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p-ADIC L-FUNCTIONS AND UNITARY COMPLETIONS OF REPRESENTATIONS OF p-ADIC REDUCTIVE GROUPS

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Abstract. In the first half of the paper we introduce the notion of the universal unitary completion of a continuous representation of a p-adic reductive group on a locally convex p-adic vector space, and prove that such a completion exists under appropriate hypotheses. The problem of studying unitary completions has been raised by Breuil in connection with his work on a possible "p-adic local Langlands correspondence" for GL_2, and we relate our construction to certain conjectures of Breuil for the group $\mathrm{GL}_2(\mathbb{Q}_p)$. In particular, we show that the universal unitary completion of the locally analytic parabolic induction of a locally algebraic character coincides with the universal unitary completion of the corresponding locally algebraic induction, provided that the character being induced satisfies a "non-critical slope" condition. (See Proposition 2.5 below.)

In the second half of the paper we consider a certain unitary Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ obtained by p-adically completing the cohomology of classical modular curves. The mere existence of this representation implies that those locally algebraic parabolically induced representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ that arise from classical finite slope newforms have a non-trivial universal unitary completion (verifying a conjecture of Breuil for these representations), while applying Proposition 2.5 in this context enables us to give a new construction of p-adic L-functions attached to p-stabilized newforms of non-critical slope. Combining our construction with a representation theoretic definition of $\mathcal{L}$-invariants due to Breuil [5], we are able to give a simple proof of the Mazur-Tate-Teitelbaum exceptional zero conjecture (in terms of Breuil's definition of the $\mathcal{L}$-invariant).

The object of this note is two-fold. In its first half we consider the following problem in the representation theory of p-adic groups: When does a topologically irreducible continuous representation of a p-adic reductive group G on a p-adic topological vector space (perhaps satisfying some additional hypotheses) admit an embedding into a unitary representation of G on a p-adic Banach space? If $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and the representation considered is admissible locally algebraic, this problem has been raised by Breuil in connection with his ideas on a "p-adic Local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$" [4, 5].

In the second half of the paper we explain a representation theoretic point of view on p-adic L-functions attached to modular forms. As an application, we prove a version of the Mazur-Tate-Teitelbaum exceptional zero conjecture [16, p. 46], using the representation theoretic definition of the $\mathcal{L}$-invariant given by Breuil in [5] (which provides a reinterpretation of the definition given by Darmon [9] and Orton [18] in the weight two and higher weight cases respectively).

The two halves of the paper are more closely related than they might appear at first glance. Indeed, the local tools discussed in the first half play a vital role in the

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argument of the second half. Conversely, the global results of the second half shed some light on the local problems discussed in the first half.

Let us now describe the contents of the note in more detail. In Section 1 we present a general framework in which to consider the problem of unitary completions of G-representations. In particular, given a G-representation on a reflexive Fréchet space V that admits a generating semi-norm (in the sense of Definition 1.13) we construct an associated G-representation V^{b}, equipped with a continuous G-equivariant injection $V^{\mathrm{b}} \to V$, whose strong dual is the universal unitary completion of the contragredient G-representation on the strong dual V_b'. (See Lemma 1.11.) In the case when V is dual to a finitely generated admissible locally algebraic representation π of G, we observe (Proposition 1.17) that V^{b} is dual to the completion $\hat{\pi}$ of π with respect to a finite type lattice. If furthermore π is irreducible and of the form "algebraic tensor absolutely cuspidal", then we show (Proposition 1.18) that $\hat{\pi}$ is non-zero if and only if the central character of π is unitary.

In Section 2 we restrict to the case $G = \mathrm{GL}_2(\mathbb{Q}_p)$, and consider irreducible admissible locally algebraic representations π whose smooth factor is principal series or special. In this case Breuil [4] has made a precise conjecture as to whether or not $\hat{\pi}$ is non-zero (see Conjecture 2.2). Briefly, this conjecture states the only possible obstruction to $\hat{\pi}$ being non-zero is presented by the exponents appearing in the Jacquet module $J_B(\pi)$ (where B is a Borel subgroup of G). (This conjecture is now known to be a theorem in many cases, thanks to recent work of Berger and Breuil [2] and Colmez [8].)

Associated to the admissible locally algebraic representation π one can construct, via locally analytic parabolic induction, an admissible locally analytic representation π^{la} that contains it. (There is one choice of π^{la} in the special case, while there are two choices in the principal series case.) Under a "non-critical slope" hypothesis, we show (Proposition 2.5) that the natural G-equivariant map $\pi \to \hat{\pi}$ extends uniquely to a G-equivariant map $\pi^{\mathrm{la}} \to \hat{\pi}$. The remainder of Section 2 elaborates on the relation of Proposition 2.5 to Conjecture 2.2, and to certain other ideas and results of [4]. The proof of Proposition 2.5 is given in Section 3.

Suppose now that the locally algebraic representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$ under consideration is of the form $W_{k-2} \otimes \pi_p$, where W_{k-2} is the contragredient to the $(k-2)$nd symmetric power of the standard representation of GL_2, and π_p is the local factor at p of a newform f of weight k. In this case one can use global considerations to verify Conjecture 2.2 for π. Indeed, in Section 4 we recall the construction of a unitary Banach space representations $\tilde{H}^1_c(K^p)$ of $\mathrm{GL}_2(\mathbb{Q}_p)$ (where K^p is the tame level of the newform f giving rise to π), which has the property that π embeds into $\tilde{H}^1_c(K^p)$. (In fact we get two embeddings, unique up to multiplication by a scalar, one for each choice of sign; see Proposition 4.3.) Since $\hat{\pi}$ is the universal unitary completion of π, the existence of this embedding certainly implies that $\hat{\pi} \neq 0$. Furthermore, assuming that we can find a p-stabilized newform g attached to f (the newform giving rise to π) that is of non-critical slope, Proposition 2.5 shows that the embeddings of π into $\tilde{H}^1_c(K^p)$ extend to embeddings of π^{la} into this space (where the choice of π^{la} is adapted to the choice of g). Using these embeddings, we are able to give a new construction of the p-adic L-function attached to g (Proposition 4.9).

In [5], Breuil has made a detailed investigation of the closure of the image of π in $\tilde{H}^1_c(K^p)$, in the case when π_p is special. In particular, he has shown that this image is classified by a certain invariant $\mathcal{L}$ (the $\mathcal{L}$-invariant of Darmon and Orton discussed above). In Section 5 we give a proof of the Mazur-Tate-Teitelbaum exceptional zero conjecture for such f using this description of the $\mathcal{L}$-invariant, and the viewpoint on p-adic L-functions developed in Section 4.

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1. Unitary completions

In this section we generalize and axiomatize some constructions from [4, 5].

Throughout this section, we fix a p-adic reductive group G – more precisely, G is the group of $\mathbb{Q}_p$-valued points of a connected, reductive, linear algebraic group defined over $\mathbb{Q}_p$. We also fix a p-adic local field K that will serve as our field of coefficients. All locally convex topological vector spaces will be assumed to be defined over K (whether or not this is explicitly stated). We let $\mathcal{O}_K$ denote the ring of integers in K. We refer to [20] for an explanation of the terminology from p-adic functional analysis that we use.

As in [4], a G-representation on a K-Banach space U will be called unitary if the topology of U may be defined by a G-invariant norm.

Definition 1.1. *Let V be a locally convex topological K-vector space equipped with a continuous G-action, and let U be a K-Banach space equipped with a unitary G-action. We say that a given continuous K-linear G-equivariant continuous map $V \to U$ realizes U as a universal unitary completion of V if any continuous K-linear G-equivariant map $V \to W$, where W is a K-Banach space equipped with a unitary G-action, factors uniquely through the given map $V \to U$.*

The usual arguments with universal properties shows that U (equipped with its map $V \to U$) is unique up to unique isomorphism. We will denote U by $\hat{V}$ if it exists, and (by abuse of language) sometimes refer to $\hat{V}$ as the universal unitary completion of V. (There should be no confusion with the traditional notation for the completion of V, since all V that we have cause to consider will be complete.) We will refer to the universal continuous G-equivariant map

$$V \to \hat{V} \tag{1.2}$$

as the canonical map from V to $\hat{V}$. If we take W in Definition 1.1 to be the closure of the image of V under the canonical map, and apply the universal mapping property of $\hat{V}$, then we find that $W = \hat{V}$, and thus that the canonical map (1.2) has dense image.

It is obvious that the formation of $\hat{V}$ (when it exists) is functorial in V.

Lemma 1.3. *The G-representation V admits a universal unitary completion if and only if the set of commensurability classes of G-invariant open lattices*[1] *in V,*

[1] In the sense used in [20] – i.e. a lattice in a K-vector space is a spanning $\mathcal{O}_K$-submodule. Note that, with this usage, a lattice need not be separated.

ordered by inclusion, contains a minimal element (i.e. if V contains a G-invariant open lattice that is absorbed by any other G-invariant open lattice).

Proof. This is clear. Briefly, if V does admit a universal unitary completion $\hat{V}$, then we take the required lattice to be the preimage under the canonical map of the unit ball of $\hat{V}$ (with respect to some G-invariant norm defining the topology of $\hat{V}$). Conversely, given such a lattice L, the completion of V with respect to the gauge of L is a universal unitary completion of V. □

It is quite possible for a G-representation on a non-zero convex space V to admit the zero space as its universal unitary completion. Indeed, it is a non-trivial question in general to determine if $\hat{V}$ is non-zero. Proposition 1.5 below gives a useful necessary and sufficient condition for this to be the case.

Let $\mathcal{C}(G, K)$ denote the space of continuous K-valued functions on G, equipped with its right regular G-action, and let $\mathcal{C}^b(G, K)$ denote the G-invariant subspace of $\mathcal{C}(G, K)$ consisting of bounded functions. The sup norm makes $\mathcal{C}^b(G, K)$ a unitary Banach G-representation.

Lemma 1.4. *If U is a non-zero unitary Banach G-representation, then there is a non-zero continuous G-equivariant map $U \to \mathcal{C}^b(G, K)$ (where the target is endowed with the topology induced by sup-norm).*

Proof. Let u' be a non-zero element of the dual space U'. By Frobenius reciprocity, u determines a non-zero continuous G-equivariant map $U \to \mathcal{C}(G, K)$; explicitly, an element $u \in U$ maps to the function $f_u : g \mapsto \langle u', gu \rangle$ on G. If the G-action on U is unitary, one sees that this map factors through the inclusion $\mathcal{C}^b(G, K) \subset \mathcal{C}(G, K)$, and that the resulting non-zero map $U \to \mathcal{C}^b(G, K)$ is continuous. □

Proposition 1.5. *If V is a convex K-vector space equipped with a continuous G-action that admits a universal unitary completion, then $\hat{V}$ is non-zero if and only if there is a non-zero continuous G-equivariant map $V \to \mathcal{C}^b(G, K)$.*

Proof. If such a map exist, then it must factor through (1.2), forcing $\hat{V}$ to be non-zero. Conversely, if $\hat{V} \neq 0$, then apply Lemma 1.4 to $\hat{V}$ to obtain a non-zero continuous G-equivariant map $\hat{V} \to \mathcal{C}^b(G, K)$. Composing this with (1.2) we obtain the required non-zero continuous G-equivariant map $V \to \mathcal{C}^b(G, K)$. □

Suppose now that V is topologically irreducible as a G-representation. If $\hat{V}$ exists and is non-zero, then the map (1.2) is non-zero (since it has dense image) and G-equivariant, and so must be injective (since its source is topologically irreducible and its target is Hausdorff). Pulling back a G-invariant norm from $\hat{V}$ to V, we see that V admits a G-invariant norm. In particular, if V has a central character, then we see that this central character must be unitary.

The Jacquet module functors defined in [11] provide additional necessary conditions for $\hat{V}$ (when it exists) to be non-zero. We suppose that V is a compact type locally analytic G-representation for which $\hat{V}$ exists. Let P be a parabolic subgroup of G, with unipotent radical N and Levi quotient M, and let $J_P(V)$ denote the Jacquet module (with respect to P) of V, as defined in [11, §3.4] – it is a compact type convex K-space equipped with a locally analytic action of M. Let N_0 be some compact open subgroup of N, let Z_M denote the center of M, and write $Z_M^+ := \{z \in Z_M \mid zN_0z^{-1} \subset N_0\}$. Let δ denote the modulus character of P (which is trivial on N, and so induces a character of $M = P/N$, which we also denote by δ; concretely, $\delta(m) = [mN_0m^{-1} : N_0])$.

Lemma 1.6. *Let χ be a locally analytic K-valued character of Z_M. If, in the above situation, the χ-eigenspace of $J_P(V)$ is non-zero, and if $\hat{V} \neq 0$, then $\mid \chi(z)\delta^{-1}(z) \mid \leq 1$ for all $z \in Z_M^+$.*

Proof. As already noted, the assumption that $\hat{V} \neq 0$ implies that V admits a G-invariant norm. The lemma thus follows from [11, Prop. 3.4.9, Lem. 4.4.2]. □

We now return to the general situation, focusing on the case when V is reflexive.

Lemma 1.7. *If V is a reflexive convex K-space equipped with a continuous G-representation, then the set of commensurability classes of G-invariant open lattices in V contains a minimal element if and only if the set of commensurability classes of G-invariant closed and bounded (equivalently compact, since V is reflexive* [20, Prop. 15.3 (iii)]*) $\mathcal{O}_K$-submodules of V_b' contains a maximal element. (We equip V_b' with the contragredient G-action.)*

Proof. This follows immediately from the theory of bipolarity [20, §13]. □

In the context of Lemma 1.7, the G-action on V_b' is again continuous, and so we may reverse the roles of V and V_b'. If V_b' admits a universal unitary completion, then Lemma 1.7 shows that V admits a compact G-invariant $\mathcal{O}_K$-submodule B that is maximal (up to commensurability); equivalently, B absorbs any G-invariant bounded $\mathcal{O}_K$-submodule of V.

Definition 1.8. *In the preceding situation, define $V^{\mathrm{b}} := K \otimes_{\mathcal{O}_K} B$, equipped with the finest convex topology for which the natural map $B \to V^{\mathrm{b}}$ is continuous. (Since any two choices of B are commensurable, we see that V^{b} is well-defined, up to natural isomorphism, independent of the choice of B.)*

The G-action on B induces a continuous G-action on V^{b}. The formation of V^{b} (when it exists) is evidently functorial in V.

We have the canonical map

$$V_b' \to (V_b')^{\widehat{}}, \tag{1.9}$$

while the inclusion of B in V induces a G-equivariant continuous K-linear map

$$V^{\mathrm{b}} \to V. \tag{1.10}$$

Lemma 1.11. *Let V be a reflexive convex K-space equipped with a continuous G-action for which V_b' admits a universal unitary completion.*

(i) The map (1.10) is continuous and injective.

(ii) The maps (1.9) and (1.10) are mutually dual.

