δ-structures as derived lifts of Frobenius

Suppose R is p-t.f., and let's compute the fibre product

$$S := R \times_{\text{Frob}\circ\text{red mod }p,\ R/p,\ \text{red. mod }p} R$$

$$S = \{(a,b) \in R \times R \mid a^p \equiv b \mod p\}$$

$$= \{(a,b) \in R \times R \mid a^p + pc = b \ \ \exists c \in R\}$$

$$= \{(a,c) \in R \times R\}$$

since R is p-t.f., c is uniquely determined.

$$(a_1, c_1) + (a_2, c_2) = \left(a_1 + a_2,\ c_1 + c_2 + \frac{a_1^p + a_2^p - (a_1 + a_2)^p}{p}\right)$$

$$(a_1, c_1) \cdot (a_2, c_2) = (a_1 a_2,\ a_1^p c_2 + a_2^p c_1 + p c_1 c_2)$$

So $S = W_2(R)$

$(a,c) \mapsto a$

The projection $W_2(R) = S \to R$

is the canonical projection.

The projection $W_2(R) = S \to R$

$(a,c) \mapsto b = a^p + c$

is a truncation $(W_2(R) \to W_1(R) = R)$

of $F : W(R) \to W(R)$, the lift

of Frob. induced by the δ-structure

(In general, $\delta : W(R) \to W(R)$ truncate to

$W_n(R) \to W_{n-1}(R)$,

and similarly F truncate to $W_n(R) \to W_{n-1}(R)$.)

So we can rewrite δ as the fiture square

$$\begin{tikzcd}
W_2(R) \arrow[r, "F"] \arrow[d, "\text{augmentation}"] & R \arrow[d, "\text{proj.}"] \\
R \arrow[r] \arrow[d, "\text{proj.}"] & R/pR \\
R/p \arrow[ur, "\text{Frob.}"'] &
\end{tikzcd}$$

If R is p-torsion-free, then, giving a morphism

$$R \to W_2(R)$$ lifting the augmentation

(i.e. making R a δ-ring) is the same as giving a diagram

$$\begin{tikzcd}
R \arrow[rr, bend left, "\varphi"] \arrow[d, equal] & & R \arrow[d] \\
R \arrow[d] \arrow[rr] & & R/p \\
R/p \arrow[urr, "\mathrm{Frob}"'] & &
\end{tikzcd}$$

i.e. giving $\varphi : R \to R$ lifting Frobenius.

So this gives a more sophisticated "explanation" as to why, for p-torsion free rings, giving a δ-structure amounts to giving a lift of Frobenius.

(4)

Now suppose R is a (not nec. p-t.f.) δ-ring.

Then, recalling that free δ-rings are in particular free rings (i.e. polynomial rings)

we may find a simplicial resolution of R by a simplicial δ-ring

$$R_\bullet \longrightarrow R$$

with ~~each~~ $R_\bullet$ being a δ-ring, and also a polynomial ring.

The δ-structure on $R_\bullet$ then amounts to a lift of Frobenius on $R_\bullet$,

$\therefore$ we get $\varphi : R_\bullet \to R_\bullet$

lifting $\mathrm{Frob} : R_\bullet \otimes \mathbb{F}_p \to R_\bullet \otimes \mathbb{F}_p$

But $\varphi : R_\bullet \longrightarrow R_\bullet$ is just a "lift" to the resolution $R_\bullet$ of the given $\varphi : R \to R$.

And $R_\bullet \otimes \mathbb{F}_p$ computes $R \overset{\mathbb{L}}{\otimes} \mathbb{F}_p$.

So the δ structure on R

induces $\varphi : R \circlearrowleft$ which not only lifts

Frob. on $R \otimes \mathbb{F}_p$, but actually lifts Frob.

on $R \overset{\mathbb{L}}{\otimes} \mathbb{F}_p$.

Conversely, giving a lift of Frob. on $R \overset{\mathbb{L}}{\otimes} \mathbb{F}_p$

to $\varphi : R \circlearrowleft$ induces a δ-structure on R.

To see this, we reinterpret the previous fibre product square as giving a

fibre product square

$$\begin{tikzcd} W_2(R_\cdot) \arrow[r] \arrow[d] & R_\cdot \arrow[d] \\ R_\cdot \arrow[r] & R_\cdot \overset{\mathbb{L}}{\otimes} \mathbb{F}_p \end{tikzcd}$$

for any simplicial (or, better, "animated") ring R.