Perfect prisms (cont'd)

We've seen that perfect prisms are necessarily classically p-adically complete (b/c they are both p-torsion free and derived p-complete), and that the ideal I is necessarily principal.

Thus perfect prisms have the form

$$(W(R), (d))$$

where R is a perfect $\mathbb{F}_p$-algebra, and d is distinguished & a non-zero divisor.

Lemma An element $d \in W(R)$ is distinguished iff, when we write $d = \sum_{n \geq 0} [a_n] p^n$,

with $a_i \in R$, the element a_1 is a unit.

Furthermore, d is automatically a non-zero divisor.

Proof. $\delta(d) = \dfrac{\varphi(d) - d^p}{p}$.

Now $\varphi(d) = \sum_{n \geq 0} [a_n^p] p^n$,

while $d^p \equiv [a_0^p] \bmod p^2$ (since $d \equiv [a_0] \bmod p$)

$\therefore \dfrac{\varphi(d) - d^p}{p} \equiv [a_1^p] \bmod p$

$\therefore$ This is a unit iff $[a_1^p]$ is a unit (b/c $W(R)$ is p-adically complete)
iff $[a_1]$ is a unit
if a_1 is a unit.

A proof that d is non-zero divisor is given in Bhatt-Scholze (Lemma 2.34), in the more general context of a p-adically separated and p-torsion free δ-ring with A/p reduced. It just uses δ-ring manipulations. I will give a different proof here (unnecessarily elaborate!) just to illustrate a different technique. (See "Another δ-ring Lemma" for their proof.)

Here is the idea: if a_0 is a unit, then $[a_0] + p[a_1] + \cdots$ is a unit, and so in particular a non-zero divisor.
On the other hand, if $a_0 = 0$, then

$[a_1] \cdot p + \cdots = p([a_1] + p \cdot [a_2] + \cdots)$ is p

times a unit (since $[a_1]$ is a unit),

and so is a non-zero divisor, since p is so.

So we want to reduce to one of these two possibilities.

Let's start in a more general context: if R is a ring and $f \in R$, we can consider R_f, the localization of R w.r.t. the multiplicative subset $\{1, f, f^2, \ldots\}$, and also the localization of R along the ideal (f), i.e. the localization $R_{\mathcal{S}}$, where $\mathcal{S} = \{x \in R \mid x \equiv 1 \bmod f\}$.

Then the natural map $R \to R_f \times R_{\mathcal{S}}$ is an embedding, and indeed

$$\mathrm{Spec}\, R_f \amalg \mathrm{Spec}\, R_{\mathcal{S}} \longrightarrow \mathrm{Spec}\, R$$

is a faithfully flat cover.

So for certain questions, we can replace R by R_f and $R_{\mathcal{S}}$ in turn. In the first, f is a unit. In the second, $f \in \mathrm{Rad}(R_{\mathcal{S}})$.

For some questions, we really want f is either a unit or zero, or at least nilpotent.

So we can consider $R/f^nR = R_s/f^nR_s$, $\exists n \geq 1$

and consider $R \to R_f \times R/f^nR$.

This is no longer flat, but can still be injective:

The kernel of $R \to R_f$ is $R[f^\infty]$, so the map to the product will be injective iff $R[f^\infty] \cap f^nR = 0$.

Now $R[f^\infty] \cap f^nR = f^n R[f^\infty]$, so this holds iff $R[f^\infty] = R[f^n]$.

<u>Eg.</u> If R is Noetherian, then $R[f^\infty] = R[f^n]$ for some $n \gg 0$, and so

$$R \hookrightarrow R_f \times R/f^n \text{ for suff. large } n.$$

We are interested here in the case when R is a perfect $\mathbb{F}_p$-algebra. Such R are seldom Noetherian (indeed, a perfect $\mathbb{F}_p$-algebra is Noetherian iff it is a ~~[illegible]~~ finite product of fields.).

Nevertheless, we have strong control of torsion in perfect $\mathbb{F}_p$-algebras.

<u>Lemma</u>: If $f \in R$, a perfect $\mathbb{F}_p$-algebra, then

$$R[f^\infty] = R[f] = R[f^{1/p^n}] \text{ for any } n \geq 0.$$

Pf: If $f^n x = 0$, then $f^n x^p = 0$,

$$\therefore f^{n/p} \cdot x = 0$$

$\square$

In particular, if $a_0 \in R$, then

$$R \hookrightarrow R/a_0R \times R_{a_0}$$

$$\therefore W(R) \hookrightarrow W(R/a_0R) \times W(R_{a_0})$$

$\therefore$ If a_1 is a unit, $[a_0] + p[a_1] + \cdots$ is a non-zero divisor in $W(R)$, since it is so in either factor, by the special cases (a_0 a unit or $a_0 = 0$) already discussed.