Proof. Claim (i) is immediate. To prove (ii), consider the strong dual of (1.10):

$$V_b' \to (V^{\mathrm{b}})_b'. \tag{1.12}$$

The results of [22] show that $(V^{\mathrm{b}})_b'$ is a Banach space, with unit ball the polar of B, and that V^{b} is naturally identified (via double duality) with $((V^{\mathrm{b}})_b')'$. Thus (1.10) and (1.12) are mutually dual. The preimage of the unit ball of $(V^{\mathrm{b}})_b'$ under (1.12) is the polar of B in V_b', which by construction of B (and the theory of bipolarity) coincides with the preimage of the unit ball of $(V_b')^{\widehat{}}$ under (1.9). Thus (1.12) and (1.9) are naturally isomorphic. □

Definition 1.13. *If V is a K-Fréchet space equipped with a continuous G-action, then a continuous semi-norm q on V is called a generating semi-norm if the set of translates gq of q (as g ranges over all elements of G) serves to define the topology of V. (Here $gq(v) := q(g^{-1}v)$ for all $v \in V$.)*

This definition could be applied to arbitrary locally convex K-spaces equipped with a continuous G-action. However, if G_0 is a compact open subgroup of G, then any continuous semi-norm on V is bounded above by a G_0-invariant continuous semi-norm [10, Lem. 6.5.4]. Thus if V admits a generating semi-norm q, we may assume that q is G_0-invariant, and so conclude (since G/G_0 is countable) that the topology of V is definable by a countable set of semi-norms. Hence V is metrizable. If V is furthermore reflexive, then it is necessarily Fréchet [3, Cor., p. IV.23].

Proposition 1.14. *If V is reflexive K-Fréchet space equipped with a continuous G-action that admits a generating semi-norm, then V'_b admits a universal unitary completion (and so V^{b} is defined).*

Proof. Let q be a generating semi-norm, and set $B_q = \{v \in V \mid q(gv) \leq 1 \text{ for all } g \in G\}$. By definition B_q is a bounded G-invariant $\mathcal{O}_K$-submodule of V. Since the G-action on V is continuous, B_q is evidently also closed. If B is any bounded G-invariant $\mathcal{O}_K$-submodule of V, and we choose $a \in K^\times$ so that $\mid s(v) \mid \leq \mid a \mid$ for all $v \in B$, then clearly $a^{-1}B \subset B_q$. Thus B_q absorbs all bounded G-invariant $\mathcal{O}_K$-submodules of V, and the proposition follows from Lemmas 1.3 and 1.7. □

Lemma 1.15. *If V is a K-Fréchet space equipped with a continuous G-action and W is a closed G-invariant subspace of V, then the restriction of any generating semi-norm on V to W is a generating semi-norm on W.*

Proof. This follows from the fact that any set of continuous semi-norms on V that defines the topology of V restricts to a set of continuous semi-norms on W that defines the topology of W (by definition of the subspace topology on W). □

Example A. We will show that universal unitary completions always exist for finitely generated G-representations equipped with their finest convex topology.

Definition 1.16. *If V is a K-vector space equipped with a representation of G, then we say that a G-invariant lattice in V is of finite type if it is finitely generated as an $\mathcal{O}_K[G]$-module.*

Clearly V contains a finite type lattice if and only if it is finitely generated (as a $K[G]$-module). Furthermore, any two finite type lattices are then commensurable.

Proposition 1.17. *If V is a finitely generated G-representation equipped with its finest convex topology, then the completion of V with respect to any finite type lattice of V is a universal unitary completion of V.*

Proof. This follows from Lemma 1.3, the fact that any lattice in V is open (since V is equipped with its finest convex topology), and the fact that any G-invariant lattice in V contains a finite-type sublattice (since V is finitely generated). □

Recall that a locally algebraic representation V of G is called admissible if it is admissible as a locally analytic representation of G; see [10, Def. 6.3.9] for example. Such a representation V is necessarily equipped with its finest convex topology [10, Cor. 6.3.7]. If V is irreducible, then it admits a central character, and as remarked above, a necessary condition for $\hat{V} \neq 0$ is that this central character be unitary.

Proposition 1.18. *Let V be an irreducible admissible locally algebraic representation of G, and write $V \xrightarrow{\sim} U \otimes W$, where U is an admissible smooth representation of G and W is a finite dimensional algebraic representation of G (as we may, by* [10, Prop. 6.3.11]*). If U is absolutely cuspidal, then $\hat{V} \neq 0$ if and only if the central character of V is unitary. (Recall that U is said to be absolutely cuspidal if it cannot be embedded into the induction from P of a smooth M-representation, for any proper parabolic subgroup P of G with Levi factor M* [6, §5]*.)*

Proof. The "only if" direction has already been noted (and does not require U to be absolutely cuspidal). Suppose now that the central character of V is unitary and that U is absolutely cuspidal. If we fix non-zero elements $\check{u}$ and $\check{W}$ in the smooth dual to U and the dual to W respectively, then $\check{u} \otimes \check{w}$ is in the topological dual to V. By Frobenius reciprocity, the dual vector $\check{u} \otimes \check{w}$ induces a G-equivariant map

$$V \to \mathcal{C}(G, K); \tag{1.19}$$

explicitly, $v \in V$ maps to the function $f_v : g \mapsto \langle \check{u} \otimes \check{w}, gv \rangle$. Note that (1.19) is non-zero, since $\check{u} \otimes \check{w} \neq 0$.

Under the hypothesis of the proposition, we claim that (1.19) factors through the inclusion of $\mathcal{C}^b(G, K)$ into $\mathcal{C}(G, K)$, and thus induces a map $V \to \mathcal{C}^b(G, K)$ (which is necessarily continuous, since V is equipped with its finest convex topology). Indeed, it suffices to show that f_v is bounded when v is of the form $u \otimes w$ for some $u \in U$ and $w \in W$. In this case we see that $f_v(g) = \langle \check{u}, gu \rangle \langle \check{w}, gw \rangle$ for all $g \in G$. Since U is absolutely cuspidal, the matrix coefficient $\langle \check{u}, gu \rangle$ is compactly supported modulo the center of G [6, Thm. 5.3.1]. Thus the same is true of f_v. Since V has a unitary central character, we conclude that f_v is bounded, as claimed. The proposition now follows from Proposition 1.5. □

It is immediate from the definition of absolutely cuspidal, together with [11, Prop. 4.3.4 (ii)], that the smooth factor of an irreducible admissible locally algebraic representation V is absolutely cuspidal if and only if $J_P(V) = 0$ for all proper parabolic subgroups P of G. The preceding proposition thus shows that in the extreme case when V has no non-vanishing Jacquet modules, the necessary condition of Lemma 1.6 is also sufficient.[2]

If the smooth factor of V is not absolutely cuspidal, then the question of whether or not $\hat{V}$ is non-zero is more subtle (as the necessary condition of Lemma 1.6 shows). This question, in the case when $G = \mathrm{GL}_2(\mathbb{Q}_p)$, will be considered in detail in the next section.

Example B. In [4, 5], the spaces $\mathrm{O}(k)$ and $\mathrm{O}(k, \mathcal{L})$ are defined (for $k \geq 2$ and $\mathcal{L} \in K$). The first is the space of rigid analytic functions on the p-adic upper half-plane $\Omega := \mathbb{C}_p \setminus \mathbb{Q}_p$, equipped with a "weight k" action of $\mathrm{GL}_2(\mathbb{Q}_p)$. The second is the space of log-rigid analytic functions on Ω, equipped with a "weight $2-k$" action of $\mathrm{GL}_2(\mathbb{Q}_p)$; one takes the branch of the p-adic logarithm to be $\log_{\mathcal{L}}$, where $\log_{\mathcal{L}}(p) = \mathcal{L}$. These are nuclear (and so reflexive) Fréchet spaces equipped with continuous actions of $\mathrm{GL}_2(\mathbb{Q}_p)$, and for fixed k and $\mathcal{L}$, differentiating $k-1$ times induces a short exact sequence

$$0 \to W_{k-2} \otimes_K |\det|^{-(k-2)/2} \to \mathrm{O}(k, \mathcal{L}) \to \mathrm{O}(k) \to 0; \tag{1.20}$$

[2] As a referee has emphasized, in the context of the Proposition 1.18 it is easy to see that $\hat{V}$ is not an admissible Banach space representation of G. In light of Breuil's ideas regarding "p-adic Local Langlands", one expects $\hat{V}$ to admit a rich family of admissible quotients.

here W_{k-2} is the contragredient to the $(k-2)$nd symmetric power of the standard two dimensional representation of $GL_2(\mathbb{Q}_p)$, embedded (with the indicated twist) into $\mathrm{O}(k, \mathcal{L})$ as the subspace of polynomial functions on Ω of degree at most $k-2$.

One easily checks that if U is a covering of $\mathbb{Q}_p$ in $\mathbb{C}_p$ satisfying condition (20) of [4, p. 34], then the formula $q(F) := ||F_{|\Omega_U}||$ (using the notation of [4]) defines a generating semi-norm (in fact norm) on each of $\mathrm{O}(k)$ and $\mathrm{O}(k, \mathcal{L})$, and thus that $\mathrm{O}(k)^{\mathrm{b}}$ and $\mathrm{O}(k, \mathcal{L})^{\mathrm{b}}$, as defined above, coincide with the same spaces as defined in [4] and [5]. For $k > 2$ and any choice of $\mathcal{L}$, the map

$$\mathrm{O}(k, \mathcal{L})^{\mathrm{b}} \to \mathrm{O}(k)^{\mathrm{b}}$$

induced by the surjection of (1.20) is a closed embedding;[3] when $k = 2$ it is surjective (with kernel equal to $W_0 \xrightarrow{\sim} K$).

Example C. Suppose that G is quasi-split, and let B be a Borel subgroup of G. Let N denote the unipotent radical of B, and fix a maximal torus T of G contained in B, so that $B = TN$. Let $\overline{B}$ be the opposite Borel to B, chosen so that $B \bigcap \overline{B} = T$. Let $\chi : T \to K^\times$ be a locally analytic character of T, which we also regard as a character of $\overline{B}$ via the natural projection $\overline{B} \to T$, and consider the locally analytic parabolic induction

$$\mathrm{Ind}_{\overline{B}}^{G} \chi := \{ f \in \mathcal{C}^{\mathrm{la}}(G, K) \mid f(\overline{b}g) = \chi(\overline{b}) f(g) \text{ for all } \overline{b} \in \overline{B}, g \in G \}.$$

The right regular G-action endows $\mathrm{Ind}_{\overline{B}}^{G} \chi$ with a continuous action of G.

Proposition 1.21. *The G-representation $\mathrm{Ind}_{\overline{B}}^{G} \chi$ is a strongly admissible locally analytic representation of G (in the sense of* [21]*), whose strong dual (which is thus a nuclear, and hence reflexive, Fréchet space) admits a generating semi-norm.*

Proof. If G_0 is a hyperspecial maximal compact open subgroup of G, then $G = \overline{B} G_0$, and so restriction of functions from G to G_0 induces a G_0-equivariant closed embedding $\mathrm{Ind}_{\overline{B}}^{G} \chi \to \mathcal{C}^{\mathrm{la}}(G_0, K)$. Since $\mathcal{C}^{\mathrm{la}}(G_0, K)$ is a strongly admissible G_0-representation (essentially by definition; see the discussion following [10, Def. 6.2.1], for example), we see that the same is true of its closed subrepresentation $\mathrm{Ind}_{\overline{B}}^{G} \chi$. Since a locally analytic G-representation is strongly admissible if and only if it is strongly admissible as a G_0-representation (again by definition), we see that $\mathrm{Ind}_{\overline{B}}^{G} \chi$ is a strongly admissible locally analytic G-representation. In particular, it is of compact type, and so its strong dual is a nuclear Fréchet space.

Let $(X, \mathbb{X}, \phi)$ be a chart centered at the identity of the p-adic analytic group N; that is, X is a compact open neighbourhood of the identity of N, $\mathbb{X}$ is an affinoid rigid analytic space over $\mathbb{Q}_p$ isomorphic to a closed polydisk, and ϕ is a locally analytic isomorphism $\phi : X \xrightarrow{\sim} \mathbb{X}(\mathbb{Q}_p)$. To save writing, we will also write simply $\mathbb{X}$ to denote this chart. As in [10, §2.1], we let $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K)$ denote the Banach space of rigid analytic K-valued functions on $\mathbb{X}$; the isomorphism ϕ allows us to see $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K)$ as a BH-subspace of $\mathcal{C}^{\mathrm{la}}(X, K)$ (in the sense of [10, Def. 1.1.4]), and thus also of $\mathcal{C}_c^{\mathrm{la}}(N, K)$ (since $\mathcal{C}^{\mathrm{la}}(X, K)$ is a closed subspace of $\mathcal{C}_c^{\mathrm{la}}(N, K)$). If $t \in T$, then conjugation by t induces a locally analytic automorphism of N, which we denote

[3] That one obtains an injection follows from the fact that $W_{k-2} \otimes_K |\det|^{-(k-2)/2}$ admits no $\mathrm{GL}_2(\mathbb{Q}_p)$-invariant norm, when $k > 2$. That this injection is in fact a closed embedding is proved in [5, Thm. 3.3.3].

by σ_t. Thus $(tXt^{-1}, \mathbb{X}, \phi \circ \sigma_{t^{-1}})$ is also a chart of N. To save writing, we will denote this chart simply by $t\mathbb{X}t^{-1}$, and will denote the corresponding BH-subspace of $\mathcal{C}^{\mathrm{la}}_c(N, K)$ by $\mathcal{C}^{\mathrm{an}}(t\mathbb{X}t^{-1}, K)$. We remark that the theory of roots shows that the set of neighbourhoods of the identity $\{tXt^{-1}\}_{t \in T}$ is cofinal in the set of all neighbourhoods of the identity.

The discussion of [13, §5] shows that $\mathcal{C}^{\mathrm{la}}_c(N, K)$ embeds as a closed subspace of $\mathrm{Ind}_{\overline{B}}^G \chi$, and hence that $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K)$ embeds as a BH-subspace of $\mathrm{Ind}_{\overline{B}}^G \chi$. One checks that if $t \in T$, then

$$t\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K) = \mathcal{C}^{\mathrm{an}}(t\mathbb{X}t^{-1}, K). \tag{1.22}$$

(Here the action of t on the left-hand side is defined via the action of t on the ambient G-representation $\mathrm{Ind}_{\overline{B}}^G \chi$.) The final sentence of the preceding paragraph, together with (1.22) and the fact that G acts transitively on the space $\overline{B}\backslash G$, shows that $\mathrm{Ind}_{\overline{B}}^G \chi = \sum_{g \in G} g\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K)$. Thus the gauge of the polar in $(\mathrm{Ind}_{\overline{B}}^G \chi)'_b$ of the unit ball $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, \mathcal{O}_K)$ of $\mathcal{C}^{\mathrm{an}}(\mathbb{X}, K)$ is a generating semi-norm for $(\mathrm{Ind}_{\overline{B}}^G \chi)'_b$. □

2. Completing parabolically induced representations of $\mathrm{GL}_2(\mathbb{Q}_p)$.

Take $G = \mathrm{GL}_2(\mathbb{Q}_p)$, and K as in Section 1. Let B denote the standard upper triangular Borel in $\mathrm{GL}_2(\mathbb{Q}_p)$, let T denote the diagonal maximal torus in B, and let $\overline{B}$ denote the lower triangular Borel (so that $\overline{B} \bigcap B = T$). Let χ be a locally algebraic K-valued character of T of the form $\chi = \theta\psi_{k-2}$, for some $k \geq 2$, where θ is smooth and ψ_{k-2} is the highest weight of the irreducible representation W_{k-2} of G (the contragredient to the $(k-2)$nd symmetric power of the standard representation of G) with respect to B. (Explicitly, $\psi_{k-2} : \begin{pmatrix} a & \\ & d \end{pmatrix} \mapsto d^{-(k-2)}$.) Let $I(\chi)$ denote the locally algebraic induction of χ from $\overline{B}$ to G; alternatively, $I(\chi)$ is the tensor product over K of W_{k-2} and the smooth induction of θ from $\overline{B}$ to G.

