Distinguished elements

We usually assume from now on that our rings A are "p-local", i.e. that $p \in \mathrm{Rad}(A)$.

(Recall that $\mathrm{Rad}(A)$, the Jacobson radical of A, is the intersection of all the maximal ideals of A. It has another characterization which is more important for our purposes:

$a \in \mathrm{Rad}(A)$ iff $1+ax$ is a unit in A $\forall x \in A$.

If A is I-adically complete, then $I \subseteq \mathrm{Rad}(A)$: if $a \in I$, then $y := ax \in I$ $\forall x \in A$, and $1+y$ is a unit: its inverse is

$1 - y + y^2 - y^3 + \dots$

If $\underset{\text{an ideal}}{I} \subseteq \mathrm{Rad}(A)$, then the diagram

$$\begin{array}{ccc} A^\times & \longrightarrow & (A/I)^\times \\ \cap\mid & & \cap\mid \\ A & \twoheadrightarrow & A/I \end{array}$$

is Cartesian (i.e. $a \in A$ is a unit iff it is a unit mod I),

and conversely, if this diagram is Cartesian, then $I \subseteq \mathrm{Rad}(A)$.

If $I \subseteq A$ is any ideal, and $S \subseteq A$ is the preimage of $(A/I)^\times$ under the quotient map, then

S is a multiplicative subset of A,

and $I \cdot A_S \subseteq \mathrm{Rad}(A_S)$.

The localization A_S is an initial object in the category of A-algebras B for which $IB \subseteq \mathrm{Rad}(B)$. [Also, it is equivalent to localize at $T = 1 + I$])

We call this process "localizing along I".

Before getting to distinguished elements, we note a few more δ-ring facts.

(i) δ and φ commute.

(This makes intuitive sense if you think of δ as

$$\delta(x) := \frac{\varphi(x) - x^p}{p},$$

and can be checked formally from the axioms for δ and the formula $\varphi(x) = x^p + p\,\delta(x)$.

You can also check it in the universal case of $\mathbb{Z}\{x\}$, which is torsion free, so that the intuitive argument is valid. ~~argument is valid~~)

(ii) If I is an ideal of A for which $p \in I$, then $\varphi(I) \subseteq I$.

(Indeed, $\varphi(a) = a^p + p\,\delta(a) \equiv a^p \mod I$, for any $a \in A$)

In particular

(iii) If A is p-local, i.e. if $p \in \mathrm{Rad}(A)$, then

$$\varphi(\mathrm{Rad}(A)) \subseteq \mathrm{Rad}(A).$$ In fact $\varphi^{-1}(\mathrm{Rad}(A)) = \mathrm{Rad}\,A$

($\varphi(a) = a^p + p\,\delta(a) \in \mathrm{Rad}(A) \iff a^p \in \mathrm{Rad}(A) \iff a \in \mathrm{Rad}(A)$, since $\mathrm{Rad}(A)$ is a radical ideal.)

Also

(iv) If A is p-local, then $a \in A^\times$ iff $\varphi(a) \in A^\times$ iff $(a, \varphi(a)) = A$.

(Since $p \in \mathrm{Rad}(A)$, we know that $a \in A^\times$ iff $\bar a := a \mod p$ is a unit in A/p, and similarly for $\overline{\varphi(a)} = \varphi(\bar a) = \bar a^p$.

Also $(a, \varphi(a)) = A$ iff $(a, \varphi(a), p) = A$ iff

$$(\bar a, \overline{\varphi(a)}) = A/p.$$
$$\quad\| \quad (\bar a, \bar a^p)$$

Thus the claim reduces to the obvious claim that

$\bar a \in (A/p)^\times$ iff $\bar a^p \in (A/p)^\times$ iff $(\bar a) = (\bar a, \bar a^p)) = A/p$)

(v) If $S \subseteq A$ is a multiplicative subset that is φ-stable (i.e. $\varphi(S) \subseteq S$) then there is a unique δ-ring structure on the localization A_S that is compatible with the δ-ring structure on A.

(Use free δ-rings to choose a surjection $B \xrightarrow{\pi} A$ of δ-rings, with B being p-torsion free. Let $T := \pi^{-1}(S)$; T is a φ-stable multiplicative subset of B. Then φ on B extends uniquely to $\tilde{\varphi}$ on B_T, and $\tilde{\varphi}$ again lifts Frob. mod p. B/c B_T is p-torsion free, we get $\tilde{\delta}$ on B_T, giving it a δ-ring structure. [And since $\tilde{\varphi}$ is the unique extension of φ, we see $\tilde{\delta}$ is the unique extension of δ.] Then $A_S = A \otimes_B B_T$ gets the coproduct δ-ring structure.)

