Simplicial set

+ pts

* joined by

+ some 1-simplices

+ 2-simplices

For each n, a set $X_n$ of n-simplices.

$\partial_i : X_n \to X_{n-1}$ boundary maps.

iso, ..., n.

NB: There are also "degeneracy" maps.

- I'll come to those in a minute.

If we actually glue together simplices according to the instructions given by $\partial_i$, we get a top. space $|X|$, the "geometric realization" of $X$. 
So simplicial sets are a combinatorial model for top. spaces and (especially) their homotopy theory.

**Product** The interval \([0,1] \times [0,1]\) is a simplicial set.

Let's form its product with itself:

\[
\begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\bullet & \rightarrow & \bullet
\end{array}
\]

This doesn't look "simplicial."

Let's triangulate it in the usual way:

\[
\begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
& & \\
\bullet & \rightarrow & \bullet
\end{array}
\]

\[
\begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
& & \\
\bullet & \rightarrow & \bullet
\end{array}
\]
We have \(3\) 1-simplices in this picture.

If we project them onto each of the axes, only the diagonal actually projects to the 1-simplex in the interval itself.

The edge projects to a point in either the horizontal or vertical axis.

This suggests adding "constant" 1-simplices (one at each point), so that all \(3\) 1-simplices in the (triangulated) product arise as products of 1-simplices in the two intervals we are taking the product of.

There are two 2-simplices in the product. If we project them onto the two axes, neither stays a 2-simplex.

E.g., the upper right 2-simplex, when we project it to the vertical axis, gets crushed along the interval \(j\). Two of its edges are also "glued" to the interval \(j\).
3rd edge project to the "constant" 1-simplex at the top vertex.

Similarly, the lower right 2-simplex projects to a "crushed" 2-simplex lying along the interval.

2 of its edges also project to this interval. The 3rd projects to the constant 1-simplex at the bottom vertex.

So we should add these "crushed", or "degenerate", 2-simplices to the ones under as well:

Now the 2-simplices in

arise as products of non-degenerate 2-simplices in

Since now has 3 1-simplices, there are 9 product 1-simplices in the product square. 5 of them are the "visible", non-degenerate 1-simplices; the other 4 are degenerate, or constant, at the 4 vertices of the square.
And since each interval factor now has two 2-simplices, both degenerate, we have 4 2-simplices in the product square.

Two of them are the "trivial", "non-degenerate" 2-simplices.

The other 2 are degenerate.

They are the 2 degenerate 2-simplices lying along the diagonal 1-simplex.
to be consistent/general

But shouldn't we also have degenerate 2-simplices along the 4 edges?

To get there, we need to add more
degenerate 2-simplices to each interval—
namely a degenerate 2-simplex at each vertex.

Each interval actually has 4 2-simplices:

- The product has 16 2-simplices:
  - 2 non-degenerate
  - \(2 \times 5 = 10\) degenerate 2-simplices along the 5 edges
  - \(4 \times 1 = 4\) degenerate 2-simplices at each vertex.

To be consistent, we should add degenerate 3-simplices, 4-simplices, etc., as well.
These products will all be degenerate, though, so we don't "see" them in the square.
In general, a simplex, or determined by an ordered set of vertices, maps b/w them are determined by maps on the vertices.

Face are ordered in an ordered set of vertices, \( \Delta_n \rightarrow \Delta_n \) for ordered vertices, \( \Delta_{n-1} \rightarrow \Delta_{n-1} \).

In general, a map \( \Delta_m \rightarrow \Delta_n \) factors as:

\[
\Delta_m \rightarrow \Delta_k \rightarrow \Delta_n
\]

degenerate \( m \)-simplices supported on a \( k \)-face of an \( n \)-simplex.

So a simplicial set \( X \) gives a collective \( X_n \) of \( n \)-simplices for each \( n \), and if \( \Delta_m \rightarrow \Delta_n \) is a set \( X_m \rightarrow X_n \), each \( n \)-simplex gives rise to correspondingly \( n \)-simplex.

