Some manipulations with simplicial rings

A philosophical remark

If $A_\cdot$ is a simplicial ring, $\bigoplus_{i\geq 0}\pi_i(A_\cdot)$ is a graded ring. We can think of $A_\cdot$ as a kind of "deformation" of the graded ring $\bigoplus_{i\geq 0}\pi_i(A_\cdot)$.

Similarly, if $M_\cdot$ is a simplicial $A_\cdot$-module, then $\bigoplus_{i\geq 0}\pi_i(M_\cdot)$ is a graded module over $\bigoplus_{i\geq 0}\pi_i(A_\cdot)$, and $M_\cdot$ is a kind of deformation of $\bigoplus_{i\geq 0}\pi_i(M_\cdot)$.

Now if we work up to weak equivalence, i.e. with "animated rings" and "animated modules", then $A_\cdot$ is not well-defined (in some sense, i.e. we can replace it by something weakly equivalent), so we have to be thoughtful before defining notions like ideals, regular sequences, localization, etc.

Eg. The degeneracy maps $A_0 \to A_i$ $(i\geq 0)$ make all the A_i's A_0-algebras, so $A_\cdot$ is a simplicial A_0-alg.

But A_0 is not intrinsic (i.e. not preserved by weak equivalence), so it doesn't make sense directly to localize at an element $f \in A_0$, or to talk about a sequence of elements in A_0.

Instead, notions such as localization / regular sequence should be defined with reference to $\pi_0(A_\bullet)$, which <u>is</u> intrinsic, even though $A_\bullet$ is <u>not</u> a simplicial $\pi_0(A_\bullet)$-alg.

<u>Ex</u>. Let $S \subseteq \pi_0(A)$ be a multiplicative subset, let

$$S^{sat} = \{ f \in \pi_0(A) \mid fg \in S \ \exists g \in \pi_0(A) \}$$

(so that S^{sat} is maximal multiplicative subset $S' \supseteq S$ for which $\pi_0(A_\bullet)_S \to \pi_0(A_\bullet)_{S'}$ is an isomorphism),

let $\tilde{S}$ = preimage of S^{sat} under $A_0 \twoheadrightarrow \pi_0(A_\bullet)$ (a saturated multiplicative subset of A_0), and let $T \subseteq \tilde{S}$ be any multiplicative subset of A_0 s.t.

$$(\underbrace{\overline{T}}_{\text{image of } T \text{ in } \pi_0(A_\bullet)})^{sat} = S^{sat}.$$

Then for any simplicial $A_\bullet$-module $M_\bullet$, the natural map

$$(M_\bullet)_T \longrightarrow (M_\bullet)_{\tilde{S}}$$

localization of A_0-modules

is a weak equivalence.

Pf: Localization is exact, and so on homotopy this induces $\pi_i(M_\bullet)_T \longrightarrow \pi_i(M_\bullet)_{\tilde{S}}$.

But the A_0-action on $\pi_i(M.)$ factors through $\pi_0(A.)$, and so this reduces to the equality

$$\pi_i(M)_T = \pi_i(M)_S \qquad (\text{both equal } \pi_i(M)_{S^{sat}}.)$$

;

$\square$

<u>Ex</u>. If $u \in \pi_0(A.)^\times$, then if $f \in A_0$ lifts u, we find that

$$A. \to (A.)_f \quad \text{is a w.e.}$$

(take $S = \{1\}$, $T = \{f\}$).

Thus we may assume that u is in the image of $A_0^\times \to \pi_0(A_0)$, if we are willing to replace $A.$ by something weakly equivalent.

<u>Ex</u>.

If $A. \to B.$ is a weak equiv., if $S \subseteq \pi_0(A.)$ is a multiplicative subset, and if we let T_A, T_B = preimage of S in A_0, B_0 resp., then

$$(A.)_{T_A} \xrightarrow[w.e.]{\sim} (B.)_{T_B}$$

~~by the same exactness of localization~~ by the same "exactness of localization" argument, and so we find that "localization of A_0 over $\operatorname{Spec} \pi_0(A_0)$" makes sense, to yield a sheaf (in the ∞-categorical sense) of animated rings over $\mathrm{Spec}\, \pi_0(A_\bullet)$ — this gives (one version of) the derived scheme attached to $A_\bullet$ (thought of as an animated ring).

