Prismatic envelopes

Prop: (A,I) bounded prism. (i.e. A/I has bounded p^∞-torsion).

B a (p,I)-completely flat δ-A algebra.

$x_1,\dots,x_r \in B$ (p,I)-completely regular relative to A.

Then $\exists$ universal map of δ-pairs over (A,I)

$$(B, J = (I, x_1, \dots, x_r)) \longrightarrow (C, IC)$$

with (C, IC) a prism.

(i.e. (C, IC) is initial among prisms over (A,I) receiving such a map).

<u>Further:</u> C is (p,I)-completely flat over A, the formation of C is compatible with base change $(A,I) \to (A', IA')$.

We write $C := B\{\frac{J}{I}\}^\wedge$.

Finally, if $B \to B'$ is a (p,I)-completely flat morphism of δ-A-algebras, then

$$B\{\tfrac{J}{I}\}^\wedge \overset{\mathbb{L}}{\underset{B}{\otimes}} B' \simeq B'\{\tfrac{J'}{I}\}^\wedge,$$

where $J' = JB' \subseteq B'$.

Proof: We work locally on (A, I), and so assume that I is principal, say $I = (d)$ with d being distinguished.

We then form $B\{\frac{x_1}{d}, \dots, \frac{x_r}{d}\}$ in the category of δ-A-algebras.

Note that $J \cdot B\{\frac{x_1}{d}, \dots, \frac{x_r}{d}\}$

$$= (I, x_1, \dots, x_r)\ B\{\frac{x_1}{d}, \dots, \frac{x_r}{d}\}$$
$$= (d, x_1, \dots, x_r)\ B\{\frac{x_1}{d}, \dots, \frac{x_r}{d}\}$$
$$= (d) \cdot B\{\frac{x_1}{d}, \dots, \frac{x_r}{d}\}$$
$$= I \cdot B\{\frac{x_1}{d}, \dots, \frac{x_r}{d}\}$$

so this satisfies the criterion of receiving a map $(B, J) \to (X, I \cdot X)$. ($X$ some δ-A-algebra)

Also, if $(B, J) \to (X, IX) = (X, (d)X)$, then $d \mid x_1, \dots, x_r$ in X, since $x_1, \dots, x_r \in J$, and so the map $B \to X$ factors through $B\{\frac{x_1}{d}, \dots, \frac{x_r}{d}\}$. This factorization is furthermore unique provided that X is d-torsion free.

So $\left(B\{\frac{x_1}{d}, \dots, \frac{x_r}{d}\}, I \cdot B\{\frac{x_1}{d}, \dots, \frac{x_r}{d}\}\right)$

would satisfy the requirements of the prop^n if it were a prism.

However, it may not be derived (p, I)-adic complete, and the ideal gen'd by I may not be invertible.

Now if (X, IX) is a prism, then the map

$$B\{\frac{x_1}{d}, \dots, \frac{x_r}{d}\} \longrightarrow X$$

will factor through the derived (p, I)-completion of the target.

We write $C := B\{\frac{x_1}{d}, \dots, \frac{x_r}{d}\}^\wedge$ to denote this derived completion. A priori, this is not a δ-ring, but rather an <u>animated</u> δ-ring. We will show that it is actually a genuine δ-ring (i.e. that all its π_i for $i > 0$ vanish),

and that $C[I] = 0$, so that $I \cdot C$ is invertible. Then (C, IC) will be the required prism.

In fact, what we will prove is that C is (p, I)-completely flat over A.

The fact that $C \xrightarrow[\text{(q.i.)}]{\cong} \pi_0(C)$ and that $C[I] = 0$

will then follow from our previous results about completely flat modules over bounded prisms.

We begin with the diagram

$$\begin{array}{ccccccc}
\mathbb{Z}_p\{z\}^{\wedge} & \xrightarrow{z \mapsto d} & A & \longrightarrow & B & \longrightarrow & C := B\{\frac{x_1}{d}, \ldots, \frac{x_r}{d}\}^{\wedge} \\
\downarrow {\scriptstyle z \mapsto \varphi(y)} & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{Z}_p\{y\} & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' := B'\{\frac{x_1}{\varphi(y)}, \ldots, \frac{x_r}{\varphi(y)}\}^{\wedge}
\end{array}$$

in which the squares are derived push-outs, i.e. given by derived $\otimes$-product followed by derived (p, I)-completion

Since $\mathbb{Z}_p\{z\} \xrightarrow{z \mapsto \varphi(z) \mapsto \varphi(y)} \mathbb{Z}_p\{z\} = \mathbb{Z}_p\{y\}$ is faithfully flat, its derived (p, z)-completion is (p, z)-completely faithfully flat. Thus so are the other vertical arrows, and hence it suffices to show that $A' \to C'$ is $(p, z) = (p, I)$-completely flat.
$= (p, d)$ •

Now consider $\hat{D} = A'\{\frac{\varphi(y)}{p}\}^\wedge$.

