Perfect $\delta$-rings and prisms

If $R$ is a perfect $F_p$-algebra, then $W(R)$ has the following description:

- $W(R)$ is $p$-adically complete and $p$-torsion free,
  so $W(R) = \varprojlim W(R)/p^nW(R)$

- $W_1(R) = W(R)/pW(R) = R$

- $\phi: W(R) \to W(R)$ is an automorphism
  (i.e. $W(R)$ is perfect).

We have the canonical lift

$[\mathcal{I}: R \to W(R)]$, a multiplicative home.

defined as follows:
If $x \in \mathbb{R}$, choose $y_n \in \mathbb{R}^p$ such that

$$x_n = \frac{1}{p^n} y_n$$

Then $\exists \overline{x} := \lim_{n \to \infty} x_n$

(If $a \equiv b \mod p^n$, then $a^p \equiv b^p \mod p^{n+1}$, since $y_{n+1}^p \equiv y_n \mod p$.
we have $x_{n+1} \equiv x_n \mod p^{n+1}$, and the limit exists.)

Also $\varphi(\overline{x}) = \overline{x^p} = \overline{x}^p$.

\[ \varphi(y_{n+1}) = y_{n+1}^p \equiv y_n \mod p, \]

\[ \therefore \varphi(x_{n+1}) = x_{n+1}^p \mod p^{n+2}. \]

Since $\varphi$ is precisely an homomorphism, $\varphi(\mathbb{R}) = \mathbb{F}_p$.

\[ W(\mathbb{R}) = \left\{ \sum_{n=0}^{\infty} \overline{x}_n p^n \mid x_n \in \mathbb{R} \right\} \]

(Given $x \in W(\mathbb{R})$, let $\overline{x} = x_n \mod p$, then $x - \overline{x} \in \mathbb{F}_p(\mathbb{R})$, and we obtain the required expansion recursively.)
Eg. To compute \([\overline{x^2}] + [\overline{y^3}]\):

\[\text{mod } p, \text{ this becomes } \overline{x^2 y^3},\]
\[\therefore \ [\overline{x^2}] + [\overline{y^3}] = [\overline{x+\overline{y}}] + p^2 \cdot \cdots \]

To find the next coefficient in the expansion:

\[\ [\overline{x^2}] + [\overline{y^3}] = [\overline{(\overline{x} + \overline{y})^p}] \text{ mod } p \]

Raise both sides to \(p^2\) power gives

\[\ [\overline{x}] + [\overline{y}] + p \sum_{i=1}^{p-1} \frac{1}{p} \left( \begin{array}{c} p^2 \end{array} \right) \left[ \overline{x_i^p y_i^p} \right]^p \]

\[\therefore [\overline{x}] + [\overline{y}] = [\overline{x+y}] \text{ mod } p^2 \]
\[\therefore [\overline{x^2}] + [\overline{y^3}] = [\overline{x+y^2}] + p \left[ \overline{\frac{\overline{x+y} - (\overline{x^p+y^p})^p}{p}} \right] + p^2 \cdot \cdots \]

So the comparison with \(W_2(\mathbb{F}_p) - \mathbb{F}_p^2\) from the existing note is that

\[(\overline{x_0}, \overline{x_1}) \leftrightarrow [\overline{x_0}] + p[\overline{x_1^p}]\]

General description

\(W_2(\mathbb{F}_p)\) for perfect \(\mathbb{F}_p\)-algebra
**Theorem:** If $A$ is a perfect $p$-adically complete $S$-ring, then $\exists$ a canonical $L$ of $S$-rings.

$A \rightarrow W(A/pA)$

So $A \rightarrow A/pA$ induce an equivalence of categories.

$$\begin{align*}
\left\{ \text{perfect } p\text{-complete } \right\} & \rightarrow \left\{ \text{perfect } S\text{-rings} \right\} \\
\left\{ \text{and } L \text{-algebras} \right\}
\end{align*}$$

**Proof:** By Witt vector adjunction, the canonical surjection

$A \rightarrow A/pA$ induces a morphism of $S$-rings

$A \rightarrow W(A/pA)$

which by construction is an isomorphism.

Since $A$ is $p$-adically complete, while $W(A/pA)$ is $p$-adically complete & $\mathcal{P}$-adic, this is necessarily an isomorphism.
As a corollary, one finds that a perfect pre-complete \( f \)-ring is automatically \( p \)-turbin free.

One can also see this directly (w/o any completeness assumption):

**Lemma:** If \( x \in A \) (\( f \)-ring) and \( px = 0 \), then \( f(x) = 0 \).

:. perfect \( f \)-rings are \( p \)-turbin free.

**Proof:** \( 0 = f(px) = p^1f(x) + f(p)f(x) \).

Now \( f(x) = x^p + px \)

:. \( 0 = f(px) = px^p + p^2x \)

= \( p^2x \) (since \( px = 0 \))

:. As \( p \geq 2 \), \( p^2x = 0 \)

:. \( 0 = f(px) = (1 - p^{2-1}) f(x) . \)

Since \( p f(x) = 0 \) as well, we get

\( f(x) = 0 \) .
We now want to analyze perfect prisms, i.e., prisms whose underlying $\mathcal{D}$-ring is perfect.

If $(A,I)$ is a perfect prism, we have just seen that it is $p$-torsion free, not only derived $p$-cyclically complete, but classically $p$-complete.

Thus, $A = W(R)$ for a perfect $\mathcal{E}$-alg. $R$. If $L(R/A)$ is such and $\phi$ is an auto.

Also, $I$ is principal, say $I = (d)$, with $d$ distinguished and a nonzero divisor.

So to classify perfect prisms, we have to understand distinguished elements in $W(R)$. 