Complete flatness & related topics

If $I \subseteq A$ is a f.g. ideal, then

$M \in D(A)$ is I-completely flat if
derived cat. of A-modules

$M \overset{\mathbb{L}}{\underset{A}{\otimes}} A/I$ is discrete (i.e. has cohom. supported in degree 0) and flat over A/I.

Eg. If M is flat over A, then $M \overset{\mathbb{L}}{\otimes} A/I$ is flat over A/I, so M is I-completely flat.
an A-module placed in degree zero

$M \overset{\mathbb{L}}{\otimes} A/I \cong M \otimes A/I$

Eg. If M is a flat A-module (in degree 0), then the derived I-completion of M is I-completely flat.

Pf: If $\hat{M}$ is the I-derived completion of M, then $M \to \hat{M}$ induces an (quasi-) isomorphism

$$M \overset{\mathbb{L}}{\underset{A}{\otimes}} A/I \xrightarrow{\cong} \hat{M} \overset{\mathbb{L}}{\underset{A}{\otimes}} A/I.$$

In Lemma 2.6 of his lecture III, Bhatt writes that this follows from the universal property of derived completion, and the fact that objects of $D(A/I)$ are automatically derived I-complete.

We give a more computational explanation when $I = (f)$. (2)

In this case,
$$\mathrm{cone}\left(M \longrightarrow \hat{M} = R\varprojlim \left(\begin{array}{ccc} f\downarrow & & \mathrm{id}\downarrow \\ M & \xrightarrow{f^n} & M \\ f\downarrow & & \mathrm{id}\downarrow \end{array} \right)\right)$$

$$= \left(R\varprojlim_{f\times} M \right)[1]$$

and mult by f is an automorphism on $R\varprojlim_{f\times} M$.

Thus applying $- \overset{L}{\underset{A}{\otimes}} A/fA$ to

$$\cdots \longrightarrow M \longrightarrow \hat{M} \longrightarrow \left(R\varprojlim_{f\times} M\right)[1] \to \cdots$$

we indeed find that $M \overset{L}{\underset{A}{\otimes}} A/fA \xrightarrow{\cong} \hat{M} \overset{L}{\underset{A}{\otimes}} A/fA$. $\square$

<u>Lemma</u> A is I-completely flat iff A is I^n-completely flat for any $n > 0$. Further A is J-completely flat for any f.g. ideal J with $I \subseteq \mathrm{Rad}(J)$.

<u>Pf</u>: $0 \to I/I^2 \to A/I^2 \to A/I \to 0$

induces
$$\cdots \longrightarrow M \overset{L}{\underset{A}{\otimes}} I/I^2 \to M \overset{L}{\underset{A}{\otimes}} A/I^2 \longrightarrow M \overset{L}{\underset{A}{\otimes}} A/I \to \cdots$$
$$M \overset{L}{\underset{A}{\otimes}} I/I^2 = \left(M \overset{L}{\underset{A}{\otimes}} A/I\right) \overset{L}{\underset{A/I}{\otimes}} I/I^2$$

$\therefore$ I-complete flatness implies that $M \overset{L}{\underset{A}{\otimes}} A/I^2$ has cohom supp. in degree 0

So if we write $N := H^0(M \overset{\mathbb{L}}{\underset{A}{\otimes}} A/I^2)$, then

N is an A/I^2-module s.t. $N \overset{\mathbb{L}}{\underset{A/I^2}{\otimes}} A/I = N/IN$

is supp. just in degree zero, and is flat over A/I. This implies that N is flat over A/I^2.