If $M_\bullet$ is a simplicial $A_\bullet$-module, thought of as giving rise to an animated $A_\bullet$-module over $A_\bullet$ as an animated ring, then the same localization process gives rise to a sheaf of animated modules over $\mathrm{Spec}\, \pi_0(A_\bullet)$ on which the sheaf of animated rings obtained by localizing $A_\bullet$ acts.

If A is a ring, and $f_1, \dots, f_r \in A$, then we have seen that the Koszul complex $\mathrm{Kos}(A; f_1, \dots, f_r)$ admits a simplicial ring structure via the formula

$$\mathrm{Kos}(A; f_1, \dots, f_r) := A \overset{\mathbb{L}}{\underset{\substack{f_i \leftarrow\!\shortmid x_i \\ \mathbb{Z}[x_1, \dots, x_r]}}{\otimes}} \mathbb{Z}_{\substack{0 \leftarrow\!\shortmid x_i}}.$$

We want to extend this to the context of $A_\bullet$ a simplicial (or animated, really) ring.

So now we should imagine $f_1, \dots, f_r \in \pi_0(A_\bullet)$

If we lift f_i to $g_i \in A_0$, then the A_0 structure on each A_i makes $A^{\cdot}$ a simplicial $\mathbb{Z}[x_1, \dots, x_n]$-algebra, and so we can define

$$\mathrm{Kos}(A_\cdot\,; f_1, \dots, f_r) := A_\cdot \overset{\mathbb{L}}{\underset{\mathbb{Z}[x_1,\dots,x_r]}{\otimes}} \mathbb{Z}.$$

$g_i \leftarrow\!\shortmid x_i$ $\qquad$ $x_i \mapsto 0$

But first we need to check this is independent of the choices of the g_i.

Check one i at a time, $\Rightarrow$ assume $r=1$, and suppose g & $h \in A_0$ both lift $f \in \pi_0(A_\cdot)$.

Consider

$$\begin{tikzcd}[column sep=small] & \underline{\mathrm{Hom}}(\Delta_1, A_\cdot) \arrow[dl, "ev_0"'] \arrow[dr, "ev_1"] & \\ A_\cdot & & A_\cdot \end{tikzcd}$$

internal (i.e. simplicial) hom

The two evaluation maps are weak equivalences.

To say that $g, h \in A_0$ lift $f \in \pi_0(A_\cdot)$ is to say there is a path $\gamma \in A_1$ s.t. $d_0(\gamma) = g$
$d_1(\gamma) = h$.

Now $\quad \underline{\mathrm{Hom}}(\Delta_1, A_\cdot)_n = \mathrm{Hom}(\Delta_n, \underline{\mathrm{Hom}}(\Delta_1, A_\cdot))$

$$= \mathrm{Hom}(\Delta_n \times \Delta_1, A_\cdot)$$

In particular, $\quad A_{\cdot 1} = \mathrm{Hom}(\Delta_1, A_\cdot)$

$$= \mathrm{Hom}(\overset{\Delta_0}{\overset{\shortparallel}{pt}} \times \Delta_1, A_\cdot)$$

$$= \underline{\mathrm{Hom}}(\Delta_1, A_\cdot)_0 \quad ,$$

$\therefore \quad \gamma \in \underline{\mathrm{Hom}}(\Delta_1, A_\cdot) \quad , \quad ev_0(\gamma) = g$
$\qquad ev_1(\gamma) = h.$

$\therefore$

$$\begin{tikzcd} & \underline{\mathrm{Hom}}(\Delta_1, A_\cdot) \overset{\Delta}{\otimes} \mathbb{Z} \arrow[dl] \arrow[dr] & \\ A_\cdot \overset{\Delta}{\otimes} \mathbb{Z} & & A_\cdot \overset{\Delta}{\otimes} \mathbb{Z} \end{tikzcd}$$

$\gamma \mapsfrom x \quad \mathbb{Z}[x] \quad x \mapsto 0$

$g \mapsfrom x \quad \mathbb{Z}[x] \quad x \mapsto 0 \qquad\qquad h \mapsfrom x \quad \mathbb{Z}[x] \quad x \mapsto 0$

has both arrows being w.e.'s (b/c $-\overset{\Delta}{\otimes}$ preserves w.e.'s). This gives the required independence of choice. $\square$

