X, \tau)) \leq c_2 |B_{\mathcal{T}}(X, \tau) \cap \mathcal{N}|, \quad (17)$$

where c_1, c_2 depend only on c and the genus.

Lemma 3.2 *Let $c > 2L_0$. Then for any $(c, 2c)$ net $\tilde{\mathcal{N}}$ in \mathcal{M}_g , there exists a constant $C_2 > 0$ such that for any $X \in \mathcal{T}_g$*

$$|\mathfrak{p}(B_{\mathcal{T}}(X, \tau)) \cap \tilde{\mathcal{N}}| \leq C_2 \tau^{3g-3}. \quad (18)$$

Proof. Fix

$$\tau_0 = \log\left(\frac{C_g^2}{\epsilon_0^2}\right)$$

(see §2.4).

Step 1. First we assume that $\tau \geq \tau_0$. Let $\mathcal{A}_X = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$ be the set of simple closed curves of extremal length $\leq \epsilon_0^2 e^{-2\tau}$ on $X \in \mathcal{T}_g$. Let $\mathcal{P}_1, \dots, \mathcal{P}_m$ be the set of all combinatorially distinct pants decompositions of S_g containing $\alpha_1, \dots, \alpha_k$.

Given a pants decomposition $\mathcal{P}_j = \{\alpha_1, \dots, \alpha_k, \alpha_{k+1}^j, \dots, \alpha_{3g-3}^j\}$ and $\mathbf{i} = (i_1, \dots, i_{3g-3}) \in \mathbb{Z}^{3g-3}$ choose $Z_{j,\mathbf{i}} \in \mathcal{T}_g$ such that:

- for each $1 \leq l \leq k$

$$e^{i_l-1} \leq \frac{\text{Ext}_{\alpha_l}(Z_{j,\mathbf{i}})}{\text{Ext}_{\alpha_l}(X)} \leq e^{i_l},$$

- for $k+1 \leq l \leq 3g-3$

$$C_g e^{i_l-1} \leq \text{Ext}_{\alpha_l^j}(Z_{j,\mathbf{i}}) \leq C_g e^{i_l}.$$

Let

$$\mathcal{I} = \{(i_1, \dots, i_{3g-3}) \in \mathbb{Z}^{3g-3} \mid \text{for } 1 \leq l \leq k, -2\tau \leq i_l \leq 2\tau, \text{ and for } k+1 \leq l \leq 3g-3, -5\tau+1 \leq i_l \leq 0\}.$$

Define the set \mathcal{Z} (with $|\mathcal{Z}| = O(\tau^{3g-3})$) by

$$\mathcal{Z} = \{Z_{j,(i_1, i_2, \dots, i_{3g-3})} \mid \mathbf{i} \in \mathcal{I}\} \subset \mathcal{T}_g.$$

Claim. Given $Y \in B_{\mathcal{T}}(X, \tau)$, there exists $Z_{j,\mathbf{i}} \in \mathcal{Z}$ such that $\mathfrak{p}(Z_{j,\mathbf{i}}) \in \mathfrak{p}(B_{\mathcal{T}}(Y, C))$, where C is a constant which only depends on g . This is a simple corollary of Theorem 2.4 and Theorem 2.2. Note that given $Y \in B_{\mathcal{T}}(X, \tau)$ and $\alpha \in \mathcal{A}_X$, α is a short simple closed curve on Y and

$$e^{-2\tau} \leq \frac{\text{Ext}_{\alpha}(Y)}{\text{Ext}_{\alpha}(X)} \leq e^{2\tau}.$$

Moreover, for any other simple close curve β in a bounded pants decomposition of Y since $d_{\mathcal{T}}(X, Y) < \tau$, if $\beta \notin \mathcal{A}_X$ then

$$\epsilon_0^2 e^{-4\tau} \leq \text{Ext}_{\beta}(Y) \leq C_g.$$

Now since any bounded pants decomposition \mathcal{P}_Y on Y has the combinatorial type of \mathcal{P}_j for some $1 \leq j \leq m$, the claim follows easily from Theorem 2.4. \square

As a result,

$$\mathfrak{p}(B_{\mathcal{T}}(X, \tau)) \cap \tilde{\mathcal{N}} \subset \bigcup_{Z_i \in \mathcal{Z}} (\mathfrak{p}(B_{\mathcal{T}}(Z_i, C)) \cap \tilde{\mathcal{N}}). \quad (19)$$

Step 2. Since $\mathcal{N} \subset \mathcal{T}_g$ is a $(c, 2c)$ net with $c > 2L_0$, by Lemma 3.1 for any $Z \in \mathcal{T}_g$

$$|B_{\mathcal{T}}(Z, C) \cap \mathcal{N}| \leq \frac{\text{Vol}(B_{\mathcal{T}}(Z, C + c))}{\text{Vol}(B_{\mathcal{T}}(Z, c))} = O(1),$$

and hence

$$|\mathfrak{p}(B_{\mathcal{T}}(Z, C)) \cap \tilde{\mathcal{N}}| = O(1). \quad (20)$$

Now since $|\mathcal{Z}| = O(\tau^{3g-3})$ the result follows from (19) and (20).

Note that τ_0 only depends on g , and therefore if $\tau \leq \tau_0$ the result follows from step 2. \square

4 Geodesics in the thin part of moduli space

In this section we prove Theorem 1.2. The main idea, which is due to Margulis, is to prove a system of inequalities, which shows that the flow (or more precisely an associated random walk) is biased toward the thick part of Teichmüller space. Variations on this theme have been used in [EMM], [EM] and [Ath].

4.1 A system of inequalities

Suppose $0 < s < 1$ (in fact we will be using $s = 1/2$ only). Let $\tau \gg 1$ be a parameter to be chosen later. (In particular we will assume $e^{-(1-s)\tau} < 1/2$.) Let A_{τ} be the operator of averaging over a ball of radius τ in \mathcal{T}_g with respect to the Teichmüller metric. So if f is a real-valued function on Teichmüller space, then

$$(A_{\tau}f)(X) = \frac{1}{\mathbf{m}(B_{\mathcal{T}}(X, \tau))} \int_{B_{\mathcal{T}}(X, \tau)} f(Y) d\mathbf{m}(Y).$$

Remark. In [EM], [Ath] and [EMM] the average is over spheres. In this context, we use balls, since Minsky's product region theorem gives us much more precise information about balls than about spheres.

Let m be the maximal number of disjoint simple closed curves on a closed surface of genus g . Choose $\mathbf{C} > e^{2m\tau}$, and pick constants $\epsilon_1 < \epsilon_2 < \dots < \epsilon_m < 1/\mathbf{C}^3$ such that for all $1 \leq i \leq m-1$,

$$\epsilon_i < \frac{\epsilon_{i+1}}{\mathbf{C}^2(2m\mathbf{C}^2)^{2/s}}. \quad (21)$$

Note that \mathbf{C} and $\epsilon_1, \dots, \epsilon_m$ are constants which depend only on τ and the genus g .

For $1 \leq i \leq m$ and $X \in \mathcal{T}_g$ let $E_i(X)$ denote the extremal length of the i 'th shortest simple closed curve on X . Let $f_0 = 1$ and for $1 \leq j \leq m$ let

$$f_j(X) = \prod_{1 \leq i \leq j} \left(\frac{\epsilon_i}{E_i(X)} \right)^s.$$

Note that f_j is invariant under the action of the mapping class group, and thus descends to a function on \mathcal{M}_g . Let

$$u(X) = \sum_{k=1}^m f_k(X). \quad (22)$$

Let $\epsilon'_j = \epsilon_j/(m\mathbf{C}^2)$. Let

$$W_j = \{X \in \mathcal{T}_g : E_{j+1}(X) > \epsilon'_j\}.$$

Note that W_0 is compact, and on W_j there are at most j short simple closed curves. If $X \notin W_{j-1}$ then X has at least j short simple closed curves, and thus if $X \in W_j \setminus W_{j-1}$ then X has exactly j short simple closed curves.

In this subsection, we prove the following:

Proposition 4.1 *Set $s = 1/2$. Then we may write*

$$(A_\tau u)(X) \leq c(X)u(X) + b(X), \quad (23)$$

where $b(X)$ is a bounded function which vanishes outside the compact set W_0 , and for all j and for all $X \notin W_{j-1}$,

$$c(X) \leq C'_j \tau^j e^{-j\tau},$$

where C'_j depends only on the genus.

We now begin the proof of Proposition 4.1. If $\alpha_1, \dots, \alpha_j$ are disjoint simple closed curves let $P(\alpha_1, \dots, \alpha_j)$ denote the product region where for all $1 \leq i \leq j$, the extremal length of α_i is at most $\mathbf{C}\epsilon_i$.

Lemma 4.2 *Suppose $1 \leq j \leq m$, $1/2 \leq s < 1$, and suppose X and τ are such that $B_\mathcal{T}(X, \tau)$ is completely contained in $P(\alpha_1, \dots, \alpha_j)$. Also suppose that for all $Y \in B_\mathcal{T}(X, \tau)$, the set $\{\alpha_1, \dots, \alpha_j\}$ coincides with the set of the j shortest curves on Y . Then*

$$(A_\tau f_j)(X) \leq c_j f_j(X),$$

where for $1/2 < s < 1$,

$$c_j = C_j(s) e^{-2j(1-s)\tau}. \quad (24)$$

and if $s = 1/2$, $c_j = C_j \tau^j e^{-j\tau}$.

Sketch of proof. We will use Minsky's product region theorem as stated in Theorem 2.3. Let \mathbf{m}' denote the product measure on $(\mathbb{H}^2)^j \times \mathcal{T}'$, and let A'_τ denote the averaging operator with respect to the product measure \mathbf{m}' , i.e. for a real-valued function f ,

$$(A'_\tau f)(X) = \frac{1}{\mathbf{m}'(B_\mathcal{T}(X, \tau))} \int_{B_\mathcal{T}(X, \tau)} f(Y) d\mathbf{m}'(Y).$$

We first establish the lemma with A_τ replaced by A'_τ .