If we combine (ii) and (v), we find

(vi) If $I \subseteq A$ is an ideal s.t. $p \in I$, and if $S' :=$ preimage in A of $(A/I)^\times$, then $A_{S'}$ has a unique δ-ring structure compatible with the one on A.

We now make the following def'n.
(And we assume that our δ-rings are p-local, i.e. that $p \in \mathrm{Rad}(A)$).

<u>Defn</u>: An element $d \in A$ is called <u>distinguished</u> if $\delta(d)$ lies in $A^\times$.

By (i) and (iv) above, d is distinguished iff $\varphi(d)$ is distinguished.

We will often assume that $d \in \mathrm{Rad}(A)$. By (iii) above, this holds iff $\varphi(d) \in \mathrm{Rad}(A)$.

The following lemma gives an ideal theoretic characterization of distinguished elements.

[Lemma 1.8 of Bhatt's 3rd lecture]

Lemma If $p, d \in \mathrm{Rad}(A)$, then d is distinguished iff $p \in (d, \varphi(d))$.

Pf: If $\delta(d)$ is a unit, then the formula $\varphi(d) = d^p + p\,\delta(d)$ shows that $p \in (d, \varphi(d))$.

Conversely, suppose $p = ad + b\varphi(d)$ $\exists\, a, b \in A$.

Applying δ, we find

$$1 - p^{p-1} \in a^p \delta(d) + b^p \varphi(\delta(d)) + \mathrm{Rad}(A)$$

$$\therefore\ 1 \in (\delta(d), \varphi(\delta(d))).$$

(iv) above now shows that $\delta(d) \in A^\times$. $\square$

Our next result will let us move the "distinguished" property from principal ideals to locally principal ideals. One subtlety in doing this is that the property of being a δ-ring does not localize over Spec A. Rather, property (V) above constrains the kinds of localizations we can perform.

For purposes of orientation, we recall some background facts related to radicals and localization:

- If A is a ring, and $I \subseteq \mathrm{Rad}(A)$, then the only open subset of Spec A that contains $V(I) := \mathrm{Spec}\, A/I$ is Spec A itself.

(Indeed, any non-empty closed subset of Spec A contains a closed pt., and thus has non-empty intersection with $V(I)$, since $I \subseteq \mathrm{Rad}\, A$, i.e. $I \subseteq$ every maximal ideal of A.)

- Another way to phrase the previous point is that if $U \underset{j}{\hookrightarrow} \operatorname{Spec} A$ is an open immersion whose image contains $V(I)$, then j is surjective. This formulation generalizes to arbitrary flat morphisms, as we explain.

- First recall that flat morphisms satisfy "going down": if $\operatorname{Spec} B \xrightarrow{f} \operatorname{Spec} A$ is flat, then the image of f is closed under generalization.

(Proof: Let $\mathfrak{q} \subseteq \mathfrak{p}$ be prime ideals of A, corresponding to pts. $y \mapsto x$ of $\operatorname{Spec} A$, and assume x is the image of f. Pulling back over $\operatorname{Spec} A_{\mathfrak{p}} \to \operatorname{Spec} A$, we may assume that x is closed. Then pulling back over $\overline{\{y\}}$ with its reduced induced structure, we may assume that A is a ~~local~~ domain, and that y ~~corresponds~~ is the generic point of $\operatorname{Spec} A$. By assumption $\operatorname{Spec} B$ has non-empty image, so $B \neq 0$. Then, by flatness, each element $a \in A \setminus \{0\}$ is a non-zero divisor on B, so if we write $S = A \setminus \{0\}$, then $B_S \neq 0$, and the image of $\operatorname{Spec} B_S \to \operatorname{Spec} B \to \operatorname{Spec} A$ is exactly the generic pt. y.)

- Now we see that if $V(I) \hookrightarrow \operatorname{Spec} A$ contains all the closed pts. of $\operatorname{Spec} A$, and if $\operatorname{Spec} B \xrightarrow{f} \operatorname{Spec} A$ is flat and contains $V(I)$ in its image, then f is in fact surjective (i.e. faithfully flat).

(Any $y \in \operatorname{Spec} A$ specializes to a closed point, namely any closed point in $\overline{\{y\}}$. Now apply "going down".)

Now suppose that $\operatorname{Spec} A = \bigcup_{i=1}^{n} D(g_i)$

is an open cover of $\operatorname{Spec} A$ by distinguished opens

(i.e. $g_i \in A$, and $D(g_i) := \operatorname{Spec} A_{g_i} \underset{\text{open}}{\hookrightarrow} \operatorname{Spec} A$)

Let $J \subseteq \operatorname{Rad}(A)$ be an ideal, and write

$\bar{A} = A/J$, $\underset{\bar{A}}{\overset{\bar{g}_i}{\cap}} = g_i \bmod J$. Then $\operatorname{Spec} \bar{A} \cap D(g_i) = D(\bar{g}_i)$,

and so $\operatorname{Spec} \bar{A} = \bigcup D(\bar{g}_i)$.

