Derived completeness

Recall that an A-module M is I-adically complete if

$$M \xrightarrow{\cong} \varprojlim M/I^n M .$$

For derived completeness, we study the complex that computes this ~~[illegible]~~ quotient, rather than just the quotient itself. We also study I one element at a time. Actually, it suffices to study the generators; the theory then works for any f.g. ideal.

So suppose $I = (f)$ is principal. ~~[illegible]~~.
Then $M/f^n M$ is the cokernel of the map

$$\underset{\substack{\uparrow \\ \text{in degree} \\ -1}}{M} \xrightarrow{f^n} \underset{\substack{\uparrow \\ \text{in degree} \\ 0}}{M}.$$

(One way to describe this complex is that it computes

$$\mathbb{Z}[x]/(x^n) \overset{\mathbb{L}}{\underset{\mathbb{Z}[x]}{\otimes}} M,$$

where we regard M as a $\mathbb{Z}[x]$-module via $x \mapsto f$.)

Then we can consider the derived inverse limit

$$R\varprojlim \left(\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] \\ M \arrow[r, "f^2"] \arrow[d, "f"] & M \arrow[d, "id"] \\ M \arrow[r, "f"] \arrow[d, "f"] & M \arrow[d, "id"] \\ M \arrow[r, "id"] & M \end{tikzcd} \right)$$

(A typical square in the inverse system of complexes here is

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] \\ M \arrow[r, "f^{n+1}"] \arrow[d, "f"] & M \arrow[d, "id"] \\ M \arrow[r, "f^n"] \arrow[d] & M \arrow[d] \\ \vdots & \vdots \end{tikzcd}$$

)

I find computing derived inverse limits confusing in general, b/c you have to take injective resolutions of projective systems of complexes, ... but for derived inverse limits of sequences, there is an explicit complex that computes them: if

$$\longrightarrow M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} \cdots \longrightarrow M_1 \xrightarrow{f_1} M_0$$

is an inverse sequence of A-modules, then

$$R\varprojlim \left(\longrightarrow M_n \to \cdots \to M_0 \right) \quad \text{is}$$

computed by the 2-step complex

$$\prod_{n=0}^{\infty} M_n \xrightarrow{\quad \delta \quad} \prod_{n=0}^{\infty} M_n$$

$$(m_n)_{n\geq 0} \longmapsto \left(m_n - f_{n+1}(m_{n+1}) \right)_{n \geq 0}$$

So if we return to the derived inverse limit on p. 2, we can compute it by the total complex of the double complex

$$\begin{tikzcd} \prod_{n\geq 0} M \arrow[r, "\delta^1"] & \prod_{n\geq 0} M \\ \prod_{n\geq 0} M \arrow[u, "\prod_{n\geq 0} f^n"] \arrow[r, "\delta^0"] & \prod_{n\geq 0} M \arrow[u, "\prod_{n\geq 0} f^n"'] \end{tikzcd}$$

Think of this double complex as being in bidegrees

$$\begin{matrix} (-1,1) & (0,1) \\ (-1,0) & (0,0) \end{matrix}$$

where the upper row computes $R\varprojlim_{id} M$,

and the lower row computes $R\varprojlim_{\times f} M$.

Of course, the first $R\varprojlim$ just equals M itself

So the derived inverse limit, i.e. the "derived f-adic completion" of M, is equal to M iff

$$R\varprojlim \left(\cdots \xrightarrow{f} M \xrightarrow{f} M \xrightarrow{f} \cdots \xrightarrow{f} M \right) = 0.$$

This vanishing is the standard def$^{\underline{n}}$ of what it means for M to be derived f-adically complete

Of course, "$R\varprojlim_{\times f} M = 0$" has to be understood as a condition in the derived category of complexes of A-modules, ie. "$= 0$" means "quasiisomorphic to 0", or "acyclic", or, concretely, that $\varprojlim_{\times f} M = 0$ and that $R^1\varprojlim_{\times f} M = 0$.