In short, \( \Delta \rightarrow \text{Cat} \) of finite ordered sets with order-preserving maps.
Then a simplicial set is a contravariant functor
\((-\text{pre} \text{-} \text{sheaf})\)

\[ X : \Delta^{\text{op}} \to \text{Sets}. \]

On itself is the presheaf represented by \( \Delta_n \) for \( n \geq 0 \).

Yoneda gives \( \Delta \to \text{Sets} \), e.g.

\[ \text{Hom}_{\text{Set}}(\Delta_n, X_n) = X_n. \]

So if all works out!

\( X \times Y \) is just the product functor,

\[ (X \times Y)_n = X_n \times Y_n, \]

and the above example illustrates how this
really matches with the intuitive "geometric" product.
More precisely, if $I I$ denoted geometric realization, as above, then $I I$ commutes with Hurewicz fibrations.

In particular, $| X \times Y | = | X | \times | Y |$.

If $T$ is a top. space,

$$\text{Sing}_T (T) = \text{set of simplices in } T \text{ in } T$$

where $\text{Sing}_T (T) = \text{Maps} (\Delta^m, T)$.

In fact, $I I$ is left adjoint to $\text{Sing}$.

(The preceding def. is Mii's, but applied to the object $\Delta^m$.)

$: I I \text{ preserves arbitrary colimits.}$

$| \text{Sing}_T (T) | \to T$ is a weak equivalence ("nerve replacement").
We also have maps \( X \to \text{Sing}_0(1 \times 1) \), which will also be a weak equivalence - a “distant replacement” - as I now try to explain.

We can do homotopy theory of simplicial sets, but not all s. sets are well-adapted to this.

E.g. \( \Delta^n \)

\[
\begin{array}{ccc}
0 & \rightarrow & 1 \\
\downarrow & & \downarrow \\
2 & \rightarrow & 3
\end{array}
\]

This is much smaller than \( \text{Sing}_s(1 \times 1) \).

The adjunction gives

\[
\Delta^2 \rightarrow \text{Sing}_s(1 \times 1)
\]

But in the latter, we have a 1-simplex \( \gamma \) joining 0 & 1 by passing around the top (indeed, many such, depending on parameterization), and degenerate 2-simplices filling in \( \gamma \), and of course many more simplices.
A 2-simplex "filling in"

with another edge and 2-face

is called a "filling of the "horn"

Simplicial sets which admit all horn fillings are called Kan complexes.

E.g., $\text{Sing}(C(-))$ is always a Kan complex.

Top. spaces have a "model structure" with weak equivalence, being weak homotopy equivalence, and fibrations being Serre fibrations.

Note that any map $T \rightarrow pt.$ is a Serre fibration, so all objects are fibrant.
7a. "matching" model structure on $S^2$ Feb. 16
("Quillen model $q$" again!)

11: $S^2$ -- $\rightarrow$ spaces

is a Quillen equivalence.

\[ \otimes \quad X \rightarrow Y \quad \text{ is a weak equiv} \]

iff

\[ |X| \rightarrow 1 \quad \text{ is.} \]

\[ \otimes \quad \text{ fibrations are just inclusions } X \rightarrow Y. \]

\[ \otimes \quad \text{ fibrations have to satisfy a homotopy lifting property in lieu of "horn-hitting"}. \]

If $X \rightarrow \Delta^n$ is a commutative diagram,

\[ \begin{array}{ccc}
X \rightarrow \\
\downarrow \quad \quad \downarrow \quad \\
X \rightarrow Y
\end{array} \]

we should be able to lift as indicated.

In particular, not all objects are fibrant; evidently the Kan complexes are.

8b. There is another natural model structure on each the "Joyal model $q$" in which has to do with co-simplicial sets; in this picture, simplicial sets are related to categories rather than spaces.
Internal hom: \( \text{Hom}(X, Y) \) is a set, defined by

\[
\text{Hom}(U, \text{Hom}(X, Y)) \\
\cong \text{Hom}(U \times X, Y)
\]

So it is enriched over sets.

If \( Y \) is Kan, then \( \text{Hom}(X, Y) \) is Kan.