We may write $X = (X_1, \dots, X_j, X')$ where X_k is in the k 'th copy of the hyperbolic plane and $X' \in \mathcal{T}'$. Because of the product region theorem $B_\mathcal{T}(X, \tau)$ is essentially $B_\mathcal{T}(X_1, \tau) \times \dots \times B_\mathcal{T}(X_j, \tau) \times B'(X', \tau)$ where for $1 \leq k \leq j$, $B_\mathcal{T}(X_k, \tau)$ is a ball of radius τ in the hyperbolic plane and $B'(X', \tau)$ a ball of radius τ in \mathcal{T}' . Also, by assumption, for any $Y \in B(X, \tau)$ the set of j shortest simple closed curves on Y is $\{\alpha_1, \dots, \alpha_j\}$. Thus, for $Y \in B(X, \tau)$, with $Y = (Y_1, \dots, Y_j, Y')$, we have

$$f_j(Y) \approx (\epsilon_1 \dots \epsilon_j)^s \prod_{k=1}^j \ell_{\min}(Y_k)^{-2s}$$

where for $Y_k \in \mathbb{H}^2$, $\ell_{min}(Y_k)$ is the flat length of the shortest simple closed curve in the torus parametrized by Y_k . We remark that the exponent is $2s$ instead of s because on the torus extremal length is the square of flat length. As a result, we have

$$(A'_\tau f_j)(X) \approx (\epsilon_1 \dots \epsilon_j)^s \prod_{k=1}^j \left(\frac{1}{\text{Vol}(B(X_k, \tau))} \int_{B(X_k, \tau)} \ell_{min}(Y)^{-2s} d\text{Vol}(Y) \right), \quad (25)$$

where Vol is the standard volume form on \mathbb{H}^2 . Now the integral in the parenthesis, i.e. an average of ℓ^{-2s} over a ball in a hyperbolic plane is essentially done in [EM, Lemma 7.4] (except that there the average is over spheres, but to get the average over balls one just makes an extra integral over the radius). One gets for $1/2 < s < 1$,

$$\frac{1}{\text{Vol}(B(X, \tau))} \int_{B(X, \tau)} \ell_{min}(Y)^{-2s} d\text{Vol}(Y) \leq c(s) e^{-2(1-s)\tau} \ell_{min}(X)$$

and for $s = 1/2$,

$$\frac{1}{\text{Vol}(B(X, \tau))} \int_{B(X, \tau)} \ell_{min}(Y)^{-1} d\text{Vol}(Y) \leq c' \tau e^\tau \ell_{min}(X).$$

Substituting these expressions into (25) completes the proof of Lemma 4.2 with A'_τ instead of A_τ . In view of the form of the function f_j ,

$$c_1 \sum_{B(X, \tau) \cap \mathcal{N}} f_j(Y) \leq \int_{B(X, \tau)} f_j(Y) d\mathbf{m}(Y) \leq c_2 \sum_{B(X, \tau) \cap \mathcal{N}} f_j(Y). \quad (26)$$

Choose $L \gg b$ (where b is as in Theorem 2.3), and choose a $(L, 2L)$ -separated net \mathcal{N}_k in each factor. Let \mathcal{N} be the product of the \mathcal{N}_k . Then \mathcal{N} is an $(L - b, 2L + b)$ -separated net in $P(\alpha_1, \dots, \alpha_j)$. Now in view of (17),

$$\mathbf{m}(B(X, \tau)) \approx |B(X, \tau) \cap \mathcal{N}| \approx \prod_{k=1}^j |B(X_k, \tau) \cap \mathcal{N}_k| \approx \prod_{k=1}^j \mathbf{m}'(B(X_k, \tau)) \approx \mathbf{m}'(B(X, \tau)),$$

where, as before, $A \approx B$ means that the ratio A/B is bounded by two constants depending only on b , L and g , and thus ultimately only on g . Similarly, using (26), we can show that

$$\int_{B(X, \tau)} f_j(Y) d\mathbf{m}(Y) \approx \int_{B(X, \tau)} f_j(Y) d\mathbf{m}'(Y).$$

Thus $(A'_\tau f_j)(X) \approx (A_\tau f_j)(X)$. □

Remark. The proof works even if at some point $Y \in B(X, \tau)$ there are short simple closed curves other than $\{\alpha_1, \dots, \alpha_j\}$ (but these other curves are longer than the maximum of the lengths of the α_j at Y). This is used in the next lemma.

Lemma 4.3 *For $1 \leq j \leq m$, let $u_j(X) = \sum_{k=j}^m f_j(X)$. Suppose $E_j(X) < \epsilon_j$. Then (assuming τ is large enough),*

$$(A_\tau u_j)(X) \leq \left(c_j + \frac{1}{2\mathbf{C}} \right) u_j(X), \quad (27)$$

where c_j are as in (24). In particular, letting $j = 1$, and noting that the set $\{X \in \mathcal{M}_g : E_1(X) > \epsilon_1\}$ is compact, we have for all $X \in \mathcal{T}_g$,

$$(A_\tau u_1)(X) \leq \left(c_1 + \frac{1}{2\mathbf{C}}\right) u_1(X) + b(\tau).$$

Proof. Note that for any $1 \leq i \leq m$, and any $X \in \mathcal{T}_g$,

$$\frac{1}{\mathbf{C}} f_i(X) \leq (A_\tau f_i)(X) \leq \mathbf{C} f_i(X) \quad (28)$$

(this is because in $B(X, \tau)$ the extremal length of any simple closed curve cannot change by more than $e^{2\tau}$).

We divide the set $\{j, j+1, \dots, m\}$ into two disjoint subsets: Let I_1 be the set of $k \in \{j, j+1, \dots, m\}$ such that

$$f_k(X) \leq \frac{u_j(X)}{2m\mathbf{C}^2}, \quad (29)$$

and let I_2 be the set of $k \in \{j, j+1, \dots, m\}$ such that the opposite inequality to (29) holds. Suppose $k \in I_1$. Then, by (28), $(A_\tau f_k)(X) \leq \frac{1}{2m\mathbf{C}} f_k(X)$, and thus

$$\sum_{k \in I_1} (A_\tau f_k)(X) \leq \frac{1}{2m\mathbf{C}} u_j(X). \quad (30)$$

Now suppose $k \in I_2$. We claim that

$$E_k(X) < (2m\mathbf{C}^2)^{1/s} \epsilon_k. \quad (31)$$

Indeed, if $k = j$ then (31) is true by assumption. If $k > j$ then

$$u_j(X) \geq f_{k-1}(X) = f_k(X) \left(\frac{E_k(X)}{\epsilon_k}\right)^s \geq \frac{u_j(X)}{2m\mathbf{C}^2} \left(\frac{E_k(X)}{\epsilon_k}\right)^s,$$

where we have used the inequality opposite to (29) in the last estimate. Thus (31) follows.

We now claim that under the assumption that $k \in I_2$ we have

$$E_{k+1}(X) \geq \mathbf{C}^2 E_k(X). \quad (32)$$

If $k = m$ this is clear from (31) (since in the case where $E_m(X)$ is small, there are no other short simple closed curves on X). Now if $k < m$, then

$$u_j(X) \geq f_{k+1}(X) = f_k(X) \left(\frac{\epsilon_{k+1}}{E_{k+1}(X)}\right)^s \geq \frac{u_j(X)}{2m\mathbf{C}^2} \left(\frac{\epsilon_{k+1}}{E_{k+1}(X)}\right)^s$$

where again we used the inequality opposite to (29) in the last estimate. Thus,

$$E_{k+1}(X) \geq \frac{\epsilon_{k+1}}{(2m\mathbf{C}^2)^{1/s}}. \quad (33)$$

Now (32) follows from (33), (31) and (21).

Now in view of (31) and (32), Lemma 4.2 can be applied to f_k . Thus, for $k \in I_2$,

$$(A_\tau f_k)(X) \leq c_k f_k(X) \leq c_j f_k(X) \quad (34)$$

where for the last inequality we assumed that τ was large enough so that $c_k < c_j$ for $k > j$. Now (27) follows from (30) and (34). \square

Proof of Proposition 4.1. Suppose $X \notin W_{j-1}$. We may write

$$u(X) = u_j(X) + \sum_{k=1}^{j-1} f_k(X).$$

Note that for $X \notin W_{j-1}$ and $k < j$, $f_k(X) \leq u_j(X)/(m\mathbf{C}^2)$. Hence, (23) follows from (27) and (28). \square

4.2 A uniform estimate for the measure of a ball

Given X and Y in \mathcal{T}_g and $\tau > 0$, let

$$F_\tau(X, Y) = \{Z \mid Z \in \Gamma_g \cdot Y, d_{\mathcal{T}}(X, Z) \leq \tau\} \subset \mathcal{T}_g.$$

In the Appendix we show:

Proposition 4.4 *Given $\epsilon > 0$, there exists $\tau_0 > 0$ such that for any $\tau > \tau_0$ and $X, Y \in \mathcal{T}_g$ we have*

$$|F_\tau(X, Y)| \leq e^{(h+\epsilon)\tau} G(Y)^2.$$

In order to prove this statement we show that in general

$$|F_\tau(X, Y)| = O(\tau^{3g-3} e^{h\tau} G(X)G(Y)). \quad (35)$$

Using these estimates we show that:

Proposition 4.5 *There exists a constant C_2 such that for any X , any $\delta'' > 0$ and any sufficiently large τ , the volume of any $B(X, \tau)$ is bounded by $C_2 e^{(h+\delta'')\tau}$.*

Proof. See Appendix. \square

Remark. Lemma 7.2 in the appendix is a stronger version of (35). The proof of this lemma uses Theorem 2.4. This statement can be generalized for other strata of moduli space of quadratic differentials.

4.3 Proof of Theorem 1.2

As in 3.2, we discretize Teichmüller space by fixing a $(c, 2c)$ separated net $\mathcal{N} \subset \mathcal{T}_g$. We note that there exists a constant κ_0 such that for all $X \in \mathcal{T}_g$,

$$\frac{1}{\kappa_0} u(X) \leq G(X) \leq \kappa_0 u(X),$$

where κ_0 depends on τ and the ϵ_i (and thus ultimately only on τ and the genus). Also, as in (6),

$$G(X) = \prod_{1 \leq i \leq m} E_i(X)^{-1/2}.$$

Trajectories of the random walk. Suppose $R \gg \tau$ and let n be the integer part of R/τ . By a *trajectory of the random walk* we mean a map $\lambda : \{0, n-1\} \rightarrow \mathcal{T}_g$ such that :

- for all $0 < k \leq n-1$ we have $d(\lambda_k, \lambda_{k-1}) \leq \tau$, and
- λ_k belongs to the net \mathcal{N} .

Let $\mathcal{P}_\tau(X, R)$ denote the set of all trajectories for which $d(\lambda_0, X) = O(1)$. It is a corollary of Proposition 4.5 that

$$|\mathcal{P}_\tau(X, R)| \leq C_2 e^{(h+\delta'')R}, \quad (36)$$

where $|\cdot|$ denotes the cardinality of a set.

We say that a trajectory is *almost closed in the quotient* if the distance in \mathcal{M}_g between the projection to \mathcal{M}_g of $\lambda(0)$ and the projection to \mathcal{M}_g of λ_{n-1} is $O(1)$.

