Examples

If R is a perfect $\mathbb{F}_p$-algebra, the canonical δ-ring structure on $W(R)$ is the unique one.

Indeed, if $a \in R$, then $[a] \in W(R)$ has p^nth roots for all $n \geq 0$,
$\therefore$ necessarily $\varphi([a]) = [a]^p$.

Other rings If k is any field of char. p, we can find a flat p-adically complete $\mathbb{Z}_p$-algebra W s.t. $W/pW \cong k$, and a lift φ of Frobenius to W.
W is unique up to $\cong$, but not up to unique isomorphism, and φ is not uniquely determined.

Eg. $W := \widehat{\mathbb{Z}_p[[x]][x^{-1}]}$ p-adic completion

$W/pW = \mathbb{F}_p((x))$.

Lots of possible choices of φ

e.g. $\varphi(x) = x^p$
$\varphi(x) = (1+x)^p - 1$
$\vdots$

One perspective: let $k = \mathbb{F}_p((x))^{perf}$

$= \varinjlim_n \mathbb{F}_p((x^{1/p^n}))$

Then we can find $W \hookrightarrow W(k)$ compatibly with the inclusion $\mathbb{F}_p((x)) \hookrightarrow k$, but this embedding is not unique.

__But__ it becomes uniquely determined once we choose φ on W and ask that it be φ-compatible as well

(b/c — since W is p-torsion free — choosing φ makes W a δ-ring, and then Witt vector adjunction ~~gives~~ shows that the map $W \twoheadrightarrow \mathbb{F}_p((x)) \hookrightarrow k$ induces / is induced by a unique φ-equivariant map $W \hookrightarrow W(k)$.)

Complete regular local rings (Bhatt–Scholze §3.11) ③

Let R be a complete regular local ring of dimension d which is in mixed characteristic, so a p-complete flat $\mathbb{Z}_p$-algebra

Let $\mathfrak{m}$ = maximal ideal (so $p \in \mathfrak{m}$), let W be a Cohen ring for $R/\mathfrak{m}$.

(So if $R/\mathfrak{m}$ is perfect, $W = W(R/\mathfrak{m})$.)

We can find an embedding

$W \hookrightarrow R$ (general theory)

and since $\mathfrak{m}/\mathfrak{m}^2$ is d-dim'l over $R/\mathfrak{m}$, by choosing lifts to R of basis elements, we get a surjection

$$W[[x_1, \dots, x_d]] \twoheadrightarrow R$$

Kernel is generated by some $f \in (p, x_1, \dots, x_d)$

(compare Krull dimensions, and use Hauptidealsatz)

Write $\mathfrak{m} = (p, x_1, \ldots, x_d)$ = max. ideal of $W[[x_1, \ldots, x_d]]$,

so $\mathfrak{m}/\mathfrak{m}^2 = \langle p, x_1, \ldots, x_d \rangle$.

Let $\bar{f}$ = image of f in $\mathfrak{m}/\mathfrak{m}^2$, and suppose $\bar{f}$ has a non-trivial coefficient of p.

image of f in $W = W[[x_1, \ldots, x_d]]/(x_1, \ldots, x_d)$ is exactly divisible by p.

Make $W[[x_1, \ldots, x_d]]$ a δ-ring via

$\varphi(x_i) = x_i^p$.

Then $W[[x_1, \ldots, x_d]] \to W$ is a δ-map, whose kernel is contained in the radical of its domain

and image of f is distinguished,

$\therefore$ f is distinguished

In general, $\bar{f} \neq 0$ (since R is regular),

$\therefore$ can always make a coordinate change

(eg $x_i \mapsto x_i + p$) to ensure that
for some i

coeff. of p in $\bar{f}$ is non-zero.

Conclusion: ~~We can find a Frobenius structure on~~

We can find $f \in W[[x_1, \ldots, x_d]]$
distinguished w.r.t. $\varphi(x_i) = x_i^p$

s.t. $W[[x_1, \ldots, x_d]]/(f) \xrightarrow{\sim} R$.

$(W[[x_1, \ldots, x_d]], (f))$ is an example of a prism

If we take $\varinjlim_{\varphi}$ to perfectify, get

and induced ~~map~~ embedding $R \hookrightarrow R_\infty$ with

perfectoid target and good properties.

<u>Special case</u> :

$$\begin{array}{c} K \\ | \\ \mathbb{Q}_p \end{array} \quad \text{finite}$$

$\mathcal{O}_K$ is regular.

Let k = residue field, π = uniformizer,

$E(u)$ = Eisenstein poly. over $W(k)$
satisfied by π.
(Choice of "u" to denote the variable is traditional)

Then $\left(\underset{\varphi(u) = u^p}{W(k)[[u]]}, \underset{u^d + \cdots + p \cdot \text{unit}}{\underset{\shortparallel}{E(u)}} \right)$ is a

special case of the preceding construction.

$\mathcal{O}_{\mathbb{C}_p}$ is the "universal" perfectoid containing the various $\mathcal{O}_K$'s.

Indeed $\mathcal{O}_{\mathbb{C}_p}$ is p-complete, p-torsion free, integrally closed in $\mathbb{C}_p = \mathcal{O}_{\mathbb{C}_p}[\frac{1}{p}]$,

and contains $p^{\frac{1}{p}}$.

This gives rise to perfect prism

$$\left(\underset{\substack{\| \\ A_{\inf}}}{W(\mathcal{O}_{\mathbb{C}_p}^\flat)} , \ker\theta \right)$$

The embedding $\mathcal{O}_K \hookrightarrow \mathcal{O}_{\mathbb{C}_p}$ lifts to an embedding of prisms

$$(W(k)[[u]], E(u)) \hookrightarrow (A_{\inf}, \ker\theta)$$