So if we restrict to \( \text{Kan} \) complexes, \( \text{Sset} \), we get a "locally Kan" simplicial category, one model for \( \infty \)-cats.

This gives the \( \infty \)-cat of "anima" (\(-\)-animated) sets.
Now we consider simplicial $\mathcal{G}$, e.g. $\mathcal{G}$, rings.

$X : \text{Set} \to \text{Grp} / \text{AbGrp} / \text{Ring}$

Forgetful functor:

$\text{s-Grp.} \to \text{s-Gp.} 
\text{s-Ab} 
\text{s-Ring}$

Left adjoint:

$\text{s-Set} \to \text{s-Grp.} 
\text{s-Ring}$

Free gp. / ab gp. / ring.

Quillen adjunction: $\ast$ right adjoint preserves fibrations (so we can left fibrations on underlying $\text{s-Set}$)

* left adjoint preserves colimitation (so only certain objects, namely free objects, are known to be colimitant objects)
Lemma: Any simplicial gap is a Kan complex.

So all objects are fibrant. But not all objects are cofibrant.

If \( X \) is a fibrant free object, then \( X \) is cofibrant (at least in the gap/ring case).

Using "free resolutions", we can replace any \( X \) by a weakly equivalent cofibrant object \( \tilde{X} \).

\[ \tilde{X} \sim X. \]

This is again a "locally Kan" simplicial category, i.e., an \( \infty \)-cat.

"Animated abelian gap?"

"Animated rings?"
Basic Explanations

Simplicial set structure:

\[ X, Y, \quad \text{s.-gps or rings} \]

Then if \( U \) is any \( s \)-set,

\[ \text{Hom}(U, Y) \text{ is not just a set, but is a s-gp/ring.} \]

and \( \text{Hom}(U, Y) \) is a \( s \)-gp or \( s \)-ring.

(Just as maps from any space to a top gp or ring form a gp or ring, via pointwise operations.)

\[ \text{Hom}(U, \text{Hom}(X, Y)) \]

\[ \text{set} \quad s \text{-set} \]

\[ \text{Hom}(U \times X, Y) \]

\[ \text{s-sets, gp or ring hom in 2nd variable} \]

\[ \text{Hom}(X, \text{Hom}(U, Y)) \]

\[ \text{s-gps, } \quad \text{s-gp or s-ring} \]

define \( \text{Hom}(X, Y) \) in terms of things we already know.
Shift to example (non-free!)

\[ \mathbb{Z}/n\mathbb{Z} \text{ simple ex. of an ab. gp} \]

hom. e.g. p.o.v.
\[ a: \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0 \]

simplcial p.o.v.:

\[ \mathbb{Z} : \quad \text{set of} \quad 0 \text{-simplices,} \]

Want to identify 0 \& 1
- Introduce a 1-simplex

Perform gp. operation on \( f \) to get \( Z : f \)

\[ Z : f \quad \overset{\partial_1}{\longrightarrow} \quad Z \]

\[ \mathcal{D}_1 (x) = n \]

\[ \mathcal{D}_0 (x) = 0 \]
degenerate 1-simplex:

$$\Delta : \mathbb{R} \rightarrow \mathbb{R} \cdot s(\tau)$$

So actually, need

$$\mathbb{R} \langle \tau, s(\tau) \rangle \xrightarrow{d_1} \mathbb{R}$$

$$d_1 \tau = 0 \quad d_1 s(\tau) = 1$$

$$d_0 \tau = 0 \quad d_0 s(\tau) = 1$$

We are trying to make a coboundary s-aba. If that is $\text{u.equiv. to } \mathbb{Z}/n\mathbb{Z}$ (discrete sgp),

so we need to fill in the loop.