Let $\delta > 0$ be a constant to be chosen later. (We will have $\delta < \epsilon'_j$ for $1 \leq j \leq m$, where the ϵ'_j are as in §4.1). For $j \in \mathbb{N}$, let $\mathcal{P}_{j,\tau}(X, \delta, R)$ denote the set of all trajectories starting within $O(1)$ of X for which at any point, there are at least j simple closed curves of length at most δ . Let $\tilde{\mathcal{P}}_j(X, \delta, R)$ denote the subset of these trajectories which are almost closed in the quotient. We use the systems of inequalities discussed in §4.1 to show:

Lemma 4.6 *Let $j \in \mathbb{N}$. For any $\epsilon' > 0$, there exists $\tau_0 > 0$ such that for $\tau > \tau_0$, $\delta > 0$ small enough (depending on τ , ϵ' and g), and any sufficiently large R (depending only on ϵ' and τ) we have:*

$$|\tilde{\mathcal{P}}_{j,\tau}(X, \delta, R)| \leq e^{(h-j+\epsilon')R}. \quad (37)$$

Proof. By the definition of u for $s = 1/2$ (see §4.1), it is easy to check that for any $Y \in \mathcal{T}_g$

$$\frac{1}{\kappa_0} u(Y) \leq G(Y) \leq \kappa_0 u(Y), \quad (38)$$

where

$$\kappa_0 \leq e^{m_0\tau}.$$

Here m_0 is a universal constant independent of τ and Y . Also, by the definition of the function u if $d_{\mathcal{T}}(X_1, X_2) = O(1)$, then

$$\frac{u(X_1)}{u(X_2)} = O(1). \quad (39)$$

Note that if $\lambda \in \tilde{\mathcal{P}}_j(X, \delta, R)$ then $G(X) \approx G(\lambda_{n-1}) \leq \kappa_0 u(\lambda_{n-1})$. Let $R = n\tau$, and let

$$q_j(X, R) = \sum_{\lambda \in \mathcal{P}_j(X, \delta, R)} u(\lambda_{n-1})$$

Therefore,

$$|\tilde{\mathcal{P}}_j(X, \delta, R)| \leq \frac{C_1}{G(X)} \sum_{\lambda \in \tilde{\mathcal{P}}_j(X, \delta, R)} \kappa_0 u(\lambda_{n-1}) = \frac{C_1 \kappa_0}{G(X)} q_j(X, R), \quad (40)$$

where $C_1 = O(1)$. For $0 < r = k\tau < R$ let

$$q_j(X, R, r) = \sum_{\lambda \in \mathcal{P}_j(X, \delta, R, r)} u(\lambda_{k-1}),$$

where the elements of $\mathcal{P}_j(X, \delta, R, r)$ are the trajectories λ belonging to $\mathcal{P}_j(X, \delta, R)$ but truncated after $k = r/\tau$ steps. Then

$$\begin{aligned} q_j(X, R, r + \tau) &= \sum_{\lambda \in \mathcal{P}_j(X, \delta, R, r + \tau)} u(\lambda_k) \\ &= \sum_{\lambda \in \mathcal{P}_j(X, \delta, R, r)} \sum_{\lambda_k \in \mathcal{N} \cap B(\lambda_{k-1}, \tau)} u(\lambda_k) \\ &\leq C \sum_{\lambda \in \mathcal{P}_j(X, \delta, R, r)} \int_{B(\lambda_{k-1}, \tau)} u(Y) \, d\mathbf{m}(y) \\ &= C \sum_{\lambda \in \mathcal{P}_j(X, \delta, R, r)} \mathbf{m}(B(\lambda_k, \tau)) (A_\tau u)(\lambda_{k-1}) \end{aligned} \quad (41)$$

where in the next to last line we use (39) and Lemma 3.1 to estimated a sum over $\mathcal{N} \cap B(\lambda_{k-1}, \tau)$ by a constant C times an integral over $B(\lambda_{k-1}, \tau)$.

Note that for $\lambda \in \mathcal{P}_j(X, \delta, R)$, the number of simple closed curves shorter then δ on λ_{k-1} is at least j . Thus, if δ is small enough, (depending on the the ϵ'_j and thus ultimately only on τ and the genus), $\lambda_{k-1} \notin W_{j-1}$. Then, from Proposition 4.1, and assuming τ is large enough so that Proposition 4.5 holds with $\delta'' < \epsilon'/3$, we get

$$q_j(X, R, r + \tau) \leq CC_2 \sum_{\lambda \in \mathcal{P}_j(X, \delta, R, r)} e^{(h+\epsilon'/3)\tau} C'_j \tau e^{-j\tau} u(\lambda_{k-1}) = CC'_j C_2 \tau e^{(h+(\epsilon'/3)-j)\tau} q_j(X, R, r). \quad (42)$$

Now iterating (42) $n = R/\tau$ times we get

$$q_j(X, R) \leq u(X) (CC_2 C'_j)^n \tau^n e^{(h-j+\epsilon'/3)n\tau} = u(X) e^{(h-j+\epsilon'/3+\frac{\log(CC_2 C'_j \tau)}{\tau})R}. \quad (43)$$

Hence, in view of (40) and (38), we get

$$|\tilde{\mathcal{P}}_j(X, \delta, R)| \leq C_1 e^{2m_0\tau} \times e^{(h-j+\epsilon'/3+\frac{\log(CC_2 C'_j \tau)}{\tau})R} \leq C_1 e^{(h-j+\epsilon'/3+\frac{\log(CC_2 C'_j \tau)}{\tau}+2m_0 \frac{\tau}{R})R}. \quad (44)$$

Now it is enough choose τ_0 so that $\frac{\log(CC_2 C'_j \tau_0)}{\tau_0} < \epsilon'/3$. Then for any $\tau \geq \tau_0$ and $R \geq \frac{6m_0\tau+3\log(C_1)}{\epsilon'}$ (44) implies (37). \square

Remark. One can use Lemma 7.2 to prove Lemma 4.6.

Let $N_j(X, \delta, R)$ be the number of conjugacy classes of closed geodesics of length at most R which pass within $O(1)$ of the point X and always have at least j simple closed curves of length at most δ .

Lemma 4.7 For any $\epsilon' > 0$ we may choose τ large enough (depending only on ϵ') so that for any $X \in \mathcal{T}_g$, any $\delta < 1/c_0$ and any sufficiently large R (depending only on ϵ' and τ) we have

$$N_j(X, \delta, (1 - \epsilon')R) < C|\tilde{\mathcal{P}}_j(X, c_0\delta, R)|, \quad (45)$$

where C and c_0 are constants which only depend on \mathcal{N} and g .

Proof. Let

$$I_X = \{\mathfrak{g} \in \Gamma_g \mid B_{\mathcal{T}}(X, \mathfrak{g} \cdot X) \leq 1\}.$$

In other words, I_X is the subset of the mapping class group which moves X by at most $O(1)$. In fact, $|I_X| \approx G(X)^2$ (See also Theorem 7.2).

Now consider a closed geodesic γ in \mathcal{M}_g which passes within $O(1)$ of $\mathfrak{p}(X)$ (recall that \mathfrak{p} denotes the projection map $\mathfrak{p} : \mathcal{T}_g \rightarrow \mathcal{M}_g$). Let $[\gamma]$ denote the corresponding conjugacy class in Γ_g . Then there are approximately $|I_X|$ lifts of $[\gamma]$ to \mathcal{T}_g which start within $O(1)$ of X . Each lift $\tilde{\gamma}$ is a geodesic segment of length equal to the length of γ .

We can mark points distance τ apart on $\tilde{\gamma}$, and replace these points by the nearest net points. (This replacement is the cause of the ϵ'). This gives a map Ψ from lifts of geodesics to trajectories. If the original geodesic γ has length at most $(1 - \epsilon')R$ and always has j simple closed curves shorter than δ , then by Theorem 2.2 the resulting trajectory belongs to $\tilde{\mathcal{P}}_j(X, c_0\delta, R)$, where c_0 only depends on \mathcal{N} .

If two geodesic segments map to the same trajectory, then the segments fellow travel within $O(1)$ of each other. In particular if \mathfrak{g}_1 and \mathfrak{g}_2 are the pseudo-Anosov elements corresponding to the two geodesics, then $d_{\mathcal{T}}(\mathfrak{g}_2^{-1}\mathfrak{g}_1X, X) = O(1)$, thus $\mathfrak{g}_2^{-1}\mathfrak{g}_1 \in I_X$.

We now consider all possible geodesics contributing to $N_j(X, \delta, (1 - \epsilon')R)$; for each of these we consider all the possible lifts which pass near X , and then for each lift consider the associated random walk trajectory. We get:

$$N_j(X, \delta, (1 - \epsilon')R)|I_X| \leq C|I_X||\tilde{\mathcal{P}}_j(X, c_0\delta, R)|$$

(the factor of $|I_X|$ on the left hand side is due to the fact that we are considering all possible lifts which pass near X , and the factor of $|I_X|$ on the right is the maximum possible number of times a given random walk trajectory can occur as a result of this process). Thus, the factor of $|I_X|$ cancels, and the lemma follows. \square

The following lemma is due to Veech [Ve].

Lemma 4.8 Suppose γ is a closed geodesic of length at most R on \mathcal{M}_g . Then for any $X \in \gamma$ the extremal length of the shortest simple closed curve on X is at least $\epsilon'_0 e^{-(6g-4)R}$, where ϵ'_0 depends only on g .

Proof. We reproduce the proof for completeness. Let \tilde{X} be some point in \mathcal{T}_g with $\mathfrak{p}(\tilde{X}) = X$. Suppose the estimate is false, and let α be a simple closed curve on \tilde{X} with extremal length less than $\epsilon'_0 e^{-(6g-4)R}$. Let \mathfrak{g} be the element of the mapping class group associated to the lift of γ passing through \tilde{X} . $\text{Ext}_{\alpha}(\tilde{X}) \leq \epsilon'_0 e^{-(6g-4)R}$. Therefore, by Theorem 2.2, for $j \in \mathbb{N}$,

$$\text{Ext}_{\mathfrak{g}^j\alpha}(\tilde{X}) = \text{Ext}_{\alpha}(\mathfrak{g}^{-j}\tilde{X}) \leq \epsilon'_0 e^{-(6g-4+2j)R}$$

In particular, for $1 \leq j \leq 3g - 2$, $\text{Ext}_{\mathfrak{g}^j \alpha}(\tilde{X}) \leq \epsilon'_0$. Therefore (assuming that ϵ'_0 is small enough), for $1 \leq j \leq 3g - 2$, the simple closed curves $\mathfrak{g}^j \alpha$ are disjoint. This is a contradiction. \square

Remark. Recall that by (5) for very short curves the hyperbolic and the extremal lengths are comparable. Therefore hyperbolic length of the shortest closed geodesic on $X \in \gamma$ is at least $\epsilon''_0 e^{-(6g-6)R}$, where ϵ''_0 depends only on g .

Proof of Theorem 1.2. Let $\epsilon' = \epsilon/8c_0$, where c_0 is the constant in Theorem 4.7. By Lemma 4.7 and Lemma 4.6 we can choose τ and δ so that (45) holds and also (37) holds with δ replaced by $c_0\delta$. We get, for sufficiently large R ,

$$N_j(X, \delta, R) < C' e^{(h-j+\epsilon'c_0/4)R}. \quad (46)$$

Finally $N_j(\delta, R)$ is at most $\sum_X N_j(X, \delta, R)$, where we have to let X vary over the net points within distance 1 of a fundamental domain for the action of the mapping class group. In view of Lemma 4.8, the number of relevant points in the net is at most polynomial in R . Thus Theorem 1.2 follows. \square

5 Recurrence of Geodesics

In this section, we discuss basic recurrence properties of closed geodesics in $\mathcal{Q}^1 \mathcal{M}_g$. Note that a Teichmüller geodesic γ is in fact a path in the space of unit area holomorphic quadratic differentials on surfaces of genus g . As in §2.2, let $\mathcal{Q}^1 \mathcal{M}_g(1, \dots, 1) \subset \mathcal{Q}^1 \mathcal{M}_g$ denote the *principal stratum*, i.e. the set of pairs (X, q) where q is a holomorphic quadratic differential on X with simple zeroes.

Notation. For a compact subset K of \mathcal{M}_g and a number $\theta > 0$ let $N^K(\theta, R)$ denote the number of closed geodesics γ of length at most R such that γ spends at least θ -fraction of the time outside K .

Similarly, for a compact subset \mathcal{K} of $\mathcal{Q}^1 \mathcal{M}_g(1, \dots, 1)$ and $\theta > 0$ we denote by $N^{\mathcal{K}}(\theta, R)$ the number of closed geodesics γ of length at most R such that γ spends at least θ -fraction of the time outside \mathcal{K} .