Write θ in the form $\theta : \begin{pmatrix} a & \\ & d \end{pmatrix} \mapsto \theta_1(a)\theta_2(d)$, for two smooth characters θ_1 and θ_2 of $\mathbb{Q}_p^\times$. Write $\alpha = \theta_1(p)$ and $\beta = p\theta_2(p)$. The central character of $I(\chi)$ takes $\begin{pmatrix} p & \\ & p \end{pmatrix}$ to $\alpha\beta p^{-(k-1)}$.

Let $\theta' \begin{pmatrix} a & \\ & d \end{pmatrix} = \mid d/a \mid \theta_2(a)\theta_1(d)$, and define χ' to be the locally algebraic character $\chi' = \theta'\psi_{k-2}$. (Note that the α and β for χ' are equal to the β and α for χ.)

If $\theta_{1|\mathbb{Z}_p^\times} = \theta_{2|\mathbb{Z}_p^\times}$ and $\alpha/\beta = p^{\pm 1}$, then we say that we are in the special case. In this case, the representation $I(\chi)$ is reducible. If $\alpha/\beta = p^{-1}$, then $I(\chi)$ contains a unique finite dimensional subrepresentation, isomorphic to a twist of W_{k-2}, and the quotient is isomorphic to a twist of W_{k-2} tensored with St (the Steinberg representation), which we denote by $\pi(\chi)$. If $\alpha/\beta = p$, then $I(\chi)$ has a unique finite dimensional quotient, isomorphic to a twist of W_{k-2}, and the kernel of the projection onto this quotient is isomorphic to $\pi(\chi')$. From now on, when we are in the special case we always assume that χ is chosen so that $\alpha/\beta = p^{-1}$.

If $\alpha/\beta \neq p^{\pm 1}$, then we say that we are in the principal series case. In this case, the representation $I(\chi)$ is irreducible, and we write $\pi(\chi) = I(\chi)$. The theory of intertwining operators shows that in this case there is an isomorphism $\pi(\chi) \xrightarrow{\sim} \pi(\chi')$.

We let $\hat{\pi}(\chi)$ denote the completion of $\pi(\chi)$ with respect to a finite type lattice. Proposition 1.17 shows that $\hat{\pi}(\chi)$ is the universal unitary completion of $\pi(\chi)$.

Lemma 2.1. *The following two conditions are necessary in order for $\hat{\pi}(\chi)$ to be non-zero:*

(i) The central character of $\pi(\chi)$ is unitary; that is, $\mid \alpha\beta \mid = \mid p^{k-1} \mid$.

(ii) Each of α and β lie in $\mathcal{O}_K$.

Proof. It was observed in Section 1 that (i) is a necessary condition for π to admit a non-zero universal unitary completion. In the special case, since $\alpha/\beta = p^{-1}$, it is clear that (i) implies (ii). Thus we assume for the remainder of the proof that we are in the principal series case. We will compute $J_B(I(\chi))$, and apply Lemma 1.6.

Since $I(\chi) \xrightarrow{\sim} (\mathrm{Ind}_{\overline{B}}^G \theta)_{\mathrm{sm}} \otimes_K W_{k-2}$ (where the left-hand factor is the smooth parabolic induction of θ from $\overline{B}$ to G), it follows from [11, Props. 4.3.4 (ii), 4.3.6] that there is an isomorphism of T-representations $J_B(I(\chi)) \xrightarrow{\sim} J_B((\mathrm{Ind}_{\overline{B}}^G \theta)_{\mathrm{sm}}) \otimes \psi_{k-2}$. The theory of Jacquet modules of smooth representations [6] shows that $J_B((\mathrm{Ind}_{\overline{B}}^G \theta)_{\mathrm{sm}})$ is two dimensional, and that it has a non-zero $\theta\delta$-eigenspace, and also a non-zero $\theta'\delta$-eigenspace. (Below we will give a direct proof that these eigenspaces are non-zero.) Thus $J_B(I(\chi))$ has non-zero $\chi\delta$- and $\chi'\delta$-eigenspaces.

If we take the compact open subgroup N_0 of Lemma 1.6 to be $\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{Z}_p\}$, then we see that the matrix $t = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ conjugates N_0 into itself. Applying Lemma 1.6 twice, with the character labelled χ in the statement of that result taken to be $\chi\delta$ and $\chi'\delta$ in turn, we find that $\mid \alpha \mid = \mid \chi(t) \mid \leq 1$ and also that $\mid \beta \mid = \mid \chi'(t) \mid \leq 1$. This proves the lemma.

We finish with a direct proof that the $\theta\delta$-eigenspace of $J_B((\mathrm{Ind}_{\overline{B}}^G \theta)_{\mathrm{sm}})$ is non-zero. Let N denote the unipotent radical of B, and let $K(\theta)$ denote K regarded as a T-module via the character θ. The discussion of [13, §5] shows that there is a natural B-equivariant embedding $\mathcal{C}_c^{\mathrm{sm}}(N, K(\theta)) \to \mathrm{Ind}_{\overline{B}}^G \theta$. (The target denotes the locally analytic parabolic induction of θ from $\overline{B}$ to G.) Since θ is a smooth character of T, this embedding factors through the inclusion $(\mathrm{Ind}_{\overline{B}}^G \theta)_{\mathrm{sm}} \to \mathrm{Ind}_{\overline{B}}^G \theta$ of the smooth parabolic induction into the locally analytic parabolic induction. Applying the left-exact functor J_B, and taking into account [11, Lemma 3.5.2], we see that the $\theta\delta$-eigenspace of $J_B((\mathrm{Ind}_{\overline{B}}^G \theta)_{\mathrm{sm}})$ is non-zero. The theory of intertwining operators provides an isomorphism $(\mathrm{Ind}_{\overline{B}}^G \theta)_{\mathrm{sm}} \xrightarrow{\sim} (\mathrm{Ind}_{\overline{B}}^G \theta')_{\mathrm{sm}}$. Replacing θ by θ' in the preceding argument shows that the $\theta'\delta$-eigenspace of $J_B((\mathrm{Ind}_{\overline{B}}^G \theta)_{\mathrm{sm}})$ is also non-zero. □

The following conjecture is due to Breuil [4].

Conjecture 2.2. *The necessary conditions of Lemma 2.1 are also sufficient: if these conditions hold, then $\hat{\pi}(\chi)$ is non-zero.*

If we take into account the method of proof of Lemma 2.1, one can could rephrase the conjecture as saying that the necessary conditions of Lemma 1.6 are also sufficient; i.e. that the characters appearing in the Jacquet module of $\pi(\chi)$ are the only possible obstruction to its unitarizability.

This conjecture has been proved in the special case by Teitelbaum [23] and Grosse-Klönne [14], and recently in the unramified principal series case by Berger and Breuil [2]. We note that there is one subcase of the principal series case (what one might call the ordinary case) in which the conjecture is trivial to prove. Namely, if α is a p-adic unit, then the character χ is unitary, and so the continuous induction (as considered in [22], for example) $I^{\mathrm{cont}}(\chi)$ of χ from $\overline{B}$ to G is a unitary Banach space representation of G containing $I(\chi)$. (Because χ is unitary, the sup norm over $\overline{B}\backslash G$ is a well-defined G-invariant norm on $I^{\mathrm{cont}}(\chi)$.) Similarly, if $\mid \alpha \mid = \mid p^{k-1} \mid$, then β is a p-adic unit, and we may embed $I(\chi) \xrightarrow{\sim} I(\chi')$ in the unitary representation $I^{\mathrm{cont}}(\chi')$.

Given χ as above, we let $I^{\mathrm{la}}(\chi)$ denote the locally analytic induction of χ from $\overline{B}$ to G; by Proposition 1.21 this is a strongly admissible locally analytic representation of G. It contains $I(\chi)$ as a closed subrepresentation. We may identify $I^{\mathrm{la}}(\chi)$ with the space of locally analytic functions $f : \mathbb{Q}_p \to K$, satisfying an appropriate regularity condition at infinity, with the G-action defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x) = \frac{\theta_1(a+cx)}{\theta_2(a+cx)}(a+cx)^{k-2}\frac{\theta_2(ad-bc)}{(ad-bc)^{k-2}} f(\frac{b+dx}{a+cx}); \tag{2.3}$$

$I(\chi)$ is then identified with the subspace of functions that are locally polynomial of degree at most $k-2$.

In the principal series case we write $\pi^{\mathrm{la}}(\chi) = I^{\mathrm{la}}(\chi)$. In the special case (recall that in this case we assume that $\alpha/\beta = p^{-1}$) we let $\pi^{\mathrm{la}}(\chi)$ denote the quotient of $I^{\mathrm{la}}(\chi)$ by the invariant subspace consisting of polynomials of degree at most $k-2$. In both cases, the closed embedding $I(\chi) \to I^{\mathrm{la}}(\chi)$ induces a closed embedding

$$\pi(\chi) \to \pi^{\mathrm{la}}(\chi). \tag{2.4}$$

The following result gives some hint as to the structure of $\hat{\pi}(\chi)$, or more precisely, of its locally analytic vectors.

Proposition 2.5. *Suppose that χ satisfies the conditions of Lemma 2.1, and that furthermore $\mid \alpha \mid > \mid p^{k-1} \mid$. Then the following equivalent conditions hold:*

(i) Any G-equivariant K-linear map $\pi(\chi) \to U$, where U is a unitary Banach space representation, extends uniquely to a continuous K-linear G-equivariant map $\pi^{\mathrm{la}}(\chi) \to U$.

(ii) The natural map $\pi(\chi) \to \hat{\pi}(\chi)$ extends uniquely to a continuous K-linear G-equivariant map $\pi^{\mathrm{la}}(\chi) \to \hat{\pi}(\chi)$.

(iii) The map (2.4) induces an isomorphism of universal unitary completions $\hat{\pi}(\chi) \xrightarrow{\sim} \hat{\pi}^{\mathrm{la}}(\chi)$.

We give our proof of this result in the following section. In [2] Berger and Breuil give an explicit description of $\hat{\pi}(\chi)$ (as a quotient of a certain parabolic induction of χ – see p. 25, Def. 4.2.4, and Thm. 4.3.1 of [2]), using which they are able to give a direct proof of this result [2, p. 42]. Nevertheless, it seems worthwhile to give our original proof, since it is more representation theoretic in nature than that of [2], and it exhibits an interesting relationship between the classical technique of Amice-Vélu and Vishik and the representation theory of parabolically induced representations.

In the remainder of this section, we elaborate further on Conjecture 2.2, Proposition 2.5, and their relations to some of the ideas and results of [4, 5], in each of the two cases: special and principal series.

The special case. The special case has been discussed extensively by Breuil in his paper [4]. We recall the key points of that discussion.

In the special case, condition (i) of Lemma 2.1 implies condition (ii), and all the $\pi(\chi)$ satisfying (ii) (that are in the special case!) are unitary twists one of the other. The same is true of all the $\pi^{\mathrm{la}}(\chi)$, and these representations are all unitary twists of the representation $\Sigma(k)$ defined in [4]; write $\pi^{\mathrm{la}}(\chi) = \Sigma(k) \otimes_K \omega$, for some unitary character ω of $\mathrm{GL}_2(\mathbb{Q}_p)$. The results of Morita [17] then show that $\pi^{\mathrm{la}}(\chi)$ is dual to the representation $\mathrm{O}(k) \otimes_K \omega^{-1}$, where $\mathrm{O}(k)$ is the $\mathrm{GL}_2(\mathbb{Q}_p)$-representation considered in Section 1.

Dualizing the closed embedding (2.4) yields a surjection

$$\mathrm{O}(k) \otimes_K \omega^{-1} \to \pi(\chi)'_b,$$

which can be identified with (the twist by ω^{-1} of) the residue map from rigid analytic functions on Ω onto the space of harmonic cocycles on the Bruhat-Tits tree of $\mathrm{GL}_2(\mathbb{Q}_p)$ considered in [23]. Proposition 2.5 implies that this surjection induces an isomorphism

$$\mathrm{O}(k)^{\mathrm{b}} \otimes_K \omega^{-1} \xrightarrow{\sim} \pi(\chi)'^{\mathrm{b}}_b,$$

a result originally proved in [23] and [14]. As we noted above, in those papers it is furthermore proved that $\mathrm{O}(k)^{\mathrm{b}} \neq 0$, and thus that Conjecture 2.2 holds in the special case. (Note that this conjecture is obviously invariant under unitary twists).

Breuil has noted [4, Prop. 4.6.5] that when $k > 2$, the unitary Banach space representation $\hat{\pi}(\chi)$ is not an admissible Banach space G-representation (in the sense of [22]). For each $k > 2$ and $\mathcal{L} \in K$, dualizing the surjection of (1.20), and twisting by ω, yields a closed embedding of strongly admissible locally analytic $\mathrm{GL}_2(\mathbb{Q}_p)$-representations

$$\pi^{\mathrm{la}}(\chi) \to \pi^{\mathrm{la}}(\chi, \mathcal{L})$$

(where the target is defined to be the strong dual of $\mathrm{O}(k, \mathcal{L}) \otimes_K \omega^{-1}$). Dualizing the closed embedding $\mathrm{O}(k, \mathcal{L})^{\mathrm{b}} \otimes_K \omega \to \mathrm{O}(k)^{\mathrm{b}} \otimes_K \omega$ yields a surjection of unitary Banach space $\mathrm{GL}_2(\mathbb{Q}_p)$-representations

$$\hat{\pi}(\chi) \xrightarrow{\sim} (\mathrm{O}(k)^{\mathrm{b}} \otimes_K \omega^{-1})' \to (\mathrm{O}(k, \mathcal{L})^{\mathrm{b}} \otimes_K \omega^{-1})'.$$

Let $\hat{\pi}(\chi, \mathcal{L})$ denote the target of this surjection. (This is equal to $B(k, \mathcal{L}) \otimes_K \omega$, where $B(k, \mathcal{L})$ is the unitary Banach space $\mathrm{GL}_2(\mathbb{Q}_p)$-representation defined in [4].)

Breuil makes the following conjecture [4, Conj. 1.1.8]:

Conjecture 2.6. *(i) The natural map $\pi^{\mathrm{la}}(\chi, \mathcal{L}) \to \hat{\pi}(\chi, \mathcal{L})$ identifies the source with the space of locally analytic vectors in the target.*

(ii) The unitary Banach space $\mathrm{GL}_2(\mathbb{Q}_p)$-representation $\hat{\pi}(\chi, \mathcal{L})$ is admissible.

Note that part (i) of this conjecture in particular implies that $\hat{\pi}(\chi, \mathcal{L})$ is non-zero. Together with part (ii), it furthermore implies that $\hat{\pi}(\chi, \mathcal{L})$ is topologically irreducible [4, Thm. 1.1.9]. In fact, Colmez [8] has recently proved that $\hat{\pi}(\chi, \mathcal{L})$

is non-zero, admissible, and topologically irreducible for all $\mathcal{L}$ (without, however, proving part (i) of the conjecture).