This completes the proof of the lemma from p. 1. $\square$

We now prove another Lemma.

Lemma If R is a perfect $\mathbb{F}_p$-algebra, and $d \in W(R)$, then TFAE:

(1) $W(R)$ is derived (p,d)-complete

(2) $W(R)$ is classically (p,d)-complete

(3) R is derived d-complete

(4) R is classically d-complete.

Pf: (4) $\Rightarrow$ (3) and (2) $\Rightarrow$ (1) are general facts.

We have seen that $R[d^\infty] = R[d]$, so that R has bounded d-power torsion. Thus also (3) $\Rightarrow$ (4).

Since R is the cokernel of the embedding $W(R) \xrightarrow{p} W(R)$ we see that (1) $\Rightarrow$ (3).

Thus it suffices to show that (4) $\Rightarrow$ (2). So suppose (4) holds.

Each $W_n(R)$ is a successive ext. of copies of R, and so is derived d-complete, since R is. But since $W_n(R)$ then also has bounded d-power torsion (as R does) we see that $W_n(R)$ is classically d-complete.

$$\text{Then } W(R) = \varprojlim_n W_n(R) = W(R)/p^n$$

$$= \varprojlim_n \varprojlim_m W(R)/(p^n, d^m)$$

is classically (p,d)-complete, as claimed. $\square$

Thus perfect prisms are given by pairs

$$(W(R), (d))$$

perfect $\mathbb{F}_p$-algebra

where $d = [a_0] + [a_1]p + \cdots$

with $a_1 \in R^\times$ and R being classically a_0-adically complete.

However, there is another description of perfect prisms, in terms of the quotient A/I $(= W(R)/(d))$, which we turn to next.

(8)

We begin with some discussion in a more general context.

Our first observation is that the fully faithful embedding

$$\{\text{perfect } \mathbb{F}_p\text{-algebras}\} \hookrightarrow \{\mathbb{F}_p\text{-algebras}\}$$

has both a left and a right adjoint, namely $R \longmapsto \varinjlim_{x \mapsto x^p} R$ and

$$R \longmapsto \varprojlim_{x \mapsto x^p} R.$$

The latter functor is the one that will be relevant to us.

More precisely, we consider the composite

$$\begin{array}{ll} \{\text{classically } p\text{-adically complete rings}\} & R \\ & \downarrow \\ \longrightarrow \{\cancel{}\, \mathbb{F}_p\text{-algebras}\} & R/pR \\ & \downarrow \\ \longrightarrow \{\text{perfect } \mathbb{F}_p\text{-algebras}\} & \varprojlim_{x \mapsto x^p} R/pR \end{array}$$

which we denote $R \mapsto R^\flat$ ("R-flat" or "R-tilt")

<u>Claim</u> The functor $R \mapsto W(R)$

$$\{\text{perfect } \mathbb{F}_p\text{-algebras}\} \longrightarrow \{\text{classically } p\text{-complete rings}\}$$

is <u>left</u> adjoint to $R \to R^\flat$.

<u>Pf</u>: If R is a perfect $\mathbb{F}_p$-algebra and S is a p-complete ring, then we have maps

$$\mathrm{Mor}(W(R), S) \xrightarrow[\text{reduction mod } p]{} \mathrm{Mor}(R, S/pS) \xrightarrow[\substack{\text{right adjoint} \\ \text{property} \\ \text{of } \varprojlim S/pS}]{\simeq} \mathrm{Mor}(R, S^\flat)$$

$$S^\flat = \varprojlim S/pS$$

We claim that the first map is also an $\cong$; the composite $\cong$ then yields the desired adjunction.

Since S is p-complete, we have

$$S = \varprojlim S/p^n S,$$

$\therefore$ $\mathrm{Mor}(W(R), S)$

$$= \varprojlim \mathrm{Mor}(W(R), S/p^n S)$$

$$= \varprojlim \mathrm{Mor}(W_n(R), S/p^n S)$$

and the first arrow in the putative ~~adjunction~~ ~~isomorphism~~ adjunction map is just the projection onto the $n=1$ term.

