Suppose $X_{/\mathbb{C}}$ is a smooth variety.

Let X^{an} denote the underlying complex manifold.

$\Omega^{\cdot}_X$ denotes the algebraic de Rham complex

It is a complex of sheaves on X.
Each Ω^i_X is an $\mathcal{O}_X$-module (locally free of finite rank), but the differentials are not $\mathcal{O}_X$-linear (Leibnitz rule!)

$\Omega^{\cdot}_{X^{an}}$ denotes the holomorphic de Rham complex; similar remarks apply.

{ There is a canonical morphism of ringed spaces ~~maps~~ $i: X^{an} \to X$, and
$\Omega^{\cdot}_{X^{an}} = i^* \Omega^{\cdot}_X$. } But since the differentials are not $\mathcal{O}_X$-linear, it takes some interpretation to make sense of how to pull-back the differentials.

Now we may form the cohomology sheaves $\mathcal{H}^i(\Omega^{\cdot}_X)$ and $\mathcal{H}^i(\Omega^{\cdot}_{X^{an}})$ — these are sheaves of $\mathbb{C}$-vector spaces on X & X^{an} resp.

Pull back induces $i^{-1}\mathcal{H}^i(\Omega^\cdot_X) \longrightarrow \mathcal{H}^i(\Omega^\cdot_{X^{an}})$.

Now the holomorphic Poincaré Lemma shows that actually

$$\mathcal{H}^i(\Omega^\cdot_{X^{an}}) = \begin{cases} \mathbb{C} & i=0 \\ 0 & i>0 \end{cases}$$

What about $\mathcal{H}^i(\Omega^\cdot_X)$.

If $x \in X$, $\mathcal{H}^i(\Omega^\cdot_X)_x$ ← stalk at x

$$= \varinjlim_{\substack{x \in U \text{ affine} \\ \text{open in } X}} H^i(\Omega^\cdot_X(U))$$

b/c $(\Omega^\cdot_X)|_U = \Omega^\cdot_U$ ⟶ $$= \varinjlim_{x\in U} H^i(\Omega^\cdot_U(U))$$

$$= \varinjlim_{x\in U} H^i_{dR}(U)$$

Grothendieck's theorem ↗

$$= \begin{cases} \mathbb{C} & \text{if } i=0 \\ \text{something complicated} & \text{if } 0<i\leq \dim X \\ 0 & \text{if } i>\dim X \end{cases}$$

Pull back map is not an ≅ in degrees $0<i\leq\dim X$

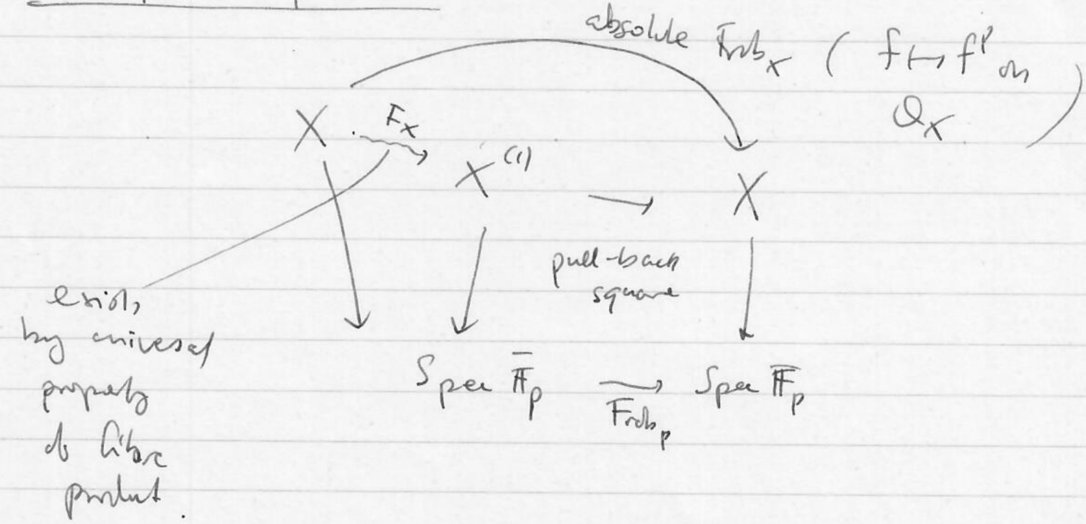
Suppose $X_{/\overline{\mathbb{F}}_p}$ is a smooth variety.