We prove the following theorems:

Theorem 5.1 *Suppose $\theta > 0$. Then there exists a compact subset K of \mathcal{M}_g and $\delta > 0$ such that for sufficiently large R ,*

$$N^K(\theta, R) \leq e^{(h-\delta)R}.$$

Theorem 5.2 *Suppose $\theta > 0$. Then there exists a compact subset $\mathcal{K} \subset \mathcal{Q}^1 \mathcal{M}_g(1, \dots, 1)$ and $\delta > 0$ such that for sufficiently large R ,*

$$N^{\mathcal{K}}(\theta, R) \leq e^{(h-\delta)R}.$$

Proof of 5.1. In view of Corollary 1.3, there exists a compact set K_1 , such that the number of geodesics which do not return to K_1 is $O(e^{(h-0.99)R})$. But then using Proposition 4.1 and [Ath, Corollary 2.7] (cf. [Ath, Theorem 2.3]), there exists a compact set K depending on K_1 , θ and there exists $\delta' > 0$, such that the number of random walk trajectories which start in K_1 and spend at least θ -fraction of the time outside of K is at most $e^{(h-\delta')R}$. (Note that even though [Ath, Corollary 2.7] is not stated with an exponential bound, the proof does in fact imply this, as is done in the statement and proof of [Ath, Theorem 2.3]). It follows that the same kind of estimate is true for the number of geodesics (see the proof of Lemma 4.7). \square

The rest of this subsection will consist of the proof of Theorem 5.2.

5.1 Choosing hyperbolic neighborhoods of points

Notation. Given $q \in \mathcal{Q}^1\mathcal{M}_g$ we let $\ell_{min}(q)$ denote the length of the shortest saddle connection on q in the flat metric defined by q . Suppose $K \subset \mathcal{M}_g$ is a compact set. For simplicity, we denote the preimage of K in \mathcal{T}_g by $\tilde{K} = \mathcal{P}^{-1}(K)$. Also, to simplify the notation, let $q_{X,Y} \in \mathcal{Q}^1(X)$ be the quadratic differential such that $g_{d_{\mathcal{T}}(X,Y)}(q) \in \mathcal{Q}^1(Y)$.

As in [ABEM, §2], we denote the strong unstable, unstable, stable and strong stable foliations of the geodesic flow by \mathcal{F}^{uu} , \mathcal{F}^u , \mathcal{F}^s and \mathcal{F}^{ss} respectively; for a given quadratic differential q ,

$$\mathcal{F}^{ss}(q) = \{q_1 \in \mathcal{Q}^1\mathcal{T}_g \mid \text{Re}(q_1) = \text{Re}(q)\}, \mathcal{F}^s(q) = \{q_1 \in \mathcal{Q}^1\mathcal{T}_g \mid [\text{Re}(q_1)] = [\text{Re}(q)]\},$$

and

$$\mathcal{F}^{uu}(q) = \{q_1 \in \mathcal{Q}^1\mathcal{T}_g \mid \text{Im}(q_1) = \text{Im}(q)\}, \mathcal{F}^u(q) = \{q_1 \in \mathcal{Q}^1\mathcal{T}_g \mid [\text{Im}(q_1)] = [\text{Im}(q)]\}.$$

We consider the distance function defined by the modified Hodge norm d_H on each horosphere \mathcal{F}^{ss} . This is closely related to the Hodge norm studied by Forni [Fo]. We also consider $d_E(\cdot, \cdot)$ the Euclidean metric on each horosphere as defined in [ABEM, §3.5]. In fact, the Euclidean norm on the tangent space of $\mathcal{Q}^1\mathcal{M}_g$ is defined using period coordinates (see §2.3). We remark that this norm depends on the choice of a triangulation on the surface. However, in a given compact subset of $\mathcal{Q}^1\mathcal{M}_g$ it is well-defined up to a multiplicative constant. So we can use it to measure the "distance" between two quadratic differentials in a compact subset of $\mathcal{Q}^1\mathcal{M}_g$. We will show:

Lemma 5.3 *Suppose $K \subset \mathcal{M}_g$ is compact. Given $\rho, \epsilon > 0$, and $1 > b > 0$, there exists $\rho_0 > 0$ (depending only on K , and b), and $L_0 = L_0(K, \epsilon, \rho, b)$ such that if $X, p_0 \in \tilde{K}$, $d_{\mathcal{T}}(p_0, p_1) < \rho_0$, $d_{\mathcal{T}}(X, p_1) = L > L_0$, and*

$$\{s \mid s \in [0, L], \ell_{min}(g_s(q_{X,p_0})) \geq \epsilon\} > bL,$$

then

$$d_E(q, q_{X,p_0}) < \rho,$$

where q is the unique quadratic differential in $\mathcal{F}^{uu}(q_{X,p_0}) \cap \mathcal{F}^s(q_{X,p_1})$.

Remark. Note that given $q_1, q_2 \in \mathcal{Q}^1\mathcal{T}_g$, $|\mathcal{F}^{uu}(q_1) \cap \mathcal{F}^s(q_2)| \leq 1$. In general, this set can be empty, but if q_2 is close enough to q_1 then $\mathcal{F}^{uu}(q_1) \cap \mathcal{F}^s(q_2) \neq \emptyset$. The proof of Lemma 5.3 implies that when ρ_0 is small and L_0 is large $\mathcal{F}^{uu}(q_{X,p_0}) \cap \mathcal{F}^s(q_{X,p_1}) \neq \emptyset$.

We will show:

Lemma 5.4 *Suppose $K \subset \mathcal{M}_g$ is compact. Given $\epsilon > 0$, there exists a constants L_0 , depending only on ϵ and K , and c_0 depending only on K with the following property.*

If

- $\gamma : [0, L] \rightarrow \mathcal{Q}^1\mathcal{T}_g$ is a geodesic segment (parametrized by arclength) such that $(\mathbf{p} \circ \pi)(\gamma(0)) \in K$, $(\mathbf{p} \circ \pi)(\gamma(L)) \in K$, and $L > L_0$,
- $\hat{\gamma} : [0, L'] \rightarrow \mathcal{Q}^1\mathcal{T}_g$ is the geodesic segment connecting $p_1, p_2 \in \mathcal{T}_g$ such that

$$d_{\mathcal{T}}(p_1, \pi(\gamma(0))) < c_0, \quad d_{\mathcal{T}}(p_2, \pi(\gamma(L))) < c_0,$$

and

•

$$|\{t \in [0, L] : \ell_{\min}(\gamma(t)) > \epsilon\}| > (1/2)L,$$

then

$$|\{t \in [0, L'] : \ell_{\min}(\hat{\gamma}(t)) > \epsilon/4\}| > (1/3)L'.$$

5.2 Hodge and Euclidean distance functions on the stable and unstable foliations

First, we briefly recall some useful decay properties of the Hodge and Euclidean distances proved in [ABEM, §3.4, §3.5]:

- **P1:** Given $\epsilon > 0$, there exists $\epsilon' > 0$ such that if $\ell_{\min}(q_1) \geq \epsilon$ and $d_E(q_1, q_2) \leq \epsilon'$ then $\ell_{\min}(q_2) \geq \epsilon/2$.
- **P2:** Assume that $\tilde{K}_1 \subset \mathcal{T}_g$ is the preimage of a compact subset $K_1 \subset \mathcal{M}_g$. Given $\rho_1 > 0$, there exists ρ_0 such that if $q_1 \in \mathcal{F}^{ss}(q_2)$, $\pi(q_1) \in \tilde{K}_1$, and $d_E(q_1, q_2) > \rho_1$, then $d_T(\pi(q_1), \pi(q_2)) > \rho_0$.
- **P3:** There exists $C_1 > 0$ such that if $d_H(q_1, q_2) < 1$, $q_1 \in \mathcal{F}^{ss}(q_2)$ and $s \geq 0$

$$d_E(g_s q_1, g_s q_2) \leq d_H(g_s q_1, g_s q_2) < C_1 d_H(q_1, q_2); \quad (47)$$

- **P4:** Moreover, given $\epsilon, b > 0$, there exist C_0, C'_0 and $a > 0$ such that for any $q_1 \in \mathcal{F}^{ss}(q_2)$ with $d_H(q_1, q_2) < 1$, and $s \geq 0$ if

$$|\{t \mid t \in [0, s], \ell_{\min}(g_t q_1) > \epsilon\}| > b s, \quad (48)$$

and $\pi(g_s q_1) \in \tilde{K}_1$, then

$$d_E(g_s q_1, g_s q_2) \leq C'_0 d_H(g_s q_1, g_s q_2) < C_0 e^{-as} d_H(q_1, q_2). \quad (49)$$

Note that in this case, by (49), there exists L_0 (depending only on \tilde{K}_1 , b and ϵ) such that for $s > L_0$, (48) implies that :

$$|\{t \mid t \in [0, s], \ell_{\min}(g_t q_2) > \epsilon/2\}| > \frac{b}{2}s. \quad (50)$$

Proof of Lemma 5.3. Let $B_E(q, r)$ denote the ball of radius r with center q in the Euclidean metric. Let $S_0 = S(X)$ denote the sphere at X , i.e. the set of unit area holomorphic quadratic differentials on the surface X , and $\gamma(t) = g_t(q_{X, p_0})$. For $q \in S_0$ near $\gamma(0) = q_{X, p_0}$, let

$$f(q) = \mathcal{F}^{uu}(q_{X, p_0}) \cap \mathcal{F}^s(q);$$

in other words, we can choose $t(q) \in \mathbb{R}$ be such that $f(q) \in \mathcal{F}^{uu}(q_{X, p_0})$ and $g_{t(q)} q$ and $f(q)$ are on the same leaf of \mathcal{F}^{ss} . Then clearly $f(q_{X, p_0}) = q_{X, p_0}$, and there exists a number $\rho_3 > 0$ depending only on K such that the restriction of f to $B_E(q_{X, p_0}, \rho_3) \cap S_0$ is a homeomorphism onto a neighborhood of

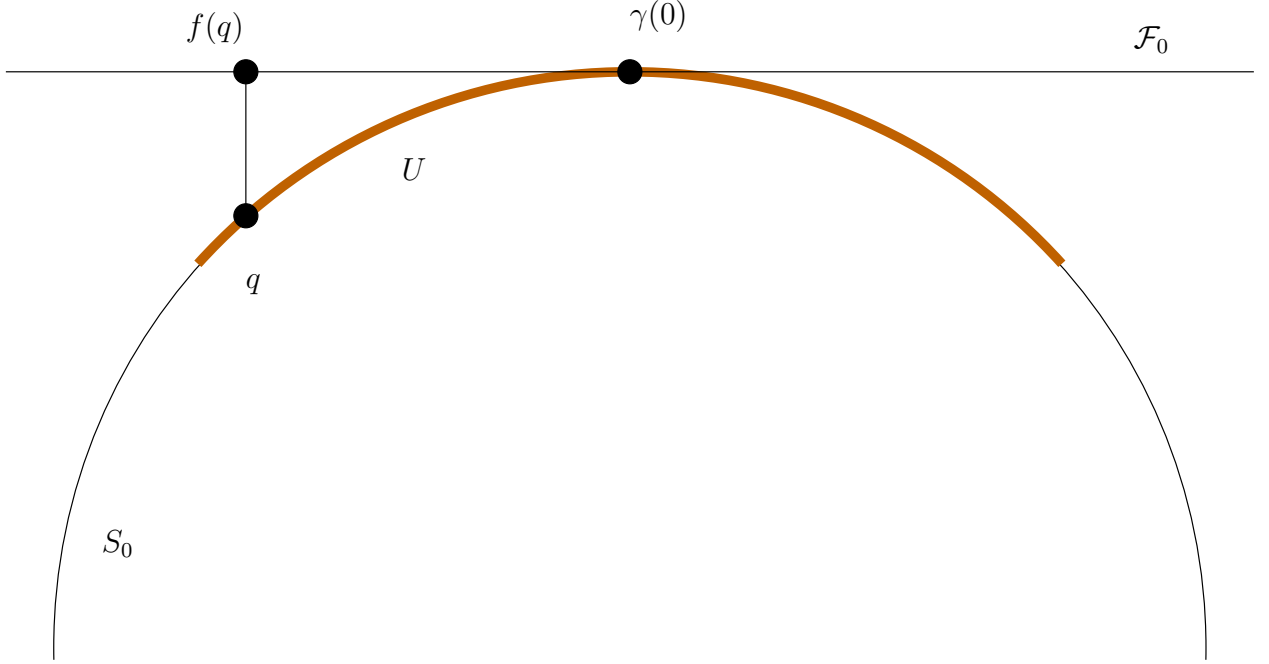


Figure 1. Proof of Lemma 5.3.