- We can analyze the double complex on p. 4 by passing to the ~~…~~ E_2 page of the associated spectral sequence, which is simply

$$\begin{array}{cc}
\underset{(-1,1)}{\overset{\varprojlim M/f^n M}{\bullet}} & \underset{(0,1)}{\overset{R^1\varprojlim M/f^n}{\bullet}} \\[2em]
\underset{(-1,0)}{\overset{\varprojlim_{\times f} M[f^n]}{\bullet}} & \underset{(0,0)}{\overset{R^1\varprojlim_{\times f} M[f^n]}{\bullet}}
\end{array}$$

with all other terms $= 0$, and all differentials (thus) $= 0$.

The $(0,1)$-term $R^1\varprojlim M/f^n$ also $= 0$, b/c the transition maps are surjective.

(You can check directly, using the complex on p. 3 that computes $R\varprojlim$, that $R^1\varprojlim$ vanishes if the transition morphisms are surjective. This is a special case of the Mittag-Leffler property, which you can also check if you'd like to.)

If we write H^{-1}, H^0 for the cohomology of the complex

$$R\varprojlim \begin{pmatrix} \begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] \\ M \arrow[r, "f^{n+1}"] \arrow[d, "f"] & M \arrow[d, equal] \\ M \arrow[r, "f^n"] \arrow[d] & M \arrow[d] \\ \vdots & \vdots \end{tikzcd} \end{pmatrix}$$

computing the derived completion, we thus find (using the preceding spectral sequence) that

$$H^{-1} = \varprojlim_{\times f} M[f^n] \quad \text{(the "}f\text{-adic Tate module" of } M\text{)}$$

while H^0 sits in a short exact sequence

$$0 \to R^1\varprojlim_{\times f} M[f^n] \to H^0 \to \underbrace{\varprojlim M/f^nM}_{\text{the "classical" } f\text{-adic completion}} \to 0$$

We can use this explicit description of H^{-1} and H^0 to understand more concretely what it means for M to be f-adically derived complete.

First, the natural morphism

$$M \longrightarrow R\varprojlim \left(\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ M & \xrightarrow{f^{n+1}} & M \\ {\scriptstyle f}\downarrow & & \| \\ M & \xrightarrow{f^n} & M \\ \downarrow & & \downarrow \\ \vdots & & \vdots \end{array} \right)$$

induces a morphism $M \to H^0$ by passing to H^0, which more concretely fits in the diagram

$$\begin{tikzcd} & & M \arrow[d] \arrow[dr, "\swarrow \text{natural map}"] & & \\ 0 \arrow[r] & R^1\varprojlim_{\times f} M[f^n] \arrow[r] & H^0 \arrow[r] & \varprojlim M/f^nM \arrow[r] & 0 \end{tikzcd}$$

Of course M is derived complete when this natural morphism
$$M \to R\varprojlim \begin{pmatrix} \vdots & & \vdots \\ \downarrow & & \downarrow \\ M & \xrightarrow{f^{n+1}} & M \\ f\downarrow & & \downarrow \\ M & \xrightarrow{f^n} & M \\ \downarrow & & \downarrow \\ \vdots & & \vdots \end{pmatrix}$$
is a quasi-isomorphism, i.e. when ~~...~~ $H^{-1} = 0$ and $M \xrightarrow{\cong} H^0$.

Before making a more general analysis, we <u>consider</u> ~~a~~ particular case.

<u>Eg.</u> Suppose that M has bounded f-power torsion, i.e. that $M[f^\infty] = M[f^n]$ for $n \gg 0$.

Then the transition maps in $\cdots \xrightarrow{\times f} M[f^{n+1}] \xrightarrow{\times f} M[f^n] \to \cdots$ are eventually zero, and both $\varprojlim$ and $R^1\varprojlim$ vanish. Thus $H^{-1} = 0$, while $H^0 =$ the classical f-adic completion, and so M is derived complete iff it is classically complete.

