Hodge-Tate comparison

Some base-change properties for $\Delta_{R/A}$

(1) If R is p-completely smooth over A/I,

if $(A,I) \to (A', I' = IA')$ is a map of bounded prisms with finite (p,I)-complete Tor-amplitude, and if we set $R' := R \hat{\otimes}_A A'$, then there are isomorphisms

$$\Delta_{R/A} \hat{\otimes}^L_A A' \xrightarrow{\sim} \Delta_{R'/A'}$$

&

$$\underbrace{\overline{\Delta}_{R/A} \hat{\otimes}^L_A A'}_{\text{derived } (p,I)\text{-completed}} \xrightarrow{\sim} \overline{\Delta}_{R'/A'}$$

Pf: Choose B a (free δ-A-alg. with a surjection

$B \to R$ factors through B/I

If J is the kernel, let $C = B\{\frac{J}{I}\}^\wedge$

prismatic envelope.

Our results on prismatic envelopes (suitably augmented as in 3.14 & 4.16 of [BS']) show that

(C, IC) is flat (i.e. (p,I)-comp. flat) over (A,I),

and that the formation of C is compatible with base-change, i.e. if $B' = B\hat{\otimes}_A A'$, and if $J' = \ker(B' \to R')$, Then $C' := B'\left\{\frac{J'}{I'}\right\}^{\wedge} = C \hat{\otimes}_A A'$.

~~Since B is p-completely free~~

Now by construction, $IC \supseteq JC$,

$$\therefore \begin{array}{ccc} & & C/IC \\ & \nearrow & \\ B/JB & & \\ \| & & \\ R & & \end{array},$$

$\therefore$ get

$$\begin{tikzcd} & C \arrow[d] \\ R \arrow[r] & C/IC \end{tikzcd}$$

an object of the prismatic site of R over A. Since B is free, it is weakly final, $\therefore$

$$\Delta_{R/A} = \mathrm{Tot.}(C^\bullet)$$

cosimplicial ring; Spec $C^\bullet$ = (p,I)-completed Čech nerve of Spec C over Spec A.

Also $\Delta_{R'/A'} = \mathrm{Tot.}((C')^\bullet) = \mathrm{Tot.}(C^\bullet \hat{\otimes}_A A')$.

So the technical pt is to check that we can pass the $\hat{\otimes}^{\mathbb{L}}$ through the totalization.

The finite Tor-dim assumption lets us do this. $\square$

(Once we have the HT comparison, we can prove this with no finite Tor dim hypothesis on $(A, I) \to (A', I')$.)

(2) If $R \to S$ is a p-completely étale map of p-completely smooth A/I algs., then

$$\overline{\Delta}_{R/A} \hat{\otimes}^{\mathbb{L}}_{R} S \xrightarrow{\sim} \overline{\Delta}_{S/A}.$$

Pf. The map $R \to S$ induces a "forgetful functor" from the prismatic site of S/A to that of R/A.

This has a right adjoint (right adjoint b/c the prismatic site is defined via Spec's; it is left adjoint from the perspective of rings) given as follows:

if
$$\begin{array}{ccc} & & B \\ & & \downarrow \\ R & \longrightarrow & B/IB \end{array},$$

then $B/IB \to B/IB \hat{\otimes}^{\mathbb{L}}_{R} S$, which is p-completely étale, deforms to a (p,I)-completely étale δ-map

$$B \longrightarrow B_S \quad \text{(by } \delta\text{-ring generalities)}$$

giving

$$\begin{array}{ccc} & & B_{\mathfrak{f}} \\ & & \downarrow \\ \mathfrak{f} = R\hat{\otimes}^{\mathbb{L}}_{\mathfrak{f}} S & \longrightarrow & B_{\mathfrak{f}}/IB_{\mathfrak{f}} = (B/I)\hat{\otimes}^{\mathbb{L}}_{R} S \end{array}$$

This right adjoint will preserve product and weekly final object, ~~and~~

Let C and $C^{\cdot}$ be as in ① ~~comp~~

Then if $\underset{C_{\mathfrak{f}}}{\overset{\shortparallel}{D}}$ is the correspondy object over $\mathfrak{f}$,

D is weekly final, and $D^{\cdot}$ ~~arises from $C^{\cdot}$ via right adjoint~~

~~computes~~ arises from $C^{\cdot}$ via the right adjoint.

Then $D^{\cdot}/ID^{\cdot} = C^{\cdot}/IC^{\cdot}\hat{\otimes}^{\mathbb{L}}_{R}\mathfrak{f}$ by construction.