Add $\Delta$ to $\tau$ - simplex

$$\Delta : \mathbb{R} \rightarrow \mathbb{R} \cdot s(\tau)$$
More general set-up:

\[ \begin{array}{c}
B & \overset{2_1}{\leftrightsquigarrow} & A \\
\downarrow & \sigma_0 \downarrow \\
\end{array} \] \quad 2_1 s = 2_0 s = i_0 A

(For \( a_1, B = 2 < \gamma, s(a_1), A = 2 \))

Given \( b_1, b_2 \in B \), we get a \("\" \text{loop}\"

\[ \begin{array}{c}
\sigma_0(b_1) + \sigma_0(b_2) & \overset{2_1}{\rightarrow} & \sigma_1(b_1) + \sigma_1(b_2) \\
\sigma_1(b_1) + \sigma_0(b_2) & \rightarrow & 2_1(s \sigma_1(b_1) + b_2) \\
\sigma_1(b_1) + 2_0(b_2) & \rightarrow & s \sigma_1(b_1) + b_2 \\
2_1(b_1) + 2_0(b_2) & \rightarrow & b_1 + b_2
\end{array} \]

(Previous example was \( b_1 = b_2 = \gamma \))
Add a 2-simpler labelled by \((b_i, b_2)\), that fills in this loop.

Actually the loop only depend on \((b_i, b_2) \in B \otimes B\), \(\text{i.e., } (b_i, b_2) \in A\), \((b_i + s_i(a), b_2 - s_i(a))\) give same loop, if \(a \in A\).

\[\begin{array}{c}
\text{define} \\
\begin{array}{c}
B \otimes B \\
A
\end{array}
\rightarrow B
\end{array}\]

Via
\[\begin{array}{c}
(b_i, b_2) \\
\rightarrow b_i + s_i(b_2) \\
\rightarrow b_i + b_2 \\
\rightarrow s_i(b_i) + b_2
\end{array}\]

But now, having added all these 2-simplices, we get unwanted 2-spheres that have to be filled in with 3-cells, and so on.
Here is part of one such:

\[ \partial_i (b_1) + \partial_0 (b_2) + \partial_0 (b_3) \]

\[ \partial_1 (b_1) + \partial_1 (b_2) + \partial_1 (b_3) \]

\[ \partial_2 (b_1) + \partial_2 (b_2) + \partial_2 (b_3) \]

\[ (b_1 + b_2, b_3) \]

\[ \text{(The other 2 faces are labelled by) (b_1, b_2, b_3) } \text{ and } (b_1, b_2, s\partial_0 (b_3)) \]

In general, we add n-cells labelled by \( d_\alpha \) and \( b_\alpha \).
\[ d_0 : (b_1, \ldots, b_n) \mapsto (s \delta_1 (b_1) + b_2, \ldots, b_n) \]

\[ d_i : (b_1, \ldots, b_n) \mapsto (b_1, \ldots, b_i + b_{i+1}, \ldots, b_n) \]

\[ \delta_i \text{ for } i = 1, \ldots, n \]

given by inserting "o" in the
position.
Now \( B = s(A) \oplus \ker d_0 \)

\[ B^\mathbb{R} = s(A) \oplus (\ker d_0) \oplus \mathbb{R} \]

so we can unwrap all this pretty explicitly.

**Lemma**

If \( \exists_1 : \ker d_0 \to A \) is injective, then

\[ B \cong A \]

is weakly equiv. to \( A / \exists_1(\ker d_0) \)

How to check such a thing?

**Thm**

If \( X \) is a simplicial e.g. qf,

let \( \text{tot}(X) \) = chain complex constructed from \( X \).

Then \( H_i(\text{tot}(X)) = \pi_i(X) \)
In our example, with $X_n = B^n$, we have that

$$\ker d_i \rightarrow \ker d_{i+1}$$

Proof:

The case of simple differentials!

Taking total complex & apply.

Since Lema gives

$$0 \rightarrow H_1(\text{tot}(\xi)) \rightarrow \ker d_i \rightarrow A$$

& all higher $H_i(\text{tot}(\xi)) = 0$. \hfill \Box

This proves the lemma, given the theorem.

In particular, it gives $\pi_1$ simplicial null for $\mathbb{Z}/n\mathbb{Z}$. 
The map is related to the

\[ \text{Dold-Kan equivalence} \]

\[ X \rightarrow \operatorname{tot}(X^\otimes) \]

gives an equivalence to \( \infty \)-cats

"animated as \( \infty \)-gps."