(If L is an A/I^2-module, we may filter L as

$$0 \to L' = IL \to L \to L'' = L/IL \to 0,$$

giving
$$\cdots \to N \overset{\mathbb{L}}{\underset{A/I^2}{\otimes}} L' \to N \overset{\mathbb{L}}{\underset{A/I^2}{\otimes}} L \to N \overset{\mathbb{L}}{\underset{A/I^2}{\otimes}} L'' \to \cdots$$

$$N \overset{\mathbb{L}}{\underset{A/I^2}{\otimes}} L' = (N \overset{\mathbb{L}}{\underset{A/I^2}{\otimes}} A/I) \overset{\mathbb{L}}{\underset{A/I}{\otimes}} L' = (N/IN) \underset{A/I}{\otimes} L' \text{ in deg } 0$$

$$N \overset{\mathbb{L}}{\underset{A/I^2}{\otimes}} L'' = (N \overset{\mathbb{L}}{\underset{A/I^2}{\otimes}} A/I) \overset{\mathbb{L}}{\underset{A/I}{\otimes}} L'' = (N/IN) \underset{A/I}{\otimes} L'' \text{ in deg } 0$$

$\therefore$ $N \overset{\mathbb{L}}{\underset{A/I^2}{\otimes}} L$ is supp. in degree zero.)

Inductively, we find M is I^n-completely flat for all $n \geq 0$.

If $\mathrm{rad}(J) \supseteq I$, then $I^n \subseteq J$ for some n (since I is f.g.), and so if

M is I^n-completely flat, we find that

$$M \overset{\mathbb{L}}{\underset{A}{\otimes}} A/J = \left(M \overset{\mathbb{L}}{\underset{A}{\otimes}} A/I^n \right) \overset{\mathbb{L}}{\underset{A/I^n}{\otimes}} A/J$$

$$= \left(\begin{matrix}\text{flat } A/I^n\text{-module} \\ \text{in degree } 0\end{matrix} \right) \overset{\mathbb{L}}{\underset{A/I^n}{\otimes}} A/J$$

$$= \left(\text{——— " ———} \right) \underset{A/I^n}{\otimes} A/J$$

$$= \text{flat } A/J\text{-module in degree } 0. \qquad \square$$

<u>Lemma</u> Let $f \in A$, and suppose that A has bounded f^∞-torsion, i.e. that $A[f^\infty] = A[f^c]$ $\exists\, c \geq 0$.

Then TFAE, for $M \in D(A)$

(1) M is derived f-complete & f-completely flat

(2) M is (quasi-isomorphic to) a classically f-adic complete A-module placed in degree zero, $M/f^n M$ is flat over $A/f^n A$ $\forall n \geq 0$, and M has bounded f^∞-torsion.

Furthermore, if these equivalent conditions hold, then

$$M \underset{A}{\otimes} A[f^n] \xrightarrow{\cong} M[f^n] \quad \forall n \geq 0.$$

Proof: Since A has bounded f-power torsion, we have that

$$\{A/f^nA\} \quad \& \quad \{A \xrightarrow{f^n} A\}$$

are ~~isomorphic~~ equivalent inverse systems.

Now $A \xrightarrow{f^n} A$ is the complex underlying the simplicial ring

$$\underset{\substack{f^n \nwarrow \\ \nwarrow x}}{A} \otimes_{\mathbb{Z}[x]} \mathbb{Z} =: A_n$$

and we find that in fact we get an equivalence of inverse systems of simplicial rings

$$\{A/f^nA\} \simeq \{A \otimes_{\mathbb{Z}[x]} \mathbb{Z}\} = \{A_n\}$$

If either (1) or (2) holds, then

(remembering that classically complete $\Rightarrow$ derived complete)

we find that $M = R\varprojlim (M \xrightarrow{f^n} M)$

$$= R\varprojlim (M \overset{L}{\underset{A}{\otimes}} \{A \xrightarrow{f^n} A\})$$

$$= R\varprojlim (M \overset{L}{\underset{A}{\otimes}} A/f^nA)$$

equivalent inverse systems

Now if (1) holds, then $M \overset{\mathbb{L}}{\underset{A}{\otimes}} A/f^nA$ has cohom supp. in degree zero, and the transition maps

$$H^0(M \overset{\mathbb{L}}{\underset{A}{\otimes}} A/f^nA) \to H^0(M \overset{\mathbb{L}}{\underset{A}{\otimes}} A/f^{n-1}A)$$