If $M_\bullet$ is a simplicial $A_\bullet$-module, we set

$$\mathrm{Kos}(M_\bullet; f_1, \dots, f_r) = M_\bullet \overset{\mathbb{L}}{\underset{A_\bullet}{\otimes}} \mathrm{Kos}(A; f_1, \dots, f_r)$$

$$\underset{\substack{\text{transitivity} \\ \text{of } \overset{\mathbb{L}}{\otimes}}}{\overset{\uparrow}{=}} \; M_\bullet \overset{\mathbb{L}}{\underset{f_i \leftarrow x_i \; \mathbb{Z}[x_1, \dots, x_r] \; x_i \mapsto 0}{\otimes}} \mathbb{Z}$$

So $\mathrm{Kos}(M; f_1, \dots, f_r)$ is a simplicial (or, maybe better, animated) $\mathrm{Kos}(A_\bullet; f_1, \dots, f_r)$-module. ~~whose underlying complex~~

If M is discrete, over the discrete ring A, then its underlying complex is just the usual Koszul complex of M w.r.t. $f_1, \dots, f_r$.

The preceding discussion relates to a more general fact:

If A is a (usual, i.e. discrete) ring, then

$$\mathrm{Hom}_{\text{rings}}(\mathbb{Z}[x], A) \xrightarrow[\substack{\text{ev. at} \\ x}]{\sim} A.$$

Analogously, if $A_\bullet$ is a simplicial ring, and if we regard $\mathbb{Z}[x]$ as a discrete simplicial ring, then $\underline{\mathrm{Hom}}(\mathbb{Z}[x], A_\bullet) \cong A_\bullet$

simplicial set of homs. of simplicial rings

Indeed, $\mathrm{Hom}(\mathbb{Z}[x], A_\bullet) = A_0$

(b/c ~~[crossed out]~~ giving $(\mathbb{Z}[x])_{n\geq 0} \to (A_n)_{n\geq 0}$ amounts to giving $\mathbb{Z}[x] \to A_0$; all the other maps are determined by the degeneracies)

$$\therefore \quad \underline{\mathrm{Hom}}(\mathbb{Z}[x], A_\bullet)_n = \mathrm{Hom}_{\text{s.sets}}(\Delta_n, \underline{\mathrm{Hom}}(\mathbb{Z}[x], A_\bullet))$$

$$= \mathrm{Hom}_{\text{s.sets}}(\Delta_n \times \mathbb{Z}[x], A_\bullet)$$

which are ring homs in the 2nd variable

$$= \mathrm{Hom}_{\text{s.rings}}(\mathbb{Z}[x], \underline{\mathrm{Hom}}(\Delta_n, A_\bullet))$$

s. set of homs of s. sets

$$= \underline{\mathrm{Hom}}(\Delta_n, A_\bullet)_0 \quad \text{by the above}$$

$$= \mathrm{Hom}_{\text{s.sets}}(\Delta_n, A_\bullet) \quad \text{by def}^{\underline{n}}$$

$$= A_n.$$

(This is just an elaborate verification that $\mathbb{Z}[x]$ = free simplicial ring on the simplicial point)

So $\pi_0\left(\underline{\mathrm{Hom}}\,(\mathbb{Z}[x], A_\cdot)\right) = \pi_0(A_\cdot)$

So maps of animated rings $\mathbb{Z}[x] \to A_\cdot$, up to equiv., are given by maps $\mathbb{Z}[x] \to \pi_0(A_\cdot)$, i.e. by element of $\pi_0(A_\cdot)$.

The previous diagram of weak equivalences

$$\begin{tikzcd} & & \underline{\mathrm{Hom}}(\Delta_1, A_\cdot) \arrow[dl] \arrow[d] \\ \mathbb{Z}[x] \arrow[urr, bend left, "x \mapsto \gamma"] \arrow[r, "x \mapsto g"'] \arrow[rr, bend right, "x \mapsto h"'] & A_0 & A_\cdot \end{tikzcd}$$

is just a concrete invocation of this fact.

Given $a \in \pi_0(A_\cdot)$, inducing $\mathbb{Z}[x] \to A_\cdot$, we define $(A_\cdot)[\frac{1}{a}] = A_\cdot \otimes_{\mathbb{Z}[x]} \mathbb{Z}[x, \frac{1}{x}]$;

This coincides with the previously discussed localization obtained by lifting a to A_0 and then localizing. Indeed, choosing such a lift is required to give a concrete meaning to the "induced map" $\mathbb{Z}[x] \to A_\cdot$