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# Completed cohomology – a survey

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This note summarizes the theory of $p$-adically completed cohomology. This construction was first introduced in paper [8] (although insufficient attention was given there to the integral aspects of the theory), and then further developed in the papers [4] and [12]. The papers [8] and [4] may give the impression that $p$-adically completed cohomology is some sort of auxiliary construction that can be used to prove theorems (of either a $p$-adic or classical nature) about automorphic forms. However, we believe that $p$-adically completed cohomology is in fact an object of fundamental importance, and that it provides the best approximation that we know of to spaces of "$p$-adic automorphic forms". (In particular, unlike the spaces that go by this name that are sometimes constructed by arithmetico-geometric means in the theory of modular curves, or more generally Shimura varieties, $p$-adically completed cohomology admits a representation of the $p$-adic group, and thus allows the introduction of representation-theoretic methods into the study of $p$-adic properties of automorphic forms.)

A systematic exposition of the theory, and of its (largely conjectural, at this point) applications to the $p$-adic aspects of the Langlands correspondence between automorphic eigenforms and Galois representations, will be given in the paper [6]. These notes provide a summary of some of the basic points of the theory, as well as one of the main conjectures of [6] (Conjecture 1.5 below). The anticipated connection of the theory with the $p$-adic aspects of the Langlands program are discussed in the final Section 1.8. We don't attempt to formulate any formal conjectures in this section, but simply try to indicate, in very general and somewhat idealized terms, what form we expect this connection to take. Our main goal for doing so here is to explain some the motivations behind making

Conjecture 1.5. It remains to be seen how closely our expectations, and our conjecture, conform to the reality of the situation.

## 1.1 Definitions

Let $G_0$ be a pro-finite group, assumed to admit a countable basis of neighbourhoods of the identity, consisting of normal open subgroups, say

$$\cdots \subset G_r \subset \cdots \subset G_1 \subset G_0.$$

Suppose given a tower of topological spaces

$$\cdots \to X_r \to \cdots \to X_1 \to X_0,$$

each equipped with an action of $G_0$, such that:

1. The maps $X_{r+1} \to X_r$ are $G_0$-equivariant.
2. $G_r$ acts trivially on $X_r$, and realizes $X_r$ as a $G_0/G_r$-torsor over $X_0$. (In particular, all the maps in the tower are finite coverings.)

In this context we may define the $p$-adically completed homology and cohomology modules attached to the tower $X_\bullet$ (having fixed the prime $p$), namely:

$$\tilde{H}_\bullet := \varprojlim_r H_\bullet(X_r, \mathbb{Z}_p)$$

and

$$\tilde{H}^\bullet := \varprojlim_s \varinjlim_r H^\bullet(X_r, \mathbb{Z}/p^s\mathbb{Z}).$$

We can also consider Borel–Moore and compactly supported variants:

$$\tilde{H}_\bullet^{BM} := \varprojlim_r H_\bullet^{BM}(X_r, \mathbb{Z}_p)$$

and

$$\tilde{H}_c^\bullet := \varprojlim_s \varinjlim_r H_c^\bullet(X_r, \mathbb{Z}/p^s\mathbb{Z}).$$

There are natural maps $\tilde{H}_\bullet \to \tilde{H}_\bullet^{BM}$ and $\tilde{H}_c^\bullet \to \tilde{H}^\bullet$.

From now on we suppose that each $X_r$ is a manifold which is homotopic to a finite simplicial complex. This ensures us that all the homology spaces (usual or Borel–Moore) and cohomology spaces (usual or compactly supported) of $X_r$ that we have written down are finitely generated over the indicated ring of coefficients, and that they are related by appropriate forms of duality.[1]

We also suppose from now on that $G_0$ is a $p$-adic analytic group. Without this hypothesis, it seems impossible to control the inverse limits involved in our constructions. On the other hand, with this hypothesis, one can control these limits in a very satisfactory way. Indeed, Lazard [14] has proved that completed group ring $\mathbb{Z}_p[[G_0]]$ is Noetherian, and Schneider and Teitelbaum [16] have explained how this result can be applied to control various analytic difficulties that arise in the study of $p$-adic Banach-space representations of $G_0$.

In particular, we have the following result (which strengthens the results stated in [8], in so far as it deals explicitly with homology as well as cohomology, and pays attention to the integral aspects of the constructions, and not just to the objects obtained after tensoring up with $\mathbb{Q}_p$), whose proof will appear in [6].

**Theorem 1.1** *1. The natural action of $\mathbb{Z}_p[[G_0]]$ on the modules $\tilde{H}_\bullet$ and $\tilde{H}_\bullet^{BM}$ makes each of them a finitely generated left module over $\mathbb{Z}_p[[G_0]]$. Furthermore, the canonical topology on each of these modules (obtained by writing it as a quotient of a finite direct sum of copies of $\mathbb{Z}_p[[G_0]]$) coincides with projective limit topology.*

*Note that this implies in particular that each of the torsion submodules $\tilde{H}_\bullet[p^\infty]$ and $\tilde{H}_\bullet^{BM}[p^\infty]$ is finitely generated over $\mathbb{Z}_p[[G_0]]$, and so in particular, is of bounded exponent.*

*2. The spaces $\tilde{H}^\bullet$ and $\tilde{H}_c^\bullet$ are $p$-adically complete and their $p$-power torsion submodules have bounded exponent.*

*3. There are short exact sequences*

$$0 \to \mathrm{Hom}_{\mathrm{cont}}(\tilde{H}_{\bullet-1}, \mathbb{Q}_p/\mathbb{Z}_p) \to \tilde{H}^\bullet \to \mathrm{Hom}_{\mathrm{cont}}(\tilde{H}_\bullet, \mathbb{Z}_p) \to 0$$

