SOME MOTIVATION

One of my favourite papers is Ken Ribet's paper *Mod p Hecke operators and congruences between modular forms* [Rib83]. In this paper he shows that congruences between modular forms can be detected using group cohomology (via the Eichler–Shimura isomorphism that relates the two). This is not an evident result, because the Eichler–Shimura isomorphism is basically a piece of complex Hodge theory, and involves sheaves, cohomology, etc., that have complex coefficients, whereas to detect congruences mod p, one has to use cohomology with integral, or perhaps mod p, coefficients. I'll describe a toy version of the kind of argument that Ribet makes, after setting up some notation.

Let X be a smooth projective geometriclly connected curve defined over a number field K, with good reduction at a prime $\mathfrak{p}$ lying over p. Let $\mathcal{O}$ denote the completion of K at $\mathfrak{p}$, with residue field k of characteristic p, and let $\mathcal{X}$ denote a smooth proper model of X over $\mathcal{O}$. Fix an embedding $K \subset \mathbb{C}$, so that we may form the singular cohomology $H^1\big(X(\mathbb{C}), \mathbb{Z}\big)$.

Let T be a correspondence on X (which then extends to a correspondence on $\mathcal{X}$), so that T induces endomorphisms both of $H^1\big(X(\mathbb{C}), \mathbb{Z}\big)$ and of $H^0(\mathcal{X}, \Omega^1_{\mathcal{X}/\mathcal{O}})$. Then we have the following result.

Theorem 1. *If p divides T thought of as an element of* $\operatorname{End}\Big(H^1\big(X(\mathbb{C}), \mathbb{Z}\big)\Big)$*, then p divides T thought of as an element of* $\operatorname{End}\big(H^0(\mathcal{X}, \Omega^1_{\mathcal{X}/\mathcal{O}})\big)$.

Proof. If p divides T as an endomorphism of $H^1\big(X(\mathbb{C}), \mathbb{Z}\big)$ then T annihilates $H^1\big(X(\mathbb{C}), \mathbb{F}_p\big)$. We may interpret this latter group as the "physical" p-torsion in $\operatorname{Jac}(X)$. Thus T annihilates the p-torsion subgroup of $\operatorname{Jac}(X)$, and so is divisble by p in $\operatorname{End}\big(\operatorname{Jac}(X)\big)$. Since X has good reduction, so does its Jacobian, and if we let $\mathcal{A}$ denote the Néron model of $\operatorname{Jac}(X)$ over $\mathcal{O}$ (results of Raynaud identify $\mathcal{A}$ concretely as the identity component of the Picard group scheme $\operatorname{Pic}^0(\mathcal{X}/\mathcal{O})$) then the endomorphism ring of the generic fibre embeds into the endomorphism ring of $\mathcal{A}$. Thus p divides T as an element of $\operatorname{End}(\mathcal{A})$, and thus p divides T as an element of $\operatorname{End}\big(\operatorname{Cot}(\mathcal{A})\big)$. But $\operatorname{Cot}(\mathcal{A}) = H^0(\mathcal{X}, \Omega^1_{\mathcal{X}/\mathcal{O}})$. □

What is the mechanism underlying this argument? It is the fact that there is a group scheme, namely the Jacobian $\operatorname{Jac}(X)$, which "underlies" the mod p singular cohomology (the latter being obtained as its p-torsion); which connects to de Rham-theoretic notions such as differential forms; and which makes sense in positive or mixed characteristic, where "physical" p-torsion is not so well-behaved (instead one has to use the language of group schemes). In fact, we don't really need the entire Jacobian (or its entire Néron model), but just the p-divisble group $\mathcal{A}[p^\infty]$.

An important aspect of Ribet's argument is that it doesn't require any bound on the ramification of $\mathcal{O}$. More modern p-adic Hodge-theoretic arguments (due e.g. to Faltings and Caruso) would give an "integral comparison" between the p-adic étale cohomology of X (which is isomorphic to $H^1\big(X(\mathbb{C}), \mathbb{Z}_p\big)$) and the de

Rham cohomology of $\mathcal{X}/\mathcal{O}$ (which would then contain $H^0(\mathcal{X}, \Omega^1_{\mathcal{X}/\mathcal{O}})$ as the non-trivial piece of its Hodge filtration), from which one could deduce the theorem; but such comparison theorems require that $e(\mathcal{O})$ (the absolute ramification degree) be $< p-1$, and in the context of modular curves, Ribet will ultimately want to take $K = \mathbb{Q}(\zeta_p)$, for which $e = p-1$ on the nose.

It's interesting to consider whether one could prove a version of the theorem for H^2, say of a surface X, relating it to $H^0(\mathcal{X}, \Omega^2)$ for some smooth proper model $\mathcal{X}$ of X. In general, in the integral comparision theorems of p-adic Hodge theory, one needs $ei < p-1$ when studying cohomology in degree i, and so the restrictions on the ramification get even tighter as we progress from H^1 to H^2. On the other hand, there is no obvious analogue of the preceding argument that we could apply, because there is no analogue of the theory of Jacobians that relates to H^2, and to 2-forms, in the way that Jacobians relate to H^1 and 1-forms. (The theory of motives, in Grothendieck's vision of it, is supposed to supply such a theory, in some sense, but my impression is that we are far from having a mixed characteristic theory a la Grothendieck that does this, even for H^2.)

On the other hand, there are other, less geometric, generalizations of the notion of group scheme, or p-divisible group, besides motives. One example is Galois representations; another is Dieudonné modules (particular examples of which can be used to model p-torsion commutative group schemes or p-divisble groups over perfect fields of characteristic p) and Breuil–Kisin modules (which are a version of Dieudonné modules which work over mixed characteristic DVRs).

Prismatic cohomology is a cohomology theory (or, really, a collection of theories, depending on which "prism" you choose) that (for one particular choice of prism) takes values in Breuil–Kisin modules. The H^1 of the theory, when applied to some $\mathcal{X}$ over $\mathcal{O}$, will recover (through the Breuil–Kisin version of Dieudonné module theory that I've already intimated) the p-divisible group of the Picard group scheme of $\mathcal{X}$. The higher cohomology doesn't have such a nice geometric interpretation (one can think of prismatic cohomology as happending on the "realization" side of the approach to motives and cohomology, rather than on Grothendieck's originally envisaged geometric side), but it still provides a bridge between p-adic or mod p étale cohomology and notions related to differential forms and de Rham theory. And the theory works without any bounds on degrees of ramification or cohomology!

References

[Rib83] Kenneth A. Ribet. "Mod p Hecke operators and congruences between modular forms". In: *Invent. Math.* 71.1 (1983), pp. 193–205.