are surjective

($0 \to f^{n-1}A/f^nA \to A/f^nA \to A/f^{n-1}A \to 0$

gives

$$\to M \overset{\mathbb{L}}{\underset{A}{\otimes}} (f^{n-1}A/f^nA) \to M \overset{\mathbb{L}}{\underset{A}{\otimes}} A/f^nA \to M \overset{\mathbb{L}}{\underset{A}{\otimes}} (A/f^{n-1}A) \to \cdots$$

$$M \overset{\mathbb{L}}{\underset{A}{\otimes}} (f^{n-1}A/f^nA) = \underbrace{(M \overset{\mathbb{L}}{\underset{A}{\otimes}} A/fA)}_{\text{cohom. supp. only in deg zero}} \overset{\mathbb{L}}{\underset{A/fA}{\otimes}} (f^{n-1}A/f^nA)$$

Now pass to long exact seq. of cohom.)

$\therefore M = R\varprojlim (M \overset{\mathbb{L}}{\otimes} A/f^nA)$ has cohom. supp in deg. zero.

Now we find that $H^0(M) \otimes A/f^nA = H^0(M \overset{\mathbb{L}}{\otimes} A/f^nA)$ is flat over A/f^nA.

Finally, we show $H^0(M) \otimes A[f^n] \xrightarrow{\sim} H^0(M)[f^n]$.

(This in particular proves that $H^0(M)$ has bounded f^∞-torsion, since A does.)

Recall $A_n := A \underset{\substack{f^n \uparrow \\ x}\,\mathbb{Z}[x]}{\otimes} \mathbb{Z}$, and write

$$M_n = M \overset{\mathbb{L}}{\underset{A}{\otimes}} A_n \quad (\text{an } A_n\text{-module})$$

$$\left(= (M \xrightarrow{f^n} M) \text{ as a complex of } A\text{-modules} \right)$$

We have the distinguished triangle

$$\cdots \to (A[f^n])[1] \to A_n \to A/f^nA \to \cdots$$

in $D(A_n)$

(just expressing that $H^{-1}(A \xrightarrow{f^n} A) = A[f^n]$
$H^0(\text{—"—}) = A/f^nA$)

If we $M_n \overset{\mathbb{L}}{\underset{A_n}{\otimes}} -$ we get
$$\overset{\shortparallel}{M \overset{\mathbb{L}}{\underset{A}{\otimes}} -}$$

$$\cdots \to (M_n \overset{\mathbb{L}}{\underset{A_n}{\otimes}} A[f^n])[1] \to M_n \to M \overset{\mathbb{L}}{\underset{A}{\otimes}} A/f^nA \to \cdots$$

$$\shortparallel$$

$$\left((M_n \overset{\mathbb{L}}{\underset{A_n}{\otimes}} A/f^n) \overset{\mathbb{L}}{\underset{A/f^n}{\otimes}} A[f^n] \right)[1]$$

$$\shortparallel$$

$$\left((M \overset{\mathbb{L}}{\underset{A}{\otimes}} A/f^nA) \overset{\mathbb{L}}{\underset{A/f^n}{\otimes}} A[f^n] \right)[1]$$

$$\shortparallel \; \left(H^0(M \overset{\mathbb{L}}{\underset{A}{\otimes}} A/f^nA) \underset{A/f^n}{\otimes} A[f^n]\right)[1]$$

$$\left(H^0(M) \underset{A}{\otimes} A[f^n]\right)[1]$$
$$\shortparallel$$
$$\left((H^0(M) \underset{A}{\otimes} A/f^nA) \underset{A/f^nA}{\otimes} A[f^n]\right)[1]$$
$$\shortparallel$$

and so passing to cohomology gives

$$~~\ldots~~ \quad H^0(M) \underset{A}{\otimes} A[f^n] \xrightarrow{\cong} H^{-1}(M_n) = M[f^n],$$

as required.