Also $\bar{\Delta}_{R/A} = \mathrm{Tot}(C^{\cdot}/IC^{\cdot})$ by weak finality of C

$\bar{\Delta}_{S/A} = \mathrm{Tot}(D^{\cdot}/ID^{\cdot})$ —— " —— of D

$= \mathrm{Tot}(C^{\cdot}/IC^{\cdot}\hat{\otimes}^{\mathbb{L}}_{R} S)$.

Now ② follows by the same homological algebra as ①;

the complete flatness of S over R lets us pass $\hat{\otimes}^{\mathbb{L}}_{R} S$ through the total complex. $\square$

We can now prove the crystalline case of the HT comparison: $(A, (p))$ a crystalline prism.
R smooth over A/pA.

Universal properties of $\Omega^{\bullet}_{R/(A/p)}$ give rise to

$$\Omega^i_{R/(A/p)} \longrightarrow H^i(\overline{\Delta}_{R/A})$$

compatible with wedge / cup product, exterior diff'l, and Bockstein.

<u>Thm</u>: This is an isomorphism.

<u>Proof</u>: By étale base change (② above) and naturality, we reduce to the case when $R = A/pA[x_1, \dots, x_n]$.

Then by ① above, applied to $\mathbb{Z}_p \to A$, we reduce to the case when $A = \mathbb{Z}_p$ and $R = \mathbb{F}_p[x_1, \dots, x_n]$.

In this case the prismatic cohom. reduces to crystalline cohom., & the above map becomes the inverse Cartier iso., which <u>is</u> an isomorphism!! □

We now consider the HT comparison Thm. in general.

~~Approach~~ So now R is a p-completely smooth A/I-alg.

Again, universal property of $\Omega^\bullet_{R/(A/I)}$ induces

$$\Omega^i_{R/(A/I)} \longrightarrow H^i(\overline{\Delta}_{R/A}) \otimes_{A/I} I^i/I^{i+1}$$

wedge $\longleftrightarrow$ cup product

exterior derivative $\longleftrightarrow$ Bockstein.

<u>Thm</u>: This is an isomorphism

<u>Pf</u>: We use p-complete étale base change to reduce to the case when $R = A/I\langle x_1, \ldots, x_n\rangle$

(p-adic comp. of $A/I[x_1, \ldots, x_n]$).

Now the statement we want to check can be ~~..~~ verified flat locally on (A, I) (here "flat" means in the prismatic sense, i.e. (p, I)-completely flat locally)

$\therefore$ can assume A is oriented, i.e. that $I = (d)$.

Let $(A_0, (d))$ = universal oriented prism

$$:= \mathbb{Z}_p\{d, \delta(d)^{-1}\}^\wedge.$$

Then R is obtained by base-change from $R_0 := A_0/(d)\langle x_1, \ldots, x_n\rangle$

We first verify the case when $A = A_0$.

The good features of this case are that

Frob : $A/p \to A/p$ is flat, and $A/(d)[p] = 0$,

i.e. d is a non-zero divisor on A/pA.

In this context (i.e. if these properties hold), we set

$B := A\left\{\frac{\varphi(d)}{p}\right\}^\wedge$ derived p-completion

Since both d (and thus $\varphi(d)$) & p are distinguished in $\pi_0(B)$, we have $\varphi(d) = u \cdot p$ for $u \in \pi_0(B)^\times$, and so B is derived (p,d)-complete.

Also $A\left\{\frac{\varphi(d)}{p}\right\} = D_A((d))$ by our pd-envelope result,

$\therefore$ is p-torsion free, $\therefore$ $B = A\left\{\frac{\varphi(d)}{p}\right\}^\wedge$ = classical completion

So B is discrete, i.e. just a usual ring, and being derived (p,d)-complete it is a prism, with prismatic ideal (p),

and $(A,(d)) \xrightarrow{\alpha} (B,(p))$ is a map of prisms.

$$\begin{tikzcd}
(A,(d)) \arrow[rr, "\alpha"] \arrow[dr, "\varphi_A"'] & & (B,(p)) \\
& (A,(d)) \arrow[ur, "\text{natural map}"'] &
\end{tikzcd}$$

(since it takes d to $\varphi(d) = u \cdot p$ in B)

Now mod p, α becomes

$$A/p \xrightarrow{\text{Frob}} A/p \longrightarrow D_{A/pA}((d))$$

Since $d^p = \underbrace{p}_{=0} \cdot \gamma_p(d)$ in $D_{A/pA}((d))$,

this factors as

$$A/p \longrightarrow A/(p,d) \xrightarrow{\text{Frob}} A/(p,d^p) \longrightarrow D_{A/pA}((d)).$$

finite Tor-amplitude b/c d is a non-zero divisor

base change of $A/p \xrightarrow{\text{Frob}} A/p$ over $A/(p,d)$, $\therefore$ faithfully flat

inclusion of a summand in a free $A/(p,d^p)$-module.

