A simple δ-ring identity

If A is a δ-ring, then of course

$$\varphi(px) = (px)^p + p\,\delta(px) \quad \forall x \in A$$

Since φ is an endomorphism of A, we can rewrite this as

$$p\,\varphi(x) = p^p \cdot x^p + p\,\delta(px) \quad \forall x \in A$$

But in fact, we can "divide" this identity by p. (*)

Lemma $\varphi(x) = p^{p-1} \cdot x^p + \delta(px) \quad \forall x \in A$

Pf:
$$\begin{aligned} \delta(px) &= \delta(p) \cdot x^p + \varphi(p) \cdot \delta(x) \\ &= (1 - p^{p-1}) \cdot x^p + p \cdot \delta(x) \\ &= -p^{p-1} \cdot x^p + x^p + p\,\delta(x) \\ &= -p^{p-1} \cdot x^p + \varphi(x). \end{aligned}$$
$\square$

(*) Since the "universal case" is p-torsion free, we can always divide such an identity by p. But its easy to give a concrete argument too.

This immediately implies various facts we've seen.

Eg 1 If $px = 0$ then $\varphi(x) = 0$.

Pf: If $px = 0$, then

$$0 = \delta(0) = \delta(px) = \varphi(x) - p^{p-1} \cdot x^p$$

$$= \varphi(x) - \underbrace{p \cdot x}_{=0} (p^{p-2} x^{p-1})$$

$$= \varphi(x). \quad \square$$

Eg 2 If $x \in p^n A$, then $\delta(x) \in p^{n-1} A$

Pf: Write $x = p^n y$.

Then $\delta(x) = \delta(p^n y) = \varphi(p^{n-1} y) - p^{p-1} (p^{n-1} y)^p$

$$= p^{n-1} \varphi(y) - p^{p-1+p(n-1)} \cdot y^p$$

$$\in p^{n-1} A. \quad \square$$

Eg 3 If $x \in A$, then

$$\delta(x^p) \in pA.$$

Pf: If $a, b \in A$, then the lemma, together with the addition law for δ, gives

$$\delta(a + pb) \equiv \delta(a) + \varphi(b) \pmod p$$
$$\equiv \delta(a) + b^p \pmod p$$

Now $\varphi(x) = x^p + p\,\delta(x)$

$$\therefore \varphi(\delta(x)) = \delta(\varphi(x))$$

$$\equiv \delta(x^p) + \varphi(\delta(x)) \pmod p$$

$$\therefore \delta(x^p) \equiv 0 \pmod p$$

$\square$

As an application, if x has a p^nth root for all n, and A is p-adically separated, then $\varphi(x) = x^p$ (~~[illegible]~~ since $\delta(x) = 0$).