Thus ~~[illegible]~~ it suffices to show that

$$\mathrm{Mor}(W(R), S/p^n S) \xrightarrow{\cong} \mathrm{Mor}(W(R), S/p^{n-1} S)$$

for all n.

This is similar to the construction of the lift $[\ ]: R \to W(R)$.

First, let $\varphi, \psi : W(R) \to S/p^n S$, (11)

and suppose that $\varphi \equiv \psi \mod p^{n-1} S$.

Let $a = [a_0] + pb \in W(R)$.

Then $\varphi(b) \equiv \psi(b) \quad (p^{n-1})$

$$\therefore \underset{\varphi(pb)}{\underset{\shortparallel}{p\varphi(b)}} \equiv \underset{\psi(pb)}{\underset{\shortparallel}{p\psi(b)}} \quad (p^n)$$

Also $\varphi([a_0^{1/p}]) \equiv \psi([a_0^{1/p}]) \quad (p^{n-1})$,

so raising each side to the p^{th} power, we get

$$\varphi([a_0]) \equiv \psi([a_0]) \quad (p^n).$$

$\therefore$ In fact $\varphi = \psi$.

Similarly, if we are given $\bar{\varphi} : W(R) \to S/p^{n-1}$, we define $\varphi : W(R) \to S/p^n$ via

$$\varphi([a_0] + pb) = \bar{\varphi}([a_0^{1/p}])^p + p\bar{\varphi}(b)$$

both well-defined mod p^n.

One easily checks that φ is a homomorphism, and so we've shown that

$$\mathrm{Mor}(W(R), S/p^n S) \to \mathrm{Mor}(W(R), S/p^{n-1} S)$$

is an $\cong$, as required. $\square$

In particular, if R is classically p-complete, we get a map

$\theta : W(R^\flat) \longrightarrow R$ (the co-unit of the adjunction)

<u>Lemma</u> θ is surjective iff $R^\flat \to R/p$ is surjective

iff $R/p \xrightarrow{x \mapsto x^p} R/p$ is surjective

<u>Pf</u>: If θ is surjective, then reducing mod p we obtain the projection $R^\flat \to R/p$, which must also be surjective. The converse follows by an easy form of Nakayama's lemma (w.r.t. the prime p).

Now the projection $R^\flat \to R/p$ factors as

$$\begin{tikzcd} R^\flat \arrow[r] \arrow[d] & R/p \\ R/p \arrow[ur, "x \mapsto x^p"'] & \end{tikzcd}$$

and so if the former map is surjective, so is the latter.

Conversely, if $x \mapsto x^p$ is surjective, then $R^\flat := \varprojlim_{x \mapsto x^p} R/p$ is defined via surjective transition maps, and so the projection to R/p is also surjective. $\square$

<u>Terminology</u>: An $\mathbb{F}_p$-algebra R is called <u>semi-perfect</u> if $x \mapsto x^p$ is surjective on R.

<u>Lemma</u>: ① An $\mathbb{F}_p$-algebra is semi-perfect iff it is the quotient of a perfect $\mathbb{F}_p$-algebra.

② A semi-perfect $\mathbb{F}_p$-algebra is perfect iff it is reduced.

Pf: (1) If $R \xrightarrow{x \mapsto x^p} R$ is surjective,

then R is a quotient of the perfect $\mathbb{F}_p$-algebra $\varprojlim_{x \mapsto x^p} R$.

Conversely, any quotient of a perfect $\mathbb{F}_p$-algebra is evidently semi-perfect

(2) A semi-perfect $\mathbb{F}_p$-algebra R is perfect iff $x \mapsto x^p$ is injective iff R is reduced

$\square$

Thus $\theta : W(R^\flat) \longrightarrow R$ is surjective iff R/p is semi-perfect.

More generally, if $\exists$ perfect $\mathbb{F}_p$-algebra A and a surjection

$$W(A) \twoheadrightarrow R,$$

then $A = W(A)/pW(A) \longrightarrow R/pR$ is surjective,

$\therefore$ R/pR is semiperfect.

Also the surjection $A \twoheadrightarrow R/pR$ induces a surjection $A = \varprojlim_{x \mapsto x^p} A \longrightarrow \varprojlim_{x \mapsto x^p} R/pR =: R^\flat$, and so the surjection $W(A) \twoheadrightarrow R$ factors as a composite of surjections $W(A) \twoheadrightarrow W(R^\flat) \overset{\theta}{\twoheadrightarrow} R$.