q_{X,p_0} in $\mathcal{F}^{uu}(q_{X,p_0})$. In particular, for any $q \in B_E(q_{X,p_0}, \rho_3) \cap S_0$, we know that $\mathcal{F}^{uu}(q_{X,p_0}) \cap \mathcal{F}^s(q) \neq \emptyset$. Let

$$U = \bigcup_{|t| < \rho_3} \{g_t q : q \in B_E(q_{X,p_0}, \rho_3/2) \cap S_0\}.$$

Claim: there exists $\rho_0 > 0$ depending only on K , and \tilde{L}_0 depending on ϵ and K such that for $L > \tilde{L}_0$,

$$\pi(g_L U) \supset B_{\mathcal{T}}(\pi(\gamma(L)), \rho_0). \quad (51)$$

This is straightforward (in view of the non-uniform hyperbolicity as in (47) and (49)) but somewhat tedious argument. Let

$$V = \bigcup_{|t| < \rho_3} \{g_{t+t(q)} f(q) : q \in B_E(q_{X,p_0}, \rho_3/2) \cap S_0\} \subset \mathcal{F}^u(q_{X,p_0}).$$

Then V is relatively open as a subset of $\mathcal{F}^u(q_{X,p_0})$. Let ∂V denote the boundary of V viewed as a subset of $\mathcal{F}^u(q_{X,p_0})$. Therefore we can choose $\rho_2 > 0$ depending only on K such that for all $q' \in \partial V$,

$$d_E(q', q_{X,p_0}) \geq \rho_2.$$

By (47) and the fact that $\partial V \subset \mathcal{F}^u(q_{X,p_0})$, this implies that for some constant ρ'_2 , (depending only on K), all $q' \in \partial V$ and all $L > 0$,

$$d_E(g_L q', \gamma(L)) \geq \rho'_2. \quad (52)$$

Note that $U \subset \bigcup_{t \in \mathbb{R}} g_t S_0 \cong \mathbb{R} \times S_0$. Let ∂U denote the boundary of U viewed as a subset of $\mathbb{R} \times S_0$. Suppose $q_1 \in \partial U$. We may write $q_1 = g_t q$ for some $q \in S_0$. Then the fact that $q_1 \in \partial U$ implies that either $d_E(q, q_{X,p_0}) = \rho_3/2$ or $|t| = \rho_3$. In either case, let $q_2 = g_{t+t(q)} f(q)$. Then $q_2 \in \partial V$, q_1 and q_2 are on the same leaf of \mathcal{F}^{ss} , and

$$d_E(q_1, q_2) \leq C,$$

where C depends only on K . Hence, by (47), we have

$$d_E(g_L q_1, g_L q_2) \leq C \cdot C_0. \quad (53)$$

In order to prove the claim, we show that there exists L_0 (depending only on K and ϵ) such that for $L > L_0$,

$$d_E(g_L q_1, \gamma(L)) \geq \rho'_2/2. \quad (54)$$

Suppose that (54) fails. Then:

- by (53), $d_E(g_L q_2, \gamma(L)) \leq C_2$, where C_2 only depends on K . On the other hand, we can choose $|t_0| \leq \rho_3$ such that $g_{t_0} q_2 \in \mathcal{F}^{ss}(q_{X,p_0})$. Using (49) for $g_{L+t_0} q_2 \in \mathcal{F}^{ss}(\gamma(L))$, we get that

$$|\{t \mid t \in [0, L], \ell_{\min}(g_t q_2) > \epsilon'\}| > \epsilon_0 s,$$

where ϵ', ϵ_0 only depend on K , and ϵ .

- Now we can apply (49) for $q_1 \in \mathcal{F}^{ss}(q_2)$, and we get $d_E(g_L q_1, g_L q_2) \leq C_0 e^{-aL} d_H(q_1, q_2)$. Therefore there exists \tilde{L}_0 such that if $L > \tilde{L}_0$, $d_E(g_L q_1, g_L q_2) \leq \rho'_2/2$. Then, in view of (52), $d_E(g_L q_2, \gamma(L)) \geq \rho'_2$. So for all $q_1 \in \partial U$ and all $L > \tilde{L}_0$, $d_E(g_L q_1, \gamma(L)) \geq \rho'_2/2$.

By **P2**, there exists ρ_0 depending only on K such that for $L > \tilde{L}_0$ and all $q_1 \in \partial U$,

$$d_{\mathcal{T}}(\pi(g_L q_1), \pi(\gamma(L))) \geq \rho_0. \quad (55)$$

Let $\phi : U \rightarrow \mathcal{T}_g$ denote the map $\phi(q) = \pi(g_L(q))$. Then ϕ is continuous, and $\phi(q_{X,p_0}) = \pi(\gamma(L))$. Now in view of (55), (51) holds. In other words, if $d_{\mathcal{T}}(p_0, p_1) < \rho_0$, and $L > \tilde{L}_0$, $\mathcal{F}^{uu}(q_{X,p_0}) \cap \mathcal{F}^s(q_{X,p_1}) = \emptyset$ is well defined, and $d_E(q, q_{X,p_0}) < \rho_3$. On the other hand, $d_H(g_L q_{X,p_0}, g_L q) < C$, where C only depends on K . Therefore, we can use **P4** to get

$$d_E(q, q_{X,p_0}) < C_0 e^{-aL} C,$$

where $C_0, a > 0$ depend on ϵ, b and K . This shows that we can find $L_0 > \tilde{L}_0$ (depending on ϵ, ρ , and K) such that if $L > L_0$ we have

$$d_E(q, q_{X,p_0}) < \rho.$$

□

Proof of Lemma 5.4. It is enough to show that given $\epsilon > 0$ and $1 > m > m' \geq 1/2$, there exist $c_0 > 0$ (depending only on K, m and m') and $L_0 > 0$ (depending only on ϵ, m, m' and K) such that if $x, p_1 \in \tilde{K}$, $d_{\mathcal{T}}(p_1, p_2) < c_0$, for $i = 1, 2$, $d_{\mathcal{T}}(X, p_i) = r_i > L_0$, and

$$\{s \mid 0 < s \leq r_1, \ell_{\min}(g_s(q_{X,p_1})) \geq \epsilon\} > r_1 m,$$

then

$$\{s \mid 0 < s \leq r_2, \ell_{\min}(g_s(q_{X,p_2})) \geq \epsilon/2\} > r_2 m'.$$

Let $q_1 = q_{X,p_1}$, and (as in the previous lemma) let $q = f(q_{X,p_2})$ be the unique point in $\mathcal{F}^{uu}(q_1) \cap \mathcal{F}^s(q_{X,p_2})$. Now let q_2 be a quadratic differential of area 1 on the geodesic joining X to p_2 such that $q_2 \in \mathcal{F}^{ss}(q)$; in particular, we have

$$q \in \mathcal{F}^{uu}(q_1), \quad q \in \mathcal{F}^{ss}(q_2). \quad (56)$$

By **P1**, we can choose $\epsilon_0 > 0$ such that if $d_E(q', q'') < \epsilon_0$, and $\ell_{\min}(q') > \epsilon$ then $\ell_{\min}(q'') > \epsilon/2$. Then by Lemma 5.3, **P4**, and **P2**, we can choose $c_0 < 1/2$ (depending only on K) and $L_0 > 0$ such that for $d_T(X, p_1) = r > L_0$, and $d_T(p_1, p_2) < c_0$ the following hold:

1. As in **P4** (for $b = m$), there exist $C_0 > 0$ and $a > 0$ such that (49) holds. Then L_0 should satisfy:

$$C_0 e^{-aL_0/6} < \epsilon_0/4,$$

2. Also, by Lemma 5.3, if L_0 is large enough

$$d_E(q_1, q) < \epsilon_0/4C_0, \quad d_E(q, q_2) \leq \epsilon_0/4C_0,$$

and

$$d_E(g_r q_1, g_r q_2) \leq 1.$$

As a result, from (47) we get

$$d_E(g_r q, g_r q_2) < \epsilon_0/4 \text{ and } d_E(g_r q_1, g_r q) < C_1,$$

where C_1 only depends on K , and ϵ .

Consider the map between the points on the geodesic $[xp_1]$ to the points on $[xp_2]$ as follows:

$$[xp_1] \rightarrow [xp_2]$$

$$g_s q_1 \rightarrow g_s q_2.$$

We can choose, $0 \leq s_0 \leq r$ such that $|\{t \mid s_0 < t < r, \ell_{\min}(g_t q_1) \geq \epsilon\}| = r(m - m')$, and let

$$\mathcal{A} = \{t \mid 0 < t < s_0, \ell_{\min}(g_s q_1) \geq \epsilon\}.$$

It is easy to check that $|\mathcal{A}| > rm'$. We claim that for $s \in \mathcal{A}$, we have

$$d_E(g_s q_1, g_s q_2) \leq \epsilon_0. \quad (57)$$

This is because:

- For any $s \in \mathcal{A}$, (48) holds for $g_r q$ and $g_r q_1$ and the interval $(0, r - s)$. Hence, by (49),

$$d_E(g_s q_1, g_s q) = d_E(g_{s-r}(g_r q_1), g_{s-r}(g_r q)) < \epsilon_0/4;$$

- by (47),

$$d_E(g_s q, g_s q_2) < \epsilon_0/4;$$

- Finally, since $q_1 \in \mathcal{F}^{uu}(q)$, $d_E(g_s q_1, g_s q_2) \leq \epsilon_0$.

Hence, for $s \in \mathcal{A}$, $\ell_{\min}(g_s q_1) \geq \epsilon$, (57), and **P1** imply that $\ell_{\min}(g_s q_2) \geq \epsilon/2$. □

5.3 Proof of Theorem 5.2

Choose $\theta_1 > 0$. Let $K \subset \mathcal{M}_g$ be such that Theorem 5.1 holds for K , and $\theta_1 = \theta$. Let \mathcal{K}_2 be a compact subset of $\mathcal{Q}^1\mathcal{T}_g(1, \dots, 1)/\Gamma_g$ such that $\mathcal{K}_2 \subset \mathfrak{p}^{-1}(K)$, and let \mathcal{K}_3 be a subset of the interior of \mathcal{K}_2 . We may choose these sets so that $\mu(\mathcal{K}_3) > (1/2)$, where μ is defined in §2.2. We also choose \mathcal{K}_2 and \mathcal{K}_3 to be symmetric, i.e if $q \in \mathcal{K}_2$ then $-q \in \mathcal{K}_2$ (and same for \mathcal{K}_3). Then there exists $\epsilon > 0$ such that for $X \in \mathcal{K}_3$, $\ell_{\min}(X) > \epsilon$. Let c_0 be as in Lemma 5.4. We now choose a (c_1, c_2) separated net \mathcal{N} on \mathcal{T}_g , which $c_1 < c_0$, $c_2 < c_0$. We may assume that $\mathcal{N} \cap \mathfrak{p}^{-1}(K)$ is invariant under the action of the mapping class group.