(\( \infty \)-cat arising from

simplified ab gp)

praises

The K-theory says that \( K_\cdot \rightarrow K_\cdot \) under

this equivalence.

Lemma: It we reduce \( \operatorname{tot}(X^\otimes) \) by a certain quasi-isomorphic

subcomplex, we even get an equal. by a

simplified ab gp. and chain complex of ab gp.
This is more elementary to state, and not so hard to prove, and gives the above oo-cat. version by inverting weak equivalence on each side. But oo-version seems more natural to me.

**Ex:** Suppose $A$ is an ab spf, place in hom. degree.

The the corresponding $X_\ast$ is a simplicial ab spf, whose underlying set is $K(A, n)$.

Why should $K(A, n)$ come from an ab spf object? B/c it is an oo-loop space!

$$(K(A, n) = \coprod K(A, n+1)).$$

Recall how to make a $K(A, 1)$:

Fix a pt.: this will be to (i.e. $X_0$ is a Hausdorff, irreducible...
Add loops labelled by elt. of $A$:

so $x_1 = A \Rightarrow x_0 = 0$

Now we want $[x+b] \sim [x] + [b]$, :
all $A_i$'s labelled by pair of elt,

so $x_2 = A \oplus A$, ...

In fact we are in a special case of the
we considered above! $A \rightarrow 0$

$x_n = \text{lin } A \oplus n$

and so $\begin{array}{c}{\text{lin } (\text{lin } (x_1))} \\ = \begin{cases} \text{lin } (A \rightarrow 0) \\ 1 \end{cases} \\ = \begin{cases} A \text{ in deg. } 1 \\ 0 \text{ in all other degrees} \end{cases} \end{array}$
This example illustrates how to prove the Dold-Kan equivalence, and why
\[ \Pi. C \rightarrow H. \]

For full proof, can combine this basic idea with Postnikov tower technique. (while a simplicial set/\ast \text{-}\text{sp.}
\text{as an \textit{ill.}}
\text{related by } K(\pi,1) / S)

\[ X \rightarrow Y \text{ is a morphism if } \]
\[ \text{simplicial, } \text{it is a homotopy iff the induced} \]
\[ \text{map } X^\circ \rightarrow Y^\circ \text{ is surjective term by term.} \]
\[ \text{comp. of identity } \quad (\text{This holds if e.g.} \]
\[ X \rightarrow Y^\circ \text{ is itself surjective term by term.} \]
If \( \{ X(n) \} \) is an inverse sequence of simplicial abelian groups, we can replace it by a sequence of fibrant transition maps:

\[
\begin{align*}
Y(n+1) & \overset{\sim}{\longrightarrow} X(n+1), \\
\downarrow \text{fib.} & \quad \downarrow \\
Y(n) & \overset{\sim}{\longrightarrow} X(n),
\end{align*}
\]

and now the transition maps for the \( Y(n) \)'s are "almost" surjective, possible only if failure of surjectivity is an \( \pi_0 \)'s.

And so \( \text{tot}(\varprojlim Y(n)) = \varprojlim \text{tot}(X(n)) \)

Using homological conventions, \( \text{tot} \text{lim} \) is in degree -1.
So if \( X(n) \) is an inverse sequence for which \( \text{Rlin} \subseteq \text{xt}(X(n)) \) is in degree \( 0 \),

then we can compute the \( \text{Rlin} \) in the simplicial world as a homotopy inverse limit.

\[ \text{Note: } \text{Rlin} \text{ satisfies a universal property in } D(A) \]

(as a limit in an \( \infty \)-categorical sense)

and so \( T_\geq 0 \text{Rlin} \subseteq \text{does also, on } D_\geq 0 (A) \).

Similarly, the homotopy inverse limit computed by replacing the \( X(n) \) by \( Y(n) \) computes an \( \infty \)-categorical limit. Thus the matching of the homotopy inverse limit with \( T_\geq 0 \text{Rlin} \subseteq \text{under the Dold-Kan equivalence also holds for general accepted reasons.} \]