Some manipulations with simplified rings

If $A_i$ is a simplified ring, $(\oplus_{i=0}^n A_i)$ is a graded ring. We can think of $A_i$ as a kind of "deformation" of the graded ring $(\oplus_{i=0}^n A_i)$.

Similarly, if $M_i$ is a simplified $A_i$-module, then $(\oplus_{i=0}^n M_i)$ is a graded module over $(\oplus_{i=0}^n A_i)$, and $M_i$ is a kind of deformation of $(\oplus_{i=0}^n M_i)$.

Now if we work up to weak equivalence, i.e., with "animated rings" and "animated modules", then $A_i$ is not well-defined (in some sense, we can replace it by something weakly equivalent), so we have to be thoughtful before defining notions like ideals, regular sequences, localization, etc.

E.g. The degeneracy maps $A_0 \to A_i$ (120) make all the $A_i$'s $A_0$-algebras, so $A_i$ is a simplified $A_0$-alg.

But $A_0$ is not intrinsic (i.e., not preserved by weak equivalence), so it doesn't make sense directly to localize at an element $f \in A_0$, or to talk about a sequence of elements in $A_0$. 
Indeed, notions such as localization / regular sequence should be defined with reference to \( \pi_0(A_\cdot) \), which is intrinsic, even though \( A_\cdot \) is not a simplicial \( \pi_0(A_\cdot) \)-alg.

**Ex.** Let \( S \subseteq \pi_0(A) \) be a multiplicative subset, let
\[ S_{\text{max}} = \{ f \in \pi_0(A) \mid fg \in S \forall g \in \pi_0(A) \} \]
(so that \( S_{\text{max}} \) is maximal multiplicative subset of \( S \)),

for which
\[ \pi_0(A_\cdot)^{S_{\text{max}}} \to \pi_0(A_\cdot)^{S} \]

is an isomorphism.

Let \( \tilde{S} = \text{preimage of } S \text{ in } \pi_0(A) \) (a saturated subset of \( \pi_0(A_\cdot) \)), and let \( T \subseteq \tilde{S} \)

be any multiplicative subset of \( A_\cdot \) s.t.
\[ (\tilde{S})_{\text{max}} = (T)_{\text{max}} \]

then \( \pi_0(A_\cdot)^T \to \pi_0(A_\cdot)^S \)

is a weak equivalence.

**Pf:** Localization is exact and so on homotopy this induces
\[ \pi_i(M_\cdot)^T \to \pi_i(M_\cdot)^S \]
But the $A_0$-action on $\pi_i(M)$ factors through $\pi_0(A_0)$, and so this reduces to the equality

$$\pi_i(M)_T = \pi_i(M)^{A_0}$$

(but equal set).

**Example.** If $u \in \pi_0(A_0)^*$, then if $f : A \to A_0$ is a $u$-equivariant map, we find that

$$A \to (A)_f$$

is a w.e.

(take $S = \mathbb{Z}, T = \mathbb{Q}$).

Thus, we may assume that $u$ is in the image of $A_0^* \to \pi_0(A_0)$, if we are willing to replace $A$ by something weakly equivalent.

**Example.** If $A \to B$ is a weak equivalence, if $S \subseteq \pi_0(A_0)$ is a multiplicative subset, and if we let $T_A, T_B$ be preimages of $S$ in $A_0, B_0$ respectively, then

$$A \to B$$

by the same "exhaustion of localization" argument, and so we find that "localization of $A_0$ over $\text{Spec } \pi_0(A_0)$" makes sense, to yield...
a sheaf (in the co-categorical sense) of animated rings over \( \text{Spec } \mathbb{R}(A) \) — this gives (one version of) the derived scheme attached to \( A \). (Thought of as an animated ring.)

If \( M \) is a simplicial \( A \)-module, thought of as giving rise to an animated \( A \)-module over \( A \) as an animated ring, then the same localization process gives rise to a sheaf of animated modules over \( \text{Spec } \mathbb{R}(A) \) on which the sheaf of animated rings obtained by localizing \( A \) acts.

If \( A \) is a ring, and \( f_1, \ldots, f_r \in A \), then we have seen that the Koszul complex
\[
\text{Kos} (A; f_1, \ldots, f_r)
\]
admits a simplicial ring structure via the formula
\[
\text{Kos} (A; f_1, \ldots, f_r) := A \otimes \bigotimes_{i=1}^{r} \mathbb{R}(A)_{f_i}.
\]

We want to extend this to the context of \( A \), a simplicial (or animated, really) ring.

So now we should imagine \( f_1, \ldots, f_r \in \mathbb{R}(A) \).
If we lift \( f \) to \( g \in A_0 \), then

the \( A_0 \) structure on each \( A_i \) makes

\( A_i \) a simplicial \( \mathbb{Z}[x_1, \ldots, x_n] \) -algebra, and so we can define

\[
Kos(A_i; x_1, \ldots, x_r) := A_i \otimes_{\mathbb{Z}[x_1, \ldots, x_r]} \mathbb{Z}
\]

But first we need to check this is independent of the choice of the \( g \).

Check one \( i \) at a time, i.e., assume \( i = 1 \), and suppose \( g, h \in A_0 \) both lift \( f \in \pi_0(A_i) \).

