Simplicial Rings

("Ring" = commutative ring w/ 1)

If B & C are A-algebras, then

$B \otimes_A C$ is an A-alg., not just a module.

What about $B \overset{L}{\otimes}_A C$?

Using Dold-Kan, we can compute this on the simplicial side:

we choose a cofibrant replacement (for C, say)

$$X_\bullet \xrightarrow[\text{weak equiv.}]{\sim} C$$

with each X_n a polynomial alg. over A.

Then $B \underset{A}{\otimes} X_{\bullet}$

is a simplicial ring, and

$$\mathrm{Tot}\left(B \underset{A}{\otimes} X_{\bullet}\right) \xleftarrow{\sim} B \overset{\mathbb{L}}{\underset{A}{\otimes}} C \quad \text{in } D(A\text{-mod}).$$

(In general, if $M_{\bullet}$, $N_{\bullet}$ are simplicial A-modules, and one, say $N_{\bullet}$, consists of flat A-modules, then

$$\underbrace{M_{\bullet} \underset{A}{\otimes} N_{\bullet}}_{\text{computed degree-wise}} \longrightarrow \mathrm{Tot}(M_{\bullet}) \overset{\mathbb{L}}{\underset{A}{\otimes}} \mathrm{Tot}(N_{\bullet})$$

via Dold-Kan. You can check directly, or use general universal property arguments in some appropriate derived / ∞-categorical context.)

Recall e.g. that if $f \in A$,

then derived f-adic completion of an A-module M is given by

$$R\underrightarrow{\lim}\,(M \xrightarrow{f^n} M)$$

$$= R\underrightarrow{\lim}\,(M \overset{L}{\underset{A}{\otimes}} A/f^n)$$

So working simplicially, we can derived f-complete an A-algebra, by doing $\overset{L}{\underset{A}{\otimes}}$ on the simplicial ring side, and then taking a homotopy limit. (Recall that, working cohomologically, $M \xrightarrow{f^n} M$ is in degrees -1 & 0, and the derived completion *doesn't* spill over into degree 1, b/c $R^1\underrightarrow{\lim}\, M/f^nM = 0$ since the transition maps are surjective. Thus, shifting to homological gradings, we have

$\tau_{\geq 0} R^{\#}\varprojlim = R\varprojlim$ in this context,

and so the homotopy inverse limit computes the correct object.)

So if B is an A-algebra (or, better, "animated" A-alg.) its derived f-completion, is a simplicial A-algebra λ $\hat{B}_\bullet$ s.t.

$\mathrm{Tot}(\hat{B}_\cdot)$ = derived completion of B as an A-module.

This extends to f.g. ideals in A.

The expressions $B \overset{L}{\otimes} A/f^n A$ appearing in the $R\varprojlim$ computing $\hat{B}_\bullet$ are also interesting simplicial rings. In general, if $a \in A$, then the Koszul complex $A \xrightarrow{a} A$ which one use to compute $- \overset{L}{\otimes} A/_a A$ has a nice simplicial ring realisation.

Consider the universal case first:

$$x \in \mathbb{Z}[x].$$

$$\left(\mathbb{Z}[\cancel{x}] \xrightarrow{\times x} \mathbb{Z}[x]\right) \underset{q.i.}{\simeq} \underset{\substack{\circlearrowleft \\ x=0}}{\mathbb{Z}},$$

so we want to think about how to freely present $\mathbb{Z}$ as a $\mathbb{Z}[x]$-algebra

$$\begin{tikzcd} \mathbb{Z}[x,y] \arrow[r, shift left=2, "y \mapsto x"] \arrow[r, shift right=2, "y \mapsto 0"'] & \mathbb{Z}[x] \arrow[l] \end{tikzcd}$$

Add a 1-simplex y which joins 0 & x.

Abstractly, we have $\begin{tikzcd} B \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & A \arrow[l, "s"] \end{tikzcd}$, where s makes B an A-algebra

So we're in a similar situation to the one we had w/ presenting $\mathbb{Z}/n\mathbb{Z}$ as a cofibrant simplicial abelian gp.

Since we have A-algebras now, we form the coproduct as A-algebra

$$\cdots \;\overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}}\; B^{\otimes_A n} \;\overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}}\; B^{\otimes_A n-1} \cdots \rightrightarrows B \rightrightarrows A$$

$$b_1 \otimes \cdots \otimes b_n \overset{d_0}{\longmapsto} s d_1(b_1) b_2 \otimes \cdots \otimes b_n$$

$$\vdots$$

$$\overset{d_i}{\longmapsto} b_1 \otimes \cdots \otimes b_i b_{i+1} \otimes \cdots \otimes b_n$$

$$\vdots$$

$$\overset{d_n}{\longmapsto} b_1 \otimes \cdots \otimes b_{n-1}\, s d_0(b_n)$$

$$s_i : b_1 \otimes \cdots \otimes b_{n-1} \longmapsto b_1 \otimes \cdots b_{i-1} \otimes 1 \otimes b_i \cdots \otimes b_n.$$

It's fairly easy to check that

$$\mathrm{Tot}\left(\left(\mathbb{Z}[x, y_1, \dots, y_n]\right)_{n\geq 0}\right) \xrightarrow[q.i]{} \mathbb{Z}\,\mathcal{O}_{x=0}$$

i.e. $\pi_0 = \mathbb{Z}\,\mathcal{O}_{x=0}$

$\pi_i = 0$ if $i \geq 1$.

Now applying $\underset{\substack{a \\ \uparrow \\ x}}{A} \underset{\mathbb{Z}[x]}{\otimes} -$, we find that if

$$B = A[y] \underset{y \mapsto 0}{\overset{y \mapsto a}{\rightrightarrows}} A ,$$

then $\mathrm{Tot}\left(\left(B^{\otimes_A n}\right)_{n\geq 0}\right) \xrightarrow[q.i]{} (A \xrightarrow{a} A)$

$B^{\otimes_A n} = A[y_1, \dots, y_n]$

so $\left(A[y_1, \dots, y_n]\right)_{n\geq 0}$ is a simplicial ring that makes the Koszul complex $(A \xrightarrow{a} A)$.

Can do the same thing for a sequence
$a_1, \ldots, a_r \in A$ — just set

$$B = A[y_1, \ldots, y_r] \underset{y_i \mapsto 0}{\overset{y_i \mapsto a_i}{\rightrightarrows}} A .$$

So we can form an animated A-algebra

$$\mathrm{Kos}(A; a_1, \ldots, a_r).$$

And if $B = A$-alg., we can describe its derived $I = (a_1, \ldots, a_r)$-adic completion as

$\hat{B}_\circ$ = homotopy inverse limit of $B \overset{L}{\underset{A}{\otimes}} \mathrm{Kos}(A; a_1, \ldots, a_r)$,

again an "animated A-alg.".