

Some manipulations with simplicial rings

(1)

A philosophical
remark

If A_* is a simplicial ring, $\bigoplus_{i \geq 0} \pi_i(A_*)$ is a graded ring. We can think of A_* as a kind of "deformation" of the graded ring $\bigoplus_{i \geq 0} \pi_i(A_*)$.

Similarly, if M_* is a simplicial A_* -module, then $\bigoplus_{i \geq 0} \pi_i(M_*)$ is a graded module over $\bigoplus_{i \geq 0} \pi_i(A_*)$, and M_* is a kind of deformation of $\bigoplus_{i \geq 0} \pi_i(M_*)$.

Now if we work up to weak equivalence, i.e. with "animated rings" and "animated modules", then A_* is not well-defined (in some sense, so we can replace it by something weakly equivalent), so we have to be thoughtful before defining notions like ideals, regular sequences, localization, etc.

Eg. The degeneracy maps $A_0 \rightarrow A_i$ ($i \geq 0$) make all the A_i 's A_0 -algebras, ~~but~~ so A_* is a simplicial A_0 -alg.

But A_0 is not intrinsic (i.e. not preserved by weak equivalence), so it doesn't make sense directly to localize at an element $f \in A_0$ or to talk about a sequence of elements in A_0 .

Instead, notions such as localization / regular sequence should be defined with reference to $\pi_0(A_0)$, which is intrinsic, even though A_0 is not a simplicial $\pi_0(A_0)$ -alg.

Ex. Let $S \subseteq \pi_0(A)$ be a multiplicative subset, let

$$S^{sat} = \{ f \in \pi_0(A) \mid fg \in S \exists g \in \pi_0(A) \}$$

(so that S^{sat} is maximal multiplicative subset $S' \supseteq S$

for which ~~$\pi_0(A_0) \xrightarrow{S} \pi_0(A_0)_S$~~ $\pi_0(A_0)_{S^{sat}} \rightarrow \pi_0(A_0)_S$,

is an isomorphism),

let $\tilde{S} =$ preimage of S^{sat} under $A_0 \rightarrow \pi_0(A_0)$

(a saturated / multiplicative subset of A_0), and let $T \subseteq \tilde{S}$

be any multiplicative subset of A_0 s.t.

$$(\tilde{S})^{sat} = S^{sat}$$

image of T in $\pi_0(A_0)$

Then for any simplicial A_0 -module M_0 , the natural map

$$(M_0)_T \rightarrow (M_0)_{\tilde{S}}$$

Localization of A_0 -modules

is a weak equivalence.

Pf: Localization is exact, and so on homotopy this induces

$$\pi_i(M_0)_T \rightarrow \pi_i(M_0)_{\tilde{S}}$$

But the A_0 -action on $\pi_i(M_0)$ factors through $\pi_0(A_0)$, and so this reduces to the equality

$$\pi_i(M)_{\overline{T}} = \pi_i(M)_S \quad (\text{both equal to } \pi_i(M)_{S, \text{sect}})$$

□

Ex. If $u \in \pi_0(A_0)^*$, then if $f \in A_0 \subseteq A_0^*$, we find that

$$A_0 \rightarrow (A_0)_f \text{ is a w.e.} \\ (\text{take } S = \{1\}, T = \{f\})$$

Thus we may assume that u is in the image of $A_0^* \rightarrow \pi_0(A_0)$, if we are willing to replace A_0 by something weakly equivalent.

Ex.

If $A_0 \rightarrow B_0$ is a weak equiv., if $S \subseteq \pi_0(A_0)$ is a multiplicative subset, and if we let $T_A, T_B = \text{preimage of } S \text{ in } A_0, B_0$ resp., then

$$(A_0)_{T_A} \xrightarrow{\text{w.e.}} (B_0)_{T_B}$$

by the same "exactness of localization" argument,

~~by the same "exactness of localization" argument,~~

and so we find that "localization of A_0 over $\text{Spec } \pi_0(A_0)$ " makes sense, to yield

a sheaf (in the ∞ -categorical sense) of animated rings over $\text{Spec } \pi_0(A_0)$ — This gives (one version of) the derived scheme attached to A_0 (thought of as an animated ring).

If M_0 is a simplicial A_0 -module, thought of as giving rise to an animated A_0 -module over A_0 as an animated ring, then the same localization process gives rise to a sheaf of animated modules over $\text{Spec } \pi_0(A_0)$ on which the sheaf of animated rings obtained by localizing A_0 acts.

If A is a ring, and $f_1, \dots, f_r \in A$, then we have seen that the Koszul complex $\text{Kos}(A; f_1, \dots, f_r)$ admit a simplicial ring structure via the formula

$$\text{Kos}(A; f_1, \dots, f_r) := A \underset{\substack{\text{fib} \\ \times_i}}{\otimes} \mathbb{Z} \underset{\substack{\uparrow \\ \times_i}}{\mathbb{Z}[x_1, \dots, x_r]} \mathbb{Z}$$

We want to extend this to the context of A_0 a simplicial (or animated, really) ring.