*(here* cont *means continuous with respect to the canonical topology on the source, and the evident topology — either discrete or $p$-adic — on the target) and*

$$0 \to \mathrm{Hom}(\tilde{H}^{\bullet+1}[p^\infty], \mathbb{Q}_p/\mathbb{Z}_p) \to \tilde{H}_\bullet \to \mathrm{Hom}(\tilde{H}^\bullet, \mathbb{Z}_p) \to 0$$

*(since the natural topology on $\tilde{H}^\bullet$ is the $p$-adic topology, and since all $\mathbb{Z}_p$-linear maps are automatically $p$-adically continuous, there is*

[1] The paper [12] provides a sheaf-theoretic description of $\tilde{H}^\bullet$ and $\tilde{H}_c^\bullet$, which allows for more flexibility in the basic set-up of the theory than we permit here. However, we will have no need for this extra generality in the applications that we have in mind.

*no need to explicitly specify any continuity conditions on the maps appearing in this exact sequence), as well as similar short exact sequences relating* $\tilde{H}_\bullet^{BM}$ *and* $\tilde{H}_c^\bullet$. *These exact sequences are compatible with the maps* $\tilde{H}_\bullet \to \tilde{H}_\bullet^{BM}$, *and the maps* $\tilde{H}_c^\bullet \to \tilde{H}^\bullet$.

4. *For each* $r \geq 0$, *there are Hochschild–Serre type spectral sequences*

$$E_2^{i,j} := H^i(G_r, \tilde{H}^j) \implies H^{i+j}(X_r, \mathbb{Z}_p)$$

*and*

$$E_2^{i,j} := H^i(G_r, \tilde{H}_c^j) \implies H_c^{i+j}(X_r, \mathbb{Z}_p),$$

*compatible with the maps* $\tilde{H}_c^\bullet \to \tilde{H}^\bullet$.

**Remark 1.2** If $M$ is any continuous $G_0$-module over $\mathbb{Z}_p$, then we may associate a family of local systems $\mathcal{M}_\bullet$ on the tower $X_\bullet$ to $M$, and there is an analogue of part (4) of the theorem for the cohomology of the local system $\mathcal{M}_r$.

**Remark 1.3** The theorem shows the advantages of working with both homology and cohomology. It is on the homology side that the algebraic aspects of the theory are most transparent, while the cohomology side is better adapted to comparison with the situation at finite levels.

It is useful to make some additional definitions on the cohomology side, as follows:

$$\begin{aligned}\hat{H}^\bullet &:= \text{ the } p\text{-adic completion of } \varinjlim_r H^\bullet(X_r, \mathbb{Z}_p) \\ &= \varprojlim_s \big(\varinjlim_r H^\bullet(X_r, \mathbb{Z}_p)/p^s H^\bullet(X_r, \mathbb{Z}_p)\big)\end{aligned}$$

and

$$\begin{aligned}T_p H^\bullet &:= \text{ the } p\text{-adic Tate module of } \varinjlim_r H^\bullet(X_r, \mathbb{Z}_p) \\ &= \varprojlim_s \varinjlim_r H^\bullet(X_r, \mathbb{Z}_p)[p^s]\end{aligned}$$

(the transition maps being given by multiplication by $p$). Note that the $p$-adic completion kills the $p$-divisible part of $\varinjlim_r H^\bullet(X_r, \mathbb{Z}_p)$, while the $p$-adic Tate module knows only about the torsion subgroup of this divisible part. There is an exact sequence

$$0 \to \hat{H}^\bullet \to \tilde{H}^\bullet \to T_p H^{\bullet+1} \to 0. \tag{1.1}$$

Since $T_pH^{\bullet+1}$ is $p$-torsion-free and $p$-adically complete, we may take the $\mathbb{Z}_p$ dual of this exact sequence to obtain a short exact sequence

$$0 \to \mathrm{Hom}(T_pH^{\bullet+1}, \mathbb{Z}_p) \to \mathrm{Hom}(\tilde{H}^\bullet, \mathbb{Z}_p) \to \mathrm{Hom}(\hat{H}^\bullet, \mathbb{Z}_p) \to 0.$$

Recall from Theorem 1.1 (3) that $\mathrm{Hom}(\tilde{H}^\bullet, \mathbb{Z}_p)$ is the $p$-torsion free quotient of $\tilde{H}_\bullet$.

There are similar constructions for compactly supported cohomology, and their formation is compatible with the maps $\tilde{H}^\bullet_c \to \tilde{H}^\bullet$.

## 1.2 Non-commutative Iwasawa theory

If $M$ is any finitely generated left $\mathbb{Z}_p[[G_0]]$-module, then we define

$$E^\bullet(M) := \mathrm{Ext}^\bullet(M, \mathbb{Z}_p[[G_0]]).$$

These are again naturally $\mathbb{Z}_p[[G_0]]$-modules. (We compute the Exts via the left $\mathbb{Z}_p[[G_0]]$-module structure on $\mathbb{Z}_p[[G_0]]$, leaving a right $\mathbb{Z}_p[[G_0]]$-module structure on $E^\bullet(M)$. We convert this back to a left-module structure via the canonical anti-involution of $\mathbb{Z}_p[[G_0]]$ induced by $g \mapsto g^{-1}$.) We note that these modules are unchanged (up to a natural isomorphism, and applying the forgetful functor from $\mathbb{Z}_p[[G_0]]$-modules to $\mathbb{Z}_p[[G_0']]$-modules) if we replace $G_0$ by any open subgroup $G_0'$.