Let's continue to suppose that R/pR is semi-perfect, that A is a perfect $\mathbb{F}_p$-algebra, and that we are given a surjection $W(A) \twoheadrightarrow R$, with kernel I
(E.g. it could be the canonical surjection
$$\theta : W(R^\flat) \twoheadrightarrow R.)$$

Then $R/p = W(A)/(I, p)$, and we have the isomorphism of inverse systems

$$\begin{tikzcd}
\cdots \arrow[r] & W(A)/(\varphi^n(I), p) \arrow[r] \arrow[d, "\varphi^{-n}", "\cong"'] & W(A)/(\varphi^{n-1}(I), p) \arrow[r] \arrow[d, "\varphi^{-(n-1)}", "\cong"'] & \cdots \arrow[r] & W(A)/(I, p) \arrow[d, equal] \\
\cdots \arrow[r] & W(A)/(I, p) \arrow[r, "\varphi"] \arrow[d, equal] & W(A)/(I, p) \arrow[r, "\varphi"] \arrow[d, equal] & \cdots \arrow[r, "\varphi"] & W(A)/(I, p) \arrow[d, equal] \\
\cdots \arrow[r] & R/pR \arrow[r, "x \mapsto x^p"] & R/pR \arrow[r] & \cdots \arrow[r] & R/pR
\end{tikzcd}$$

where the maps in the top row are the canonical quotient maps corresponding to the inclusions

$$(I,p) \supseteq (\varphi(I),p) \supseteq \cdots \supseteq (\varphi^{n-1}(I),p) \supseteq (\varphi^n(I),p) \supseteq \cdots$$

If we let $\bar{I}$ = image of I in $A = W(A)/pW(A)$,

then we may also rewrite this inverse system as

$$\longrightarrow A/\bar{I}^{[p^n]} \to A/\bar{I}^{[p^{n-1}]} \to \cdots \to A/\bar{I}^{[p]} \to A/\bar{I} = R/pR \ ;$$

here $J^{[p^n]}$ = ideal gen'd by x^{p^n} $\forall x \in J$

if J is an ideal in an $\mathbb{F}_p$-algebra.

(If the algebra is semi-perfect, then in fact $J^{[p^n]} = \{x^{p^n} \mid x \in J\}$.)

The sequence $\cdots \subseteq \bar{I}^{[p^n]} \subseteq \bar{I}^{[p^{n-1}]} \subseteq \cdots \subseteq \bar{I}^{[p]} \subseteq \bar{I}$

is cofinal with the sequence $\bar{I}^n$, $n \geq 0$.

Thus the surjection $A \twoheadrightarrow R^\flat$ induces an identification

$$\text{classical } \bar{J}\text{-adic completion of } A \xrightarrow{\sim} R^\flat .$$

(Apologies for the constantly shifting notation!!)

Suppose now that (A, I) is a perfect prism, so A/pA is a perfect $\mathbb{F}_p$-algebra, and

$$(A, I) \cong (W(A/pA), (d))$$

for some distinguished element d. Write $R := A/I$.
R is classically p-complete – see "Another δ-ring lemma".
Then the preceding discussion shows that

$$d\text{-adic completion of } A/pA \xrightarrow{\sim} R^\flat .$$

On the other hand, our earlier discussion (p. 6) shows that A/pA is d-adically classically complete.
Thus $A/pA \xrightarrow{\sim} R^\flat$, and so we have a canonical isomorphism

$$(A, I) \xrightarrow{\sim} (W(R^\flat), \ker \theta) .$$

Thus $(A, I) \mapsto R := A/I$

induces a fully-faithful embedding

$\{$perfect prisms$\} \hookrightarrow \{$classically p-complete rings$\}$

The essential image is (by definition, if you like) the category of <u>perfectoid rings</u>.

We can give an intrinsic characterization of the image (Prop. 2.10 of Bhatt's lecture 4).

~~Prop: A p-complete ring R, [illegible] R/pR is semi-perfect, is perfect~~

~~$A \otimes_p$~~

Prop: A classically p-complete ring R is perfectoid iff

- R/pR is semi-perfect
- $\theta : W(R^\flat) \to R$ has a principal ideal as kernel
- $\exists\, \varpi \in R$ s.t. $\varpi^p = p \times$ a unit in R.

<u>Proof</u>: We have seen that the first two conditions are necessary, since if R is perfectoid, then $R \cong A/I$ for some perfect prism (A, I), and then (as we've seen)

$$(A, I) \cong (W(R^\flat), \ker\theta),$$

showing that $\ker\theta$ is principal, since I is.