We now make a more general analysis. (9)

We consider the two short exact sequences of inverse systems,

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & M[f^{n+1}] \arrow[r, hook] \arrow[d, "f"] & M \arrow[r, "\times f^{n+1}"] \arrow[d, "f"] & f^{n+1}M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M[f^n] \arrow[r, hook] \arrow[d] & M \arrow[r, "\times f^n"] \arrow[d] & f^n M \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & f^{n+1}M \arrow[r, hook] \arrow[d] & M \arrow[r] \arrow[d, equal] & M/f^{n+1}M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f^n M \arrow[r, hook] & M \arrow[r] & M/f^n M \arrow[r] & 0
\end{tikzcd}$$

Passing to $R\varprojlim$'s, we obtain long exact sequences

$$0 \to \varprojlim_{\times f} M[f^n] \to \varprojlim_{\times f} M \to \bigcap_{n\geq 0} f^n M \to R^1\varprojlim_{\times f} M[f^n] \to R^1\varprojlim_{\times f} M \to R^1\varprojlim_{\subset} f^n M \to 0$$

recall this $= H^{-1}$

and

$$0 \to \bigcap_{n\geq 0} f^n M \to M \to \varprojlim M/f^n M \to R^1\varprojlim f^n M \to 0.$$

The boundary map in the first of these lets us improve the diagram at the bottom of p. 7:

$$\begin{tikzcd}
0 \arrow[r] & \bigcap_{n\geq 0} f^n M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & \varprojlim M/f^n M \arrow[r] \arrow[d, equal] & R^1\varprojlim f^n M \arrow[r] & 0 \\
0 \arrow[r] & R^1\varprojlim_{\times f} M[f^n] \arrow[r] & H^0 \arrow[r] & \varprojlim M/f^n M \arrow[r] & 0 &
\end{tikzcd}$$

A consideration of this diagram and the preceding long exact sequences allows us to expand on the concept of derived completeness. For example, if $H^{-1} = 0$, then

$\varprojlim_{\times f} M$ is the kernel of the first vertical arrow in the above diagram, and so

The vanishing of $\varprojlim_{\times f} M$ (which also entails the vanishing of H^{-1}) is equivalent to asking that the morphism $M \to H^0$ be injective. Its cokernel is then equal to $R^1\varprojlim_{\times f} M$, sitting in the short exact sequence

$$0 \to \operatorname{coker}\left(\bigcap_{n\geq 0} f^n M \to R^1\varprojlim M[f^n]\right)$$

$$\longrightarrow \operatorname{coker}(M \to H^0) \longrightarrow \operatorname{coker}(M \to \varprojlim M/f^nM) \to 0$$

which is exactly the short exact sequence

$$0 \to R^1\varprojlim_{\times f} M[f^n] \Big/ \operatorname{image}\left(\bigcap_{n\geq 0} f^n M\right) \to R^1\varprojlim_{\times f} M$$

$$\to R^1\varprojlim f^n M \to 0$$

arising from the second half of the long exact sequence at the bottom of p. 9.

Eg. If M is f-adically separated,

i.e. $\bigcap_{n \geq 0} f^n M = 0$,

then also $\varprojlim_{\times f} M = 0$

(any element (m_n) in this inverse limit has each of its terms laying in $\bigcap_{n \geq 0} f^n M$)

and so also $H^{-1} = 0$.

Thus M is derived complete iff $M = H^0$,

which implies in particular that

$M \longrightarrow \varprojlim M/f^n M$ is surjective.

But by assumption, its kernel $\bigcap_{n \geq 0} f^n M = 0$,

and so we see {f-adically separated + derived complete}

$\Longrightarrow$ classically complete.

Conversely, if M is classically f-adically complete, then M is derived complete. (And of course M is f-adically separated!) This is not so obvious (to me) from manipulating the preceding exact sequences, but can be seen as follows: if $M \xrightarrow{\cong} \varprojlim M/f^n M$, then (because the transition morphisms are surjective) we have

$$M \xrightarrow{\cong} R\varprojlim M/f^n M.$$

Now using the fact that an inverse limit of an inverse limit can be computed as a single ("diagonal") inverse limit, we find that

$$R\varprojlim_{\times f} M \cong R\varprojlim_{\times f} M/f^n M,$$

and in the right-hand side, the transition morphisms are eventually zero, so that $R\varprojlim_{\times f} M = 0$, as required.