So now we should imagine $f_1, \dots, f_r \in \pi_0(A_0)$

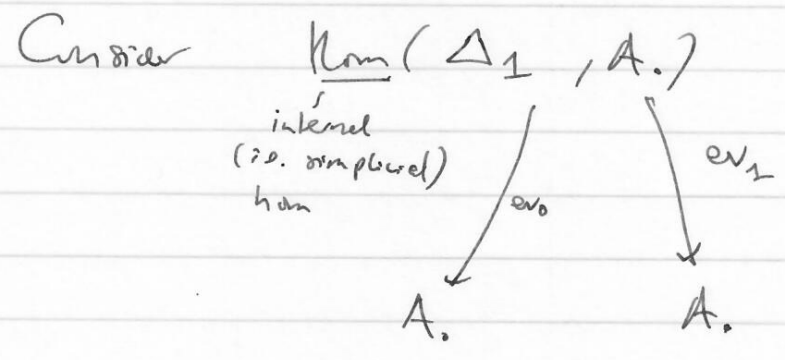
If we lift f_i to $g_i \in A_0$, then
 The A_0 structure on each A_i makes
 A_i a simplicial $\mathbb{Z}[x_1, \dots, x_n]$ -algebra, and
 so we can define

$$Kos(A_\bullet; f_1, \dots, f_r) := A_\bullet \otimes_{\mathbb{Z}[x_1, \dots, x_n]}^{\mathbb{Z}} \mathbb{Z}$$

$\begin{matrix} & & \mathbb{Z} & & \\ & & \otimes & & \\ & & \mathbb{Z}[x_1, \dots, x_n] & & \\ & \nearrow^{g_i} & & \searrow^{g_i} & \\ & x_i & & x_i & \end{matrix}$

But first we need to check this is
 independent of the choices of the g_i .

Check one i at a time, i.e. assume $i=1$,
 and suppose $g \neq h \in A_0$ both lift $f \in \pi_0(A_\bullet)$.



The two evaluation maps are weak equivalences.

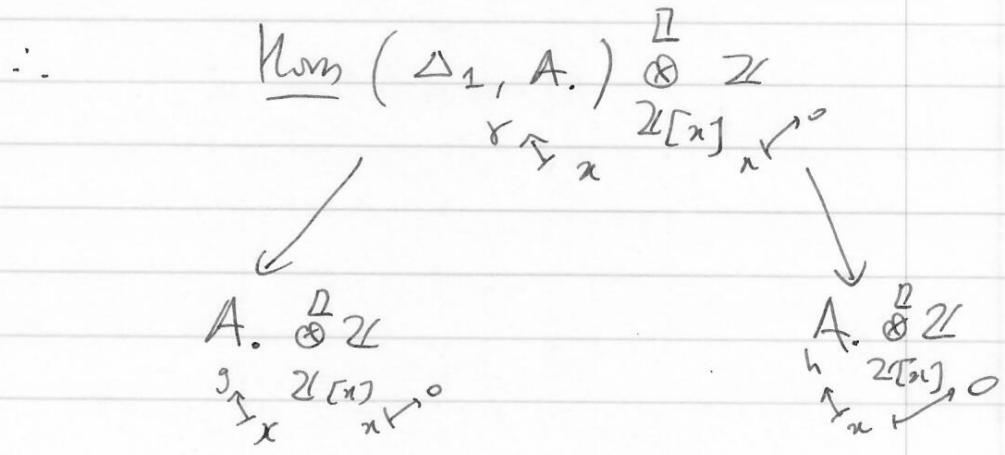
To say that $g, h \in A_0$ lift $f \in \pi_0(A_\bullet)$ is to
 say there is a path $\gamma \in A_1$ s.t. $d_0(\gamma) = g$
 $d_1(\gamma) = h$.

~~Now~~

$$\begin{aligned} \text{Now } \underline{\text{Hom}}(\Delta_1, A.)_n &= \text{Hom}(\Delta_n, \underline{\text{Hom}}(\Delta_1, A.)) \\ &= \text{Hom}(\Delta_n \times \Delta_1, A.) \end{aligned}$$

$$\begin{aligned} \text{In particular, } A_{\bullet 1} &= \text{Hom}(\Delta_1, A.) \\ &= \text{Hom}(\overset{\Delta_0}{\text{pt}} \times \Delta_1, A.) \\ &= \underline{\text{Hom}}(\Delta_1, A.)_0, \end{aligned}$$

$$\therefore \gamma \in \underline{\text{Hom}}(\Delta_1, A.), \quad \begin{aligned} \text{ev}_0(\gamma) &= g \\ \text{ev}_1(\gamma) &= h. \end{aligned}$$



has both arrows being w.e.'s (b/c $\overset{\square}{\otimes}$ preserves w.e.'s). This gives the required independence of choice. \square