Suppose $X \in \mathcal{T}_g$, and as before, let $S(X)$ denote the set of area 1 holomorphic quadratic differentials on X . Let

$$U(X, T) = \{q \in S(X) : |\{t \in [0, T] : g_t q \in \mathcal{K}_2^c\}| > (1/2)T\}.$$

Let

$$V(X, T) = \bigcup_{0 \leq t \leq T} \pi(g_t U(X, t)),$$

so that $V(X, T)$ is the subset of $B_{\mathcal{T}}(X, T)$ consisting of points $Y \in B_{\mathcal{T}}(X, T)$ such that the geodesic from X to Y spends more than half the time outside \mathcal{K}_2 .

By [ABEM, Theorem 2.6], for any $\theta_1 > 0$, there exists $T > 0$ such that for any $\tau > T$ and any $X \in \mathcal{N} \cap \mathfrak{p}^{-1}(K)$,

$$\mathbf{m}(\text{Nbhd}_{c_2}(V(X, \tau) \cap \mathfrak{p}^{-1}(K))) \leq \theta_1 e^{h\tau},$$

where $\text{Nbhd}_a(A)$ denotes the set of points within Teichmüller distance a of the set A . Then, since K is compact and θ_1 is arbitrary, this implies that for any $\theta_2 > 0$ there exists $T > 0$ such that for any $\tau > T$ and any $X \in \mathcal{N} \cap \mathfrak{p}^{-1}(K)$,

$$|\mathcal{N} \cap V(X, \tau) \cap \mathfrak{p}^{-1}(K)| \leq \theta_2 e^{h\tau}.$$

By the compactness of K and [ABEM, Theorem 1.2 and Theorem 5.1] there exists $C_1 > 1$ such that for τ sufficiently large and any $X \in \mathcal{N} \cap \mathfrak{p}^{-1}(K)$,

$$C_1^{-1} e^{h\tau} \leq |\mathcal{N} \cap B_{\mathcal{T}}(X, \tau)| \leq C_1 e^{h\tau}.$$

Thus, for any $\theta_3 > 0$ there exists $T > 0$ such that for $\tau > T$,

$$|\mathcal{N} \cap V(X, \tau) \cap \mathfrak{p}^{-1}(K)| < \theta_3 |\mathcal{N} \cap B_{\mathcal{T}}(X, \tau)|. \quad (58)$$

From now on we assume that τ is sufficiently large so that (58) holds.

Let $K'_1 = \text{Nbhd}_{c_2}(K)$, and let $\mathcal{G}(R)$ denote the set of closed geodesics in \mathcal{M}_g of length at most R , and let $\mathcal{G}^{K'_1}(\theta_3, R) \subset \mathcal{G}(R)$ denote the subset which contributes to $N^{K'_1}(\theta_3, R)$. In view of Theorem 5.1, it is enough to show that there exists $\delta_0 > 0$ such that for R sufficiently large,

$$|\mathcal{G}(R) \setminus \mathcal{G}^{K'_1}(\theta_3, R)| \leq e^{(h-\delta_0)R}. \quad (59)$$

As in §4, we associate a random walk trajectory $\Phi(\gamma)$ to each closed geodesic $\gamma \in \mathcal{G}(R)$. Let $\mathcal{P}_1(R) = \Phi(\mathcal{G}(R) \setminus \mathcal{G}^{K'_1}(\theta_3, R))$ denote the set of resulting trajectories. Note that by construction, every trajectory in \mathcal{P}_1 spends at most θ_3 fraction of the time outside K .

Suppose $\Lambda = (\lambda_1, \dots, \lambda_n)$ be a trajectory in $\mathcal{P}_1(R)$. Let $J(\Lambda)$ denote the number of j , $1 \leq j \leq n$ such that

$$\lambda_j \in \mathfrak{p}^{-1}(\mathbb{K}^c) \quad \text{or} \quad \lambda_{j+1} \in V(\lambda_j, \tau) \cup \mathfrak{p}^{-1}(\mathbb{K}^c). \quad (60)$$

Let $\mathcal{P}_2(R) \subset \mathcal{P}_1(R)$ denote the trajectories for which $J(\Lambda) < (\theta/2)n$. Then, as long as θ_3 is chosen sufficiently small, the Law of Large Numbers implies that there exists $\delta_1 > 0$ such that for $n = R/\tau$ sufficiently large,

$$|\mathcal{P}_1(R) \setminus \mathcal{P}_2(R)| \leq e^{(h-\delta_1)R}.$$

Since every trajectory in $\mathcal{P}_1(R)$ intersects the compact set \mathbb{K} , for any $\Lambda \in \mathcal{P}_1(R)$, the cardinality of $\Phi^{-1}(\Lambda)$ is bounded by a constant C depending only on \mathbb{K} . Thus,

$$|\Phi^{-1}(\mathcal{P}_1(R) \setminus \mathcal{P}_2(R))| \leq C e^{(h-\delta_1)R}.$$

To complete the proof we will show that there exists a compact set $\mathcal{K} \subset \mathcal{Q}^1\mathcal{M}_g(1, \dots, 1)$ such that any geodesic $\gamma \in \Phi^{-1}(\mathcal{P}_2(R))$ spends at least $1 - \theta$ fraction of the time in \mathcal{K} . Indeed, suppose $\Lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}_2(R)$ and $\gamma \in \Phi^{-1}(\Lambda)$. Then there exist points $q_j = \gamma(t_j) \in \mathcal{Q}^1\mathcal{M}_g$ such that $q_j \in \gamma$ and for $1 \leq j \leq n$, $d_{\mathcal{T}}(\pi(q_j), \lambda_j) \leq c_2 < c_0$. Now suppose $j \notin J(\Lambda)$. Then by Lemma 5.4, there exists $\epsilon' > 0$ depending only on \mathbb{K} and ϵ , and a point $t_j < t' < t_{j+1}$ such that $\ell_{\min}(\gamma(t')) > \epsilon'$. But then for all $t_j < t < t_{j+1}$, $\ell_{\min}(\gamma(t)) \geq e^{-\tau} \epsilon'$, so the entire geodesic γ spends at least $(1 - \theta)$ -fraction of the time in the compact set $\{q \in \mathcal{Q}^1\mathcal{M}_g : \ell_{\min}(q) > e^{-\tau} \epsilon'\}$. \square

6 A Closing Lemma

In this section, we use the properties of the geodesic flow discussed in §5 to prove the following closing lemma:

Lemma 6.1 (Closing Lemma) *Let \mathcal{K} be a compact subset of $\mathcal{Q}^1\mathcal{M}_g(1, \dots, 1)$ consisting of non-orbifold points of $\mathcal{Q}^1\mathcal{M}_g$. Given $q \in \mathcal{Q}^1\mathcal{M}_g(1, \dots, 1)$ and $\epsilon > 0$, there exist constants $L_0 > 0$, $\epsilon' > 0$, and open neighborhoods $U \subset U'$ of q with the following property.*

Suppose that $\gamma : [0, L] \rightarrow \mathcal{Q}^1\mathcal{M}_g$ is a geodesic segment (parametrized by arclength) such that

(a) $\gamma(0), \gamma(L) \in U$ and

(b)

$$|\{t \in [0, L] : \gamma(t) \in \mathcal{K}\}| > (1/2)L,$$

(c) $L > L_0$.

Let γ_1 be the closed path in \mathcal{M}_g which is the union of γ and the geodesic segment from $\pi(\gamma(L))$ to $\pi(\gamma(0))$ in $\pi(U)$. Then there exists a unique closed geodesic γ' with the following properties:

(a') γ' and γ_1 have lifts in \mathcal{T}_g which stay ϵ close (with respect to the Teichmüller metric on \mathcal{T}_g).

(b') The length of γ' is within ϵ of L ,

(c') γ' passes through U' .

Remark. A stronger version of this statement can be found in [H2] (§6).

Outline of Proof. Consider the stable and unstable foliations for the geodesic flow as in §5.1. Our goal is to show that if U is small enough, the first return map on these foliations will define a contraction with respect to the modified Hodge distance function. As a result, we find a fixed point for the first return map in U' (which is the same as a closed geodesic going through U').

From Lemma 5.4, we can find open neighborhoods $U \subset U'' \subset U'$ of q , and $L_0 > 0$ such that the following properties hold :

- if $\gamma : [0, L] \rightarrow \mathcal{Q}^1\mathcal{M}_g$ satisfies properties (a), (b) and (c) then in view of the hyperbolicity statement (49) the time L geodesic flow restricted to the neighborhood U'' expands along the leaves of \mathcal{F}^{uu} and contracts along the leaves of \mathcal{F}^{ss} , in the metric d_H ,
- for any $q_1, q_2 \in U'$, if $q_1 \in \mathcal{F}^{ss}(q_2)$ or $q_1 \in \mathcal{F}^{uu}(q_2)$ then $d_H(q_1, q_2) \leq \epsilon$.

We can apply the contraction mapping principle to \mathcal{F}^{ss} to find $q_0 \in U'$ such that $g_L(q_0) \in \mathcal{F}^{uu}q_0$. Now we can consider the first return map of the map g_{-t} on $\mathcal{F}^{uu}(q_0)$. Again, we can use (49), Lemma 5.4 and the contraction mapping principle to find a fixed point for the geodesic flow in U' . \square

6.1 Remark on proof of Theorem 1.1

Note that by the bound proved in Theorem 5.2, we only need to consider the set of closed geodesics going through a fixed compact subset of \mathcal{M}_g . We remark that in Lemma 6.1 if we remove the assumption that \mathcal{K} consists of non-orbifold points then there are at most c_0 closed geodesics satisfying conditions (a'), (b') and (c'), where c_0 is a constant depending only on g .

Roughly speaking, since

- by Theorem 2.1, the geodesic flow on $\mathcal{Q}^1\mathcal{M}_g(1, \dots, 1)$ is mixing, and
- on a fixed compact subset of $\mathcal{Q}^1\mathcal{M}_g(1, \dots, 1)$ the geodesic flow is uniformly hyperbolic,

the derivation of Theorem 1.1 from Lemma 6.1 can be done following the work of Margulis [Mar]. See also §20.6 in [KH]. We skip the argument since it has been done carefully in §5 and §6 of [H2].

7 Appendix: Proof of Proposition 4.5

Given X and Y in \mathcal{T}_g and $\tau > 0$, let $F_\tau(X, Y)$ denote the intersection of $B_{\mathcal{T}}(X, \tau)$ with $\Gamma_g \cdot Y$. Proposition 4.5 will follow from the following:

Proposition 7.1 *Given $\epsilon > 0$, there exists $\tau_0 > 0$ such that for any $\tau > \tau_0$ and $X, Y \in \mathcal{T}_g$ we have*

$$|F_\tau(X, Y)| \leq e^{(h+\epsilon)\tau} G(Y)^2,$$

where G is defined in (6).