Thus for each $\pi(\chi)$ in the special case with $k > 2$, there exists a family of topologically irreducible quotients of $\hat{\pi}(\chi)$ depending on the parameter $\mathcal{L}$. According to Breuil's p-adic local Langlands philosophy, these representations $\hat{\pi}(\chi, \mathcal{L})$ correspond to those irreducible potentially semi-stable representations of $G_{\mathbb{Q}_p}$ with Hodge-Tate weights $0, k-1$ that are not potentially crystalline. (Such representations are classified by their associated Weil group representation and Hodge-Tate weights, encoded in the character χ, together with their $\mathcal{L}$-invariant, which should be encoded in the parameter $\mathcal{L}$ on which $\hat{\pi}(\chi, \mathcal{L})$ depends.)

When $k = 2$, one may make analogous constructions to those above so as to obtain for each $\mathcal{L} \in K$ a closed embedding of strongly admissible locally analytic $\mathrm{GL}_2(\mathbb{Q}_p)$-representations $\pi^{\mathrm{la}}(\chi) \to \pi^{\mathrm{la}}(\chi, \mathcal{L})$ and a map $\hat{\pi}(\chi) \to \hat{\pi}(\chi, \mathcal{L})$ of unitary Banach space $\mathrm{GL}_2(\mathbb{Q}_p)$-representations. In this case, though, this latter map is not a surjection, but rather is a closed embedding (since the map $\mathrm{O}(2, \mathcal{L})^{\mathrm{b}} \to \mathrm{O}(2)^{\mathrm{b}}$ is not a closed embedding, but is rather a surjection). When $k = 2$, the analogue of Conjecture 2.6 was proved by Breuil [4, §4.5]; note though that $\hat{\pi}(\chi, \mathcal{L})$ is not topologically irreducible in this case (for example, because it contains $\hat{\pi}(\chi)$ as a closed subrepresentation).

We conclude our discussion of the special case by noting that the space $\pi^{\mathrm{la}}(\chi, \mathcal{L})$ has a concrete interpretation as a space of functions [4, Prop. 2.2.1]. As remarked above, we may identify $\pi^{\mathrm{la}}(\chi)$ with the space of locally analytic functions on $\mathbb{Q}_p$, having a pole of order at most $k-2$ at infinity, and with the $\mathrm{GL}_2(\mathbb{Q}_p)$-action being given by the formula (2.3). We may identify $\pi^{\mathrm{la}}(\chi, \mathcal{L})$ with the space of locally analytic functions on $\mathbb{Q}_p$, whose singularity at infinity is at worst the sum of a pole of order $k-2$ and a term of the form $P(x)\log_{\mathcal{L}}(x)$, where $P(x)$ is a polynomial of degree at most $k-2$ (with $\mid x \mid >> 0$). The $\mathrm{GL}_2(\mathbb{Q}_p)$-action is given by a modification of the formula (2.3): if a function $f(x)$ has a logarithmic singularity of the form $P(x)\log_{\mathcal{L}} x$ for $\mid x \mid >> 0$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} f(x)$ is defined by adding the following term to the expression in (2.3):

$$-\frac{1}{2}\frac{\theta_1(a+cx)}{\theta_2(a+cx)}(a+cx)^{k-2}\frac{\theta_2(ad-bc)}{(ad-bc)^{k-2}}P(\frac{b+dx}{a+cx})\log_{\mathcal{L}}(\frac{ad-bc}{(a+cx)^2}). \tag{2.7}$$

The principal series case. Suppose that $\pi(\chi)$ is in the principal series case, and that neither α nor β is a p-adic unit.

Proposition 2.5, together with the isomorphism $\pi(\chi) \xrightarrow{\sim} \pi(\chi')$ discussed above, shows that both $\pi^{\mathrm{la}}(\chi)$ and $\pi^{\mathrm{la}}(\chi')$ map into $\hat{\pi}(\chi)$. If $\pi^{\mathrm{la}}(\chi, \chi')$ denotes the sum of $\pi^{\mathrm{la}}(\chi)$ and $\pi^{\mathrm{la}}(\chi')$ amalgamated along their common closed subspace $\pi(\chi)$, then these maps glue to give a map

$$\pi^{\mathrm{la}}(\chi, \chi') \to \hat{\pi}(\chi). \tag{2.8}$$

The following conjecture is due to Breuil. It is extracted from the introduction to [4], and from a letter from Breuil to the author.

Conjecture 2.9. *(i) The map (2.8) identifies the source with the locally analytic vectors in the target.*

(ii) The unitary Banach space $\mathrm{GL}_2(\mathbb{Q}_p)$*-representation* $\hat{\pi}(\chi)$ *is admissible.*

Part (i) of this conjecture in particular implies that $\hat{\pi}(\chi) \neq 0$, i.e. Conjecture 2.2 for $\pi(\chi)$. An argument analogous to that which proves [4, Thm. 1.1.9] shows that this conjecture also implies the topological irreducibility of $\hat{\pi}(\chi)$. As was noted above, in the case when θ is unramified, Conjecture 2.2 has recently been proved [2]. In this reference Berger and Breuil furthermore prove that for such χ the unitary Banach space representation $\hat{\pi}(\chi)$ is admissible and topologically irreducible (without, however, establishing part (i) of Conjecture 2.9).

According to Breuil's local Langlands philosophy, the representations $\hat{\pi}(\chi)$ (for χ in the principal series case with α and β both non-units) correspond to those irreducible potentially crystalline representations of $G_{\mathbb{Q}_p}$ for which the associated representation of the Weil group is abelian.[4]

3. Proof of Proposition 2.5

We first note that Propositions 1.17, 1.21, and 1.14 (together with Lemma 1.15, in the special case) show that each of $\pi(\chi)$ and $\pi^{\mathrm{la}}(\chi)$ admit a universal unitary completion. The equivalence of the conditions in the statement of Proposition 2.5 is then clear.

We turn to proving (i). In fact, we will prove the analogous statement in which $\pi(\chi)$ and $\pi^{\mathrm{la}}(\chi)$ are replaced by $I(\chi)$ and $I^{\mathrm{la}}(\chi)$. In the principal series case this is simply a change of notation. In the special case, the representations $\pi(\chi)$ and $\pi^{\mathrm{la}}(\chi)$ are quotients of $I(\chi)$ and $I^{\mathrm{la}}(\chi)$ respectively by the same finite dimensional subspace, and so the statement for $I(\chi)$ and $I^{\mathrm{la}}(\chi)$ implies the corresponding statement for $\pi(\chi)$ and $\pi^{\mathrm{la}}(\chi)$.

We begin by recalling (from Example C of Section 1) that for any character $\chi = \theta\psi_{k-2}$ of T as in Section 2,

$$I^{\mathrm{la}}(\chi) := \{f \in \mathcal{C}^{\mathrm{la}}(G,K) \mid f(\overline{b}g) = \chi(\overline{b})f(g) \text{ for all } \overline{b} \in \overline{B}, g \in G\}.$$

(As always, we regard χ as a character of $\overline{B}$ via the natural projection $\overline{B} \to T$.) The support of any element of $I^{\mathrm{la}}(\chi)$ is clearly invariant under left-multiplication by elements of $\overline{B}$, and so may be identified with a closed subset of the quotient $\overline{B}\backslash G$. Since this quotient is compact, and since functions in $I^{\mathrm{la}}(\chi)$ are locally analytic, we see that the support of any element of $I^{\mathrm{la}}(\chi)$ is a compact open subset of $\overline{B}\backslash G$. If Ω is an open subset of $\overline{B}\backslash G$, then we let $I^{\mathrm{la}}(\chi)(\Omega)$ denote the closed subspace of $I^{\mathrm{la}}(\chi)$ consisting of functions whose support lies in Ω.

Let N denote the unipotent radical of B; we identify N with $\mathbb{Q}_p$ via the map $x \mapsto n_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. Write $N_0 := \{n_x \mid x \in \mathbb{Z}_p\} \subset N$. The embedding $N \to \overline{B}\backslash G$ realizes the source as an open subset of the target, and thus realizes N_0 as a compact open subset of the target.

If for any integer $r \geq 1$ we set

$$G_0(r) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p) \mid c \equiv 0 \bmod p^r\},$$

[4]Actually we have omitted the crystalline representation with Hodge-Tate weights $0, k-1$ with $k > 2$ for which a_p (the trace of Frobenius) is equal to $p^{(k-2)/2} + p^{k/2}$, as well as all of its twists. Breuil's philosophy accounts for these representations as well, in a rather surprising way – see [4, Rem. 1.3.4].

then $G_0(r)$ is a compact open subgroup of G that admits an Iwahori decomposition with respect to B and $\overline{B}$. Namely, if we write $\overline{N}(r) = \{\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \mid x \in p^r\mathbb{Z}_p\}$ and $T_0 := \mathrm{GL}_2(\mathbb{Q}_p) \bigcap T = \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in \mathbb{Z}_p^\times\}$, then $G_0(r) = N_0 T_0 \overline{N}(r)$.

Write $T^+ := \{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in T \mid \dfrac{a}{d} \in \mathbb{Z}_p\}$. If $t \in T^+$, then one sees that $tN_0t^{-1} \subset N_0$, while $t^{-1}\overline{N}(r)t \subset \overline{N}(r)$ for any $r \geq 1$. Taking into account these formulas, the inclusion $T_0 \subset T^+$, and the fact that $G_0(r)$ is a subgroup of G for each $r \geq 1$, we see that if for any $r \geq 1$ we write $G^+(r) := N_0T^+\overline{N}(r)$, then $G^+(r)$ is a submonoid of G. Write $B^+ := G^+(r) \bigcap B = N_0T^+$; it is a submonoid of B, independent of the choice of r, that generates B as a group.

One sees that $\overline{B}N_0G^+(1) \supset \overline{B}N_0$, and thus that $I^{\mathrm{la}}(\chi)(N_0)$ is a $G^+(1)$-invariant closed subspace of $I^{\mathrm{la}}(\chi)$. Write $I(\chi)(N_0) := I(\chi) \bigcap I^{\mathrm{la}}(\chi)(N_0)$ (the intersection taking place in the ambient space $I^{\mathrm{la}}(\chi)$).

Lemma 3.1. *Let U be a K-Banach space equipped with a continuous G-action. For any integer $r \geq 1$, the restriction maps $\mathcal{L}_G(I^{\mathrm{la}}(\chi), U) \to \mathcal{L}_{G^+(r)}(I^{\mathrm{la}}(\chi)(N_0), U)$ and $\mathcal{L}_G(I(\chi), U) \to \mathcal{L}_{G^+(r)}(I(\chi)(N_0), U)$ are both isomorphisms.*

Proof. Let V be any closed subspace of $I^{\mathrm{la}}(\chi)$ satisfying the following two conditions: (i) V is G-invariant; (ii) V is local: that is, if $F \in V$ and if Ω is a compact open subset of $\overline{B}\backslash G$, then the restriction $F_{|\Omega}$ of F to Ω also lies in V. (The two examples of such a V that we have in mind are $I^{\mathrm{la}}(\chi)$ itself and $I(\chi)$.) Then $V(N_0) := V \bigcap I^{\mathrm{la}}(\chi)(N_0)$ is $G^+(1)$-invariant, and we will prove that the restriction map

$$\mathcal{L}_G(V, U) \to \mathcal{L}_{G^+(r)}(V(N_0), U) \tag{3.2}$$

is an isomorphism for any $r \geq 1$.

Since the G-translates of N_0 cover $\overline{B}\backslash G$, and since V is local, we see that $V(N_0)$ generates V as a G-representation. The map (3.2) is thus certainly injective, and we must show that it is also surjective.

Let $V(N)$ denote the B-invariant closed subspace of V consisting of elements whose support is contained in N. Since V is local, the natural map $K[N] \otimes_{K[N_0]} V(N_0) \to V(N)$ is an isomorphism, and hence (taking into account the fact that B^+ generates B as a group) restriction induces an isomorphism

$$\mathcal{L}_B(V(N), U) \xrightarrow{\sim} \mathcal{L}_{B^+}(V(N_0), U). \tag{3.3}$$

Let $\mathcal{L}_B(V(N), U)^\circ$ denote the subspace of the source of (3.3) consisting of maps ϕ satisfying the following condition:

(3.4) For each $\overline{n} \in \overline{N}$, there exists a neighbourhood Ω of the identity in N such that, for any $F \in V$ whose support lies in Ω, the element $\overline{n}F$ lies in $V(N)$, and $\phi(\overline{n}F) = \overline{n}\phi(F)$.

Note that $\mathcal{L}_{G^+(r)}(V(N_0), U)$ is a subspace of the target of (3.3).

Claim 3.5. The image of $\mathcal{L}_{G^+(r)}(V(N_0), U)$ under the inverse of the isomorphism (3.3) (which consists of elements of the source whose restrictions to $V(N_0)$ are $\overline{N}(r)$-equivariant) is contained in $\mathcal{L}_B(V(N), U)^\circ$.

Choose $t \in T^+$ such that $\overline{n} \in t\overline{N}(r)t^{-1}$, and take $\Omega = tN_0t^{-1}$. Since $tG_0(r)t^{-1}$ is a subgroup of G, we see that tN_0t^{-1}, regarded as a compact open subset of $\overline{B}\backslash G$, is invariant under the action of $t\overline{N}(r)t^{-1}$. Thus if F is any element of V with support contained in Ω, the same is true of $\overline{n}F$. In particular, we see that $\overline{n}F \in V(N)$, as required.

Now let ϕ be an element of $\mathcal{L}_B(V(N), U)$ whose restriction to $V(N_0)$ is $\overline{N}(r)$-equivariant. If the support of F is contained in Ω, we compute that

$$\phi(\overline{n}F) = t\phi(t^{-1}\overline{n}tt^{-1}F) = t(t^{-1}\overline{n}t)\phi(t^{-1}F) = \overline{n}\phi(F).$$

(The first and third equalities hold because ϕ is B-equivariant by assumption. The second equality holds because $t^{-1}F$ is supported on N_0, since F was assumed to be supported on Ω, while $t^{-1}\overline{n}t$ lies in $\overline{N}(r)$, and ϕ is assumed to be $\overline{N}(r)$-equivariant on $V(N_0)$.) This proves the claim.

Claim 3.6. The image of the restriction map $\mathcal{L}_G(V, U) \to \mathcal{L}_B(V(N), U)$ is equal to $\mathcal{L}_B(V(N), U)^\circ$.

Since any element of $\mathcal{L}_G(V, U)$ is $G^+(r)$-equivariant when restricted to $V(N_0)$, it follows from Claim 3.5 that the restriction to $V(N)$ of any element of $\mathcal{L}_G(V, U)$ lies in $\mathcal{L}_B(V(N), U)^\circ$. We must show that any element of $\mathcal{L}_B(V(N), U)^\circ$ may be extended to an element of $\mathcal{L}_G(V, U)$.

For $g \in G$, let $gV(N)$ denote the closed subspace of V obtained by translating $V(N)$ by g. If w denotes a representative of the non-trivial element of the Weyl group of G, then since N and Nw together cover $\overline{B}\backslash G$, and since V is local, the closed embeddings of $V(N)$ and $wV(N)$ into V induce a continuous surjection

$$V(N) \bigoplus wV(N) \to V, \tag{3.7}$$

which must be strict, since the source and target are of compact type. More generally, the closed embeddings of $gV(N)$ into V for each $g \in G$ together induce a continuous K-linear surjection

$$\bigoplus_{g \in G} gV(N) \to V. \tag{3.8}$$

(More canonically, this is the G-equivariant map $K[G] \otimes_K V(N) \to V$ induced by the inclusion of $V(N)$ into V and the G-action on V.) Since (3.7) is a strict surjection, the same is true of (3.8).