Here is another argument, which gives some insight into the meaning of $R^1 \varprojlim$:

Since M is classically complete, it is f-adically separated, and so (as we've already seen)

$$\varprojlim_{\times f} M = 0.$$

Now let's consider $R^1 \varprojlim_{\times f} M$.

This is the cokernel of $(x_n)_{n \geq 0} \mapsto (x_n - f x_{n+1})_{n \geq 0}$.

Consider some tuple $(y_n)_{n \geq 0} \in \prod_{n \geq 0} M$.

Let

$$\left.\begin{aligned} x_0 &= y_0 + f y_1 + f^2 y_2 + \cdots, \\ x_1 &= y_1 + f y_2 + f^2 y_3 + \cdots, \\ &\;\vdots \\ x_n &= y_n + f y_{n+1} + f^2 y_{n+2} + \cdots \end{aligned}\right\} \text{all these expressions are well-defined, since } M \text{ is classically } f\text{-adically complete}$$

Then $(x_n)_{n \geq 0} \mapsto (y_n)_{n \geq 0}$, $\therefore$ $R^1 \varprojlim_{\times f} M = 0$, and so M is derived complete.

In conclusion, "derived completeness" is a relaxation of the condition of classical completeness, in which we still ask that

$$M \longrightarrow \varprojlim M/f^n M \quad \text{be surjective,}$$

but we allow certain kinds of non-separatedness.

Eg. If $M \hookrightarrow N$ is an inclusion of classically complete modules, the cokernel need not be classically complete, but it will be derived complete.

Indeed, N/M will be classically complete iff it is f-adically separated iff M is f-adically closed in N.

Eg. $$\widehat{\bigoplus_{n\geq 0} p^n \mathbb{Z}_p} \hookrightarrow \widehat{\bigoplus_{n \geq 0} \mathbb{Z}_p}$$ ("$\wedge$" denotes p-adic completion)

has non-closed image: $(1, p, p^2, \dots)$ is in the closure of the image, but not in the image.

(Indeed, this is the map on unit balls induced by the most basic example of a non-finite rank compact morphism between p-adic Banach space. The closure of the image is the product $\prod_{n\geq 0} p^n \mathbb{Z}_p$ (with its product topology), which is a compact subset of $\widehat{\bigoplus_{n\geq 0} \mathbb{Z}_p}$, analogous to the Hilbert cube sitting inside the unit ball of an infinite dimensional Hilbert space.)

~~[illegible]~~ This example also gives rise to an interesting short exact sequence

$$0 \to \frac{\prod_{n\geq 0} p^n \mathbb{Z}_p}{\widehat{\bigoplus_{n\geq 0}} p^n \mathbb{Z}_p} \to \frac{\widehat{\bigoplus_{n\geq 0} \mathbb{Z}_p}}{\widehat{\bigoplus_{n\geq 0} p^n \mathbb{Z}_p}} \to \frac{\widehat{\bigoplus_{n\geq 0} \mathbb{Z}_p}}{\prod p^n \mathbb{Z}_p} \to 0$$

in which the outer two terms are classically p-adically complete, while the middle term is only derived p-adically complete.

One more remark: Note that $1 - af$ (for any $a \in A$) acts invertibly on each $M[f^n]$ and $M/f^n M$ — its inverse is given by the geometric series $1 + af + \cdots + (af)^{n-1}$ and so $1-af$ acts invertibly on the cohomology of

$$R\varprojlim \left(\begin{array}{ccc} \vdots & & \vdots \\ \downarrow & & \downarrow \\ M & \xrightarrow{f^{n+1}} & M \\ {\scriptstyle f}\downarrow & & \| \\ M & \xrightarrow{f^n} & M \\ \downarrow & & \downarrow \\ \vdots & & \vdots \end{array} \right),$$

and thus acts invertibly on this complex itself (thought of as lying in the derived category).

In particular, if A is derived complete, then $f \in \mathrm{Rad}(A)$.