Suppose now that (2) holds, so $M = H^0(M)$ is classically f-adically complete with bounded f^∞-torsion, and M/f^nM is A/f^n-flat for each $n \geq 0$.

As noted at the start of the proof,

$$\{ M \overset{\mathbb{L}}{\underset{A}{\otimes}} A/f^nA \} \approx \{ M \overset{\mathbb{L}}{\underset{A}{\otimes}} A_n \} = \{ M \xrightarrow{f^n} M \}$$

$$\approx \{ M/f^nM \}$$

where the second equivalence follows from the fact that M has bounded f^∞-torsion.

$$\therefore \quad \{ M \overset{\mathbb{L}}{\underset{A}{\otimes}} A/fA \} = \{ M \overset{\mathbb{L}}{\underset{A}{\otimes}} A/f^nA \overset{\mathbb{L}}{\underset{A/f^nA}{\otimes}} A/fA \}$$

from the previous equivalence of inverse systems. $\longrightarrow$ $\approx \{ M/f^nM \overset{\mathbb{L}}{\underset{A/f^nA}{\otimes}} A/fA \}$

$\underbrace{\text{b/c } M/f^nM \text{ is } A/f^nA\text{-flat}}$ $\nearrow$ $= \{ M/fM \}$

So $M \overset{\mathbb{L}}{\underset{A}{\otimes}} A/fA = M/fM$ is concentrated in degree 0, and flat over A/fA. $\square$

Now suppose that $I \subseteq A$ is an invertible ideal, and that A/I has bounded p^∞-torsion (so that the same is true of A/I^n $\forall n \geq 0$).

We have already seen that if A is derived-(p,I) complete, then it is actually classically complete. (In the context of studying basic properties of bounded prisms.)

We now generalize this result: let $M \in D(A)$ be derived (p,I)-complete and (p,I)-completely flat. Then cohom. of M is supp. in degree 0, $H^0(M)$ is classically (p,I)-complete, $H^0(M)[I^n]=0$ for all $n \geq 0$, and $H^0(M)/I^n H^0(M)$ has bounded p^∞-torsion for all $n \geq 0$.

Pf: Since $\mathrm{Rad}(I^n,p) \supseteq (I,p)$, we see that M is (I^n,p)-completely flat. Since

$$\left(M \overset{L}{\underset{A}{\otimes}} A/I^n\right) \overset{L}{\underset{A/I^n}{\otimes}} A/(I^n,p) = M \overset{L}{\underset{A}{\otimes}} A/(I^n,p)$$

, we find that $M \overset{L}{\underset{A}{\otimes}} A/I^n$ is p-completely flat in $D(A/I^n)$.

The previous lemma (recalling that A/I^n has bounded p^∞-torsion) shows that

$$M \overset{L}{\underset{A}{\otimes}} A/I^n \text{ has cohom.}$$

supported in degree 0, and that $H^0(M \overset{L}{\underset{A}{\otimes}} A/I^n)$ has bounded p^∞-torsion, & is classically p-complete.

Since M is derived-(p,I)-complete, it is derived I-complete. Since I is invertible,

$M \otimes \{I^n \hookrightarrow A\}$ computes $M \overset{L}{\otimes} A/I^n$, and so

$$M = R\varprojlim \{ M \overset{L}{\otimes} A/I^n \} = R\varprojlim \{ H^0(M \overset{L}{\otimes} A/I^n) \}$$

and the transition maps in ↗ are surjective.

(Since $M \overset{L}{\underset{A}{\otimes}} A/I^n = \left(M \overset{L}{\underset{A}{\otimes}} A/I^{n+1}\right) \overset{L}{\underset{A/I^{n+1}}{\otimes}} A/I^n$

$$H^0(M \underset{A}{\otimes} A/I^n) = H^0\left(M \overset{L}{\otimes} A/I^{n+1}\right) \overset{L}{\underset{A/I^{n+1}}{\otimes}} A/I^n \quad).$$

we conclude that M has cohom. supp in degree 0.