Now fix $\phi \in \mathcal{L}_B(V(N), U)^\circ$, and for each $g \in G$, define $\phi_g : gV(N) \to U$ via $\phi_g(gF) = g\phi(F)$. The maps ϕ_g induce a continuous K-linear map

$$\bigoplus_{g \in G} \phi_g : \bigoplus_{g \in G} gV(N) \to U. \tag{3.9}$$

(More canonically, this is the G-equivariant map $K[G] \otimes_K V(N) \to U$ induced by the map $\phi : V(N) \to U$ and the G-action on U.) We must show that (3.9) factors through (3.8), to yield a map $\tilde{\phi} : V \to U$. The map $\tilde{\phi}$ will then be a G-equivariant extension of ϕ by construction, and will also be continuous, since (3.8) is strict.

To prove the required factorization, we must show that if $(g_1F_1, \ldots, g_lF_l)$ is any finite sequence with $g_i \in G$ and $F_i \in V(N)$ such that $\sum_i g_iF_i = 0$ (the sum taking place in V), then $\sum_i g_i\phi(F_i) \stackrel{?}{=} 0$. Since V is local, it suffices to show that each point x of $\overline{B}\backslash G$ has a compact open neighbourhood Ω_x such that if Ω'_x is a neighbourhood of x contained in Ω_x then

$$\sum_i g_i\phi(F_{i|\Omega'_x g_i^{-1}}) \stackrel{?}{=} 0. \tag{3.10}$$

Indeed, we can then partition $\overline{B}\backslash G$ into a disjoint union of such neighbourhoods, say $\overline{B}\backslash G = \coprod_{j=1}^s \Omega'_{x_j}$, and writing $F_i = \sum_{j=1}^s F_{i|\Omega'_{x_j}g_i^{-1}}$, we conclude that $\sum_i g_i\phi(F_i) = \sum_i \sum_{j=1}^s g_i\phi(F_{i|\Omega'_{x_j}g_i^{-1}}) = \sum_{j=1}^s \sum_i g_i\phi(F_{i|\Omega'_{x_j}g_i^{-1}}) = 0$, as required.

If $x = \overline{B}g$ for some $g \in G$, then replacing $(g_1F_1, \ldots, g_lF_l)$ by $(gg_1F_1, \ldots, gg_lF_l)$, we see that it suffices to treat the case when x equals the identity coset, which we identify with the identity e of N, under the open immersion $N \to \overline{B}\backslash G$. If e is not in the domain of g_iF_i, for some i, then taking the sought-after neighbourhood Ω_e of e to be sufficiently small, we will have that $(g_iF_i)_{|\Omega_e} = 0$, or equivalently, $F_{i|\Omega_e g_i^{-1}} = 0$, and hence the ith summand will contribute zero to the sum (3.10). Thus we may furthermore assume that e is in the support of each g_iF_i for each i. Also, by choosing Ω_e so as to be contained in N, we may assume without loss of generality that g_iF_i is supported on N.

Since the support of F_i lies in N by assumption, we see that $g_i \in \overline{B}N = \overline{N}B$, and so we may write $g_i = \overline{n}_i b_i$ for some $\overline{n}_i \in \overline{N}$, $b_i \in B$. Take Ω_e to be the intersections of the neighbourhoods Ω corresponding to the elements $\overline{n}_i^{-1}$ in condition (3.4). Then

$$\begin{aligned}\phi(F_{i|\Omega_e g_i^{-1}}) = \phi(g_i^{-1}(g_iF_i)_{|\Omega_e}) &= \phi(b_i^{-1}\overline{n}_i^{-1}(g_iF_i)_{|\Omega_e}) \\ &= b_i^{-1}\overline{n}_i^{-1}\phi((g_iF_i)_{|\Omega_e}) = g_i^{-1}\phi((g_iF_i)_{|\Omega_e}).\end{aligned}$$

(Here the third equality follows from the fact that ϕ is B-equivariant, together with hypothesis (3.4).) Thus

$$\sum_i g_i\phi(F_i|\Omega_e g_i^{-1}) = \sum_i g_ig_i^{-1}\phi((g_iF_i)_{|\Omega_e}) = \phi((\sum_i g_iF_i)_{|\Omega_e}) = 0,$$

since $\sum_i g_iF_i = 0$ by assumption.

The surjectivity of (3.2) follows from Claims 3.5 and 3.6 taken together. □

Now let U be a K-Banach space equipped with a unitary G-representation, and fix some integer $r \geq 1$. Restriction of continuous linear maps yields the following commutative diagram:

$$\begin{array}{ccc} \mathcal{L}_G(I^{\mathrm{la}}(\chi), U) & \longrightarrow & \mathcal{L}_G(I(\chi), U) \\ \downarrow & & \downarrow \\ \mathcal{L}_{G^+(r)}(I^{\mathrm{la}}(\chi)(N_0), U) & \longrightarrow & \mathcal{L}_{G^+(r)}(I(\chi)(N_0), U). \end{array} \tag{3.11}$$

Our goal is to prove that top horizontal arrow is an isomorphism under the hypotheses of Proposition 2.5. Lemma 3.1 implies that both vertical arrows are isomorphisms, and so it suffices to prove that the bottom horizontal arrow is an isomorphism. We turn our efforts to this task.

We first recall an important result due to Vishik and Amice-Vélu (Lemma 3.14 below).

Let $\mathcal{A}$ denote the Tate algebra of rigid analytic functions on the closed unit disk over K; thus $\mathcal{A} := \{\sum_{i=0}^{\infty} a_i x^i \mid a_i \in K, \lim_{i\to\infty} \mid a_i \mid = 0\}$. Let $||{-}||_{\mathcal{A}}$ denote the usual Gauss norm on $\mathcal{A}$; i.e. $|| \sum_{i=0}^{\infty} a_i x^i||_{\mathcal{A}} = \sup\{\mid a_i \mid\}_{i\geq 0}$. With respect to this norm $\mathcal{A}$ becomes a Banach space over K.

We let $\mathcal{A}^{\leq k-2}$ denote the finite dimensional subspace of $\mathcal{A}$ consisting of polynomials of degree $\leq k-2$, and let $\mathcal{C}^{\mathrm{lp}\leq k-2}(\mathbb{Z}_p, K)$ denote the closed subspace of $\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, K)$ consisting of functions that are locally polynomial of degree $\leq k-2$.

If $F \in \mathcal{A}$, $b \in \mathbb{Z}_p$, and $m \geq 0$, we write $F_{|b+p^m\mathbb{Z}_p}$ to denote the element of $\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, K)$ defined by F on the compact open subset $b + p^m\mathbb{Z}_p$ of $\mathbb{Z}_p$, and by zero at all other points of $\mathbb{Z}_p$. If $F \in \mathcal{A}^{\leq k-2}$ then $F_{|b+p^m\mathbb{Z}_p}$ lies in $\mathcal{C}^{\mathrm{lp}\leq k-2}(\mathbb{Z}_p, K)$.

Definition 3.12. *If $\phi \in \mathcal{L}(\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, K), U)$ (respectively $\phi \in \mathcal{L}(\mathcal{C}^{\mathrm{lp}\leq k-2}(\mathbb{Z}_p, K), U)$), then ϕ is called α-tempered (for some $\alpha \in K^\times$) if for all $F(x) \in \mathcal{A}$ (respectively $F(x) \in \mathcal{A}^{\leq k-2}$), all $b \in \mathbb{Z}_p$, and all $m \geq 0$, we have*

$$||\phi\left(F(\frac{x-b}{p^m})\right)||_U \leq C|\alpha|^{-m}||F(x)||_{\mathcal{A}}$$

for some (equivalently any) choice of norm $||{-}||_U$ defining the topology on U and some positive constant C (both independent of the choice of F, b, or m). We let $\mathcal{L}(\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, K), U)^{\alpha-\mathrm{temp}}$ (respectively $\mathcal{L}(\mathcal{C}^{\mathrm{lp}\leq k-2}(\mathbb{Z}_p, K), U)^{\alpha-\mathrm{temp}}$) denote the subspace of $\mathcal{L}(\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, K), U)$ (respectively of $\mathcal{L}(\mathcal{C}^{\mathrm{lp}\leq k-2}(\mathbb{Z}_p, K), U)$) consisting of α-tempered maps.

We remark that α-tempered distributions are also said to be "tempered of order $\leq h$", if h is the p-adic valuation of α.

Restricting maps from $\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, K)$ to $\mathcal{C}^{\mathrm{lp}\leq k-2}(\mathbb{Z}_p, K)$ induces a map

$$\mathcal{L}(\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, K), U)^{\alpha-\mathrm{temp}} \to \mathcal{L}(\mathcal{C}^{\mathrm{lp}\leq k-2}(\mathbb{Z}_p, K), U)^{\alpha-\mathrm{temp}} \tag{3.13}$$

Lemma 3.14. *If $\alpha \in K^\times$ satisfies $\mid \alpha \mid > \mid p^{k-1} \mid$, then the restriction map (3.13) is an isomorphism.*

Proof. This is the classical result of Vishik and Amice-Vélu. (See [1] and [24], and also the proof of [16, Thm., p. 13].) □

We return to the consideration of $I^{\mathrm{la}}(\chi)$, for some character χ as in Section 2. Restricting functions in $I^{\mathrm{la}}(\chi)$ to N_0 induces isomorphisms

$$I^{\mathrm{la}}(\chi)(N_0) \xrightarrow{\sim} \mathcal{C}^{\mathrm{la}}(N_0, K) \xrightarrow{\sim} \mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, K) \tag{3.15}$$

(where the second isomorphism is induced via the identification of $\mathbb{Z}_p$ and N_0 given by $x \mapsto n_x$). Restricting (3.15) to $I(\chi)(N_0)$ induces an isomorphism

$$I(\chi)(N_0) \xrightarrow{\sim} \mathcal{C}^{\mathrm{lp}\leq k-2}(\mathbb{Z}_p, K) \tag{3.16}$$

The isomorphism (3.15) allows us to transport the $G^+(1)$-action on the source to a $G^+(1)$-action on $\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, K)$. Concretely, if $F(x) \in \mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, K)$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G^+(1)$, then one computes for any $x \in \mathbb{Z}_p$ that

(3.17)
$$(gF)(x) = \begin{cases} 0 \text{ if } \dfrac{b+dx}{a+cx} \notin \mathbb{Z}_p \\ (a+cx)^{k-2} \dfrac{\theta_1(a+cx)}{\theta_2(a+cx)} \dfrac{\theta_2(ad-bc)}{(ad-bc)^{k-2}} F(\dfrac{b+dx}{a+cx}) \text{ if } \dfrac{b+dx}{a+cx} \in \mathbb{Z}_p. \end{cases}$$

Since $I(\chi)$ is a G-invariant closed subspace of $I^{\mathrm{la}}(\chi)$, the isomorphism (3.16) shows that $\mathcal{C}^{\mathrm{lp}\leq k-2}(\mathbb{Z}_p, K)$ is a $G^+(1)$-invariant closed subspace of $\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, K)$; this also may be checked directly from the formula (3.17) for the $G^+(1)$-action on this space.

It is useful to also define an action of $G_0(1)$ on $\mathcal{A}$. Namely, if $F(x) \in \mathcal{A}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_0(1)$, then we define

$$(gF)(x) = (a+cx)^{k-2} \frac{\theta_2(ad-bc)}{(ad-bc)^{k-2}} F(\frac{b+dx}{a+cx}) \tag{3.18}$$

Note that the norm $||{-}||_{\mathcal{A}}$ is invariant under this $G_0(1)$-action, and that $\mathcal{A}^{\leq k-2}$ is a $G_0(1)$-invariant subspace of $\mathcal{A}$. Comparing formulas (3.17) and (3.18), we find that if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_0(1)$ and $F(x) \in \mathcal{A}$, then

$$g(F(x)_{|\mathbb{Z}_p}) = \frac{\theta_1(a+cx)}{\theta_2(a+cx)} (gF(x))_{|\mathbb{Z}_p} \tag{3.19}$$

for any $x \in \mathbb{Z}_p$.

Lemma 3.20. *Suppose that χ becomes unitary when restricted to the center of G (i.e. the subgroup of T consisting of diagonal matrices), or equivalently, that $I^{\mathrm{la}}(\chi)$ has a unitary central character. As usual write $\alpha = \chi(p)$, and let $\mathcal{C}$ denote either $\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, K)$ or $\mathcal{C}^{\mathrm{lp}\leq k-2}(\mathbb{Z}_p, K)$; correspondingly, let $\mathcal{A}_{\mathcal{C}}$ denote either $\mathcal{A}$ or $\mathcal{A}^{\leq k-2}$, respectively.*

If U is a K-Banach space and $\phi \in \mathcal{L}(\mathcal{C}, U)$ then the following are equivalent:

(i) ϕ is α-tempered.

(ii) There exists a positive constant C such that $||\phi(b(F_{|\mathbb{Z}_p}))||_U \leq C||F||_{\mathcal{A}}$ for all $F \in \mathcal{A}_{\mathcal{C}}$ and $b \in B^+$. (Here $||{-}||_U$ is some fixed choice of norm defining the topology of U.)

Proof. Write $B' := \{\begin{pmatrix} p^m & -b \\ 0 & 1 \end{pmatrix} \mid m \geq 0, b \in \mathbb{Z}_p\}$; note that B' is a submonoid of B^+. If $F \in \mathcal{A}_{\mathcal{C}}$, then we compute using (3.17) that

$$\begin{pmatrix} p^m & -b \\ 0 & 1 \end{pmatrix} F(x)_{|\mathbb{Z}_p} = \alpha^m F(\frac{x-b}{p^m})_{|b+p^m\mathbb{Z}_p}.$$

Thus (fixing a norm $||{-}||_U$ defining the topology of U) we find that ϕ is α-tempered if and only if there exists a positive constant C such that

$$||\phi(b'(F_{|\mathbb{Z}_p}))||_U \leq C||F||_{\mathcal{A}} \tag{3.21}$$

for all $F \in \mathcal{A}$ and $b' \in B'$.