We define the *codimension* of $M$ (or $\mathrm{codim}\, M$ for short) to be the minimal value of $i$ such that $E^i(M) \neq 0$. If $G_0$ is commutative, then this agrees with the usual notion of codimension of support of the coherent sheaf associated to $M$ on $\mathrm{Spec}\, \mathbb{Z}_p[[G_0]]$. In general, even if $G_0$ is non-commutative (which is typically the case), this notion behaves entirely analogously to the usual notion of codimension of support of a sheaf in algebraic geometry [18]. In addition to the codimension of $M$, it is sometimes convenient to speak of the *dimension* of $M$ (or the *Iwasawa dimension* of $M$, for emphasis), which we define to be the quantity $1 + \dim G_0 - \mathrm{codim}\, M$. (If we think of $\mathrm{codim}\, M$ as being the codimension of the support of the — in general non-existent — coherent sheaf associated to $M$ on the — in general non-existent — Spec of $\mathbb{Z}_p[[G_0]]$, then the Iwasawa dimension of $M$ would be the dimension of its support.)

We also remark that if we shrink $G_0$ sufficiently (i.e. replace $G_0$ by $G_r$ for a large enough value of $r$), then $\mathbb{Z}_p[[G_0]]$ admits a skew-field of fractions, say $\mathcal{L}$, and hence we can speak of a finitely generated $\mathbb{Z}_p[[G_0]]$-module $M$ being torsion-free (the natural map $M \to \mathcal{L} \otimes M$ is injective) or torsion ($\mathcal{L} \otimes M = 0$), and we can speak of the rank of the (torsion-free part of) $M$ (i.e. the rank of the $\mathcal{L}$-vector space $\mathcal{L} \otimes M$). Having positive rank is equivalent to being of codimension 0, while being torsion (or equivalently, being of rank 0) is equivalent to being of positive codimension.

## 1.3 Poincaré duality

We suppose that $X_0$ (and hence every $X_r$) is equidimensional of some dimension $d$. We then have two spectral sequences that express Poincaré duality in the $p$-adically completed situation (the detailed consructions of which will appear in [6]). Note that both work with homology, rather than cohomology. (The reason for this is that the functors $E^i$ intervene.)

Here are the spectral sequences:

$$E_2^{i,j} := E^i(\tilde{H}_j) \implies \tilde{H}_{d-i-j}^{BM}$$

and

$$E_2^{i,j} := E^i(\tilde{H}_j^{BM}) \implies \tilde{H}_{d-i-j}.$$

They are compatible with the maps $\tilde{H}_\bullet \to \tilde{H}_\bullet^{BM}$.

We point out the amplitude of the $\delta$-functor $E^\bullet$ is given by the dimension of the group $G_0$. In applications, this can often be quite a bit larger than the dimension $d$ of the spaces $X_r$, which can lead to interesting tension in these spectral sequences.

## 1.4 A simple example of everything so far

An illustrative example is given by taking $G_0 = \mathbb{Z}_p$, $G_r := p^r\mathbb{Z}_p$ for $r > 0$, and $X_r := \mathbb{R}/p^r\mathbb{Z}$ for each $r \geq 0$. We let $G_0/G_r := \mathbb{Z}_p/p^r\mathbb{Z}_p \xrightarrow{\sim} \mathbb{Z}/p^r\mathbb{Z}$ act on $X_r$ in the obvious manner by translations. Thus each $X_r$ is a circle, and each map $X_{r+1} \to X_r$ is a degree $p$ covering map. Note that $\mathbb{Z}_p[[G_0]] = \mathbb{Z}_p[[T]]$.

Clearly $\tilde{H}_0 = \mathbb{Z}_p$, with trivial $G_0$-action, while $\tilde{H}_1 = 0$ (the inverse limit of a sequence of copies of $\mathbb{Z}_p$ under the multiplication by $p$ map obviously vanishes). Similarly $\tilde{H}^0 = \mathbb{Z}_p$, while $\tilde{H}^1 = 0$. (Since the spaces $X_r$ are compact, Borel–Moore homology agrees with usual homology, and compactly supported cohomology agrees with usual cohomology.)

Since $G_r$ is procyclic, we see that $H^0(G_r, \mathbb{Z}_p) = H^1(G_r, \mathbb{Z}_p) = \mathbb{Z}_p$, and thus we do indeed recover the cohomology of the circle $X_r$ from the completed cohomology via the Hochschild–Serre type spectral sequence.

Similarly, using the resolution

$$0 \to \mathbb{Z}_p[[T]] \xrightarrow{T\cdot} \mathbb{Z}_p[[T]] \to \mathbb{Z}_p \to 0,$$

we compute that $E^0(\mathbb{Z}_p) = 0$, while $E^1(\mathbb{Z}_p) = \mathbb{Z}_p$. This is immediately seen to be consistent with the Poincaré duality spectral sequence.

## 1.5 Congruence quotients of symmetric spaces

We now suppose that $\mathbb{G}$ is a connected reductive linear algebraic group over $\mathbb{Q}$. We let $\mathbb{A}$ denote the adèle ring over $\mathbb{Q}$, let $\mathbb{A}^\infty$ denote the ring of finite adèles, and let $\mathbb{A}^{\infty,p}$ denote the ring of prime-to-$p$ finite adèles. We fix a compact open subgroup $K^{\infty,p}$ of $\mathbb{G}(\mathbb{A}^{\infty,p})$ (the tame level), take $G_0$ to be a sufficiently small compact open subgroup of $\mathbb{G}(\mathbb{Q}_p)$, and let $(G_r)_{r\geq 1}$ be a sequence of normal open subgroups of $G_0$ that form a neighbourhood basis of the identity in $G_0$.

We let $K_\infty^\circ$ denote the connected component of the identity of a maximal compact subgroup $K_\infty$ of $\mathbb{G}(\mathbb{R})$, and let $A_\infty^\circ$ denote the connected component of the identity of the group of real points $A_\infty$ of a maximal $\mathbb{Q}$-split torus in the centre of $\mathbb{G}$.