$\therefore$ the composite has finite Tor-amplitude,

$\therefore$ $\alpha : A \longrightarrow B$ has p-completely finite Tor-amplitude,

so we get $\widehat{\alpha^* \Delta_{R/A}} = \Delta_{R_B/B}$

by base-change property ①.

Also, one finds that $\hat{\alpha}^*$ is <u>conservative</u> on derived (p,d)-complete objects of $D(A)$.

(By p-derived Nakayama, it's enough to check that if $X \to Y$ is a morphism of derived d-complete objects in $D(A/p)$, s.t. $(B/pB) \hat{\otimes}^{L}_{A/pA} X \to (B/pB) \hat{\otimes}^{L}_{A/pA} Y$ is is iso., then $X \to Y$ is an $\cong$.

Since $A/pA \to B/pB$ factors as $A/pA \to A/(p,d)$ followed by faithfully flat maps, it's enough to check the corresponding statement for $(X \to Y) \mapsto (A/(p,d) \otimes^{L}_{A/p} X \to A/(p,d) \otimes^{L}_{A/p} Y)$.

But this is just derived Nakayama for (d) !!)

Now, since $\Omega^i_{R/A}$ are (topologically) free over R, we may choose $\eta : \oplus \Omega^i_{R/A}\{-i\}[-i] \longrightarrow \overline{\Delta}_{R/A}$ inducing the HT comparison maps on cohom.

Applying $\hat{\alpha}^*$, we know we get an iso., by the crystalline case. Since $\hat{\alpha}^*$ is conservative, we find that η is an iso., and we're done.

Finally, we want to base-change from this case $(A = A_0)$ to the general case. If we knew general base-change results for $\Delta_{R/A}$, we'd be done — but these will only be deduced as a consequence of the theorem we're trying to prove!!

Consider

$$\begin{tikzcd}
 & A_0 = \mathbb{Z}_p\{d, \delta(d)^{-1}\}^\wedge \arrow[r] \arrow[d, "\alpha"] & A \arrow[d, "\beta"] \\
 & B_0 = A_0\left\{\frac{\varphi(d)}{p}\right\}^\wedge & \\
\mathbb{Z}_p \arrow[r] \arrow[rr, bend right, "\gamma"'] & A_0\left\{\frac{\varphi(d)}{p}\right\}^\wedge \arrow[r] & B = A \hat{\otimes}^{L}_{A_0} B_0
\end{tikzcd}$$

derived (p,d) completion

"simplicial prism"

$(\mathbb{Z}_p, (p)) \to (B_0, (p))$, $(p) = (\alpha(d))$ map of prisms.

Since $\hat{\alpha}^*$ is conservative, so is $\hat{\beta}^*$.

But now, using an explicit ∞-simplicial model for $\overline{\Delta}_{R/A}$, we show that $\hat{\gamma}^* \overline{\Delta}_{\mathbb{F}_p[x_1, \dots, x_n]} = \hat{\beta}^* \overline{\Delta}_{R/A}$.

(We build an explicit cosimplicial model for $\Delta_{A_0\langle x_1, \dots, x_n\rangle / A_0}$ which is manifestly base-change compatible, and use that γ, and also α, and thus β, have finite p-complete, resp. (p,d)-complete, Tor dim 1, so as to interchange completed pull-back with the formation of total complexes.)

Here it is:

$$\begin{array}{l} B = A\{X_1, \dots, X_n\}^{\wedge} \leftarrow \text{derived } (p,I)\text{-completion} \\ \quad \Big\downarrow \; x_i \mapsto x_i, \;\; \delta^j(x_i) \mapsto 0 \text{ if } j>0. \\ A/I\langle x_1, \dots, x_n\rangle \end{array}$$

and $C = B\{\frac{J}{I}\}^{\wedge}$. ($(p,d)$-completed)

Then $C^{\bullet}$ ← cosimplicial Čech nerve of C

is the model for $\Delta_{A/I\langle x_1, \dots, x_n\rangle / A}$

that we want (manifestly base-change compatible.)

So now we reduce to the (already proved) case of

$(A, I) = (\mathbb{Z}_p, (p))$.

$\square$