Simplicial Ring

("Ring" = commutative ring at t)

If \( B \) \& \( C \) are \( A \)-algebras, then \( B \otimes_A C \) is an \( A \)-algebra, not just a module.

What about \( B \otimes_A C \)?

Using Dold-Kan, we can compute this on the simplicial side:

we choose a cofibrant replacement (for \( i \), say) \( X_0 \xrightarrow{\sim} C \) weak equiv.

with each \( X_n \) a polynomial alg. over \( A \).
Then \( \mathcal{B} \otimes X. \)

is a simplicial ring, and

\[
\text{Tot}(1 \otimes X.) \otimes \mathcal{B} \otimes C \quad \text{in} \quad D(A_{\text{-mod}}).
\]

(In short, if \( M. \), \( N. \) are \( A \)-modules, and one, say \( N. \), consists of flat \( A \)-modules, then \( M. \otimes N. \) -

\[
\text{computed degree-wise}
\]

\[
\text{via} \quad D(A_{\text{-mod}}). \quad \text{You can check directly, or use general universal property arguments in some appropriate derived/co-categorical context.}
\]
Recall e.g. that if \( f : M \rightarrow M \) is given by

\[
\begin{align*}
R\lim_{i} & \quad (M \xrightarrow{f} M) \\
& = \lim_{i} (M \otimes A/p^n)
\end{align*}
\]

So working simplicially, we can derive \( f \)-complete.

In an \( A \)-algebra, by doing \( \otimes \) on the simplicial ring side, and then taking a homotopy limit. (Recall that, working cohomologically, \( \lim_{i} \) is in degree \(-1\) \( \otimes \) 0, and the derived completion doesn't spill over into degree 1.)

Let \( R\lim M/p^nM = 0 \) since the transition maps are surjective. Thus, shifting to homological grading, we have...
To mimic \( \lim \) in No context, and so the homotopy inverse limit approxima the correct object.

So if \( B \) is an \( A \)-algebra, its derived f-completion, (or, better, "animated" \( A \)-alg)
is a simplicial \( A \)-algebra \( \hat{B} \) s.t.

\[
\text{Tot}(\hat{B}) = \text{derived completion of } B \text{ as an } A\text{-module.}
\]

This extends to f.g. ideals in \( A \).

The expression, \( B \otimes_{A/f^nA} \) appearing in the \( \lim \) computing \( \hat{B} \) are also interesting simplicial rings. In general, if \( B \) is an \( A \)-algebra, then the (almost Koszul) map

\[
A \xrightarrow{a} A
\]

which we use to compute \( \hat{B} \) has a nice simplicial ring realization.
Consider the universal case first:

\[ x \in \mathbb{Z}[x]. \]

\[ (\mathbb{Z}[x] \xrightarrow{x^x} \mathbb{Z}[x]) \xrightarrow{q_i} \mathbb{Z} \quad \text{for } i \in k, \]

so we want to think about how to present \( \mathbb{Z} \) as a \( \mathbb{Z}[x] \)-algebra.

\[ \mathbb{Z}[x, y] \xrightarrow{y \mapsto x} \mathbb{Z}[x] \]

Add a 1 - simple \( y \) which joins \( 0 \) and \( x \).

Abstractly, we have \( B \xleftarrow{d_0} A \) \xrightarrow{d_1} \), where \( s \) makes \( B \) an \( A \)-algebra.
So we're in a similar situation to the one we had with presenting $W/V$ as a cofibrant simplicial abelian group.

Since we have $A$-algebras now, we turn the coproduct into an $A$-algebra.

\[
\begin{array}{ccc}
\rightarrow & B^A \& \rightarrow & B^A_{n-1} & \cdots \\
\rightarrow & \uparrow & & \uparrow \\
& b_1 \& \rightarrow & b_1 \& \rightarrow & b_1 \& \rightarrow & \cdots \& \rightarrow \cdots \\
& \vdots & & \downarrow & & \downarrow \& & \downarrow \\
& b_1 \& \rightarrow & b_1 \& \rightarrow & b_1 \& \rightarrow & \cdots \& \rightarrow \cdots \\
& \vdots & & \downarrow & & \downarrow \& & \downarrow \\
& b_1 \& \rightarrow & b_1 \& \rightarrow & b_1 \& \rightarrow & \cdots \& \rightarrow \cdots \\
& \vdots & & \downarrow & & \downarrow \& & \downarrow \\
& b_1 \& \rightarrow & b_1 \& \rightarrow & b_1 \& \rightarrow & \cdots \& \rightarrow \cdots \\
\end{array}
\]
It's fairly easy to check that

$$\text{Tot} \left( \begin{array}{c} 2 \circ \cdots \cdots \circ 2 \\ 2n \end{array} \right) \rightarrow 2^S_{n=0}$$

i.e.

$$\xrightarrow{\pi^i} \cup \delta_{x=0}$$

$$\pi_i = 0 \quad \text{for } i \geq 1.$$

Now applying $A \otimes -$ to $\times_{x=0} 2^n$ we have

Then

$$\xrightarrow{\pi^i} \xrightarrow{\otimes} A$$

Hence

$$\text{Tot} \left( \begin{array}{c} (B \otimes \cdots \otimes B) \\ 2n \end{array} \right) \rightarrow (A \xrightarrow{\pi} A)$$

$$\xrightarrow{A[0, \cdots, 0]}$$

So

$$\left( A[0, \cdots, 0] \right)_{n=0}$$

is a simplicial ring that makes the Koszul complex $A \xrightarrow{\pi} A$.\]
Can do the same thing for a sequence \( a_i, \ldots, a_n \) — just set

\[
B = A[y_i, \ldots, y_r] \xrightarrow{y_i \mapsto a_i} A \xrightarrow{y_i \mapsto 0} \]

So we can form an animated \( A \)-algebra

\[
\text{Kos}(A; a_i, \ldots, a_n).
\]

And if \( B = A \)-alg., we can describe its derived \( \mathcal{I} = (a_i, \ldots, a_n) \)-adic completion as

\[
B = \text{homotopy invariant } B \xrightarrow{\text{Kos}(A; a_i, \ldots, a_n)} A
\]

again as "animated \( A \)-alg."