We define

$$X_r := \mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A})/A_\infty^\circ K_\infty^\circ G_r K^{\infty,p}.$$

The $X_r$ form a tower with an action of $G_0$ satisfying the axioms of Section 1, and so the preceding theory applies. Furthermore, the usual argument about limits of (co)homology of arithmetic quotients in the adèlic setting shows that $\tilde{H}_\bullet$, $\tilde{H}^\bullet$, etc. all inherit not just an action of $G_0$, but of the entire $p$-adic group $\mathbb{G}(\mathbb{Q}_p)$. They also inherit an action of a suitably completed Hecke algebra $\mathbb{T}$ (built up out of spherical Hecke operators at primes $\ell \neq p$ over which $\mathbb{G}$ and the tame level $K^{\infty,p}$ are unramified), and an action of the component group $\pi_0 := (A_\infty K_\infty)/(A_\infty^\circ K_\infty^\circ)$.

The following theorem is the main result of [4]. (The result about $T_p H^{\bullet+1}$ being torsion, even in the middle dimension in the discrete series case, is not stated there, but follows from the proof.)

**Theorem 1.4** *If, in the preceding setting, the group $\mathbb{G}$ is semi-simple, then the modules $\tilde{H}_n$ are torsion $\mathbb{Z}_p[[G_0]]$-modules, unless the group $\mathbb{G}(\mathbb{R})$ admits discrete series and $n$ is the "middle dimension"* (*i.e. one half* *of the dimension of the $X_r$). In this latter case,* $\mathrm{Hom}(\hat{H}^n, \mathbb{Z}_p)$ *(and hence $\tilde{H}_n$) has positive rank, while* $\mathrm{Hom}(T_p H^{n+1}, \mathbb{Z}_p)$ *is torsion over* $\mathbb{Z}_p[[G_0]]$.

We believe that this is actually only the beginning of the story with regard to the codimensions of the various $\tilde{H}_\bullet$. In the following section, we will go on to describe how we expect the rest of the story to play out. But first, we will close this section by discussing the boundary long exact sequence.

Each $X_r$ embeds as an open subset of its Borel-Serre compactification $\overline{X}_r$. We let $\partial X_r$ denote $\overline{X}_r \setminus X_r$; it is the boundary of $\overline{X}_r$. As $r$ varies the spaces $\overline{X}_r$ form a tower to which the $p$-adically completed cohomology machinery applies, and hence we may define completed (co)homology spaces $\tilde{H}_\bullet(\partial)$ and $\tilde{H}^\bullet(\partial)$. There are long exact sequences

$$\cdots \to \tilde{H}_\bullet(\partial) \to \tilde{H}_\bullet \to \tilde{H}_\bullet^{BM} \to \tilde{H}_{\bullet-1}(\partial) \to \cdots \tag{1.2}$$

and

$$\cdots \to \tilde{H}^{\bullet-1}(\partial) \to \tilde{H}_c^\bullet \to \tilde{H}^\bullet \to \tilde{H}^\bullet(\partial) \to \cdots . \tag{1.3}$$

The complement $\partial X_r$ is a union of strata $\partial X_{\mathbb{P},r}$ indexed by the (equivalence classes of) proper parabolic subgroups $\mathbb{P}$ of $\mathbb{G}$. For fixed $\mathbb{P}$ and varying $r$ the strata $\partial X_{\mathbb{P},r}$ again form a tower to which the $p$-adically completed cohomology machine applies, allowing us to define $\tilde{H}_\bullet(\partial_\mathbb{P})$ and $\tilde{H}^\bullet(\partial_\mathbb{P})$. There is a Mayer–Vietoris spectral sequence relating the various $\tilde{H}_\bullet(\partial_\mathbb{P})$ (resp. $\tilde{H}^\bullet(\partial_\mathbb{P})$) to $\tilde{H}_\bullet(\partial)$ (resp. $\tilde{H}^\bullet(\partial)$).

If we let $\mathbb{P} = \mathbb{M}\mathbb{N}$ be a Levi decomposition of the proper parabolic $\mathbb{P}$, then each $\partial_{\mathbb{P},r}$ is a bundle over a union of congruence quotients associated to $\mathbb{M}$, whose fibre is a compact nilmanifold (of dimension equal to $\dim \mathbb{N}$). A generalization to a nilmanifolds of the example of Section 1.4 shows that these nilmanifolds contribute to completed (co)homology only in degree 0, and so $\tilde{H}_\bullet(\partial_\mathbb{P})$ and $\tilde{H}^\bullet(\partial_\mathbb{P})$ are supported in the same degrees as the $p$-adically completed (co)homology associated to $\mathbb{M}$. To describe the precise relationship between the $p$-adically completed cohomology for $\partial_\mathbb{P}$ and for $\mathbb{M}$, it is convenient to take a direct limit over all tame levels. We then find (at least when $p$ is odd) that:

$$\begin{aligned}&\varinjlim_{\text{tame levels}} \tilde{H}^\bullet(\partial_{\mathbb{P}}) \\ &\quad \xrightarrow{\sim} \operatorname{Ind}_{\mathbb{P}(\mathbb{A}_f)}^{\mathbb{G}(\mathbb{A}_f)} \Big( \varinjlim_{\text{tame levels}} \text{ invariants under } \ker(\pi_0^{\mathbb{M}} \to \pi_0) \\ &\qquad\qquad \text{of the } \tilde{H}^\bullet \text{ associated to } \mathbb{M} \Big), \qquad (1.4)\end{aligned}$$

where the induction is $p$-adically completed at $p$, and smooth at primes away from $p$. (We have written $\pi_0^{\mathbb{M}}$ to denote the analogue for the Levi subgroup $\mathbb{M}$ of the component group $\pi_0$ which was defined above for $\mathbb{G}$. Since $\pi_0$ and $\pi_0^{\mathbb{M}}$ are finite 2-groups, there is the possibility of extra complications when $p = 2$, which we won't attempt to address here.)