Clearly (ii) implies condition (3.21). Conversely, suppose that (3.21) holds. Any element of B^+ may be written in the form $zb't_0$, where z lies in the center of G, $b' \in B'$, and $t_0 \in T_0$. Writing $t_0 = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, and taking into account (3.21), (3.19), the fact that T_0 acts isometrically on $\mathcal{A}$, and that $\chi_{|Z_G}$ is unitary by assumption, we deduce that

$$||\phi(zb't_0(F_{|\mathbb{Z}_p}))||_U = ||\chi(z)\frac{\theta_1(a)}{\theta_2(a)}\phi(b'(t_0F)_{|\mathbb{Z}_p})||_U = ||\phi(b'(t_0F)_{|\mathbb{Z}_p})||_U \\ \leq C||t_0F||_{\mathcal{A}} = ||F||_{\mathcal{A}}$$

for any $F \in \mathcal{A}_{\mathcal{C}}$. Thus (3.21) is equivalent to condition (ii), and the lemma follows. $\square$

Lemma 3.22. *In the situation of Lemma 3.20, suppose that U is equipped with a unitary B^+-action. Then $\mathcal{L}_{B^+}(\mathcal{C}, U) \subset \mathcal{L}(\mathcal{C}, U)^{\alpha-\text{temp}}$.*

Proof. Fix a B^+-invariant norm $||–||_U$ defining the topology of U. If $\phi \in \mathcal{L}_{B^+}(\mathcal{C}, U)$ then the map $\mathcal{A}_{\mathcal{C}} \to U$ defined by $F \mapsto \phi(F_{|\mathbb{Z}_p})$ is continuous, and so there is a positive constant C such that $||\phi(F_{|\mathbb{Z}_p})||_U \leq C||F||_{\mathcal{A}}$ for all $F \in \mathcal{A}_{\mathcal{C}}$. For any $b \in B^+$, $F \in \mathcal{A}_{\mathcal{C}}$, we compute that

$$||\phi(bF_{|\mathbb{Z}_p})||_U = ||b\phi(F_{|\mathbb{Z}_p})||_U = ||\phi(F_{|\mathbb{Z}_p})||_U \leq C||F||_{|\mathbb{Z}_p}$$

(using the B^+-equivariance of ϕ and the B^+-invariance of $||–||_U$.) Thus ϕ satisfies condition (ii) of Lemma 3.20, and so by that lemma is α-tempered. $\square$

Lemma 3.23. *In the situation of Lemma 3.20, suppose that U is equipped with a unitary G-action, and that $\mid \alpha \mid < \mid p^{k-1} \mid$. If r is chosen so that the conductor of each of $\theta_{1|\mathbb{Z}_p^\times}$ and $\theta_{2|\mathbb{Z}_p^\times}$ divides p^r, then the map (3.13) (which is an isomorphism by Lemma 3.14) restricts to an isomorphism $\mathcal{L}_{G^+(r)}(\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, K), U) \to \mathcal{L}_{G^+(r)}(\mathcal{C}^{\mathrm{lp}\leq k-2}(\mathbb{Z}_p, K), U)$.*

Proof. Since $B^+ \subset G^+(r)$, Lemma 3.22 gives the inclusions

$$\mathcal{L}_{G^+(r)}(\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, K), U) \subset \mathcal{L}(\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, K), U)^{\alpha-\text{temp}}$$

and

$$\mathcal{L}_{G^+(r)}(\mathcal{C}^{\mathrm{lp}\leq k-2}(\mathbb{Z}_p, K), U) \subset \mathcal{L}(\mathcal{C}^{\mathrm{lp}\leq k-2}(\mathbb{Z}_p, K), U)^{\alpha-\text{temp}}.$$

The isomorphism of Lemma 3.14 thus restricts to an injection

$$\mathcal{L}_{G^+(r)}(\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, K), U) \to \mathcal{L}_{G^+(r)}(\mathcal{C}^{\mathrm{lp}\leq k-2}(\mathbb{Z}_p, K), U), \tag{3.24}$$

which we must show is again an isomorphism.

Given ϕ lying in the target of (3.24), let $\tilde{\phi}$ denote its unique extension to an element of $\mathcal{L}(\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, K), U)^{\alpha-\text{temp}}$. We must show that $\tilde{\phi}$ is again $G^+(r)$-equivariant. In other words, for any $g \in G^+(r)$, we must show that

$$g^{-1}\tilde{\phi}(gF) \stackrel{?}{=} \tilde{\phi}(F) \tag{3.25}$$

for all $F \in \mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, K)$. Since ϕ is $G^+(r)$-equivariant, both sides of (3.25) restrict to ϕ on $\mathcal{C}^{\mathrm{lp} \leq k-2}(\mathbb{Z}_p, K)$. Thus to show that they coincide, it suffices to show that

$$F \mapsto g^{-1}\tilde{\phi}(gF) \tag{3.26}$$

is also an α-tempered map.

Fix a G-invariant norm $||{-}||_U$ defining the topology of U. Since $\tilde{\phi}$ is α-tempered, Lemma 3.20 (ii) gives a positive constant C such that

$$||\tilde{\phi}(bF_{|\mathbb{Z}_p})||_U \leq C||F||_{\mathcal{A}} \tag{3.27}$$

for all $b \in B^+$, $F \in \mathcal{A}$. If $g \in G^+(r)$ and b in B^+, then we may write $gb = b'\overline{n}$ for some $b' \in B^+$, $\overline{n} \in \overline{N}(r)$, and so compute

$$\begin{aligned} ||g^{-1}\tilde{\phi}(gbF_{|\mathbb{Z}_p})||_U = ||\tilde{\phi}(gbF_{|\mathbb{Z}_p})||_U &= ||\tilde{\phi}(b'\overline{n}F_{|\mathbb{Z}_p})||_U \\ &= ||\tilde{\phi}(b'(\overline{n}F)_{|\mathbb{Z}_p})||_U \leq C||\overline{n}F||_{\mathcal{A}} = C||F||_{\mathcal{A}}. \end{aligned}$$

(Here we have used the G-invariance of $||{-}||_U$, as well as (3.19) applied to $\overline{n}$ and F (taking into account our choice of r), the inequality (3.27) applied to $\overline{n}F$, and the fact that $\overline{N}(r)$ acts isometrically on $\mathcal{A}$.) Again appealing to Lemma 3.20, we see that (3.26) is α-tempered, as required. □

Taking into account the isomorphisms (3.15) and (3.16), we see that Lemma 3.23 implies that under the hypotheses of Proposition 2.5, the bottom horizontal arrow of (3.11) is an isomorphism, as required, and so Proposition 2.5 is proved.

4. p-adic L-functions

Let $\mathbb{A}$ denote the adèles of $\mathbb{Q}$, and let $\mathbb{A}_f$, $\mathbb{A}_f^p$, have their usual meaning. Fix a prime-to-p natural number N, and let K^p be the following compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f^p)$:

$$K^p = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \prod_{\ell \neq p} \mathrm{GL}_2(\mathbb{Z}_\ell) \mid c \equiv 0 \bmod N, d \equiv 1 \bmod N\}.$$

Let us embed $\mathbb{C}^\times$ into $\mathrm{GL}_2(\mathbb{R})$ in the usual way, by choosing a real linear isomorphism $\mathbb{C} \xrightarrow{\sim} \mathbb{R}^2$, and regarding $\mathbb{C}^\times$ as the subgroup of the group of real linear automorphisms of $\mathbb{C}$ consisting of complex linear automorphisms.

If K_p is a compact open subgroup of $\mathrm{GL}_2(\mathbb{Q}_p)$ then the quotient

$$Y(K^pK_p) := \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\mathbb{C}^\times K^pK_p$$

is a classical modular curve. For example, if we take K_p to be the kernel of the reduction map $\mathrm{GL}_2(\mathbb{Z}_p) \to \mathrm{GL}_2(\mathbb{Z}/p^n)$, then this quotient is isomorphic to the modular curve that classifies elliptic curves over $\mathbb{C}$ with a $\Gamma_1(N)$-level structure and a full level p structure (the Weil pairing of which is allowed to have arbitrary determinant). In general, if we take K_p to be a compact open subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$, and use the isomorphism $\mathrm{GL}_2(\mathbb{R})/\mathbb{C}^\times \xrightarrow{\sim} \mathbb{C}\backslash\mathbb{R}$ together with strong approximation for SL_2, we obtain an isomorphism

$$Y(K^pK_p) = \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\mathbb{C}^\times K^pK_p \xrightarrow{\sim} \Gamma_1(N)\backslash(X \times \mathrm{GL}_2(\mathbb{Z}_p)/K_p),$$

where X denotes the upper half-plane of $\mathbb{C}\setminus\mathbb{R}$, and as usual $\Gamma_1(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid a \equiv d \equiv 1 \bmod N, c \equiv 0 \bmod N\}$.

Define $M(K^p) = \varprojlim_s \varinjlim_{K_p} H^1_c(Y(K^pK_p), \mathcal{O}_K/p^s)$, where s and K_p range over the directed sets of non-negative integers and compact open subgroups of $\mathrm{GL}_2(\mathbb{Q}_p)$ respectively, and define $\tilde{H}^1_c(K^p) = K \otimes_{\mathcal{O}_K} M(K^p)$. By construction, $M(K^p)$ is a p-adically complete and separated $\mathcal{O}_K$-module. Thus $\tilde{H}^1_c(K^p)$ is naturally a K-Banach space, if we equip it with the norm whose unit ball is equal to $M(K^p)$ (in other words, the gauge of $M(K^p)$). The group $\mathrm{GL}_2(\mathbb{Q}_p)$ acts on right on the projective system of arithmetic quotients $Y(K^pK_p)$, and hence on $M(K^p)$. Thus $\mathrm{GL}_2(\mathbb{Q}_p)$ acts unitarily on the Banach space $\tilde{H}^1_c(K^p)$. This representation is further equipped with actions of the Hecke algebra $\mathcal{H}^p$ of K^p in $\mathrm{GL}_2(\mathbb{A}_f^p)$ and of the connected component group $\pi_0 = \{\pm 1\}$ of $\mathrm{GL}_2(\mathbb{R})$, that commute with one another and with the $\mathrm{GL}_2(\mathbb{Q}_p)$-action.

Proposition 4.1. *The unitary Banach space representation $\tilde{H}^1_c(K^p)$ is admissible.*

Proof. This is [5, Lem. 2.1.2]. It is also a special case of [12, Prop. 2.2.6]. $\square$

We let $\tilde{H}^1_c(K^p)^{\pm}$ denote the ± 1-eigenspace for π_0 on $\tilde{H}^1_c(K^p)$. Each of these spaces is again an admissible unitary Banach space representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, equipped with a commuting $\mathcal{H}^p$-action.

Let Γ denote the normalizer in $\mathrm{GL}_2(\mathbb{Q})$ of K^p. (We regard $\mathrm{GL}_2(\mathbb{Q})$ as being embedded diagonally in $\mathrm{GL}_2(\mathbb{A}_f^p)$.) There is a natural map $\Gamma \to \pi_0 \times (\mathcal{H}^p)^\times \times \mathrm{GL}_2(\mathbb{Q}_p)$: the projection onto the first factor is given by the sign of the determinant, the projection onto the second factor is given by mapping an element in $\mathrm{GL}_2(\mathbb{A}_f^p)$ that normalizes K^p to its double coset in $\mathcal{H}^p$, and the projection onto the third factor is given by the inclusion $\mathrm{GL}_2(\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Q}_p)$. In this way we may regard $\tilde{H}^1_c(K^p)$, and hence its dual $\tilde{H}^1_c(K^p)'$, as a Γ-module.

The following proposition describes the theory of modular symbols, as it applies to the completed cohomology space $\tilde{H}^1_c(K^p)$. Let D^0 denote the $\mathcal{O}_K$-module of degree zero divisors on $\mathbb{P}^1(\mathbb{Q})$. The natural action of $\mathrm{GL}_2(\mathbb{Q})$ on D^0 induces in particular an action of Γ on D^0. Recall that $\mathbb{P}^1(\mathbb{Q})$ appears as the boundary of the standard completion of the upper half-plane X. For any compact open subgroup K_p of $\mathrm{GL}_2(\mathbb{Q}_p)$, we let $X(K^pK_p)$ denote the usual compactification of $Y(K^pK_p)$; thus $X(K^pK_p) = Y(K_pK^p) \bigcup C(K_pK^p)$, where $C(K_pK^p)$ is the set of cusps. (If $K_p \subset \mathrm{GL}_2(\mathbb{Z}_p)$, then $C(K_pK^p) = \Gamma_1(N)\backslash(\mathbb{P}^1(\mathbb{Q}) \times \mathrm{GL}_2(\mathbb{Z}_p)/K_p)$.)

Proposition 4.2. *The standard pairings between one dimensional compactly supported cohomology classes on the open curves $Y(K^pK_p)$, and one dimensional homology classes taken relative to the set of cusps $C(K^pK_p)$ on the compactified curves $X(K^pK_p)$, induces a Γ-equivariant $\mathcal{O}_K$-linear map $D^0 \to (\tilde{H}^1_c(K^p))'$, whose image lies in the closed unit ball of the target (taken with respect to norm that is dual to the norm on $\tilde{H}^1_c(K^p)$ given by the gauge of $M(K^p)$).*

Proof. Given elements $a, b \in \mathbb{P}^1(\mathbb{Q})$, the geodesic in the upper half-plane that joins a and b gives rise to an element of $H_1(X(K^pK_p), C(K^pK_p); \mathcal{O}_K/p^s)$, for any choice of K_p and s, and we obtain a map $D^0 \to H_1(X(K^pK_p), C(K^pK_p); \mathcal{O}_K/p^s)$ by mapping $a - b$ to this element. The pairing between this relative homology group and the corresponding compactly supported cohomology group thus induces a $\mathcal{O}_K$-bilinear pairing

$$D^0 \times H^1_c(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\mathbb{C}^\times K^p K_p, \mathcal{O}_K/p^s) \to \mathcal{O}_K/p^s$$

for each choice of K^p and s. The functorial nature of this pairing shows that it is Γ-equivariant, and that it is compatible with change of K^p and s. Thus, passing to the inductive limit in K^p and the projective limit in s, we obtain a Γ-equivariant $\mathcal{O}_K$-bilinear pairing

$$D^0 \times M(K^p) \to \mathcal{O}_K,$$

and consequently a Γ-equivariant $\mathcal{O}_K$-linear map as in the statement of the proposition. □

Let us fix for the duration of this section a newform f of weight k and tame conductor N[5] that is defined over K. If we fix an embedding $K \to \mathbb{C}$, then f gives rise to an automorphic representation $\pi = \pi_\infty \otimes_K \pi_p \otimes_K \pi^p$ of $\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{Q}_p) \times \mathrm{GL}_2(\mathbb{A}^p_f)$, where the infinite component π_∞ is defined over $\mathbb{C}$, while the finite component $\pi_p \otimes \pi^p$ is defined over K. Since f is assumed to have tame conductor N, the space $(\pi^p)^{K^p}$ is thus a one dimensional $\mathcal{H}^p$-representation, on which $\mathcal{H}^p$ acts through a character that we will denote by λ. If ϵ_N denotes the conductor-N part of the nebentypus of f, then the central character of π_p takes $\begin{pmatrix} p & \\ & p \end{pmatrix}$ to $\epsilon_N(p)p^{k-2}$.

If V is any representation of $\mathrm{GL}_2(\mathbb{Q}_p)$, then for each choice of sign ± 1, we let $V^{\pm,\lambda}$ denote the $\mathrm{GL}_2(\mathbb{Q}_p)$-representation V equipped with commuting actions of π_0 and $\mathcal{H}^p$, defined as follows: the non-trivial element of π_0 acts through the chosen sign ± 1, while $\mathcal{H}^p$ acts through the character λ.

