Quiz #3. Solutions

Problem 1. This problem deals with the function $f(x) = \log(1+x)$.
(a) Prove (carefully) by induction that $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ for all $n \geq 1$.
Solution: The base case $n = 1$ is clear: $f'(x) = \frac{1}{x}$. Now suppose that $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ for some n. Then $f^{(n+1)}(x)$ is equal to

$$((-1)^{n-1}(n-1)!(1+x)^{-n})' = (-1)^{n-1}(n-1)!(-n)(1+x)^{-(n+1)} = \frac{(-1)^n n!}{(1+x)^{n+1}}.$$

(b) Let P_n be the n^{th} Taylor polynomial of f centered at 0. Prove that $P_n(x) = \sum\limits_{k=1}^{n} \frac{(-1)^{k-1}}{k} x^k$.
Solution: By part (a), we have $\frac{f^{(k)}(0)}{k!} = \frac{(-1)^{k-1}(k-1)!}{k!} = \frac{(-1)^{k-1}}{k}$.
(c) Let $R_n = f - P_n$ be the n^{th} remainder term. Use the integral form of R_n to show that

$$R_n(x) = \int_0^x \frac{(-1)^n}{(1+t)^{n+1}}(x-t)^n \, dt \qquad (***)$$

Solution: We have

$$R_n(x) = \int_0^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, dt = \int_0^x \frac{(-1)^n n!}{(1+t)^{n+1}} \cdot \frac{(x-t)^n}{n!} \, dt = \int_0^x \frac{(-1)^n}{(1+t)^{n+1}}(x-t)^n \, dt.$$

(d) Deduce from (c) that $|R_n(x)| \leq \frac{x^{n+1}}{(n+1)}$ for all $x > 0$. What can you about $|R_n(x)|$ for $x < 0$, based on (***)?
Solution: If $x > 0$, then $|R_n(x)| = \int\limits_0^x \frac{(-1)^n}{(1+t)^{n+1}}(x-t)^n \, dt \leq \int\limits_0^x \frac{1}{(1+t)^{n+1}}(x-t)^n \, dt$.
When $t \in [0, x]$, we clearly have $\frac{1}{(1+t)^{n+1}} \leq 1$. Therefore,

$$|R_n(x)| \leq \int_0^x (x-t)^n \, dt = -\frac{(x-t)^{n+1}}{n+1}\Big|_{t=0}^{t=x} = \frac{x^{n+1}}{n+1}.$$

When $x < 0$, we can only say that $\frac{1}{(1+t)^{n+1}} \leq \frac{1}{(1+x)^{n+1}}$, and similar argument (be careful with absolute values) yields $|R_n(x)| \leq \frac{x^{n+1}}{(n+1)(1+x)^{n+1}}$. This is not a good upper bound when x is close to -1; a much better bound follows from formula (!!!) (see below).

As stated in the problem, there is another formula for the remainder term

$$R_n(x) = \int_0^x (-1)^n \frac{t^n}{1+t}\, dt \tag{!!!}$$

which yields a better bound for $R_n(x)$ when $-1 < x < 0$, (!!!) namely

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(1+x)(n+1)} \text{ when } -1 < x < 0 \qquad (+++).$$

(see Problem 20.11)

(e) Deduce the formula (!!!) from (***) using integration techniques as follows: write $\frac{(x-t)^n}{(1+t)^{n+1}}$ as $\frac{1}{1+t}\left(\frac{x-t}{1+t}\right)^n$ and make a sub $u = \frac{x-t}{1+t}$. Then solve the last equation for t, compute dt in terms of du, and substitute in (***).

Solution: If $u = \frac{x-t}{1+t}$, then $t = \frac{x-u}{1+u} = \frac{x+1}{1+u} - 1$, and therefore $dt = -\frac{x+1}{(1+u)^2}du$. Note that $t = 0$ corresponds to $u = x$, and $t = x$ corresponds to $u = 0$ (limits of integration are switched). Therefore,

$$R_n(x) = \int_0^x (-1)^n \frac{1}{1+t}\left(\frac{x-t}{1+t}\right)^n dt = \int_x^0 (-1)^n \frac{1+u}{x+1} u^n \left(-\frac{x+1}{(1+u)^2}du\right) = \int_0^x (-1)^n \frac{u^n}{1+u}\, du,$$

which is what we wanted to prove.

(f) Use formula (!!!) and the corresponding bounds for $|R_n(x)|$ to approximate $\log(2)$ with accuracy 10^{-2} by evaluating the 5th Taylor polynomial of $f(x) = \log(1+x)$.

Solution: If we try to approximate $\log(2)$ by plugging $x = 1$ into Taylor polynomials of $\log(1+x)$, we would need 100th polynomial to get accuracy 10^{-2}. Instead we use the fact that $\log(2) = -\log(1/2)$. To approximate $\log(1/2)$ we plug $x = -1/2$ into Taylor polynomials of $\log(1+x)$. If we use 5th Taylor polynomial, then according to (+++), the error we get is $|R_n(-1/2)| \leq \frac{|-1/2|^{n+1}}{(n+1)/2}$ which is clearly less then 10^{-2}.

Therefore, $\log(2) \approx -P_5(\frac{1}{2}) = -\sum_{k=1}^{5} (-1)^{k-1} \frac{(-1/2)^k}{k} = \sum_{k=1}^{5} \frac{1}{k \cdot 2^k}$.

(g) (BONUS): find as good an approximattion for $\log(2)$ as you can by evaluating the 5th Taylor polynomial of $\log(1+x)$ at several points (use a trick similar to Problem 20.6).

Solution: The idea is to write 2 as a product of rational numbers all of which are close to 1. For instance, we can write $2 = \frac{4}{3} \cdot \frac{3}{2} = \frac{4/3}{2/3}$. Therefore, $\log 2 = \log(4/3) - \log(2/3)$. To approximate $\log(4/3)$ (respectively, $\log(2/3)$) we plug in $1/3$ (respectively, $-1/3$) into Taylor polynomials of $\log(1+x)$. Since both $1/3$ and $-1/3$ are close to 0, error terms go to 0 very fast.

An even better approximation is given by the formula

$$\log 2 = 2\log(24/25) + 3\log(9/8) - 4\log(9/10).$$

This formula does not come from the sky, but its (more or less) natural derivation is based on some linear algebra.

Problem 2. Let $\{a_n\}$ be a non-decreasing sequence which has a convergent subsequence $\{a_{n_k}\}_{k=1}^{\infty}$. Prove that the entire sequence $\{a_n\}$ converges.

Solution: Since $\{a_n\}$ is a non-decreasing sequence, to prove that it converges, we only need to show that $\{a_n\}$ is bounded above (Theorem 22.2).

We know that the subsequence $\{a_{n_k}\}$ is bounded since it is convegrent, and let $L = \sup\{a_{n_k} : k \in \mathbb{N}\}$. We claim that L is also the upper bound for $\{a_n\}$ (this will finish the proof).

Suppose the opposite: $\exists\ n \in \mathbb{N}$ such that $a_n > L$. Choose any k such that $n_k > n$. Since $\{a_n\}$ is nondecreasing, we have $a_{n_k} \geq a_n > L$. This contradicts the definition of L as the upper bound for the subsequence $\{a_{n_k}\}$ $\square$.