We note that much of this general set-up for congruence quotients has been described by Richard Hill in [12, §4].

## 1.6 Conjectures on codimensions

We maintain the set-up of the preceding section. We write $G_\infty := \mathbb{G}(\mathbb{R})$, and we begin by defining two quantities associated to $\mathbb{G}$:

$$l_0 := \text{ rank of } G_\infty - \text{ rank of } A_\infty K_\infty,$$

and

$$q_0 = (\text{ dimension of } G_\infty - \text{ dimension of } A_\infty K_\infty - l_0\,)/2 = (d - l_0)/2,$$

where $d$ denotes the dimension of the quotients $X_r$. Note that if $\mathbb{G}$ is semi-simple, then these quantities coincide with the quantities denoted by the same symbols in Borel–Wallach [2]. Namely, $l_0$ denotes the "defect" of $G_\infty$ with regard to possessing discrete series, while $q_0$ denotes the first "interesting" dimension for the (co)homology of $X_r$.

We now state our basic conjecture on codimensions.

**Conjecture 1.5** *1. If $n < q_0$, then the codimension of $\tilde{H}_n$ is greater than $l_0 + q_0 - n$.*
*2. The codimension of $\tilde{H}_{q_0}$ equals $l_0$.*
*3. If $n < q_0$, then the codimension of $\tilde{H}_n^{BM}$ is greater than $l_0 + q_0 - n$.*
*4. The codimension of $\tilde{H}_{q_0}^{BM}$ equals $l_0$.*
*5. $\tilde{H}_{q_0}$ is p-torsion-free.*
*6. $\tilde{H}_{q_0}^{BM}$ is p-torsion-free.*
*7. $\tilde{H}_n = 0$ if $n > q_0$.*

*8.* $\tilde{H}_n^{BM} = 0$ *if* $n > q_0$.

In fact, these conjectures are not independent, as the following theorem makes clear.

**Theorem 1.6** *1. If parts* (1) *and* (2) (*resp. parts* (3) *and* (4)) *of Conjecture* 1.5 *hold for* $\mathbb{G}$*, and for all the proper Levi subgroups of* $\mathbb{G}$*, then parts* (3) *and* (4) (*resp. parts* (1) *and* (2)) *of the conjecture also holds for* $\mathbb{G}$ (*and also for all its Levi subgroups*)*, as do parts* (7) *and* (8).

*2. If part* (7) (*resp. part* (8)) *of Conjecture* 1.5 *holds for* $\mathbb{G} \times \mathbb{G}$*, then part* (5) (*resp. part* (6)) *of the conjecture holds for* $\mathbb{G}$.

*Proof* Suppose that parts (1) and (2), or parts (3) and (4), of the conjecture hold for all the proper Levi subgroups of $\mathbb{G}$. Applying the theorem inductively, we conclude that parts (1), (2), (3), (4), (5), and (6) of the conjecture hold for these Levi subgroups. Formula (1.4) then allows us to bound from below the codimensions of the $p$-adically completed homology spaces associated to the various boundary strata. A comparison of the invariants $l_0$ and $q_0$ for $\mathbb{G}$ and for its Levi subgroups, along with a consideration of the Mayer–Vietoris spectral sequence that computes the $p$-adically completed homology of the boundary in terms of the $p$-adically completed cohomology of its various strata, as well as of the long exact sequence (1.2), then shows that parts (1) and (2) of the conjecture for $\mathbb{G}$ are equivalent to parts (3) and (4) for $\mathbb{G}$. Looking at the Poincaré duality spectral sequence then shows that parts (7) and (8) of the conjecture also hold.

Part (2) of the theorem follows by applying a Kunnëth-type theorem to compare the completed cohomology for $\mathbb{G}$ to that for $\mathbb{G} \times \mathbb{G}$. $\square$

**Remark 1.7** In particular, the preceding theorem shows that if either parts (1) and (2), or part (3) and (4), of Conjecture 1.5 hold for every group $\mathbb{G}$, then all parts of the conjecture hold for all groups $\mathbb{G}$.

**Remark 1.8** Richard Hill has also made a conjecture about the vanishing of completed cohomology of arithmetic quotients, namely [12, Conj. 3]. Conjecture 1.5 implies Hill's conjecture, but is quite a bit stronger in general. The various vanishing results proved in [12, §5], being consistent with [12, Conj. 3], are thus also consistent with our general conjecture.

**Remark 1.9** Eric Urban has made a conjecture somewhat analogous to Conjecture 1.5 concerning the dimensions of irreducible components of "eigenvarieties" [17, Conj. 5.7.3]. In fact, since "eigenvarieties" can also be constructed from completed cohomology via applying the locally analytic Jacquet module functor of [7] (see [8]), it seems likely that the eventual relationship of Conjecture 1.5 to Urban's conjecture will be more than one of mere analogy. On the other hand, neither conjecture seems to be an immediate logical consequence of the other, and a precise understanding of their mutual relationship remains to be found.

There are several different heuristics and motivations behind Conjecture 1.5 (as well as a small amount of actual evidence); some of these heuristics are explained in Section 1.8. For now, let us remark that it is consistent with Künneth, and it is consistent with the Poincaré duality spectral sequence. Of course, in the case when $\mathbb{G}$ is semi-simple, it is also consistent with Theorem 1.4, which shows at least that the codimension of $\tilde{H}_n$ is zero (respectively, positive) exactly when it is predicted to be so by the conjecture.

We now discuss some illustrative examples.