We have the algebraic de Rham complex $\Omega^{\bullet}_X$
(no direct analogue of holomorphic/analytic version).

Let $X^{(1)}$ = Frob. twist of X;

then $X \xrightarrow{F_X} X^{(1)}$ (relative Frobenius) is a map of varieties$/\overline{\mathbb{F}}_p$.

<u>Fibre product picture</u>



<u>Concrete picture</u>: If $f \in \overline{\mathbb{F}}_p[x_1, \dots, x_n]$,

let $f^{(1)} = \mathrm{Frob}_p(f)$ ← apply Frob_p to coeffs., i.e. raise coeffs. to p^{th} power

Then if X is cut out by $f_1, \dots, f_r \in \overline{\mathbb{F}}_p(x_1, \dots, x_n)$, $X^{(1)}$ is cut out by $f_1^{(1)}, \dots, f_r^{(1)}$.

$X \to X^{(1)}$ via $(x_1, \dots, x_n) \mapsto (x_1^p, \dots, x_n^p)$.

Note that if $f_i(x_1, \dots, x_n) = 0$, then

$$\left(f_i(x_1, \dots, x_n)\right)^p = 0^p = 0$$

$$\parallel$$

$$f_i^{(1)}(x_1^p, \dots, x_n^p) \quad !!$$

~~[struck-out text]~~ Note that F_X is a homeo. on underlying top space: if $U \subseteq_{\text{open}} X$,

$$F_X(U) = U^{(1)} \subseteq_{\text{open}} X^{(1)}.$$

If $f \in \mathcal{O}_{X^{(1)}}(U^{(1)})$, then $F_X^* f$ ($= f \circ F_X$ concretely)

$$\cap$$

$$\mathcal{O}_X(U),$$

and $d(F_X^* f) = 0$ (ultimately b/c $dx^p = px^{p-1}dx = 0$)

Since F_X is a homeo, $F_X^{-1}\mathcal{O}_{X^{(1)}} \subseteq \mathcal{O}_X$
(↑ sheaf of subrings)

Prop 2

$$F_X^{-1}\mathcal{O}_{X^{(1)}} = \ker\left(d : \mathcal{O}_X \to \Omega^1_X\right)$$

↑ as sheaves on X.

(One possible)
Pf. If $X = \mathbb{A}^n$, check by hand using
$$dx^n = nx^{n-1}\,dx.$$

In general, cover X by open U which admit $U \xrightarrow[\pi]{\text{étale}} \mathbb{A}^n$

(possible b/c X is smooth), use good behavior of étale morphisms w.r.t. Frob, diff'l forms;

eg. $\pi^*\Omega^1_{\mathbb{A}^n} = \Omega^1_U$,

while
$$\begin{array}{ccc} U & \xrightarrow{F_U} & U^{(1)} \\ \pi\downarrow & & \downarrow\pi^{(1)} \\ \mathbb{A}^n & \xrightarrow{F_{\mathbb{A}^n}} & (\mathbb{A}^n)^{(1)} = \mathbb{A}^n \end{array}$$
is Cartesian.

□

So $(\Omega^\bullet_X, d)$ is a complex of $F_X^{-1}\mathcal{O}_X$-modules
(↑ diff'ls are $F_X^{-1}\mathcal{O}_X$-linear)

and $\mathcal{H}^0 = F_X^{-1}\mathcal{O}_X$.