If M is a simplicial A -module, we set

$$\text{Kos}(M; f_1, \dots, f_r) = M \otimes_A \text{Kos}(A; f_1, \dots, f_r)$$

$$\begin{array}{c}
 = \\
 \uparrow \\
 \text{transmits} \\
 \downarrow \\
 \otimes
 \end{array}
 \quad
 \begin{array}{c}
 M \otimes \mathbb{Z} \\
 \downarrow \quad \downarrow \\
 \mathbb{Z}[x_1, \dots, x_r] \rightarrow \mathbb{Z} \\
 x_i \quad x_i
 \end{array}$$

So $\text{Kos}(M; f_1, \dots, f_r)$ is a simplicial (or, maybe better, animated)

$\text{Kos}(A; f_1, \dots, f_r)$ -module. ~~which is a complex~~

If M is discrete, over the discrete ring A , then its underlying complex is just the usual Koszul complex of M w.r.t. f_1, \dots, f_r .

The preceding discussion relates to a more general fact:

If A is a (usual, i.e. discrete) ring, then

$$\text{Hom}_{\text{rings}}(\mathbb{Z}[x], A) \xrightarrow[\text{ev. at } x]{\cong} A$$

Analogously, if A_0 is a simplicial ring, and if we regard $\mathbb{Z}[x]$ as a discrete simplicial ring,

then $\underline{\text{Hom}}(\mathbb{Z}[x], A_0) \cong A_0$.

↑
simplicial set of homs. of simplicial rings

Indeed, $\text{Hom}(\mathbb{Z}[x], A_0) = A_0$

(b/c ~~maps~~ giving $(\mathbb{Z}[x])_{n \geq 0} \rightarrow (A_n)_{n \geq 0}$

amounts to giving $\mathbb{Z}[x] \rightarrow A_0$; all the other maps are determined by the degeneracies)

$$\therefore \underline{\text{Hom}}(\mathbb{Z}[x], A_0)_n \cong \text{Hom}_{s\text{-sets}}(\Delta_n, \underline{\text{Hom}}(\mathbb{Z}[x], A_0))$$

$$= \text{Hom}_{s\text{-sets}}(\Delta_n \times \mathbb{Z}[x], A_0)$$

which are ring homs in the 2nd variable

$$= \text{Hom}_{s\text{-rings}}(\mathbb{Z}[x], \underline{\text{Hom}}(\Delta_n, A_0))$$

s. set of Hom of s. sets

$$= \underline{\text{Hom}}(\Delta_n, A_0)$$

by the above

$$= \text{Hom}_{s\text{-sets}}(\Delta_n, A_0)$$

by defn

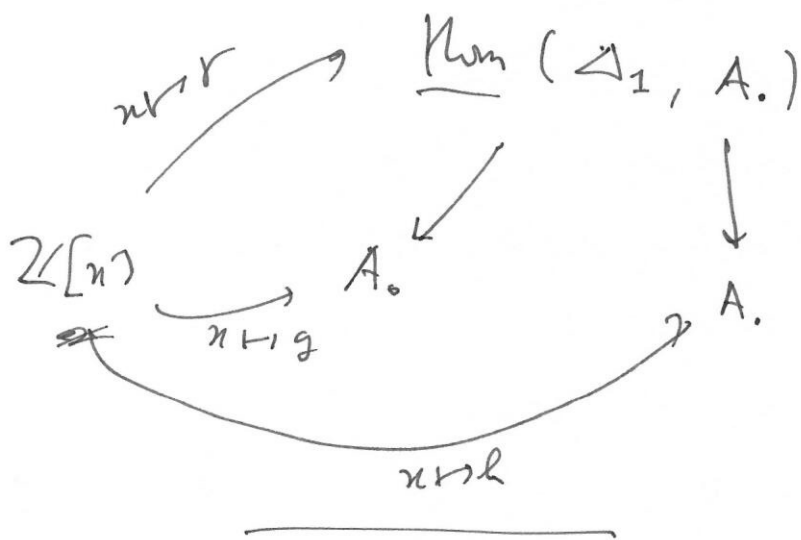
$$= A_n$$

(This is just an elaborate verification that $\mathbb{Z}[x]$ = free simplicial ring on the simplicial point)

$$\text{So } \pi_0 \left(\varinjlim (\mathbb{Z}[x], A_0) \right) = \pi_0(A_0)$$

So maps of animated rings $\mathbb{Z}[x] \rightarrow A_0$,
up to equiv., are given by maps $\mathbb{Z}[x] \rightarrow \pi_0(A_0)$,
i.e. by elements of $\pi_0(A_0)$.

The previous ^{diagram of} weak equivalences



is just a concrete invocation of this fact.

Given $a \in \pi_0(A_0)$, inducing $\mathbb{Z}[x] \rightarrow A_0$,

we define $(A_0)_{[a]} = A_0 \otimes_{\mathbb{Z}[x]} \mathbb{Z}[x, \frac{1}{x}]$;

This coincides with the previously discussed localization obtained by taking q to A_0 and then localizing. Indeed, choosing such a Pft is required to give a concrete meaning to the "induced map" $\mathbb{Z}[x] \rightarrow A_0$.