**Example 1.10** If $\mathbb{G}$ is a torus, then $q_0 = 0$, and a generalization of the analysis of the example in Section 1.4 shows that in this case, Conjecture 1.5 is equivalent to Leopoldt's conjecture. (See [12, Cor. 5].)

**Example 1.11** If $\mathbb{G} = \mathrm{SL}_2$, then $l_0 = 0$ and $q_0 = 1$, and one finds that $\tilde{H}_0 = \mathbb{Z}_p$ has codimension 3, while $\tilde{H}_1$ has codimension zero. (Apply Theorem 1.4.)

**Example 1.12** If $K$ is a quadratic imaginary field, and $\mathbb{G} = \mathrm{SL}_{2/K}$ (regarded as a group over $\mathbb{Q}$ by restriction of scalars, as usual), then $l_0 = 1$ and $q_0 = 1$, and one finds that $\tilde{H}_0 = \mathbb{Z}_p$, and so has codimension 6, that $\tilde{H}_1$ has codimension 1, and that $\tilde{H}_2 = 0$. (Apply Theorem 1.4 together with the Poincaré duality spectral sequence. Note that the term $E^6(\tilde{H}_0) = E^6(\mathbb{Z}_p) = \mathbb{Z}_p$ plays a crucial role in showing that $\tilde{H}_1 \neq 0$.)

**Example 1.13** If $\mathbb{G}$ is semi-simple and simply connected and satisfies the congruence subgroup property, then $\tilde{H}_1$ is finite.

**Example 1.14** If $G = \mathrm{Sp}_4$, then $l_0 = 0$ and $q_0 = 3$. One finds that $\tilde{H}_0 = \mathbb{Z}_p$, $\tilde{H}_1$ is finite (since $\mathbb{G}$ satisfies the congruence subgroup property), $\tilde{H}_2$ has codimension at least 1 (by Theorem 1.4), and $\tilde{H}_3$ has codimension 0 (again by Theorem 1.4). The Poincaré duality spectral sequence then shows that $\tilde{H}_n = 0$ for $n \geq 4$.

**Remark 1.15** It is not clear in general how one might go about computing the codimension of support of (or any other information about) the completed cohomology. However, suppose that one knew that the $\mathbb{Z}_p$-cohomology of each $X_r$ was $p$-torsion-free (or, more generally, of bounded exponent). The Tate module term in the exact sequence (1.1) would then vanish, and hence there would be an isomorphism $\hat{H}^\bullet \xrightarrow{\sim} \tilde{H}^\bullet$. Now $\hat{H}^\bullet$ is the $p$-adic completion of the classical cohomology, and this classical cohomology may be described in terms of classical automorphic forms (at least after tensoring with $\mathbb{Q}_p$ over $\mathbb{Z}_p$). Thus it seems not totally inconceivable that one might be able to prove results about (e.g. the codimension of support of) $\hat{H}^\bullet$. (This is done for modular curves in [9], although admittedly the arguments there rely on the full strength of the $p$-adic Langlands program for $\mathrm{GL}_2(\mathbb{Q}_p)$.) Consequently, it seems worth investigating and attempting to establish torsion-freeness results for the cohomology of congruence quotients, say in the Shimura variety context.

## 1.7 Mod $p$ analogues

Returning to the notation of Section 1.1, we observe that we may define mod $p$ analogues of completed homology and cohomology. Namely, we write

$$\tilde{H}_{\bullet,\mathbb{F}_p} := \varprojlim_r H_\bullet(X_r, \mathbb{F}_p)$$

and

$$H^\bullet_{\mathbb{F}_p} := \varinjlim_r H^\bullet(X_r, \mathbb{F}_p).$$

We can also consider Borel–Moore and compactly supported variants:

$$\tilde{H}^{BM}_{\bullet,\mathbb{F}_p} := \varprojlim_r H^{BM}_\bullet(X_r, \mathbb{F}_p)$$

and

$$H^\bullet_{c,\mathbb{F}_p} := \varinjlim_r H^\bullet_c(X_r, \mathbb{F}_p).$$

There are natural maps $\tilde{H}_{\bullet,\mathbb{F}_p} \to \tilde{H}^{BM}_{\bullet,\mathbb{F}_p}$ and $H^\bullet_{c,\mathbb{F}_p} \to H^\bullet_{\mathbb{F}_p}$. Note that in the case of cohomology, there is no completion at all. We then have the following result (whose proof will appear in [6]), which provides the analogue of Theorem 1.1 for $\mathbb{F}_p$-coefficients, as well as universal coefficient type results relating the completed (co)homology to its mod $p$ variant.

**Theorem 1.16** *1. There is a natural action of the completed group ring $\mathbb{F}_p[[G_0]]$ on each of $\tilde{H}_{\bullet,\mathbb{F}_p}$ and $\tilde{H}^{BM}_{\bullet,\mathbb{F}_p}$, which makes each of these* *spaces a finitely generated left module over $\mathbb{F}_p[[G_0]]$. Furthermore, the canonical topology on each of these modules (obtained by writing it as a quotient of a finite direct sum of copies of $\mathbb{F}_p[[G_0]]$) coincides with projective limit topology.*

2. *The spaces $H^\bullet_{\mathbb{F}_p}$ and $H^\bullet_{c,\mathbb{F}_p}$ are admissible smooth representations of $G_0$ over $\mathbb{F}_p$.*
3. *There are natural isomorphisms*

$$H^\bullet_{\mathbb{F}_p} \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont}}(\tilde{H}_{\bullet,\mathbb{F}_p}, \mathbb{F}_p)$$

*(here* cont *means continuous with respect to the canonical topology on the source, and the evident topology — either discrete or p-adic — on the target) and*

$$\tilde{H}_{\bullet,\mathbb{F}_p} \xrightarrow{\sim} \mathrm{Hom}(H^\bullet_{\mathbb{F}_p}, \mathbb{F}_p)$$