Let S_i = preimage of $\{1, \bar{g}_i, \bar{g}_i^2, \ldots\}$ in A.

Then $A_{S_i} / J \cdot A_{S_i} = (A/J)_{S_i} = \bar{A}_{\bar{g}_i}$,

$\therefore$ the morphism $\operatorname{Spec} A_{S_i} \to \operatorname{Spec} A$ contains $D(\bar{g}_i)$ in its image, and hence the morphism

$\coprod \operatorname{Spec} A_{S_i} \to \operatorname{Spec} A$ contains $\operatorname{Spec} \bar{A}$ in its image. It is also flat (all localizations are flat!) and $J \subseteq \operatorname{Rad}(A)$ by assumption, so that $\operatorname{Spec} \bar{A}$ contains all closed points of $\operatorname{Spec} A$. Thus $\coprod \operatorname{Spec} A_{S_i} \to \operatorname{Spec} A$ is faithfully flat.

Of course $g_i \in S_i$ (but typically S_i will be larger than just $\{1, g_i, g_i^2, \dots\}$), and so

$\operatorname{Spec} A_{S_i} \longrightarrow \operatorname{Spec} A$ factors through $D(g_i)$.

So $\coprod \operatorname{Spec} A_{S_i}$ is a replacement for $\coprod D(g_i)$; it is only a faithfully flat cover, rather than an open cover, but it has some important properties:

- $J \cdot A_{S_i} \subseteq \operatorname{Rad}(A_{S_i})$

(If $s \in S_i$, $x \in J$, then $s + x \equiv \bar{g}_i^n \bmod J$ for some n, by def$^{=}$ of S_i, $\therefore$ $s + x \in S_i$, $\therefore$ $1 + \frac{x}{s} \in A_{S_i}^\times$.)

- If A is a δ-ring and if $p \in J$, then A_{S_i} has a δ-ring structure extending that on A.

(This follows from (v), b/c $\varphi(S_i) \subseteq S_i$; indeed, since $p \in J$, we see that φ induces the pth power map on $\bar{A}$. Since $\{1, g_i, g_i^2, \dots\}$ is stable under the pth power map, S_i is φ-stable.)

We can now prove

Corollary (Cor. 1.9 of Bhatt's 3rd lecture)

Let A be a δ-ring, let $I \subseteq A$ be locally principal, and suppose that $(p, I) \subseteq \mathrm{Rad}(A)$. Then TFAE:

(1) $p \in (I, \varphi(I))$

(2) $\exists$ a finite collection of localizations $\{A_{s_i}\}$ of A s.t. $\coprod \operatorname{Spec} A_{s_i} \longrightarrow \operatorname{Spec} A$ is surjective, s.t. each A_{s_i} admits a δ-ring structure ~~extending~~ compatible with the δ-ring structure on A, and s.t. for each i, the ideal $I \cdot A_{s_i}$ is principal of the form (d_i) for a distinguished element d_i with $(p, d_i) \subseteq \mathrm{Rad}(A_{s_i})$.

Proof: (2) $\Rightarrow$ (1) follows by faithfully flat descent: indeed, the condition $p \in (I, \varphi(I))$ may be checked on a faithfully flat cover, and so it suffices to verify it for each A_{g_i}, where it follows from our assumption that $I \cdot A_{g_i} = (d_i)$, and our earlier lemma applied to the distinguished element d_i of the δ-ring A_{g_i}.

(1) $\Rightarrow$ (2) : By assumption we can find an open cover
$$\operatorname{Spec} A = \bigcup_{i=1}^{n} D(g_i) \quad \text{s.t.}$$
$I \cdot A_{g_i}$ is principal for each i.

Now set $J = (p, I)$, and apply our preceding construction to obtain
$$\coprod \operatorname{Spec} A_{g_i} \longrightarrow \operatorname{Spec} A.$$
Since $\operatorname{Spec} A_{g_i} \longrightarrow \operatorname{Spec} A$ factors through $D(g_i) = \operatorname{Spec} A_{g_i}$, we see that $I \cdot A_{g_i}$ is principal, say of the form (d_i). Our construction ensures that $J \cdot A_{g_i} \subseteq \operatorname{Rad}(A_{g_i})$, i.e. that $(p, d_i) \subseteq \operatorname{Rad}(A_{g_i})$. Base-changing the hypothesis that $p \in (I, \varphi(I))$, we find that $p \in (d_i, \varphi(d_i))$. Our earlier lemma then shows that d_i is distinguished. $\square$