What about the higher $\mathcal{H}^i(\Omega^\cdot_X)$?

Thm (Cartier)

$$\mathcal{H}^i(\Omega^\cdot_X) \cong F_X^{-1}\Omega^i_{X^{(1)}}$$

as $F^{-1}\mathcal{O}_{X^{(1)}}$ - sheaves.

Equivalently, $(F_X)_* \mathcal{H}^i(\Omega^\cdot_X) \cong \Omega^i_{X^{(1)}}$

as $\mathcal{O}_{X^{(1)}}$ - sheaves.

Since this is an isomorphism of coherent $\mathcal{O}_{X^{(1)}}$-sheaves, it is equivalent to giving an isomorphism of $\mathcal{O}_{X^{(1)}}(U^{(1)})$-modules

$$(F_U)_* \ H^i(\Omega^\cdot(U)) \cong \Omega^i_{U^{(1)}}(U^{(1)})$$

for each $U \underset{\text{affine open}}{\subseteq} X$, compatibly w.r.t. localization.

Again equivalently, this amounts to giving natural $\cong$'s

$$\text{closed } i\text{-forms on } U \,/\, \text{exact } i\text{-forms on } U \xrightarrow[\mathcal{C}^i]{\cong} i \text{ forms on } U^{(1)}$$

which is twisted linear, in the sense that

$$\mathcal{C}^i\big((F_X^* f)\cdot\omega\big) = f \cdot \mathcal{C}^i(\omega) \quad \text{if } f \in \Gamma(U^{(1)}, \mathcal{O}_{U^{(1)}}).$$

The map $\mathcal{C}^i$: closed i-forms on U

$\longrightarrow$ i-forms on $U^{(1)}$

is called the ith Cartier operator.

The thm. tells us it is surjective, with kernel equal to the exact forms.

To describe the Cartier operator, it is easier to describe the inverse

$$\Omega^i_{X^{(1)}} \xrightarrow{\cong} (F_X)_* \mathcal{H}^i(\Omega^\cdot_X)$$

As i varies, both sides admit a strictly graded commutative $\mathcal{O}_{X^{(1)}}$-alg. structure, and the isomorphism will be compatible with this structure.

In fact, both sides also admit a differential:

On $\Omega^\cdot_{X^{(1)}}$ we have d, the exterior derivative.

On $\mathcal{H}^\cdot(\Omega^\cdot_X)$ (and hence on $(F_X)_* \mathcal{H}^\cdot(\Omega^\cdot_X)$)

we have the Bockstein differential β, as we now explain.

If U is affine open in X, then since it is smooth over $\overline{\mathbb{F}}_p$, we may lift it to $\tilde{U}$ over $W_2(\overline{\mathbb{F}}_p)$.

Then we have a s.e.s. of complexes of sheaves

$$0 \to \Omega^\cdot_U \xrightarrow{p\times} \Omega^\cdot_{\tilde{U}} \xrightarrow{\text{red'n mod } p} \Omega^\cdot_U \to 0$$

inducing Bockstein's $\beta : \mathcal{H}^i(\Omega^\cdot_U) \to \mathcal{H}^{i+1}(\Omega^\cdot_U)$

This is ind. of the lift $\tilde{U}$, and so patch to give β on $\mathcal{H}^\cdot(\Omega^\cdot_X)$.

The universal property of $(\Omega^\cdot_{X^{(1)}}, d)$ implies that the map

$$\mathcal{O}_{X^{(1)}} \longrightarrow (F_X)_* \mathcal{H}^0(\Omega^\cdot_X)$$

extends to $(\Omega^\cdot_{X^{(1)}}, d) \to ((F_X)_* \mathcal{H}^\cdot(\Omega^\cdot_X), \beta)$

a homomorphism of dga's.

This is the (inverse) Cartier isomorphism.