*(since the natural topology on $\tilde{H}^\bullet$ is the $p$-adic topology, and since all $\mathbb{Z}_p$-linear maps are automatically $p$-adically continuous, there is no need to explicitly specify any continuity conditions on the maps appearing in this exact sequence), as well as similar isomorphisms relating $\tilde{H}^{BM}_{\bullet,\mathbb{F}_p}$ and $H^\bullet_{c,\mathbb{F}_p}$. These isomorphisms are compatible with the maps $\tilde{H}_{\bullet,\mathbb{F}_p} \to \tilde{H}^{BM}_{\bullet,\mathbb{F}_p}$, and the maps $H^\bullet_{c,\mathbb{F}_p} \to H^\bullet_{\mathbb{F}_p}$.*

4. *For each $r \geq 0$, there are Hochschild–Serre type spectral sequences*

$$E_2^{i,j} := H^i(G_r, H^j) \implies H^{i+j}(X_r, \mathbb{F}_p)$$

*and*

$$E_2^{i,j} := H^i(G_r, H^j_c) \implies H_c^{i+j}(X_r, \mathbb{F}_p),$$

*compatible with the maps $H^\bullet_{c,\mathbb{F}_p} \to H^\bullet_{\mathbb{F}_p}$.*

5. *There are short exact sequences*

$$0 \to \tilde{H}_\bullet/p\tilde{H}_\bullet \to \tilde{H}_{\bullet,\mathbb{F}_p} \to \tilde{H}_{\bullet-1}[p] \to 0$$

*and*

$$0 \to \tilde{H}^\bullet/p\tilde{H}^\bullet \to H^\bullet_{\mathbb{F}_p} \to \tilde{H}^{\bullet+1}[p] \to 0,$$

*as well as similar isomorphisms relating $\tilde{H}^{BM}_{\bullet,\mathbb{F}_p}$ to $\tilde{H}^{BM}_\bullet$ and $H^\bullet_{c,\mathbb{F}_p}$ to $\tilde{H}^\bullet_c$.*

If $M$ is any finite generated left $\mathbb{F}_p[[G_0]]$-module, we define $E^\bullet_{\mathbb{F}_p}(M) := \mathrm{Ext}^\bullet(M, \mathbb{F}_p[[G_0]])$. We can then define the $\mathbb{F}_p$-*codimension* of $M$ (or $\mathrm{codim}_{\mathbb{F}_p} M$, for short) to be the minimal $i$ such that $E^i_{\mathbb{F}_p}(M) \neq 0$. Of course, we may also regard $M$ as a left $\mathbb{Z}_p[[G_0]]$-module, and there are short exact sequences

$$0 \to E^{\bullet}(M) \to E^{\bullet}_{\mathbb{F}_p}(M) \to E^{\bullet+1}(M) \to 0$$

(as follows by applying $\mathrm{Ext}^{\bullet}(M,-)$ to the short exact sequence $0 \to \mathbb{Z}_p[[G_0]] \xrightarrow{p\cdot} \mathbb{Z}_p[[G_0]] \to \mathbb{F}_p[[G_0]] \to 0$), from which one deduces that the $\mathbb{F}_p$-codimension of $M$ is one less than the codimension of $M$. In particular, we find that the Iwasawa dimension of $M$ is equal to $\dim G_0 - \mathrm{codim}_{\mathbb{F}_p} M$.

If $M$ is a finitely generated $\mathbb{Z}_p[[G_0]]$-module, then $M/pM$ is a finitely generated $\mathbb{F}_p[[G_0]]$-module, and we have the following simple lemma.

**Lemma 1.17** *If $M$ is a finitely generated $\mathbb{Z}_p[[G_0]]$-module which is furthermore $p$-torsion free, then the codimension of $M$ and the $\mathbb{F}_p$ -codimension of $M/pM$ coincide.*

*Proof* Computing $E^i(M)$ using a projective resolution of $M$, one immediately finds (using the $p$-torsion-freeness of $M$) that

$$E^{\bullet}(M)/pE^{\bullet}(M) \xrightarrow{\sim} E^{\bullet}_{\mathbb{F}_p}(M/pM).$$

The lemma follows from this. $\square$

Returning to the case when $M$ is a finitely generated $\mathbb{F}_p[[G_0]]$-module, we note that $V := \mathrm{Hom}_{\mathrm{cont}}(M, \mathbb{F}_p)$, when equipped with the contragredient action of $G_0$, is then an admissible smooth representation of $G_0$ over $\mathbb{F}_p$. (We also recall that, conversely, $M$ can be recovered from $V$ via the double duality isomorphism $M = \mathrm{Hom}(V, \mathbb{F}_p)$.) One nice aspect of the mod $p$ situation is that the Iwasawa dimension of $M$ can be recovered from the behaviour of the corresponding smooth representation $V$ "at finite level", as the following theorem shows. (For the first statement of the following result, see [1]; the proof of the second statement will appear in [6].)

