Yet more divided power $S$-ring material.

Let $B$ be an $\mathbb{F}_p$-alg. with $x \in B$ a non-zero divisor.

Then $D(x)(B) = B[[y_1, \ldots, y_k, \ldots]] / (py_1 - x, -1, y_{k+1}, \ldots)$

where $y_k = \frac{x^{p^k}}{(p^k)!}$ is a unit.

$$= B / (xp) [[y_1, \ldots, y_k, \ldots]] / (y_1^p, \ldots, y_k^p, \ldots)$$

(We saw the analogous statement when $B$ is p.t.f. and $x$ is a non-zero divisor in $B/pB$.)

If $B$ is furthermore an $A$-alg., and if $B / \alpha$ is $A$-Nat, then $D(x)(B)$ is $A$-Nat.

(Enough to see that $B / (\alpha)$ is $A$-Nat; for this, consider $B$

\begin{align*}
B & \subseteq D(x) \\
\subseteq B / (\alpha) \\
\subseteq B / (x) \\
\vdots \\
\subseteq B / (x^k) \\
\vdots \\
\subseteq D(x)
\end{align*}

If $x$ is a non-zero divisor in $B$.}
As usual, this generalizes to \((x_1, \ldots, x_r)\) being a regular sequence in \(B\):

i.e. if \(x_1, \ldots, x_r \in B\) is a reg. sequence, and \(B / (x_1, \ldots, x_r)\) is \(A\)-flat, then \(D(x_1, \ldots, x_r) (B)\) is \(A\)-flat.

We can rephrase this as follows: if \(x_1, \ldots, x_r \in B\) is regular relative to \(A\), then \(D(x_1, \ldots, x_r) (B)\) is \(A\)-flat.

This rephrasing lets us promote this statement as follows:

if \(A\) is an artinian ring,

if \(A \rightarrow B\) is \(p\)-completely flat,

if \((x_1, \ldots, x_r) \subset \pi_0 (B)\) is \(p\)-com. regular relative to \(A\),

then \(D(x_1, \ldots, x_r) (B)\) is \(p\)-completely flat over \(A\).

Proof: To check \(p\)-completely flatness, have to check

\[ \text{Kos}(A; p) \otimes \mathbf{A} \]

and can check this after \(\pi_0 (\text{Kos}(A; p)) \otimes \mathbf{A} \).

Altogether, this means we can check after \(\pi_0 (A \otimes \mathbf{A}) \otimes \mathbf{A} \).
This base change induces the derived $p$-completion, and reduces us to considering

$$\pi_0(A^f \otimes \mathbb{F}_p) \to D \left( \pi_0(A^f \otimes \mathbb{F}_p) \otimes B \right)_{(x_1, \ldots, x_r)}$$

This now is flat

$$\pi_0(A^f \otimes \mathbb{F}_p) \to B,$$

with $(x_1, \ldots, x_r)$ being a regular sequence relative to $\pi_0(A^f \otimes \mathbb{F}_p)$

so the theorem follows from our earlier result.

Recall that we proved: if $A$ is a $p$-tilt $S$-ring, and $x_1, \ldots, x_r \in A$ form a regular sequence on $A/pA$, then

$$A \left\{ \frac{\psi(t)}{p}, \ldots, \frac{\omega(t)}{p} \right\} \cong D \left( \frac{x_1, \ldots, x_r}{p} \right)(A).$$

We combine this with the preceding result to prove the following:

If $A$ is a $p$-complete animated $S$-ring, if $B$ is a $p$-completely flat animated $S$-$A$, -$Alg$., and if $x_1, \ldots, x_r \in \pi_0(B)$ is $p$-complety regular relative to $A$,

then $C := B \left\{ \frac{x_1}{p}, \ldots, \frac{x_r}{p} \right\}$ derived $p$-completion is $p$-completely flat over $A$. 
Proof: Two steps: (i) Consider

\[ C' := \text{codom} \{ \frac{\phi(x_i)}{p}, \ldots, \frac{\phi(x_n)}{p} \} \]

by the companion 2 3 4 slice and D.P. envelope that we recalled.

\[ D(x_1, \ldots, x_n) (B_i) \quad \text{which is } p\text{-completely flat over } A. \]

by what we've already proved.

(ii) Replace the \( x_i \) by \( \phi(x_i) \):

\[
\begin{array}{ccc}
A. & \rightarrow & A_\phi x_1, \ldots, x_n \rightarrow B. \rightarrow B_\phi \left\{ \frac{x_1}{p}, \ldots, \frac{x_n}{p} \right\} \\
\downarrow & & \downarrow \\
A. & \rightarrow & A_\phi \left\{ x_1, \ldots, x_n \right\} \rightarrow B. \rightarrow B_\phi \left\{ \frac{\phi(x_1)}{p}, \ldots, \frac{\phi(x_n)}{p} \right\} \\
\end{array}
\]

map is \( \phi \)-compatibly flat.

The two squares are both pushout (i.e., \( \oplus \)-product) squares,

all vertical arrows are \( p \)-completely flat,

\( p \)-completing gives:

\[
\begin{array}{ccc}
A. & \rightarrow & C. \\
\downarrow & & \downarrow \text{p-completely filled} \\
A. & \rightarrow & C'. : A. \rightarrow C'. \text{ p-completely flat as required.}
\end{array}
\]