Proof of Proposition 4.5 assuming Proposition 7.1. Suppose $X \in \mathcal{T}_g$ is arbitrary, and suppose $Y \in \tilde{\mathcal{N}}$. Then, by Proposition 7.1, for τ sufficiently large,

$$|B_{\mathcal{T}}(X, \tau) \cap \Gamma_g Y| \leq e^{(h+\epsilon/2)\tau} G(Y)^2.$$

Note that by Theorem 2.4 (see the calculation in (11))

$$|\Gamma_g \cdot Y \cap B_{\mathcal{T}}(Y, c_2)| \geq c G(Y)^2. \quad (61)$$

Thus, since distinct points in \mathcal{N} are at least c_2 apart,

$$|B_{\mathcal{T}}(X, \tau) \cap \Gamma_g Y \cap \mathcal{N}| \leq c e^{(h+\epsilon/2)\tau}.$$

In view of (18), this implies that

$$|B_{\mathcal{T}}(X, \tau) \cap \mathcal{N}| \leq c' \tau^{6g-6} e^{(h+\epsilon/2)\tau} \leq e^{(h+\epsilon)\tau}.$$

Now this implies Proposition 4.5 in view of Lemma 3.1. \square

Given $X, Y \in \mathcal{T}_g$, and $\mathcal{B} \subset \mathcal{C}_X$ let

$$F_R(X, Y, \mathcal{B}) = \{\mathfrak{g} \cdot Y \mid \mathfrak{g} \in \Gamma_g, d_{\mathcal{T}}(X, \mathfrak{g} \cdot Y) \leq R, \mathcal{B} \subset \mathcal{C}_X \cap \mathcal{C}_{\mathfrak{g} \cdot Y}\} \subset F_R(X, Y).$$

Here, as in §2, given $Z \in \mathcal{T}_g$ the set \mathcal{C}_Z consists of simple closed curves of extremal length $\leq \epsilon_0^2$ on Z .

In general, we have:

Theorem 7.2 *Given $X, Y \in \mathcal{T}_g$*

$$|F_R(X, Y, \mathcal{B})| \leq C_1 R^{3g-3} e^{(h-2|\mathcal{B}|)R} G(X)G(Y), \quad (62)$$

where C_1 only depends on g .

In the proof we use the following lemma from [ABEM]:

Lemma 7.3 *Let $\alpha = \{\alpha_1, \dots, \alpha_{3g-3}\}$ be a bounded pants decomposition on X and suppose $Y_0 \in B_{\mathcal{T}}(X, R)$. If $h_{\alpha_1}^{m_1} \dots h_{\alpha_{3g-3}}^{m_{3g-3}}(Y_0) \in B_{\mathcal{T}}(X, R)$ then for $1 \leq i \leq 3g-3$*

$$|m_i| \cdot \sqrt{\text{Ext}_{\alpha_i}(Y_0)} \leq C \frac{e^R}{\sqrt{\text{Ext}_{\alpha_i}(X)}}.$$

Here C is a constant which only depends on g .

Proof of Theorem 7.2. To simplify the notation, given $X, Y \in \mathcal{T}_g$ define

$$D_{\alpha}(X, Y) = \log \left(\sqrt{\frac{\text{Ext}_{\alpha}(X)}{\text{Ext}_{\alpha}(Y)}} \right).$$

Choose bounded pants decompositions $\mathcal{P}_X = \{\alpha_1, \dots, \alpha_{3g-3}\}$ and $\mathcal{P}_Y = \{\beta_1, \dots, \beta_{3g-3}\}$ on X and Y . Without loss of generality, we can assume that

$$\alpha_1 = \beta_1, \dots, \alpha_b = \beta_b \in \mathcal{B},$$

where $b = |\mathcal{B}|$.

Given $\mathbf{r} = (r_1, \dots, r_{3g-3}), \mathbf{s} = (s_1, \dots, s_{3g-3}) \in \mathbb{Z}^{3g-3}$ define

$$M_{R,(\mathbf{r},\mathbf{s})}(X, Y, \mathcal{B}) = M_R(X, Y, \mathcal{B}) \cap \{Z = \mathfrak{g} \cdot Y \mid D_{\alpha_i}(X, \mathfrak{g}Y) \in [s_i, s_i + 1], D_{\beta_i}(\mathfrak{g}^{-1}X, Y) \in [r_i, r_i + 1]\}.$$

By the definition, if $M_{R,(\mathbf{r},\mathbf{s})}(X, Y, \mathcal{B}) \neq \emptyset$ then for $1 \leq i \leq 3g-3$ we have $|r_i|, |s_i| \leq R$.

Claim. Given $\mathbf{r}, \mathbf{s} \in \mathbb{Z}^{3g-3}$ with $|r_i|, |s_i| \leq R$

$$|M_{R,(\mathbf{r},\mathbf{s})}(X, Y, \mathcal{B})| \leq e^{(h-2|\mathcal{B}|)R} G(X)G(Y). \quad (63)$$

In order to prove this claim, fix $Y_0 = \mathfrak{g}_0 \cdot Y \in M_{R,(\mathbf{r},\mathbf{s})}(X, Y, \mathcal{B})$, and consider the pants decomposition $\alpha = \cup_{i=1}^{3g-3} \alpha_i$. Note that given $Z = \mathfrak{g} \cdot Y \in M_{R,(\mathbf{r},\mathbf{s})}(X, Y, \mathcal{B})$,

$$\alpha(Z) = \mathfrak{g}_0 \mathfrak{g}^{-1}(\mathcal{P}_X) = \cup_{i=1}^{3g-3} \mathfrak{g}_0 \mathfrak{g}^{-1}(\alpha_i)$$

defines a pants decomposition. Moreover, we have:

- $\mathfrak{g}_0 \mathcal{P}_Y$ is a bounded pants decomposition on $Y_0 = \mathfrak{g}_0 \cdot Y$.
- Also,

$$\sqrt{\text{Ext}_{Y_0}(\alpha(Z))} = \sqrt{\text{Ext}_Z(\alpha)} = O(e^R). \quad (64)$$

- Note that

$$\cup_{i=1}^b \alpha_i \subset \alpha(Z),$$

and hence for $i \leq b$

$$i(\mathfrak{g}_0 \beta_i, \alpha(Z)) = 0. \quad (65)$$

- On the other hand, since $Z = \mathfrak{g} \cdot Y$ is in $M_{R,(\mathbf{r},\mathbf{s})}(X, Y, \mathcal{B})$ we have:

$$\frac{\sqrt{\text{Ext}_{\beta_j}(\mathfrak{g}^{-1}X)}}{\sqrt{\text{Ext}_{\beta_j}(Y)}} = O(e^{r_i}).$$

As a result, we get

$$i(\mathfrak{g}_0 \mathfrak{g}^{-1}(\alpha_i), \mathfrak{g}_0 \beta_j) = i(\alpha_i, \mathfrak{g}(\beta_j)) \leq \sqrt{\text{Ext}_{\alpha_i}(X)} \sqrt{\text{Ext}_{\beta_j}(\mathfrak{g}^{-1}X)} = O(e^{r_j}),$$

and hence

$$i(\alpha(Z), \mathfrak{g}_0 \cdot \beta_j) = O(e^{r_j}). \quad (66)$$

- By (64) Theorem 2.4 gives rise to an upper bound on twisting parameter of $\alpha(z)$ around $\mathfrak{g}_0 \beta_j$ on Y_0 :

$$tw_{Y_0}(\alpha(Z), \mathfrak{g}_0 \beta_j) = tw_Y(\mathfrak{g}^{-1}(\alpha), \beta_j) \leq \frac{e^R}{\sqrt{\text{Ext}_{\beta_j}(Y)}}. \quad (67)$$

Now, in view of the Dehn-Thurston parametrization of multicurves, (65), (66) and (67) imply:

$$|\{\alpha(Z) \mid Z \in M_{R,(\mathbf{r},\mathbf{s})}(X, Y, \mathcal{B})\}| \leq e^{r_{b+1} + \dots + r_n} e^{(\frac{b}{2} - b)R} \prod_{i=1}^{3g-3} \frac{1}{\sqrt{\text{Ext}_{\beta_i}(Y)}} \quad (68)$$

- On the other hand given $Z_1 = \mathfrak{g}_1 \cdot Y, Z_2 = \mathfrak{g}_2 \cdot Y \in \Gamma_g \cdot y$, we have $\alpha(Z_1) = \alpha(Z_2)$ if and only if $\mathfrak{g}_1^{-1} \mathfrak{g}_2(\mathcal{P}_X) = \mathcal{P}_X$. If $\mathfrak{g}_1^{-1} \mathfrak{g}_2(\alpha_i) = \alpha_i$ for $1 \leq i \leq 3g-3$ then there are $m_1, \dots, m_{3g-3} \in \mathbb{Z}$ such that

$$\mathfrak{g}_1 = h_{\alpha_1}^{m_1} \dots h_{\alpha_{3g-3}}^{m_{3g-3}} \cdot \mathfrak{g}_2.$$

Since for any $Z \in M_{R,(\mathbf{r},\mathbf{s})}(X, Y, \mathcal{B})$

$$\frac{1}{\sqrt{\text{Ext}_{\alpha_i}(Z)}} = O\left(\frac{1}{e^{s_i}}\right),$$

from Lemma 7.3 we get

$$|\{Z \in M_{R,(\mathbf{r},\mathbf{s})}(X, Y, \mathcal{B}) \mid \alpha(Z) = \alpha(Z_0)\}| = O\left(\frac{e^{(\frac{h}{2}-b)R} 3^{g-3}}{e^{|\mathbf{s}|}} \prod_{i=1}^{3g-3} \frac{1}{\sqrt{\text{Ext}_{\alpha_i}(X)}}\right). \quad (69)$$

Therefore from (68) and (69) we get

$$|M_{R,(\mathbf{r},\mathbf{s})}(X, Y, \mathcal{B})| = O\left(e^{|\mathbf{r}|-|\mathbf{s}|} G(X)G(Y)e^{(h-2|\mathcal{B}|)R}\right).$$

On the other hand, since $M_{R,(\mathbf{r},\mathbf{s})}(X, Y, \mathcal{B}) = M_{R,(\mathbf{s},\mathbf{r})}(Y, X, \mathcal{B})$, we have

$$|M_{R,(\mathbf{r},\mathbf{s})}(X, Y, \mathcal{B})| = O\left(e^{|\mathbf{s}|-|\mathbf{r}|} G(X)G(Y)e^{(h-2|\mathcal{B}|)R}\right).$$

Therefore, we get (63).

Finally, we have

$$M_R(X, Y, \mathcal{B}) = \bigcup M_{R,(\mathbf{r},\mathbf{s})}(X, Y, \mathcal{B}),$$

where $\mathbf{r} = (r_1, \dots, r_{3g-3})$ and $\mathbf{s} = (s_1, \dots, s_{3g-3})$ with $|r_i|, |s_i| \leq R$. Hence, in view of (63) we get (62). \square

Proof of Proposition 7.1. To simplify the notation, given $X \in \mathcal{T}_g$ let

$$\mathbf{e}_\alpha(X) = \sqrt{\text{Ext}_\alpha(X)}.$$

As before, we say α is short on X if $\mathbf{e}_\alpha(X) \leq \epsilon_0$, and let \mathcal{C}_X denote the set of all short simple closed curves on X . Choose a $(c, 2c)$ net \mathcal{N} in \mathcal{T}_g as in §3.2.