Consider

\[
\text{Hom}(A_1, A_0) \quad \text{in } \text{set of morphisms} \quad \text{en}_1 \quad \text{en}_0 \quad \text{in } A_0
\]

The two evaluation maps are weak equivalences.

To say that \( g, h \in A_0 \) lift \( f \in \pi_0(A_i) \) is to say there is a path \( \gamma \in A_i \) s.t.

\[
d_0(\gamma) = g \quad d_1(\gamma) = h.
\]
Now \( \text{Hom}(\Delta_1, A.)_n = \text{Hom}(\Delta_n, \text{Hom}(\Delta_1, A.)) \)
\[= \text{Hom}(\Delta_n \times \Delta_1, A.) \]

In particular, \( A_{-1} = \text{Hom}(\Delta_1, A.) \)
\[= \text{Hom}(p^1 \times \Delta_1, A.) \]
\[= \text{Hom}(\Delta_1, A.)_0 \]

\[\therefore \ y \in \text{Hom}(\Delta_1, A.), \quad \text{ev}_0(y) = 0 \quad \text{ev}_1(y) = 1.\]

\[\therefore \quad \text{Hom}(\Delta_1, A.) \otimes \mathbb{Z} \]
\[\mathbb{Z}[x] \]

\[A. \otimes \mathbb{Z} \]
\[\mathbb{Z}[x] \]

\[A. \otimes \mathbb{Z} \]
\[\mathbb{Z}[x] \]

has both arrows being w.e.'s (hence preserving w.e.'s). This gives the required independence of shape. \(\Box\)
If \( M \) is a simplicial \( A \)-module, we set

\[
\text{Kos}(M, t_1, \ldots, t_r) = M \otimes_{\mathbb{Z}} \text{Kos}(A, t_1, \ldots, t_r)
\]

\[
= M \otimes_{\mathbb{Z}} \mathbb{K}_{\mathbb{Z}}^1 t_1, \ldots, t_r
\]

So \( \text{Kos}(t_1, \ldots, t_r) \) is a simplicial (or, maybe better, anisotropic) \( \text{Kos}(A, t_1, \ldots, t_r) \)-module.

If \( M \) is discrete, over the discrete ring \( A \), then its underlying complex is just the usual Koszul complex of \( M \) with respect to \( t_1, \ldots, t_r \).

The preceding discussion relates to a more general fact:

If \( A \) is a (usual, i.e. discrete) ring, then

\[
\text{Hom}_A \left( \mathbb{Z} \otimes \mathbb{K}_{\mathbb{Z}}^1 t_1, \ldots, t_r, A \right) \cong A
\]

Analogously, if \( A_0 \) is a simplicial ring, and we regard \( \mathbb{Z} \otimes \mathbb{K}_{\mathbb{Z}}^1 t_1, \ldots, t_r \) as a discrete simplicial ring,
Then $\text{Hom} \left( \mathbb{Z}[x], A_\star \right) \cong A_\star$.

Indeed, $\text{Hom} \left( \mathbb{Z}[x], A_\star \right) = A_\star$.

(b/c, string $(\mathbb{Z}[x])_{x_0} \to (A_n)_{x_0}$ amounts to giving $\mathbb{Z}[x] \to A_\star$; all the other maps are determined by the degeneracies.)

\[ \therefore \quad \text{Hom} \left( \mathbb{Z}[x], A_\star \right)_{x} = \text{Hom} \left( A_n, \text{Hom} \left( \mathbb{Z}[x], A_\star \right) \right) \]

s.set

\[ = \text{Hom} \left( A_n \times \mathbb{Z}[x], A_\star \right) \]

which are ring homs in the 2nd variable.

\[ = \text{Hom} \text{se} \left( \mathbb{Z}[x], \text{Hom} \left( A_n, A_\star \right) \right) \]

s.set of Hom s.set

\[ = \text{Hom} \left( A_n, A_\star \right) \text{ by the claim} \]

\[ = \text{Hom} \left( A_n, A_\star \right) \text{ by det} \]

\[ = A_n \]

(This is just an elaborate verilization that $\mathbb{Z}[x]$ = free simplicial ring on the simplicial point.)
So \( \pi_0 ( \text{Hom} ( \mathbb{Z}[x], A_0) ) = \pi_0 (A_0) \)

So maps \( b \) animate rings \( \mathbb{Z}[x] \to A_0 \),
up to equiv., are given by maps \( \mathbb{Z}[x] \to \pi_0 (A_0) \),
\text{i.e. by elements of } \pi_0 (A_0).

The previous weak equivalence,

\[
\begin{array}{ccc}
\pi_0 (A_0) & \longrightarrow & \text{Hom} (\mathbb{Z}[x], A_0) \\
\downarrow & & \downarrow \\
\mathbb{Z}[x] & \longrightarrow & A_0
\end{array}
\]

is just a concrete
\text{invocation of this fact.}

Given \( a \in \pi_0 (A_0) \), inducing \( \mathbb{Z}[x] \to A_0 \),
we define \( (A_0)[\frac{1}{a}] = A_0 \otimes \mathbb{Z}[x, \frac{1}{a}] \);

This coincides with the previously discussed localisation
obtained by lifting \( a \) to \( A_0 \) and then localising. Indeed,
choosing such a lift is required to give a concrete meaning to
the “induced map” \( \mathbb{Z}[x] \to A_0 \).