Proposition 4.3. *For each choice of sign, we obtain a $\pi_0 \times \mathcal{H}^p \times \mathrm{GL}_2(\mathbb{Q}_p)$-equivariant embedding $(W_{k-2} \otimes_K \pi_p)^{\pm,\lambda} \to \tilde{H}^1_c(K^p)^{\pm}$, unique up to scalar multiples.*

Proof. For each choice of compact open subgroup K_p of $\mathrm{GL}_2(\mathbb{Q}_p)$, write $\mathcal{V}_{\check{W}_{k-2}} := \mathrm{GL}_2(\mathbb{Q})\backslash(\mathrm{GL}_2(\mathbb{A})/\mathbb{C}^\times K^p K_p \times \check{W}_{k-2})$; this is a local system of K-vector spaces on the modular curve $\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\mathbb{C}^\times K^p K_p$. If we write $H^1_c(K^p, \mathcal{V}_{\check{W}_{k-2}}) := \varinjlim_{K_p} H^1_c(\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\mathbb{C}^\times K^p K_p, \mathcal{V}_{\check{W}_{k-2}})$, then [12, (4.3.7)] shows that there is an isomorphism

$$W_{k-2} \otimes_K H^1_c(K^p, \mathcal{V}_{\check{W}_{k-2}}) \xrightarrow{\sim} \tilde{H}^1_c(K^p)_{W_{k-2}-\mathrm{lalg}}, \tag{4.4}$$

while classical Eichler-Shimura theory shows that the space

$$\mathrm{Hom}_{\mathrm{GL}_2(\mathbb{Q}_p)}(\pi_p, H^1_c(K^p, \mathcal{V}_{\check{W}_{k-2}})^{\mathcal{H}^p=\lambda})$$

is two dimensional, and that under the action of π_0 it decomposes as the direct sum of one dimensional $\pm$-eigenspaces. The proposition follows. □

The homomorphism $\Gamma \to \pi_0 \times (\mathcal{H}^p)^\times \times \mathrm{GL}_2(\mathbb{Q}_p)$ described above allows us to define a Γ-action on $(W_{k-2} \otimes_K \pi_p)^{\pm,\lambda}$, and hence a contragredient Γ-action on $((W_{k-2} \otimes_K \pi_p)^{\pm,\lambda})'_b$ and on $((W_{k-2} \otimes_K \pi_p)^{\pm,\lambda})'^{\mathrm{b}}_b$.

[5]i.e. conductor equal to N times a power of p

Corollary 4.5. *For each choice of sign we obtain a* Γ*-equivariant map* $D^0 \to ((W_{k-2} \otimes_K \pi_p)^{\pm,\lambda})'^{\mathrm{b}}_b$.

Proof. Since the $\mathrm{GL}_2(\mathbb{Q}_p)$-action on $\tilde{H}^1_c(K^p)^{\pm}$ is unitary, Lemma 1.11 shows that the dual of the map of Proposition 4.3 factors through the natural map

$$((W_{k-2} \otimes_K \pi_p)^{\pm,\lambda})'^{\mathrm{b}}_b \to ((W_{k-2} \otimes_K \pi_p)^{\pm,\lambda})'_b$$

of (1.10). Composing the resulting map $(\tilde{H}^1_c(K^p)^{\pm})' \to ((W_{k-2} \otimes_K \pi_p)^{\pm,\lambda})'^{\mathrm{b}}_b$ with the map of Proposition 4.2 yields the corollary. $\square$

In the case when π_p is special and unramified, the maps of Corollary 4.5 were constructed by Breuil in [5, Lem. 4.2.2].

Suppose now that π_p is principal series or special. Then we may choose χ as in Section 2 such that $W_{k-2} \otimes_K \pi_p = \pi(\chi)$. We let α, β, θ_1, θ_2 and χ' have the same meanings as in Section 2. A consideration of central characters shows that $\alpha\beta = \epsilon_N(p)p^{k-1}$. Note that the necessary conditions of Lemma 2.1 are satisfied for a representation $\pi(\chi)$ arising in this way from a newform.

Proposition 4.6. *Conjecture 2.2 is true for those representations* $\pi(\chi)$ *that arise from newforms that are principal series or special locally at* p.

Proof. Proposition 4.3 shows that $\pi(\chi)$ embeds into the non-zero unitary representation $\tilde{H}^1_c(K^p)^{\pm}$ (for either choice of sign and an appropriate choice of K^p). $\square$

Of course in the unramified principal series or special case, the results of [2], [14], and [23] establish Conjecture 2.2 without any global restriction on χ.

Now suppose furthermore that the newform f is of finite slope at p. Then we may assume that θ_1 is unramified (by replacing χ by χ' if necessary). If in fact p divides the conductor of f and we are in the principal series case, then θ_2 is ramified, and so the condition that θ_1 be unramified uniquely determines χ. If p divides the conductor of f and we are in the special case, we again have a unique choice for χ, since in the special case we insist that $\alpha/\beta = p^{-1}$. In either of these cases, the U_p-eigenvalue of f is equal to α. If the conductor of f is prime to p, then both θ_1 and θ_2 are unramified, and we have two choices for χ (namely a pair χ and χ', in the notation of Section 2). Fixing one of the two choices for χ corresponds to choosing one of the two p-old U_p-eigenforms attached to f; namely, the one with U_p-eigenvalue equal to α.

We let g denote f in the case when p divides the conductor of f, and let g denote the p-oldform attached to f with U_p-eigenvalue α, in the case when the conductor of f is prime-to-p. In all cases g is a U_p-eigenform with eigenvalue α.

Corollary 4.7. *If* $\mid \alpha \mid > \mid p^{k-1} \mid$, *then the embeddings*

$$\pi(\chi)^{\pm,\lambda} = (W_{k-2} \otimes_K \pi_p)^{\pm,\lambda} \to \tilde{H}^1(K^p)^{\pm}$$

of Proposition 4.3 extend uniquely to $(\pi_0, \mathrm{GL}_2(\mathbb{Q}_p), \mathcal{H}^p)$*-equivariant maps*

$$\pi^{\mathrm{la}}(\chi)^{\pm,\lambda} \to \tilde{H}^1_c(K^p)^{\pm}.$$

Proof. This follows from Proposition 2.5. $\square$

Dualizing the maps of Corollary 4.7, and taking into account Proposition 4.2, we obtain Γ-equivariant maps $D^0 \to (\pi^{\mathrm{la}}(\chi)^{\pm,\lambda})'$ (one for each choice of sign), lifting the maps of Corollary 4.5. In particular, the element $\infty - 0$ of D^0 gives rise to a pair of functionals

$$\mu_g^{\pm} : \pi^{\mathrm{la}}(\chi) \to K. \tag{4.8}$$

If we identify $I^{\mathrm{la}}(\chi)$ as a space of locally analytic functions on $\mathbb{Q}_p^{\times}$, as in Section 2, then we obtain a closed embedding $\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^{\times}, K) \to \pi^{\mathrm{la}}(\chi)$, by identifying elements of the source as locally analytic functions on $\mathbb{Q}_p$ that are supported on $\mathbb{Z}_p^{\times}$. Restricting the functionals $\mu_g^{\pm}$ to $\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p^{\times})$ thus gives rise to a pair of distributions on $\mathbb{Z}_p^{\times}$.

Proposition 4.9. *The distributions on $\mathbb{Z}_p^{\times}$ obtained by restricting $\mu_g^{\pm}$ coincide (up to a non-zero scalar multiple) with the Mazur-Tate-Teitelbaum distributions* [16] *that compute the p-adic L-function of g.*[6]

Proof. It follows from Lemma 3.22 that $\mu_g^{\pm}$ are α-tempered, while by their very construction, the Mazur-Tate-Teitelbaum distributions defining the p-adic L-function of g are α-tempered. (See [16, Thm., p. 13].) Thus it suffices to show that the distributions $\mu_g^{\pm}$ coincide with the Mazur-Tate-Teitelbaum distributions when evaluated on functions that are locally polynomial of degree $\leq k-2$.

As in Section 3, let $\mathcal{A}^{\leq k-2}$ denote the space of polynomials in one variable over K of degree $\leq k-2$. Suppose that $b \in \mathbb{Z}_p^{\times}$, that p^m is a positive power of p, and that $F(x)$ is an element of $\mathcal{A}^{\leq k-2}$. As in Section 3, we let $F(x)_{|b+p^m\mathbb{Z}_p}$ denote the element of $\mathcal{C}^{\mathrm{lp}\leq k-2}(\mathbb{Z}_p, K)$ defined by $F(x)$ on $b + p^m\mathbb{Z}_p$, and defined to be zero elsewhere. Fix a choice of sign $\pm$, let $\langle -,- \rangle_1^{\pm} : D^0 \times \tilde{H}_c^1(K^p) \to K$ denote the Γ-equivariant pairing provided by Proposition 4.2, and let $\phi_1^{\pm} : \pi(\chi) \to \tilde{H}_c^1(K^p)^{\pm}$ denote the embedding of Proposition 4.3. Using the formula (3.17) and the fact that $\begin{pmatrix} p^m & -b \\ 0 & 1 \end{pmatrix}$, when thought of as an element of $\mathrm{GL}_2(\mathbb{Q})$, lies in Γ (in fact, even in K^p), we compute that

$$\begin{aligned} \int_{b+p^m\mathbb{Z}_p^{\times}} F(x) d\mu_g^{\pm} &= \langle \infty - 0, \phi_1^{\pm}(F(x)_{|b+p^m\mathbb{Z}_p}) \rangle_1^{\pm} \\ &= \langle \infty - 0, \alpha^{-m} \begin{pmatrix} p^m & -b \\ 0 & 1 \end{pmatrix} \phi_1^{\pm}(F(b+p^m x)_{|\mathbb{Z}_p}) \rangle_1^{\pm} \\ &= \alpha^{-m} \langle \begin{pmatrix} p^{-m} & bp^{-m} \\ 0 & 1 \end{pmatrix} (\infty - 0), \phi_1^{\pm}(F(b+p^m x)_{|\mathbb{Z}_p}) \rangle_1^{\pm} \\ &= \alpha^{-m} \langle \infty - \frac{b}{p^m}, \phi_1^{\pm}(F(b+p^m x)_{|\mathbb{Z}_p}) \rangle_1^{\pm}. \end{aligned} \tag{4.10}$$

To analyze this expression, we must unwind the definition of the pairing $\langle -,- \rangle_1^{\pm}$, which we do through a series of commutative diagrams.

[6] We take the auxiliary integer "M" of that paper to be 1; this is no loss of generality, since the Mazur-Tate-Teitelbaum distributions on $\mathbb{Z}_p^{\times} \times \mathbb{Z}_M^{\times}$ associated to g may be constructed out of the Mazur-Tate-Teitelbaum distributions on $\mathbb{Z}_p^{\times}$ attached to the collection of modular forms obtained by twisting g by all characters of $\mathbb{Z}_M^{\times}$. We have also scaled the distribution of [16] by an appropriate complex period $\Omega_g^{\pm}$.

We identify W_{k-2} with a space of functions on G in the usual way: write $\check{w}_0 = (0,1)^{k-2} \in \mathrm{Sym}^{k-2} K^2 \xrightarrow{\sim} \check{W}_{k-2}$, and define $w(g) = \langle \check{w}_0, gw \rangle$ for $g \in G$. If we write $I(\theta) := (\mathrm{Ind}_B^G \theta)_{\mathrm{sm}}$, then multiplication of functions induces an isomorphism (whose existence has been recalled several times already)

$$W_{k-2} \otimes_K I(\theta) \xrightarrow{\sim} I(\chi). \tag{4.11}$$

Restricting the functions $w(g)$ to those g of the form $n_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ for $x \in \mathbb{Z}_p$, we also obtain an identification of W_{k-2} with the space $\mathcal{A}^{\leq k-2}$. Multiplying elements of W_{k-2}, thought of as elements of $\mathcal{A}^{\leq k-2}$ in this way, by locally constant functions on $\mathbb{Z}_p$ induces an isomorphism

$$W_k \otimes_K \mathcal{C}^{\mathrm{sm}}(\mathbb{Z}_p, K) \xrightarrow{\sim} \mathcal{C}^{\mathrm{lp}\leq k-2}(\mathbb{Z}_p, K). \tag{4.12}$$

(As in Section 3, we let $\mathcal{C}^{\mathrm{lp}\leq k-2}(\mathbb{Z}_p, K)$ denote the subspace of $\mathcal{C}^{\mathrm{la}}(\mathbb{Z}_p, K)$ consisting of functions that are locally polynomial of degree $\leq k-2$.)

We may apply the discussion of Section 3 to the representation $I(\theta)$ just as well as to $I(\chi)$. The isomorphism (3.16) thus yields an isomorphism

$$\mathcal{C}^{\mathrm{sm}}(\mathbb{Z}_p, K) \xrightarrow{\sim} I(\theta)(N_0), \tag{4.13}$$

which fits in the commutative diagram of isomorphisms

$$\begin{array}{ccc} W_{k-2} \otimes_K \mathcal{C}^{\mathrm{sm}}(\mathbb{Z}_p, K) & \xrightarrow[(4.12)]{\sim} & \mathcal{C}^{\mathrm{lp}\leq k-2}(\mathbb{Z}_p, K) \\ {\scriptstyle \mathrm{id}_{W_{k-2}} \otimes (4.13)} \downarrow \wr & & {\scriptstyle (3.16)} \downarrow \wr \\ W_{k-2} \otimes_K I(\theta)(N_0) & \xrightarrow[(4.11)]{\sim} & I(\chi)(N_0). \end{array} \tag{4.14}$$

We employ the notation introduced in the proof of Proposition 4.3. As recalled in that proof, for each choice of sign $\pm$, classical Eichler-Shimura theory gives a $(G(\mathbb{Q}_p), \mathcal{H}^p)$-equivariant embedding $\phi_2^{\pm} : (\pi_p)^{\pm,\lambda} \to H_c^1(K^p, \mathcal{V}_{\check{W}_{k-2}})^{\pm}$, unique up to multiplication by a scalar. (Here $H_c^1(K^p, \mathcal{V}_{\check{W}_{k-2}})^{\pm}$ denotes the $\pm$-eigenspace for the action of π_0 on $H_c^1(K^p, \mathcal{V}_{\check{W}_{k-2}})$.) By its construction, the embedding $\phi_1^{\pm}$ of Proposition 4.3 is obtained by composing the map (4.4) with $\mathrm{id}_{W_{k-2}} \otimes \phi_2^{\pm}$. Thus we obtain a commutative diagram

$$\begin{array}{ccc} W_{k-2} \otimes I(\theta) & \xrightarrow[(4.11)]{\sim} & I(\chi) \\ \downarrow & & \downarrow \\ W_{k-2} \otimes \pi_p & \xrightarrow{\sim} & \pi(\chi) \\ {\scriptstyle \mathrm{id}_{W_{k-2}} \otimes \phi_2^{\pm}} \downarrow & & {\scriptstyle \phi_1^{\pm}} \downarrow \\ W_{k-2} \otimes H_c^1(K^p, \mathcal{V}_W)^{\pm} & \longrightarrow & \tilde{H}_c^1(K^p)^{\pm}, \end{array} \tag{4.15}$$

in which the upper vertical arrows are the defining surjections (either isomorphisms in the principal series case, or surjections with finite dimensional kernels in the special case), and the bottom horizontal arrow is induced by (4.4).