Ex

$X = \mathbb{A}^1 = \operatorname{Spec} \overline{\mathbb{F}}_p[x]$, so $X^{(1)} = \mathbb{A}^1$ too, and

$$F_X : \mathbb{A}^1 \to \mathbb{A}^1$$
$$x \mapsto x^p$$

Take $\tilde{X} = \mathbb{A}^1_{/W_2(\overline{\mathbb{F}}_p)} = \operatorname{Spec} W_2(\overline{\mathbb{F}}_p)[x]$

To compute $\mathcal{C}^{-1}(dx)$, we compute

$$\beta(F_X^* x) = \beta(x^p) = \frac{1}{p} d(x^p) = x^{p-1} dx,$$

and so deduce that $\mathcal{C}^{-1}(dx) = x^{p-1} dx$ (mod exact forms)

↑ closed on $\mathbb{A}^1$, of course, but not exact

More generally, $\mathcal{C}^{-1}(x^n dx) = x^{np+p-1} dx = x^{(n+1)p-1} dx$ (mod exact forms) if $n \geq 0$

$\Omega^1(\mathbb{A}^1) = \overline{\mathbb{F}}_p \langle x^n dx \rangle_{n \geq 0}$, and

$\overline{\mathbb{F}}_p \langle x^{mp-1} dx \rangle_{m > 0}$ spans $\Omega^1(\mathbb{A}^1)$ mod exact forms,

so this confirms the Cartier $\cong$ in this case.

If we work on $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\} = \operatorname{Spec} \overline{\mathbb{F}}_p[x, x^{-1}]$, we get a more elegant, and more celebrated, formula:

$$\mathcal{C}^{-1}\left(\frac{dx}{x}\right) = \frac{1}{x^p} \mathcal{C}^{-1}(dx) = \frac{x^{p-1} dx}{x^p} = \frac{dx}{x},$$

(mod exact forms)

<u>Ex</u>. If $f: X \longrightarrow \mathbb{G}_m$, then since $\mathcal{C}^{-1}$ is natural,

and since $d\log f^{(1)} := \frac{d f^{(1)}}{f^{(1)}} = (f^{(1)})^* \frac{dx}{x}$,

we find that $\mathcal{C}^{-1}(d\log f^{(1)}) = f^*\left(\frac{dx}{x}\right) = d\log f$.

(mod exact forms)

Ex. If X is a curve, and t is a uniformizer at a pt P of X, then $\mathcal{C}^{-1}\left(\frac{dt^{(1)}}{t^{(1)}}\right) = \frac{dt}{t}$ mod exact forms

from which we deduce that

$$\text{res}_P(\mathcal{C}^{-1}(\omega^{(1)})) = \text{res}_{P^{(1)}}(\omega^{(1)})$$ if ω is a meromorphic diff'l on X, inducing $\text{Frob}_p^* \omega =: \omega^{(1)}$ on $X^{(1)}$

(and $P^{(1)} = \text{Frob}_p(P)$).

Now if X is proper, we can compute $\mathcal{C} : H^0(X, \Omega^1) \to H^0(X^{(1)}, \Omega^1)$

We relate the computation to Serre duality on "Hodge cohom":

$$H^0(X, \Omega^1) \times H^1(X, \mathcal{O}) \longrightarrow H^1(X, \Omega^1) = \overline{\mathbb{F}}_p$$

perfect pairing.

If $X = U_0 \cup U_1$ is an affine cover of X, and $U_{01} = U_0 \cap U_1$, then computing cohom Čech-wise, we find that

$$H^0(U_{01}, \mathcal{O}) \twoheadrightarrow H^1(X, \mathcal{O})$$

and $H^0(U_{01}, \Omega^1) \longrightarrow H^1(X, \Omega^1) = \overline{\mathbb{F}}_p$

given explicitly by $\omega \longmapsto \sum'_{P \in X \setminus U_{01}} \mathrm{res}_P(\omega)$.