Since (the image of) $\varphi(y)$ and p are both distinguished in $\pi_0(\hat{D})$, and since $p \mid \varphi(y)$ in $\pi_0(\hat{D})$, we find that $\varphi(y) = p \cdot u$ for some $u \in \pi_0(\hat{D})^\times$.

$\therefore$ if we form the ~~[illegible]~~ diagram

$$\begin{array}{ccccc} A' & \longrightarrow & B' & \longrightarrow & C' \\ \downarrow & & \downarrow & & \downarrow \\ \hat{D} = A'\{\frac{\varphi(y)}{p}\}^\wedge & \longrightarrow & B'' & \longrightarrow & C'' := B''\{\frac{x_1}{\varphi(y)}, \dots, \frac{x_r}{\varphi(y)}\}^\wedge \end{array}$$

Then C'' admits the alternate description

$$C'' = B''\{\frac{x_1}{p}, \dots \frac{x_r}{p}\}^\wedge .$$

Now $A'\{\frac{\varphi(y)}{p}\}^\wedge \longrightarrow B''\{\frac{x_1}{p}, \dots, \frac{x_r}{p}\}^\wedge$ is (p,z)-completely flat,

since $A' \to B''$ is, and since $x_1, \dots, x_r$ is (p,z)-completely regular ~~[illegible]~~ relative to A'.
(We proved this in the previous set of notes.)

$\therefore$ ~~we~~ it suffices to show that (p,z), or equivalently (p,y), complete flatness can be checked after applying $\hat{D} \overset{\mathbb{L}}{\underset{A'}{\otimes}} -$. For this, it suffices to factor $A' \to \mathrm{Kos}(A'; p, y)$ through $\hat{D}$.

Indeed, if we do this, then

$$\mathrm{Kos}(A'; p, y) \overset{\mathbb{L}}{\underset{A'}{\otimes}} C' = \mathrm{Kos}(A'; p, y) \overset{\mathbb{L}}{\underset{\hat{D}}{\otimes}} (\hat{D} \underset{A'}{\otimes} C')$$

so since $\hat{D} \underset{A'}{\otimes} C'$ is (p, y)-comp. flat over $\hat{D}$, we find that $\mathrm{Kos}(A'; p, y) \overset{\mathbb{L}}{\underset{A'}{\otimes}} C'$ is (p, y)-comp. flat over $\mathrm{Kos}(A'; p, y)$. But this is the same as being simply flat over $\mathrm{Kos}(A'; p, y)$, which is what it means for C' to be (p, y)-comp flat over A'.

It remains to obtain the factorization

$$A' \to \hat{D} \to \mathrm{Kos}(A'; p, y).$$

$$\hat{D} = A'\left\{\frac{\varphi(y)}{p}\right\}^{\wedge}$$

We can construct it in the universal case,

so

$$
\begin{array}{ccccc}
A' & \rightsquigarrow & \mathbb{Z}_p\{y\} & = & \mathbb{Z}_p[\underset{y_0}{\overset{\shortparallel}{y}}, y_1, \dots, y_n, \dots] \\
\downarrow & & & & \downarrow \\
\tilde{D} & \rightsquigarrow & \mathbb{Z}_p\{y, \frac{\varphi(y)}{p}\}^{\wedge} & & \\
& & \shortparallel & & \\
& & D_{(y)}(\mathbb{Z}_p\{y\})^{\wedge} & = & D_{(y)}(\mathbb{Z}_p[y])[\underset{y_0}{\overset{\shortparallel}{y_1}}, \dots, y_n, \dots] \\
& & & & \vdots \\
\mathrm{Kos}(A'; p, y) & \rightsquigarrow & \mathrm{Kos}(\mathbb{Z}_p\{y\}; p, y) & = & \mathbb{F}_p[y_1, \dots, y_n, \dots]
\end{array}
$$

We fill in the dotted arrow via $\frac{y^n}{n!} \mapsto 0 \quad \forall\, n > 0$.

This constructs C. The base change and flat localization properties are clear from the construction.

□