Pulling back a compactly compacted cohomology class to the upper half-plane, and then integrating over a geodesic joining two cusps, we obtain a pairing $D^0 \times H^1_c(K^p, \mathcal{V}_{\check{W}_{k-2}}) \to \check{W}_{k-2}$, or equivalently, a pairing

$$\langle -, - \rangle_2 : D^0 \times W_{k-2} \otimes_K H^1_c(K^p, \mathcal{V}_{\check{W}_{k-2}}) \to K.$$

Passing to $\pm$-eigenspaces under the action of π_0, this yields pairings

$$\langle -, - \rangle_2^{\pm} : D^0 \times W_{k-2} \otimes_K H^1_c(K^p, \mathcal{V}_{\check{W}_{k-2}})^{\pm} \to K.$$

The diagram

$$\begin{array}{ccc} D^0 \times W_{k-2} \otimes_K H^1_c(K^p, \mathcal{V}_{\check{W}_{k-2}})^{\pm} & \longrightarrow & D^0 \times \tilde{H}^1_c(K^p)^{\pm} \\ & \searrow^{\langle -,- \rangle_2^{\pm}} & \Big\downarrow {\scriptstyle \langle -,- \rangle_1^{\pm}} \\ & & K \end{array} \tag{4.16}$$

in which the horizontal arrow is induced by (4.4), also commutes; indeed, the diagonal arrow is constructed by trivializing $\mathcal{V}_W$, and then integrating over geodesics joining cusps, while an examination of the proof of [12, Thm. 2.2.19] shows that the horizontal arrow is similarly constructed by trivializing $\mathcal{V}_W$ (after choosing an integral model and reducing modulo powers of p), while the vertical arrow is again constructed by integrating over geodesics joining cusps.

Let $1_{|\mathbb{Z}_p}$ denote the constant function 1 with domain $\mathbb{Z}_p$. Using the commutative diagrams (4.14), (4.15), and (4.16), we compute that

$$\alpha^{-m} \langle \infty - \frac{b}{p^m}, \phi_1^{\pm}(F(b + p^m x)_{|\mathbb{Z}_p}) \rangle_1^{\pm} = \alpha^{-m} \langle \infty - \frac{b}{p^m}, F(b + p^m x) \otimes \phi_2^{\pm}(1_{|\mathbb{Z}_p}) \rangle_2^{\pm}.$$

Combining this with (4.10), we see that

$$\int_{b + p^m \mathbb{Z}_p} F(x) d\mu_g^{\pm} = \alpha^{-m} \langle \infty - \frac{b}{p^m}, F(b + p^m x) \otimes \phi_2^{\pm}(1_{|\mathbb{Z}_p}) \rangle_2^{\pm} \tag{4.17}$$

for any $F(x) \in \mathcal{A}^{\leq k-2} \xrightarrow{\sim} W_{k-2}$.

If we fix an embedding $K \to \mathbb{C}$, then the modular form g defines an element of $c_g \in H^1_c(K^p, \mathcal{V}_{W_{k-2}}) \otimes_K \mathbb{C}$, defined to be the cohomology class of the holomorphic $\mathcal{V}_{W_{k-2}}$-valued one-form $g(\tau) d\tau (\tau, 1)^{k-2}$. (Here $(\tau, 1)^{k-2}$ is regarded as an element of $\mathrm{Sym}^{k-2} \mathbb{C}^2 \xrightarrow{\sim} \check{W}_{k-2} \otimes_K \mathbb{C}$.) If we let $c_g^{\pm}$ denote the $\pm$-eigencomponent of c_g, then $\phi_2^{\pm}(1_{|\mathbb{Z}_p}) = \Omega^{\pm} c_g^{\pm}$ for some non-zero scalars $\Omega^{\pm}$. (To see this, note that $\phi_2^{\pm}(1_{|\mathbb{Z}_p})$ is a U_p-eigenclass with eigenvalue α, and a $\mathcal{H}^p$-eigenclass with character λ, while the same is true of $c_g^{\pm}$.) From (4.17) we compute

$$\int_{b + p^m \mathbb{Z}_p} F(x) d\mu_g^{\pm} = \Omega^{\pm} \alpha^{-m} \langle \infty - \frac{b}{p^m}, F(b + p^m x) \otimes c_g^{\pm} \rangle_2^{\pm}. \tag{4.18}$$

If we don't pass to $\pm$-eigencomponents, then we compute that

$$\langle \infty - \frac{b}{p^m}, F(b + p^m x) \otimes c_g \rangle_2 = \int_{b/p^m}^{\infty} g(\tau) \langle (\tau, 1)^{k-2}, F(b + p^m x) \rangle d\tau, \tag{4.19}$$

where the pairing $\langle(\tau,1)^{k-2}, F(b+p^m x)\rangle$ denotes the natural pairing $\check{W}_{k-2}\otimes_K \mathbb{C}\times W_{k-2}\to\mathbb{C}$. We furthermore compute that

$$\langle(\tau,1)^{k-2}, F(b+p^m x)\rangle = \langle\begin{pmatrix}1 & \tau\\ 0 & 1\end{pmatrix}\check{w}_0, F(b+p^m x)\rangle$$
$$= \langle\check{w}_0, \begin{pmatrix}1 & -\tau\\ 0 & 1\end{pmatrix}F(b+p^m x)\rangle = F(b-p^m\tau)$$

(where the final equality follows from the way in which the isomorphism $\mathcal{A}^{\leq k-2}\xrightarrow{\sim} W_{k-2}$ was defined). Substituting this into (4.19), we find that

$$\langle\infty - \frac{b}{p^m}, F(b+p^m x)\otimes c_g\rangle_2 = \int_{b/p^m}^{\infty} g(\tau)F(b-p^m\tau)d\tau,$$

for any $F\in\mathcal{A}^{\leq k-2}$. Comparing this formula, along with (4.18), to the construction of [16], we find that $\mu_g^{\pm}$ does indeed coincide (up to multiplication by a non-zero scalar) with the corresponding distribution that computes the p-adic L-function of g. □

Let us remark that Proposition 4.9 and the maps (4.8) show that the distributions defining the p-adic L-function of g extend from distributions on $\mathbb{Z}_p^\times$ to distributions on $\mathbb{Q}_p$ (or more precisely, to elements in the dual of $I^{\mathrm{la}}(\chi)$). In the case when π_p is special, this was noted in [5, §5.2].

5. $\mathcal{L}$-invariants and the Mazur-Tate-Teitelbaum conjecture

In the discussion of the preceding section, suppose that the finite slope newform f is special at p. Corollary 4.7 yields a pair of Γ-equivariant morphisms

$$D^0 \to (\pi(\chi)^{\pm,\lambda})_b^{\prime\mathrm{b}} \xrightarrow{\sim} (\pi^{\mathrm{la}}(\chi)^{\pm,\lambda})_b^{\prime\mathrm{b}}, \tag{5.1}$$

which up to twisting we may regard as a pair of Γ-equivariant morphisms $D^0\to \mathrm{O}(k)^{\mathrm{b}}$. By applying the functor $\mathrm{Hom}_\Gamma(D^0,–)$ to the short exact sequence (1.20), and analyzing the result of applying the coboundary map

$$\mathrm{Hom}_\Gamma(D^0,\mathrm{O}(k))\to \mathrm{Ext}^1_\Gamma(D^0, W_{k-2}\otimes_K |\det|^{-(k-2)/2})$$

to the morphisms (5.1), Breuil in [5] proves the following result:

Theorem 5.2. *There is a unique $\mathcal{L}\in K$ so that each of the maps of (5.1) arises from a Γ-equivariant map $D^0\to(\pi^{\mathrm{la}}(\chi,\mathcal{L})^{\lambda,\pm})_b^{\prime\mathrm{b}}$.*

Theorem 5.2 has a dual version, which states that there is a unique value of $\mathcal{L}$ for which one can find $\pi_0\times\mathcal{H}^p\times\mathrm{GL}_2(\mathbb{Q})_p$-equivariant maps

$$\pi^{\mathrm{la}}(\chi,\mathcal{L})^{\pm,\lambda}\to \tilde{H}^1_c(K^p)^{\pm} \tag{5.3}$$

extending the maps of Corollary 4.7. Furthermore, Breuil shows that these extensions (5.3) are unique.[7]

[7]Breuil actually proves this statement with Γ replaced by the intersection $\mathrm{GL}_2(\mathbb{Q})\bigcap K^p$. This weakens the equivariance claim, but strengthens the uniqueness claim.

Let $\mathcal{L}_f$ denote the unique value of $\mathcal{L}$ that satisfies the conclusion of Theorem 5.2. Theorem 5.2 then implies that the element $\infty - 0 \in D^0$ induces functionals

$$\mu_f^{\pm} : \pi^{\mathrm{la}}(\chi, \mathcal{L}_f) \to K \tag{5.4}$$

extending the functionals (4.8). (We write the subscript as f rather than g, since in the special case $f = g$.)

Breuil observes in [5] that a result of Orton [18] proves that when k is even, f has trivial nebentypus, and the U_p-eigenvalue of f equals $p^{(k-2)/2}$, then the element $-\mathcal{L}_f$ satisfies the Mazur-Tate-Teitelbaum conjecture for f. We close this note by explaining how this follows formally from Theorem 5.2, even without the assumption of trivial nebentypus.

Recall that $\pi^{\mathrm{la}}(\chi, \mathcal{L}_f)$ can be identified with a certain space of locally analytic functions on $\mathbb{Q}_p^\times$; in particular, the function $x^i \log_{\mathcal{L}_f}(x)$ supported on $\mathbb{Q}_p \setminus p\mathbb{Z}_p$ belongs to this space of functions, for $0 \leq i \leq k-2$.

Proposition 5.5. *(i) Let χ be a locally algebraic character as in Section 2 with k even and for which $\pi(\chi)$ is in the special case, such that $\alpha = p^{(k-2)/2}$ (so that $\beta = p^{k/2}$, and the central character of $\pi(\chi)$ is trivial on $\begin{pmatrix} p & \\ & p \end{pmatrix}$). If $\mu \in \pi^{\mathrm{la}}(\chi)'$ is a continuous functional on $\pi^{\mathrm{la}}(\chi)$ that is invariant under the action of $\begin{pmatrix} p & \\ & 1 \end{pmatrix}$, then*

$$\int_{\mathbb{Z}_p^\times} x^{(k-2)/2} d\mu = 0.$$

(ii) If in addition to the hypotheses of (i) the functional μ extends in a $\begin{pmatrix} p & \\ & 1 \end{pmatrix}$-invariant fashion to a continuous functional on $\pi^{\mathrm{la}}(\chi, \mathcal{L})$ for some $\mathcal{L} \in K$, then

$$\int_{\mathbb{Z}_p^\times} x^{(k-2)/2} \log_{\mathcal{L}} x \, d\mu = -\mathcal{L} \int_{\mathbb{Z}_p} x^{(k-2)/2} \, d\mu.$$

Proof. If one computes $\int_{\mathbb{Z}_p} x^{(k-2)/2} d\mu$ by writing the domain of integration as the union of $\mathbb{Z}_p^\times$ and $p\mathbb{Z}_p$, one obtains the formula of (i). (Use the matrix $\begin{pmatrix} p & \\ & 1 \end{pmatrix}$ to "scale" $p\mathbb{Z}_p$ back to $\mathbb{Z}_p$.)

Assume now that the hypothesis of (ii) holds. If one takes into account the formulas (2.3) and (2.7) giving the action of $\mathrm{GL}_2(\mathbb{Q}_p)$ on elements of $\pi^{\mathrm{la}}(\chi, \mathcal{L})$, as well as the fact that the polynomial $x^{(k-2)/2} \in I(\chi)$ has zero image in $\pi(\chi)$, and computes the value of

$$\int_{\mathbb{Q}_p \setminus p\mathbb{Z}_p} x^{(k-2)/2} \log_{\mathcal{L}} x \, d\mu$$

by writing the domain of integration as the union of $\mathbb{Z}_p^\times$ and $\mathbb{Q}_p \setminus \mathbb{Z}_p$, one finds that

$$\int_{\mathbb{Z}_p^\times} x^{(k-2)/2} \log_{\mathcal{L}} x \, d\mu = \mathcal{L} \int_{\mathbb{Q}_p \setminus p\mathbb{Z}_p} x^{(k-2)/2} \, d\mu. \tag{5.6}$$

(Use the matrix $\begin{pmatrix} 1 & \\ & p \end{pmatrix}$ to "scale" $\mathbb{Q}_p \setminus \mathbb{Z}_p$ back to $\mathbb{Q}_p \setminus p\mathbb{Z}_p$.) Finally, again taking into account the fact that the element $x^{(k-2)/2} \in I(\chi)$ has zero image in $\pi(\chi)$, as well as the vanishing statement of (i), one finds that

$$\int_{\mathbb{Q}_p \setminus p\mathbb{Z}_p} x^{(k-2)/2}\, d\mu = -\int_{\mathbb{Z}_p} x^{(k-2)/2}\, d\mu.$$

Combined with (5.6), this gives the formula of (ii). □

Corollary 5.7. *Suppose that f is a newform on $\Gamma_1(N) \bigcap \Gamma_0(p)$ of even weight k, such that $\alpha = p^{(k-2)/2}$ (and hence such that $\epsilon_N(p) = 1$). Then*

$$\int_{\mathbb{Z}_p^\times} x^{(k-2)/2} d\mu_f = 0 \tag{5.8}$$

and

$$\int_{\mathbb{Z}_p^\times} x^{(k-2)/2} \log_{\mathcal{L}_f} x\, d\mu_f = -\mathcal{L}_f \int_{\mathbb{Z}_p} x^{(k-2)/2}\, d\mu_f. \tag{5.9}$$

Proof. This follows immediately from Proposition 5.5 and the fact that $\infty - 0 \in D^0$ is invariant under $\begin{pmatrix} p & \\ & 1 \end{pmatrix}$ (which lies in Γ). □

By proposition 4.9, the left-hand side of (5.8) is the value of the p-adic L-function of f at $k/2$. Thus (5.8) exhibits the "exceptional zero" of the p-adic L-function. Again by Proposition 4.9, the left-hand side of (5.9) gives the derivative of the p-adic L-function of f at $k/2$, while the proof of Proposition 4.9 shows that the integral on the right-hand side of (5.9) gives the (suitably normalized) special value of the classical L-function of f at $k/2$. Thus we have established the Mazur-Tate-Teitelbaum conjecture for f, if we define the $\mathcal{L}$-invariant of f to be $-\mathcal{L}_f$.

Note that this $\mathcal{L}$-invariant is obviously "local" in the original sense of [16], in so far as it is invariant under twisting by characters of conductor prime to p that are trivial on p. In fact, it follows immediately from Breuil's construction that $\mathcal{L}_f$ is invariant under arbitrary twisting of f by Dirichlet characters. If f is any newform of even weight k that is special at p, then the twist f_ϵ of f by a Dirichlet character ϵ will satisfy the hypothesis of Corollary 5.7 for some choice of ϵ. Assuming that we can choose ϵ such that furthermore $L(f_\epsilon, 1) \neq 0$, we see that the value of $\mathcal{L}_f$ can be "detected" via the Mazur-Tate-Teitelbaum formula (5.9) (applied to f_ϵ). This allows a comparison between the Breuil-Darmon-Orton definition of $\mathcal{L}_f$ and other definitions of $\mathcal{L}_f$ with respect to which the Mazur-Tate-Teitelbaum conjecture has been proved – namely, the "Coleman $\mathcal{L}$-invariant", in terms of which the Mazur-Tate-Teitelbaum conjecture has been proved by Stevens, and the "Fontaine $\mathcal{L}$-invariant", in terms of which the Mazur-Tate-Teitelbaum conjecture has been proved by Kato-Kurihara-Tsuji. (The introduction to [7] provides a useful survey of these definitions of the $\mathcal{L}$-invariant, and the various means of comparing them.)[8]

[8]In fact, it is an easy consequence of Colmez [8] and of Kato's explicit reciprocity law for the p-adic L-function of f that $\mathcal{L}_f$, as defined above, agrees with the Fontaine-Mazur definition of the $\mathcal{L}$-invariant of f via the local-at-p Galois representation attached to f [15].

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