The perfect pairing $H^0(X, \Omega^1) \times H^1(X, \mathcal{O}) \longrightarrow \overline{\mathbb{F}}_p$

is then induced by $H^0(X, \Omega^1) \times H^0(U_{01}, \mathcal{O}) \longrightarrow \overline{\mathbb{F}}_p$

$$\omega \times f \longmapsto \sum'_P \mathrm{res}_P (f \cdot \omega)$$

$\therefore$ If $\omega \in H^0(X, \Omega^1)$ and $g \in H^0(U_{01}^{(1)}, \mathcal{O})$ (representing a class in $[g] \in H^1(X^{(1)}, \mathcal{O})$)

we find that

$$\langle \mathcal{C}(\omega), [g] \rangle = \sum'_{P \in X \setminus U_{01}} \mathrm{res}_{P^{(1)}} \, g \cdot \mathcal{C}(\omega)$$

$$= \sum'_P \mathrm{res}_{P^{(1)}} \, \mathcal{C}(F_X^*(g) \cdot \omega)$$

$$= \sum'_P \mathrm{res}_P \, F_X^*(g) \cdot \omega$$

$$= \langle \omega, [F^* g] \rangle = \langle \omega, F_X^*(g) \rangle$$

Thus $C_e : H^0(X, \Omega^1) \to H^0(X^{(1)}, \Omega^1)$ is the transpose to $F_X^* : H^1(X^{(1)}, \mathcal{O}) \to H^1(X, \mathcal{O})$.

Eg. If X is an elliptic curve,

$F_X^* : H^1(X^{(1)}, \mathcal{O}) \to H^1(X, \mathcal{O})$ is

either $\cong$ / zero iff X is ordinary / s.s.

$\therefore$ X is s.s. iff $C_e : H^0(X, \Omega^1) \to H^0(X^{(1)}, \Omega^1)$ is zero

iff the invariant diff'l on X is exact!!

On the other hand, if X is ordinary, then

$$C_e(\omega) = \alpha\omega^{(1)} \quad \text{for some } \alpha \in \overline{\mathbb{F}}_p^\times$$

where ω = inv. diff'l on X and $\omega^{(1)} := \mathrm{Frob}_p^* \omega$ = inv. diff'l on $X^{(1)}$.

Then $C_e(\beta\omega) = \alpha\beta\,\omega^{(1)} = \frac{\alpha\beta}{\mathrm{Frob}_p(\beta)}(\beta\omega)^{(1)}$ if $\beta \in \overline{\mathbb{F}}_p^\times$

Choosing β s.t. $\alpha = \mathrm{Frob}_p(\beta)/\beta^{-1} = \beta^{p-1}$, we find that we may scale ω, uniquely up to an elt. of $\mathbb{F}_p^\times = \mu_{p-1}$,
$C_e(\omega) = \omega^{(1)}$

A converse to one of our previous calculations then implies that ω is logarithmic, i.e. that $\omega = d\log f$ $\exists$ ratl fn. f on X.

Eg $p=2$, X: $y^2 + axy + ~~y~~ = x^3 + x^2$

$$\omega = \frac{dx}{ax+1}$$

If $a=0$, $\omega = dx$.

If $a \neq 0$, $a\omega = \frac{d(ax+1)}{ax+1} = d\log(ax+1)$

Eg. $p=3$, X: $y^2 = x^3 + bx^2 + x$

$$\omega = \frac{dx}{2y} = \frac{dy}{2bx+1} = \frac{dy}{2(bx-1)}$$

If $b=0$, $\omega = dy$

If $b \neq 0$, and $\beta^2 = b$, then $\frac{d(\beta x + y - \beta^{-1})}{\beta x + y - \beta^{-1}} = \frac{(\beta + \frac{bx-1}{y})dx}{\beta x + y - \beta^{-1}}$

$$= \beta \frac{dx}{y}$$

~~$\frac{d(\beta x+y)}{\beta x+y} = \frac{(\beta + bx/y)dx}{\beta x+y} = \beta \frac{dx}{y}$ "$d\log(\beta x+y)$"~~