S-rings

φ : A \rightarrow A \text{ is a Frobenius lift if } φ(x) = x^p \text{ mod } pA \quad \forall x \in A.

\text{S : A \rightarrow A (not a homomorphism!!!) gives A a S-ring structure if \ S satisfies the obvious identities necessary to ensure that } φ(x) = x^p + ps(x) \text{ is a Frobenius lift.}

\text{i.e., } S(0) = S(1) = 0

\text{. } S(x+y) = S(x) + S(y) - \frac{(xy)^p - x^p - y^p}{p}

\text{i.e., } = S(x+y) - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} x^i y^{p-i-1}
\[ \delta (x,y) = x \rho \delta (y) + \delta (x/y) \]

\[ (\rho (x) \delta (y) + \delta (x/y)) \]

The equality of the two expressions follows directly from the formula for \( \rho \).

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A \( \delta \)-ring is a ring with extra structure, and so the forgetful functor

\[ \delta \text{-rings} \to \text{rings} \]

has a left adjoint \( A \to A^\delta \), and \( \delta \)-rings admit all limits; these are compatible with limits on the underlying rings (or, indeed, on the underlying sets).
If $S$ is a set, we can form the free ring $\mathbb{Z} [S]$ on $S$ (a polynomial ring with variable being the elements of $S$) and then also form $\mathbb{Z} [ F ] := \mathbb{Z} [ S ]^S$, the free $F$-ring on $S$.

E.g., $\mathbb{Z} \{ x \}$ is the polynomial ring

$$\mathbb{Z} [ x, f x, f^2 x, \ldots, f^n x, \ldots ]$$

($f$ is defined by $f(f^n x) = (f^n x)^2 + p f^m x$).

$\mathbb{Z} \{ f \}$ has an analogous description.
It is less obvious, but true, that the forgetful functor from \textit{S}-rings to rings also preserves all colimits (in particular, \textit{S}-rings are cocomplete).

Thus, it is a left adjoint as well as a right adjoint. Its right adjoint functor is denoted \textit{W} : rings \rightarrow \textit{S}-rings, the "Witt vector" functor.

Since the left adjoint of \textit{W} is the forgetful functor, which preserves all limits and reflects isomorphisms, we see that \textit{W} is conservative, i.e., if \textit{A} is a ring, then giving
A 8-ring structure is the same as giving a "co-achin map"

\[ A \xrightarrow{k} W(A) \]

(a homomorphism of rings)

s.t. \[ A \xrightarrow{k} W(A) \] and \( k \) is the identity

and s.t.

\[ \begin{array}{ccc}
A & \xrightarrow{k} & W(A) \\
\downarrow{k} & & \downarrow{comultiplicatic} \\
W(A) & \xrightarrow{k} & W(W(A))
\end{array} \]

commutes.
Now \( W(A) \) admits a description:

\[
W(A) = \text{Map}_{\text{Sets}} (\{x\}, W(A))
\]

\[
= \text{Map}_{\text{sets}} (\mathbb{Z}[\{x\}], W(A))
\]

\[
= \text{Map}_{\text{sets}} (\mathbb{Z}[\{x\}, A])
\]

\[
= \text{Map}_{\text{sets}} (\mathbb{Z} [\mathbb{N}, \mathbb{N}, \mathbb{N}, \ldots], A)
\]

\[
= \text{Map}_{\text{sets}} (\mathbb{N}, A)
\]

\[
= A^\mathbb{N}
\]

The ring structure on \( W(A) \) \( A^\mathbb{N} \)

isn't immediately clear from this description,

but we see that \( f (a_0, \ldots, a_n, \ldots) \)

\[
= (a_1, a_2, \ldots, a_n, \ldots)
\]
For any $\mathfrak{f}$-ring $B$, we have

$$B = \text{Maps}_{\mathfrak{f}}(\{x\}, B) = \text{Maps}_{\mathfrak{f}}\text{-rings}(\mathbb{Z} \{x\}, B)$$

Therefore, $\mathbb{Z} \{x\}$ is a "co-$\mathfrak{f}$-ring" object in $\mathfrak{f}$-rings, since $\text{Maps}_{\mathfrak{f}}\text{-rings}(\mathbb{Z} \{x\}, -)$ is a $\mathfrak{f}$-ring, not just a set.