Then in view of (61), given $Z \in \mathcal{T}_g$, $\mathcal{B} \subset \mathcal{C}_Z$, $r > 0$ and $W \in \tilde{\mathcal{N}}$, the inequality (62) implies that:

$$\begin{aligned} |B_{\mathcal{T}}(Z, \tau) \cap \mathfrak{p}^{-1}(W) \cap \mathcal{N}_{\mathcal{B}}| &\leq C_1 e^{(h-|\mathcal{B}|)r} r^{3g-3} G(Z)G(W)/G(W)^2 \\ &\leq C r^{3g-3} e^{(h-|\mathcal{B}|)r} G(Z)/G(W), \end{aligned} \quad (70)$$

where $\mathcal{N}_{\mathcal{B}} = \{Z_1 \in \mathcal{N} \mid \forall \alpha \in \mathcal{B}, \mathbf{e}_\alpha(Z_1) \leq \epsilon_0\} \subset \mathcal{N}$, and C is a universal constant independent of X , Z and r .

We order the elements in \mathcal{C}_X , $(\alpha_1, \dots, \alpha_k)$, so that

$$\mathbf{e}_{\alpha_1}(X) \geq \dots \geq \mathbf{e}_{\alpha_s}(X) \geq e^{-\tau} > \dots \mathbf{e}_{\alpha_k}(X).$$

Then $3g - 3 \geq k \geq s$. Let $\tau_0 = 0$, and for $i \geq 1$,

$$\tau_i = -\log(\mathbf{e}_{\alpha_i}(X)).$$

Also for $1 \leq i \leq s$, $d_i = \tau_i - \tau_{i-1}$, $m_i = h - k + i - 1$, $d_{s+1} = \tau - \tau_s$, and $m_{s+1} = h - k$. Then it is easy to verify that

1.

$$\sum_{i=1}^{s+1} d_i = \tau, \quad \sum_{i=1}^{s+1} m_i d_i = (h - k + s)\tau - \tau_1 \dots - \tau_s, \quad G(X) = e^{\tau_1 + \dots + \tau_k};$$

2. By Theorem 2.2, for $1 \leq i \leq s$, $\alpha_i, \dots, \alpha_k$ are all short in $B_{\mathcal{T}}(X, \tau_i)$.

3. Since $\alpha_{s+1}, \dots, \alpha_k$ stay short in $B_{\mathcal{T}}(X, \tau)$, again by Kerckhoff's theorem, for any $Y \in B_{\mathcal{T}}(X, \tau)$

$$e^{\tau_{s+1} + \dots + \tau_k} \leq G(Y) e^{(k-s)\tau}. \quad (71)$$

Consider

$$\mathcal{P} = \{(X = Z_0, Z_1, \dots, Z_s, \mathbf{g} \cdot Y = Z_{s+1}) \mid Z_1, \dots, Z_s \in \mathcal{N}, \forall i, d_{\mathcal{T}}(Z_i, Z_{i-1}) \leq d_i, \mathbf{g} \in \Gamma_g\},$$

$$\mathcal{Z} = \{(W_1, \dots, W_s) \mid W_i \in \tilde{\mathcal{N}} \cap \mathfrak{p}(B_{\mathcal{T}}(X, \tau))\},$$

and finally

$$\mathcal{P}(W_1, \dots, W_s) = \{(X, Z_1, \dots, Z_s, \mathbf{g} \cdot Y) \in \mathcal{P} \mid \pi(Z_i) = W_i\}.$$

Note that for any $(X, Z_1, \dots, Z_s, Z_{s+1} = \mathbf{g} \cdot Y) \in \mathcal{P}$, $Z_i \in B_{\mathcal{T}}(X, \tau_i)$.

On the other hand, since we can approximate a geodesic by points in the net \mathcal{N} , we have

$$|F_{\tau}(X, Y)| \leq |\mathcal{P}|,$$

also by the definition,

$$\mathcal{P} \subset \bigcup_{W \in \mathcal{Z}} \mathcal{P}(W).$$

By equation (18) we have

$$|\mathcal{Z}| = O(\tau^{c_g}),$$

where $c_g \leq (12g - 12)s < (12g - 12)^2$.

Let $Z_0 = X$. Then for a given $Z = Z_{i-1} \in \mathcal{N} \cap B_{\mathcal{T}}(X, \tau_{i-1})$, $B_{\mathcal{T}}(Z, d_i) \subset B_{\mathcal{T}}(X, \tau_i)$. This implies that $h - m_i$ simple closed curves (i.e, the curves in $\mathcal{B}_i = \{\alpha_i, \dots, \alpha_k\}$) are short on $B_{\mathcal{T}}(Z, d_i)$.

Now, since $d_i \leq \tau$, equation (70) for $\mathcal{B} = \mathcal{B}_i$, $W = W_i$, and $Z = W_{i-1}$ implies that

$$|\mathfrak{p}^{-1}(W_i) \cap \mathcal{N} \cap B_{\mathcal{T}}(Z_{i-1}, d_i)| = |\mathfrak{p}^{-1}(W_i) \cap \mathcal{N}_{\mathcal{B}_i} \cap B_{\mathcal{T}}(Z_{i-1}, d_i)| \leq C\tau^{3g-3} e^{m_i d_i} G(W_{i-1}) / G(W_i). \quad (72)$$

Given $(Z_1, \dots, Z_s) \in \mathcal{Z}$, we can apply (72) for $(X, W_1), \dots, (W_{s-1}, W_s)$, and apply (62) for (W_s, Y) . We get

$$|\mathcal{P}(W_1, \dots, W_s)| \leq C\tau^{3g-3} e^{m_1 d_1} G(X) / G(W_1) \dots C\tau^{3g-3} e^{m_s d_s} G(W_{s-1}) / G(W_s) \times$$

$$\begin{aligned}
& \times C\tau^{3g-3}e^{m_{s+1}d_{s+1}}G(W_s)G(Y) = C^s\tau^{(3g-3)s}\frac{e^{(h-k+s)\tau}}{e^{\tau_1+\dots+\tau_s}}G(X)G(Y) = \\
& = C^s\tau^{(3g-3)s}e^{(h-k+s)\tau} \times e^{\tau_{s+1}+\dots+\tau_k}G(Y) \leq C^{3g-3}\tau^{(3g-3)^2}e^{h\tau}G(Y)^2.
\end{aligned}$$

We are using (71) to obtain the last inequality. Now we have,

$$|\mathcal{P}| \leq |\mathcal{Z}| e^{h\tau}G(Y)^2 \leq C'\tau^{c'_g}e^{h\tau}G(Y)^2,$$

where $c'_g = c_g + (3g - 3)^2 = O(g^2)$, and $C' = C^{3g-3}$. □

References

- [AY] P. Arnoux and J. Yoccoz, *Construction de diff'eomorphismes pseudo-Anosov*. C. R. Acad. Sci. Paris Ser. I Math., **29**(1981), 75–78.
- [AGY] A. Avila, S. Gouezel, and J.-C. Yoccoz, *Exponential mixing for the Teichmüller flow*. Publ. Math. IHES, **104**(2006), 143–211.
- [AR] A. Avila, and M. Resende, *Exponential Mixing for the Teichmüller flow in the Space of Quadratic Differentials*. Preprint.
- [Ath] J. Athreya, *Quantitative recurrence and large deviations for Teichmüller geodesic flow*. Geom. Dedicata **119**(2006), 121–140.
- [ABEM] J. Athreya, A. Bufetov, A. Eskin and M. Mirzakhani, *Lattice Point Asymptotics and Volume Growth on Teichmüller space*. Preprint
- [Be] L. Bers, *An extremal problem for quasiconformal maps and a theorem by Thurston*. Acta Math. **141**(1978), 73–98.
- [Bu] A. Bufetov, *Logarithmic asymptotics for the number of periodic orbits of the Teichmüller flow on Veech's space of zippered rectangles*. Mosc. Math. J. **9**(2009), no. 2, 245–261.
- [Bu] P. Buser, *Geometry and spectra of compact Riemann surfaces*, Progr. Math., 106, Birkhuser, Boston, 1992.
- [EMM] A. Eskin, G. Margulis, and S. Mozes, *Upper bounds and asymptotics in a quantitative version of the Oppenheim conjecture*. Ann. of Math. (2) **147**(1998), no. 1, 93–141.
- [EM] A. Eskin and H. Masur, *Asymptotic formulas on flat surfaces*. Ergodic Theory Dynam. Systems. **21**(2001), 443–478.
- [FaMa] B. Farb and D. Margalit. *A primer on mapping class groups*. <http://www.math.utah.edu/~margalit/primer>.
- [FLP] A. Fathi, F. Laudenbach, and V. Ponaru, *Travaux de Thurston sur les surfaces*. Asterisque, Vols. 66 and 67 (1979).
- [Fo] G. Forni, *Deviation of ergodic averages for area-preserving flows on surfaces of higher genus*. Ann. of Math. (2) **155**(2002), no. 1, 1–103.

- [Hu] J. Hubbard, *Teichmüller theory and applications to geometry, topology, and dynamics I*, Matrix Editions.
- [H1] U. Hamenstädt, *Bernoulli Measures for the Teichmüller Flow*. Preprint.
- [H2] U. Hamenstädt, *Bowen's construction for the Teichmüller flow*. Preprint.
- [H3] U. Hamenstädt, *Dynamics of the Teichmüller flow on compact invariant sets*. *J. Mod. Dynamics* **4**(2010), 393–418.
- [HP] J. L. Harer and R. C. Penner, *Combinatorics of Train Tracks*, volume 125, *Annals of Math. Studies*. Princeton University Press, 1992.
- [Iv] N. V. Ivanov, *Coefficients of expansion of pseudo-Anosov homeomorphisms*. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, **167**(1988), 111–116.
- [KH] A. Katok, B. Hasselblat, *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995.
- [Ker] S. Kerckhoff, *The asymptotic geometry of Teichmüller space*. *Topology*. **19**(1980), 23–41.
- [Mar] G.A. Margulis, *On some aspects of the theory of Anosov flows*, Ph.D. Thesis, 1970, Springer, 2003.
- [M] B. Maskit, *Comparison of hyperbolic and extremal lengths*. *Ann. Acad. Sci. Fenn.*, **10**(1985), 381–386.
- [Mas] H. Masur, *Interval exchange transformations and measured foliations*. *Ann. of Math. (2)* **115**(1982), no. 1, 169–200.
- [MasSm] H. Masur and J. Smillie, *Hausdorff Dimension of Sets of Nonergodic Measured Foliations*. *Ann. of Math., Second Series*, **134**(1991), 455–543.
- [Mi] Y. Minsky, *Extremal length estimates and product regions in Teichmüller space*. *Duke Math. J.* **83**(1996), no. 2, 249–286.
- [Ra1] K. Rafi, *Closed geodesics in the thin part of moduli space*. In preparation.
- [Ra2] K. Rafi, *Thick-thin decomposition of quadratic differentials*. preprint, 2005.
- [Th] W. Thurston. *On the geometry and dynamics of diffeomorphisms of surfaces*. *Bull. Amer. Math. Soc. (N.S.)*, **19**(1988), 417–431.
- [Ve] W. Veech, *The Teichmüller geodesic flow*. *Ann. of Math. (2)* **124**(1986), no. 3, 441–530.