Now in fact, in this situation, $I = (d)$, where $d = [a_0] + p[a_1] + \cdots$ with $a_1 \in (R^\flat)^\times$.

Let $\overline{\omega} = \varphi^{-1}(d) \bmod I$

$$= [a_0^{\frac{1}{p}}] + p[a_1^{\frac{1}{p}}] + \cdots \bmod I.$$

Then $\overline{\omega}^p \equiv [a_0] \bmod (p^2, I)$

Now $[a_0] \equiv -p[a_1] \bmod (p^2, I)$ (since $I = (d)$)

$$\therefore \overline{\omega}^p \equiv p \cdot \underbrace{\begin{pmatrix}\text{image in } R \text{ of} \\ [a_1]\end{pmatrix}} \bmod p^2 \text{ in } R = W(R^\flat)/I.$$

$$= p \cdot u + p^2 x = p(\underset{\substack{\uparrow \\ \text{a unit in } R, \text{ since } p \in \mathrm{Rad}(R)}}{u + px})$$

Thus the third condition is necessary too.

Conversely, if these conditions hold, then we can write $\ker\theta = (d)$ for some d, and $W(R^\flat)/(d) = R$.

~~Let $x \in W(R^\flat)$ [illegible]~~.

We have seen that ~~[illegible]~~ the classical d-adic completion of $R^\flat$ maps isomorphically to $R^\flat$ (top of p. 17) and so $R^\flat$ is classically d-adically complete.

The lemma on p. 6 then shows that $W(R^\flat)$ is classically (p,d)-complete.

So far we have used the first two assumptions. It remains to show that d is distinguished; for this we use the 3rd assumption.

Let $x \in W(R^\flat)$ lift ϖ.

Since $(d,p) \subseteq \mathrm{Rad}\, W(R^\flat)$ (since it is (p,d)-complete) we see that the unit ~~[illegible]~~ appearing in the factorization $\varpi^p = p \cdot \text{unit}$ may be lifted to a unit $v \in W(R^\flat)$.

The equation $\varpi^p = p \cdot$ unit in R
then lifts to the divisibility

$$d \mid x^p - pv \quad \text{in } W(R^\flat).$$

Now if $x = [a_0] + p[a_1] + \cdots$,
we see $x^p = [a_0^p] + p^2[a_1^p] + \cdots$

$$\equiv [a_0^p] \mod p^2$$

$$\therefore \; x^p - pv \equiv [a_0^p] + p \cdot v \mod p^2.$$

Thus $x^p - pv$ is distinguished in $W(R^\flat)$.

Since $d \mid x^p - pv$ and $p, d \in \operatorname{Rad} W(R^\flat)$,

we find that $d = (x^p - pv) \times$ unit in $W(R^\flat)$,
so that d is distinguished, as required. $\square$

If R is p-torsion free, we can give another characterization of the kernel of θ being principal which is more intrinsic to R.

Firstly, if R is p-torsion free ~~[illegible]~~

assumed p-complete, with R/p semi-perfect and if $I = \ker(\theta : W(R^\flat) \to R)$, then

$$I \cap p^n W(R^\flat) = p^n I.$$

$\therefore$ $\bar{I} :=$ image of I in $R^\flat = \ker(R^\flat \to R/pR)$

$$= I/pI$$

Also we have short exact sequences

$$0 \to I/p^n I \hookrightarrow W_n(R^\flat) \twoheadrightarrow R/p^n R \to 0$$

so that passing to the inverse limit we get

$$0 \to \text{p-adic completion of } I \hookrightarrow W(R^\flat) \to R,$$

so that I is p-adically complete.

Thus I is principal iff $\bar{I}$ is principal, i.e. if $\ker(R^\flat \to R/pR)$ is principal.

Now if this kernel is principal, then a ~~[illegible]~~ consideration of the factorization

$$R^\flat \twoheadrightarrow R/pR \xrightarrow{x \mapsto x^p} R/pR$$

shows that $R/pR \xrightarrow{x \mapsto x^p} R/pR$ has principal kernel.

Suppose conversely that this map has principal kernel, say (y) for some $y \in R/pR$.

Let $(y^{\frac{1}{p^n}})$ be a compatible system of $p^{n\text{th}}$ roots of y in R/pR. Then

$$\mathrm{Ker}\left(R/pR \xrightarrow{x \mapsto x^{p^{n+1}}} R/pR \right) \text{ is generated by } y^{\frac{1}{p^n}}$$

(Indeed, arguing by induction, suppose that

$z^{p^{n+1}} = 0$. Then $z^{p^n} = ay$ $\exists a \in R/pR$.