$$\therefore \quad M \overset{L}{\underset{A}{\otimes}} A/I^n = H^0(M) \underset{A}{\otimes} A/I^n$$

$$= \left(H^0(M) \underset{A}{\otimes} I^n \longrightarrow H^0(M) \right)$$

and so, since $\underbrace{H^{-1}(M \overset{L}{\underset{A}{\otimes}} A/I^n)}_{=0}$, we see

$$H^0(M)[I^n] = 0,$$

while $H^0(M)/I^n = H^0(M \overset{L}{\underset{A}{\otimes}} A/I^n)$ has bounded p^∞-torsion

Finally, since $M \overset{= H^0(M)}{\ }$ is derived (p, I)-complete, we find

$$H^0(M) = R\varprojlim_{m,n} \left\{ \left(H^0(M) \otimes I^n \to H^0(M)\right) \underset{\mathbb{Z}}{\otimes} \left(\mathbb{Z} \xrightarrow{p^m} \mathbb{Z}\right) \right\}$$

$$= R\varprojlim_{m,n} \left\{ H^0(M)/I^n \xrightarrow{p^m} H^0(M)/I^n \right\}$$

bounded p^∞-torsion

$$= R\varprojlim_{m,n} \left\{ H^0(M)/(p^n, I^n) \right\},$$ proving the claimed classical completeness. □

In the context of the preceding lemma, suppose that M is a flat A-module.

Then $\hat{M}$ ~ derived (p, I)-completion is (p, I)-completely flat, (see example on p. ①)

$\therefore$ by what we just proved $\hat{M}$ is the classical (p, I)-completion of M.

Thus this classical completion is (p, I)-completely flat, and $\hat{M}[I^n] = 0$ $\forall n \geq 0$, while $\hat{M}/I^n\hat{M}$ has bounded p^∞-torsion $\forall n \geq 0$.

So far we've talked about flatness and complete flatness when A is a ring and M is either an A-module or an object of $D(A)$.

(To be precise, we can declare an object $M \in D(A)$ to be flat if $H^0(M)$ is flat over A and $H^i(M) = 0$ for $i \neq 0$, i.e. if M is q.i. to a flat A-module in degree 0.)

We now extend these notions to simplicial (or better, animated) modules over simplicial (or better, animated) rings. (These notions already appeared somewhat implicitly in the proof of the lemma on p. 4.).

If A is a simplicial ring, and M is a simplicial A-module, we say that M is flat over A if $\pi_0(M)$ is flat over $\pi_0(A)$, and if the natural maps $\pi_i(A) \otimes_{\pi_0(A)} \pi_0(M) \to \pi_i(M)$ are iso's, for $i \geq 0$.

Similarly, we say that M is faithfully flat over A if $\pi_0(M)$ is faithfully flat over $\pi_0(A)$, and if
$$\pi_i(A) \underset{\pi_0(A)}{\otimes} \pi_0(M) \xrightarrow{\sim} \pi_i(M) \quad \forall i > 0.$$

If A is discrete, i.e. $A \underset{w.e}{\xrightarrow{\sim}} \pi_0(A)$, then M is (faithfully) flat over A iff M is discrete, i.e. $M \underset{w.e.}{\xrightarrow{\sim}} \pi_0(M)$, and $\pi_0(M)$ is (faithfully) flat over $\pi_0(A)$.

Some properties: (i) If $A \to B$ is a morphism of simplicial rings, and if M is (faithfully) flat over A, then $B \overset{\mathbb{L}}{\otimes} M$ is (faithfully) flat over B.

(ii) If $A \to B$ is faithfully flat (in the sense just defined, thinking of the induced A-module structure on A), and if $B \overset{\mathbb{L}}{\otimes} M$ is (faithfully) flat over B, then M is (faithfully) flat over A.

(iii) If $\pi_0(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} M$ is (faithfully) flat over $\pi_0(A)$, then M is (faithfully) flat over A.