Example $S^* : \mathbb{Z} \{x\} \to \mathbb{Z} \{x\}$

$$S : \mathbb{Z}[x_0, x_1, \ldots] \to \mathbb{Z}[x_0, x_1, \ldots]$$

maps $x_i$ to $x_{i+1}$.
To describe the other "co-operadings" on \( X \times X \), we need to consider co-products in \( S \)-rings.

Since the forgetful functor respects colimits, these are just coproducts in rings, i.e., tensor products.

If \( A \rightarrow B \rightarrow C \) are homomorphisms of \( S \)-rings, then the \( S \)-map \( B \otimes C \rightarrow A \) is given by

\[
S(b \otimes c) = b' \otimes S(c) + S(b) \otimes c' \circ c
\]

(\( = c \cdot b \otimes c + S(b) \otimes c' \))
Now \( \mathcal{V} \{ x \} \to \mathcal{V} \{ x \} \otimes \mathcal{V} \{ x \} \)

\( \mathcal{V} \)

is the initial ring, with \( \phi = \text{id} \),

\( s(n) = \frac{n-n^p}{p} \)

\( \mathcal{V} \{ x_0, y_1, \ldots \} \)

\( \mathcal{V} \{ y_0, y_1, \ldots, y_{2k+1} \} \)

\( \mathcal{V} \{ y_{2k+1}, y_{2k+2}, \ldots \} \)

is given by

\[ x_0 \to y_0 + z_0 \]

\[ x_1 \to s(y_0 + z_0) \]

\[ s(x_0) \]

\[ s(y_0) + s(z_0) + \frac{y_0 + z_0 - (y_0 + z_0)^p}{p} \]

\[ y_1 + z_1 + \frac{y_0 + z_0 - (y_0 + z_0)^p}{p} \]

\( \ldots \)

etc.
\( x^* : \mathcal{X} \ni x \rightarrow \mathcal{U} \setminus \{x\} \setminus \mathcal{U} \)

is given by \( x_0 \rightarrow y_0 \).

\( x_1 \rightarrow f(y_0, z_0) \)

\( f(x_0) \rightarrow y_0 \cdot f(z_0) + f(y_0) \cdot f(z_0) \)

\( y_0 \cdot z_1 + y_1 \cdot (z_0 + p \cdot z_1) \)

\( y_0 \cdot z_1 + y_1 \cdot z_0 + p \cdot z_1 y_1 \)

etc.
From this, returning to the formula

\[ w(A) = \operatorname{Maps}(\mathbb{N} \to \mathbb{Z}, W(A)) \]

we compute at least the "beginning" of the ring structure on \( W(A) \):

\[
(a_0, a_1, \ldots) + (b_0, b_1, \ldots) = (a_0 + b_0, a_1 + b_1, \ldots)
\]

\[
= (a_0 + b_0, a_1 + b_1, a_0p + b_0p - (a_0 + b_0)p, \ldots)
\]

\[
(a_0, a_1, \ldots) \cdot (b_0, b_1, \ldots) = (a_0b_0, a_0b_1 + a_1b_0, p + pa_1b_1, \ldots)
\]
The inclusion \( \mathbb{Z} \mathbb{[x_0, \ldots, x_n]} \hookrightarrow \mathbb{Z} \mathbb{[x_0, \ldots, x_{n-1}, \ldots]} \)

induces a projection

\[
(\alpha_0, \alpha_1, \ldots) \in A^N = \text{Maps}_{\mathbb{Z}} \left( \mathbb{Z}[x_0, \ldots, x_n], A \right) \\
\downarrow \\
A^n = \text{Maps}_{\mathbb{Z}} \left( \mathbb{Z}[x_0, \ldots, x_{n-1}], A \right)
\]

The above inclusion is in fact an inclusion of "co-rings" (though not of "co-rings")

and so \( A^N \to A^n \) is a homomorphism of rings (which we call a dandle)

\( W(A) \to W_n(A) \).
Above, we computed $W_2(A)$ explicitly.

Of course, $W_1(A) = A$ itself, and $W(A) \rightarrow W_1(A) = A$ is the counit.

Now one checks that giving a section $A \rightarrow W_2(A)$ of the augmentation $W_2(A) \rightarrow A$ is equivalent to giving $A$ a $W_1(A)$-module structure. The formula is

$$a \mapsto (a, \delta(a))$$

$$A \uparrow W_2(A)$$
This a more finitistic incarnation of the comonadic description of $E$-rings, and using this is one way of proving that the forgetful functor commutes with colimits (and thus of establishing this comonadic description, the existence of the adjoint $W^\dagger$, etc.) in the first place.