Choose b s.t. $b^{p^n} = a$.

Then $(z - b y^{\frac{1}{p^n}})^{p^n} = 0$, so by induction,

$$z = b y^{\frac{1}{p^n}} + c y^{\frac{1}{p^{n-1}}} \quad \exists c \in R/pR$$

i.e. $z \in (y^{\frac{1}{p^n}})$.)

Thus [illegible] $(0, y, y^{\frac{1}{p}}, \ldots, y^{\frac{1}{p^n}}, \ldots) \in R^b$ generates the kernel of $R^b \to R/pR$

<u>To recap</u>: If R is classically p-complete & p-torsion free, and if R/pR is semi-perfect, then $\ker(\theta : W(R^\flat) \to R)$ is principal iff $\ker(R/pR \xrightarrow{x \mapsto x^p} R/pR)$ is principal.

Now suppose that R is perfectoid.

Then both kernels above <u>are</u> principal.

Let $(y) = \ker(R/pR \xrightarrow{x \mapsto x^p} R/pR)$.

Choose $\varpi \in R$ s.t. $\varpi^p = p \cdot u$ $\quad u \in R^\times$.
(As we've seen that we may.)

Then $\varpi \bmod p$ lies in the kernel (y),

$\therefore \quad \varpi = ay' + bp \quad$ (here y' is some lift of y to R)

$\therefore \; pu = \varpi^p = a^p (y')^p + c p^2$

$\therefore \; a^p \cdot (y')^p = p\underbrace{(u - cp)}_{\text{a unit in } R}$

$\therefore \; (y')^p \mid p.$

On the other hand, $y^p = 0$ in R/pR, $\therefore$ ~~$p \mid y$~~ $p \mid (y')^p$.

Write $(y')^p = ps$, $\quad p = (y')^p t$

Then $p = pst$

$\therefore 1 = st$ b/c R is p-torsion free

Now in terms of our earlier quantities, $t = a^p \cdot (u - cp)^{-1}$

$\therefore a^p$ is a unit
$\therefore a$ is a unit
$\therefore (\varpi) = (y')$

$\therefore$ in fact $\ker(R/pR \xrightarrow{x \mapsto x^p} R/pR)$

is generated by ϖ mod p.

This has an additional consequence:

Suppose that $x \in R[\frac{1}{p}] = R[\frac{1}{\varpi}]$ and $x^p \in R$.

Choose n minimal s.t. $\varpi^n \cdot x \in R$, and suppose $n > 0$.

Then $x = \frac{z}{\varpi^n}$ $z \in R$, $n > 0$

$$x^p = \frac{z^p}{\varpi^{pn}}$$

$\therefore$ $z^p = \varpi^{pn} \cdot x^p \equiv 0 \mod p$ b/c $n \geq 1$ & $\varpi^p = p \cdot u$

$\therefore$ $z \mod p \in (\varpi \mod p)$

$\therefore$ $z \in (\varpi)$ (recall that $\varpi | p$)

$\therefore$ $\frac{z}{\varpi} \in R$

$\therefore$ $x = \frac{z/\varpi}{\varpi^{n-1}}$, contradicting the choice of n.

Thus if R is perfectoid and p-torsion free, then if $x \in R[\frac{1}{p}]$ satisfies $x^p \in R$, we have that in fact $x \in R$.

Conversely, if R is classically p-complete, p-torsion free, if R/pR is semi-perfect, if R contains an element ϖ s.t. $\varpi^p = pu$, and if R satisfies the preceding additional condition, then R is perfectoid.

To prove this, one shows that

$\ker\left(R/pR \xrightarrow{x \mapsto x^p} R/pR \right)$

is generated by ϖ mod p.

Indeed, $\varpi^p - pu \equiv 0 \mod p$,

so $(\varpi \bmod p) \subseteq \ker\left(R/pR \xrightarrow{x \mapsto x^p} R/pR \right)$.

Conversely, if $x \in R$ and $p \mid x^p$,

then $\varpi^p \mid x^p$, $\therefore \left(\frac{x}{\varpi}\right)^p \in R$,

$\therefore \frac{x}{\varpi} \in R$ by our final hypothesis,

and so $x \in \varpi R$.

This completes our discussion of perfectoid rings (for now!).