Pf: (i) certainly seems reasonable, as does (ii). In fact, if we have (iii), then it suffices to check (i) in the case of $A \to \pi_0(A)$. Indeed, we then find that $\pi_0(B) \overset{\mathbb{L}}{\underset{B}{\otimes}} (B \overset{\mathbb{L}}{\underset{A}{\otimes}} M) = \pi_0(B) \overset{\mathbb{L}}{\underset{A}{\otimes}} M$ $= \pi_0(B) \overset{\mathbb{L}}{\underset{\pi_0(A)}{\otimes}} (\pi_0(A) \overset{\mathbb{L}}{\underset{A}{\otimes}} M)$, and so (iii) allows us to deduce the general case of (i) from the discrete case.

Similarly, assuming that we have (i) for $B \to \pi_0(B)$, and (iii), then we see that the discrete case of (ii) implies the general case.

We saw a special case of (iii) in the argument on p. 7 & the top of p. 8: our ring A was A_n, with $\pi_0(A_n) = A/f^n$, and M was $M_n := M \overset{L}{\underset{A}{\otimes}} A_n$.

From the assumption that $\pi_0(M_n) = M/f^n M$ is flat over A/f^n, we deduced that
$$\begin{array}{ccc} \pi_1(A) \underset{\pi_0(A)}{\otimes} \pi_0(M_n) & \to & \pi_1(M_n) \\ \| & & \| \\ A[f^n] \underset{A/f^n}{\otimes} M/f^n & & M[f^n] \end{array}$$
is an iso.

The general case of (iii) is deduced from a more general, iterated, truncation argument, by considering $\cdots \tau_{\leq n} A \subseteq \tau_{\leq (n-1)} A \subseteq \cdots \subseteq A$

and the induced filtration $\cdots \tau_{\leq n} A \overset{L}{\underset{A}{\otimes}} M \cdots$ on M,

which we show coincides with the filtration $\cdots \tau_{\leq n} M \cdots$ on M.

The case $A \to \pi_0(A)$ of (i) is proved similarly.

One way to think of it is to consider the analogous graded statement:

$$\pi_0(A) \overset{L}{\underset{\bigoplus_{i\geq 0} \pi_i(A)}{\otimes}} \bigoplus_{i\geq 0} \pi_i(M)$$

flat over $\pi_0(M)$

$$= \pi_0(A) \underset{\bigoplus_{i\geq 0} \pi_i(A)}{\otimes} \underbrace{\left(\bigoplus_{i\geq 0} \pi_i(A)\right) \underset{\pi_0(A)}{\otimes} \pi_0(M)}_{\text{flat over } \bigoplus_{i\geq 0} \pi_i(A)}$$

$$= \pi_0(M),$$

and then pass to the simplicial case by a filtration/spectral sequence argument.

$\square$

Now consider the case when A is a discrete ring, $I = (f_1, \dots, f_r)$ is a f.g. ideal $\subseteq A$, and

$\mathrm{Kos}(A; f_1, \dots, f_r)$ is the simplicial ring underlying the corresponding Koszul complex.

Then $\pi_0(\mathrm{Kos}(A; f_1, \dots, f_r)) = A/(f_1, \dots, f_r) = A/I$.

So if $M_\bullet$ is a simplicial A-module, then

$$\underbrace{\mathrm{Kos}(A; f_1, \dots, f_r) \overset{\mathbb{L}}{\underset{A}{\otimes}} M_\bullet}_{\overset{!}{=}\ \mathrm{Kos}(M_\bullet; f_1, \dots, f_r)} \text{ is flat over } \mathrm{Kos}(A; f_1, \dots, f_r)$$

iff $\pi_0(\mathrm{Kos}(A; f_1, \dots, f_r)) \overset{\mathbb{L}}{\underset{\mathrm{Kos}(A; f_1, \dots, f_r)}{\otimes}} \mathrm{Kos}(M_\bullet; f_1, \dots, f_r)$

$\quad \| \nwarrow$ transitivity of $\otimes$

$A/I \overset{\mathbb{L}}{\otimes} M_\bullet$

is flat over A/I.

