Prisms

A $S$-pair is a pair $(A,I)$ with

$A$ a $S$-ring and $I \subseteq A$ an ideal.

Def: A prism is a $S$-pair $(A,I)$ invertible, i.e., Zariski locally generated by

$s.t. I$ is a non-zero divisor

$A$ is derived $(p,I)$-complete,

and $p \subseteq (I^p,I^p)$.

Types of prisms

* perfect: $I : A \twoheadrightarrow A$

* bounded: $A/I$ has bounded $p$-power torsion

i.e. $(A/I)[p^\infty] = (A/I)[p^N] \cong \mathbb{Z}[p^N]

* crystalline: $I = (p)$.

Remark: The completion $p \subseteq (I,I^p)$ can be promoted to $p \subseteq (I^p,I^p)$.

Indeed, since $A$ is $(p,I)$-complete, we have

$(p,I) \subseteq \text{Red}(A)$, so our result on distinguished elements applies to our met $I : A \twoheadrightarrow A'$ faithfully flat and
a morphism of $\mathcal{S}$-rings, s.t.

$$I' = IA' = (d) \text{ with } d \text{ distinguished, i.e.}$$

with $\delta(d) \in A^*$. (We also can ensure $(p, d) \subseteq \text{Rad}(A')$, which we don't need now, but will use later.)

Then $\varphi(d) = d^p + p \delta(d)$, i.e. $p \in (d^p, \varphi(d))$.

By faithfully flat descent, $p \in (I'^p, \varphi(d'))$.

We need another distinguished element fact:

**Lemma (Lemma 1.7 of Bhattacharyya's lecture 3)**

If $d \in A$ (a $\mathcal{S}$-ring) is distinguished, if $(f, p) \subseteq \text{Rad}(A)$, and if $d = fg$, then $g$ is a unit; i.e. $(d) = (f)$.

**Pf:**

$$\delta(fd^p) = \delta(f) \cdot g^p + \varphi(f) \cdot \delta(g)$$

Since $f \in \text{Rad}(A)$, so is $\varphi(f)$ while $\delta(d) \in A^*$ by assumption, then $\varphi(f) \cdot g^p = \delta(d) - \varphi(f) \cdot \delta(g)$.

Thus $g \in A^*$.
We now have the following lemma:

**Lemma** If \( d \in A^* \) is distinguished, then \( y \in A^* \)

*Remark:* Of course, \( c_0(d) = d^p + p \cdot s(d) \), \( p = -s(d)^{-1} \cdot d^p + s(d) \cdot q(d) \), so in this case \( y = s(d)^{-1} \) is a unit. The lemma ensures that \( y \) is the general behavior of the coefficient \( y \).

**Proof** If \( y \) is a unit mod \((p,d)\), then it is a unit (since \((p,d) \subseteq \text{Rad}(A)\)) by way of obtaining a contradiction, assume that \((p,d,y)\) is a proper ideal, and let \( B = \text{localization of } A \) along this ideal.

Then \( B \) is a non-zero \( S \)-ring with \((p,d,y) \subseteq S \cdot B\), and

\[
p = x \cdot d^p + y \cdot q(d) = x \cdot d^p + y \cdot d^p + p d s(d) = 0
\]

\[
\therefore \ p (1 - y s(d)) = d (x+y) \cdot d^p
\]

Since \( y \in \text{Rad}(B) \)

Now \( p \) is distinguished (in any \( S \)-ring!!)
and \( p \not\in \text{Rad} B \), so by the preceding lemma (Bhatt's lemma 1.7) we have

\((x+y), d\hat{p}^{-1}\) is a unit. Since \( p\not\in \mathbb{Z} \),

we find that \( d \) is a unit, contradicting \( d \not\in \text{Rad}(B) \) (since \( B \not\neq 0 \)).

Thus \( y \) must have been a unit in \( A \), as claimed.

Corollary (Lemma 3.5 in Bhatt's 3rd lecture)

If \((A, I)\) is a prism, then \( \mathfrak{p}(I)A \)

is principal and generated by a distinguished element.

Proof: Since \( p \notin (I' \mathfrak{p}(I)) \), we can write

\[ p = a + b \quad a \in I', \quad b \in \mathfrak{p}(I)A \]

Choose faithfully that \( A \to A' \) as above.

Then \( I \cdot A' = (d) \), \( a = x \cdot d \hat{p} \) \( b = y \hat{p}(d) \in \mathfrak{p}(A') \).

The preceding lemma shows that \( y \in (A')^{-1} \):

\[ b \cdot A' = (\ell(x) \cdot A') = \mathfrak{p}(A') \cdot A' \]
by faithfully flat descent, we have
\[ b \cdot A = \Phi(T) \cdot A \]

Rigidity: If \( \Phi(A,T) \to (B,J) \) is a
morphism of primes, then \( I \cdot B = J \).

If, change \( A \to A' \) as before. Consider
\[
A' \to A' \otimes B \to B' \to B'' \to A' - A'
\]

This is flat, Spec\( B' \) - Spec\( B \)

We have \( B \to B'' \) faithfully flat with
\( (p, J) \in \text{Rad} B'' \) and \( I \cdot B'' \) generl by distinguished element.
\[
A \to A'' \text{ factor through } A', \quad I \cdot B'' \text{ also generj by distinguished element, and } I \cdot B'' \subset J \cdot B''.
\]
Lemma 1.7 Then shows that
\[ I \rightarrow B' \rightarrow B', \]
and so by faithfully flat descent, we have
\[ I \cdot B = \mathbb{J}, \text{ as claimed.} \]

If \( (A, I) \) is a monad, and \( B \) is a \( J \cdot A \)-algebra that is also \( (p, I) \)-derived complete, then \( (B, I_B) \) will be a monad provided that \( IB \) is invertible.

For this, we need \( I \otimes B \stackrel{\sim}{\rightarrow} IB \), which holds precisely when \( B[I] = 0 \).

(For any \( A \)-module \( M \), the inclusion \( M[I] \rightarrow M \) induces an inclusion \( M[I] \otimes I \rightarrow M \otimes I \).

(When \( I \) is flat over \( A \),)

being invertible \( M[I] \otimes I \rightarrow M \otimes I \)

and in fact \( M[I] \otimes I \rightarrow \text{ker}(M \otimes I \rightarrow I, M) \)

Claim : \( M[I] \otimes I \rightarrow \text{ker}(M \otimes I \rightarrow I, M) \)

Pf : Check locally over \( \text{Spec } A \). Assume \( I \) is principal, say \( I = \langle f \rangle \) in \( A \).

which case it is clear: it amounts to the claim that \( M[I] \otimes I \rightarrow \text{ker}(M \otimes I \rightarrow I, M) \).
If $A$ is a bounded prism, then $A$ is classically $(p, I)$-algebraically complete.

**Pf.** By assumption, $A$ is derived $(p, D)$-complete, i.e., $A \cong \text{Rlim}_{m_n} (\ldots)$

Here we use the fact that $I$ is locally principal, generated by a non-zero divisor. $I_m \to A$

\[ \cong \text{Rlim}_{m_n} \left( \frac{A}{I_m} \overset{p_n}{\to} \frac{A}{I_{m+n}} \right) \]

\[ \cong \text{Rlim}_{m_n} \left( \frac{A}{(I_m, p^n)} \right) \]

\[ \cong \text{Em}_{m_n} \frac{A}{(I_m, p^n)} \quad \text{b/c the transition maps are surjective.} \]