But (using Dold-Kan to pass to complexes of A-modules), this is exactly what it means for $M_\bullet$ to be I-completely flat.

So now we can define "completely flat" when $A_\bullet$ is a simplicial (or, better, animated) ring:

If $I = (f_1, \dots, f_r) \subseteq \pi_0(A_\bullet)$, then $M_\bullet$ is <u>I-completely flat</u> over $A_\bullet$ if $\mathrm{Kos}(M_\bullet; f_1, \dots, f_r)$ is flat over $\mathrm{Kos}(A_\bullet; f_1, \dots, f_r)$

~~[illegible]~~ Note that if $M_\cdot$ is $A_\cdot$-flat, then

$$\mathrm{Kos}(M_\cdot; f_1, \dots, f_r) = \mathrm{Kos}(A_\cdot; f_1, \dots, f_r) \otimes_{A_\cdot} M_\cdot \text{ is}$$

$\mathrm{Kos}(A; f_1, \dots, f_r)$ - flat.

So flatness $\Rightarrow$ I-complete flatness.

Similarly, if $M_\cdot$ is $A_\cdot$-flat, its I-derived completion $\left(= \mathrm{ho.}\varprojlim_N \mathrm{Kos}(M_\cdot; f_1^N, \dots, f_r^N) \right)$

will be I-completely flat.

(This generalizes the eg. on p. 1.)

Indeed, if $\hat{M}_\cdot$ = I-derived completion of $M_\cdot$, then

$$\mathrm{Kos}(M_\cdot; f_1, \dots, f_r) \xrightarrow[\text{w.e}]{\sim} \mathrm{Kos}(\hat{M}_\cdot; f_1, \dots, f_r),$$

and so this follows from the preceding result.

Some variants

If $A_\bullet \to B_\bullet$, and if $x_1, \dots, x_r \in \pi_0(B_\bullet)$, then we obtain an induced map $A_\bullet \to \mathrm{Kos}(B_\bullet; x_1, \dots, x_r)$.

If A and B are discrete, then this map is flat iff (i) $(x_1, \dots, x_r)$ is regular on B
(this is vanishing of higher π_i)
& (ii) $B/(x_1, \dots, x_n)$ is A-flat

(this is the flatness on π_0.)

We say that (in the general case of $A_\bullet \to B_\bullet$) $x_1, \dots, x_r \in \pi_0(B_\bullet)$ is regular relative to $A_\bullet$ if

$$A_\bullet \longrightarrow \mathrm{Kos}(B_\bullet; x_1, \dots, x_r) \quad \text{is flat.}$$

~~We say that $x_1, \dots, x_r \in \pi_0(B_\bullet)$ is completely~~ If $I = (f_1, \dots, f_n) \subseteq \pi_0(A_\bullet)$, we say $x_1, \dots, x_r$ is I-completely regular relative to $A_\bullet$ if the induced morphism

$$\mathrm{Kos}(A_\bullet; f_1, \dots, f_n) \longrightarrow \mathrm{Kos}(A_\bullet; f_1, \dots, f_n) \overset{L}{\underset{A_\bullet}{\otimes}} \mathrm{Kos}(B_\bullet; x_1, \dots, x_r)$$

$$= \mathrm{Kos}(B_\bullet; f_1, \dots, f_n, x_1, \dots, x_r)$$

is flat.

Preservation of flatness under $\overset{L}{\otimes}$. Show that "regular relative to $A_\cdot$" $\Rightarrow$ "I-completely regular relative to $A_\cdot$".

Similarly, if $x_1, \dots, x_r$ are regular relative to $A_\cdot$ on $B_\cdot$, then they are I-completely regular relative to $A_\cdot$ on the derived I-completion of $B_\cdot$.

(Again, use that $Kos(\hat{B}_\cdot; f_1, \dots, f_r, x_1, \dots, x_r)$

just as we did in the analysis of flatness and derived completion on p. 16

$= Kos(B_\cdot; f_1, \dots, f_r, x_1, \dots, x_r)$.)