**Theorem 1.18** *Let $M$ be a finitely generated $\mathbb{F}_p[[G_0]]$-module, and write $V := \mathrm{Hom}_{\mathrm{cont}}(M, \mathbb{F}_p)$, so that $V$ is an admissible smooth $G_0$-representation over $\mathbb{F}_p$. If the Iwasawa dimension of $M$ is equal to $d$, then we have that $\dim H^0(G_r, V) \sim \lambda \cdot [G_0 : G_r]^{\frac{d}{\dim G_0}}$, for some $\lambda > 0$, while $\dim H^i(G_r, V) = O([G_0 : G_r]^{\frac{d}{\dim G_0}})$ for all $i > 0$.*

**Remark 1.19** If $G$ is (the group of $\mathbb{Q}_p$-points of) a reductive group over $\mathbb{Q}_p$, if $V$ is an irreducible admissible smooth representation of $G$ over $\mathbb{C}$, and if $\{G_r\}_{r \geq 0}$ is a neighbourhood basis of the identity of $G$

consisting of compact open subgroups, then $\dim H^0(G_r, V) \sim \lambda \cdot [G_0 : G_r]^{\frac{d_{\mathrm{GK}}}{\dim G}}$, for some $\lambda > 0$, where $d_{\mathrm{GK}}$ is the Gelfand–Kirillov dimension of $V$. Theorem 1.18 thus suggests an analogy between Gelfand–Kirillov dimensions of admissible smooth representations in characteristic zero, and Iwasawa dimensions of duals to admissible smooth representations in characteristic $p$. It remains to be seen how fruitful this analogy is.

Turning now to the context of Sections 1.5 and 1.6, we believe that the analogue of Conjecture 1.5 above, with codimensions replaced by $\mathbb{F}_p$-codimensions, should hold for $\tilde{H}_{\bullet,\mathbb{F}_p}$ and $\tilde{H}^{BM}_{\bullet,\mathbb{F}_p}$. In fact, Conjecture 1.5 and its mod $p$ analogue are far from independent. The following result gives one example of how they are related.

**Proposition 1.20** *If $\tilde{H}_{q_0}$ is $p$-torsion-free, then $\tilde{H}_{q_0}$ has codimension equal to $l_0$ if and only if $\tilde{H}_{q_0,\mathbb{F}_p}$ has $\mathbb{F}_p$-codimension equal to $l_0$.*

*Proof* This follows immediately from part (5) of Theorem 1.16, together with Lemma 1.17. □

We also have the following interpretation of (a somewhat weakened form) of the mod $p$ analogue of Conjecture 1.5 in terms of the rate of growth of $\mathbb{F}_p$-cohomology up the congruence tower $X_r$.

**Proposition 1.21** *The following are equivalent:*

1. *$\tilde{H}_{i,\mathbb{F}_p}$ has codimension $> l_0$ for $i < q_0$, and $\tilde{H}_{q_0,\mathbb{F}_p}$ has codimension $l_0$;*
2. *$\dim H^i(X_r, \mathbb{F}_p) = O\big(V(X_r)\big)^{1-\frac{l_0+1}{\dim \mathbb{G}}}$ if $i < q_0$, while*
$$\dim H^{q_0}(X_r, \mathbb{F}_p) \asymp V(X_r)^{1-\frac{l_0}{\dim \mathbb{G}}}.$$

*Proof* The equivalence of the two conditions follows from Theorem 1.18 together with part (4) of Theorem 1.16. □

Of course, there is an analogous statement relating the codimension of the modules $\tilde{H}^{BM}_{i,\mathbb{F}_p}$ to the rate of growth of compactly supported $\mathbb{F}_p$-cohomology up the tower.

**Remark 1.22** If we assume that not only the equivalent conditions (1) and (2) hold, but that the analogous statements also hold for each Levi factor of $\mathbb{G}$, then we may use a Poincaré duality argument, together with a comparison of $\tilde{H}^{BM}_{\bullet,\mathbb{F}_p}$ and $\tilde{H}_{\bullet,\mathbb{F}_p}$, to deduce that furthermore $\dim H^i(X_r, \mathbb{F}_p) = O\big(V(X_r)\big)^{1-((l_0+1)/d)}$ if $i > l_0 + q_0$.

## 1.8 Heuristics related to the $p$-adic Langlands programme

Throughout this section we fix an isomorphism $\imath : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$. If $\mathbb{G}$ is a reductive group over $\mathbb{Q}$, then the Langlands programme [13], when coupled with the conjecture of Fontaine–Mazur [10], posits the existence of a correspondence of the form

$$\begin{gathered}\{\text{algebraic automorphic Hecke eigenforms on } \mathbb{G}(\mathbb{A})\} \\ \longleftrightarrow \\ \{\text{continuous representations } G_{\mathbb{Q}} \to \mathbb{G}^L(\overline{\mathbb{Q}}_p) \text{ which are unramified} \\ \text{outside finitely many primes and are de Rham at } p\},\end{gathered} \tag{1.5}$$

where $G_{\mathbb{Q}}$ denotes the absolute Galois group of $\mathbb{Q}$, and $\mathbb{G}^L$ denotes the $L$-group of $G$ (thought of as a reductive group over $\mathbb{Q}$). There are many subtleties involved in making a conjecture of this form precise, beginning perhaps with the precise definition of *algebraic* in the context of automorphic forms, and we won't attempt to discuss them here, or to state a precise conjecture, referring instead to the recent paper [3], which gives this issue the careful consideration that it deserves.

We content ourselves by remarking that the matching (such as it exists) between automorphic forms and Galois representations is supposed to be made in the following manner (which depends on our chosen isomorphism $\imath : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_p$): an algebraic automorphic Hecke eigenform $f$ and a $p$-adic Galois representation $\rho$ match if, for some sufficiently large finite set of primes $S$, containing $p$ together with all the primes dividing either the conductor of $f$ or the conductor of $\rho$, we have that for each prime $\ell \notin S$, the $\ell$th Langlands parameter computed by applying the Satake isomorphism (suitably normalized) to the system of Hecke eigenvalues at the prime $\ell$ attached to $f$, which is a semi-simple conjugacy class in $\mathbb{G}^L(\mathbb{C})$, coincides (under the isomorphism $\mathbb{G}^L(\mathbb{C}) \xrightarrow{\sim} \mathbb{G}^L(\overline{\mathbb{Q}}_p)$ induced by $\imath$) with the semi-simple part of the conjugacy class in $\mathbb{G}^L(\mathbb{C})$ obtained by applying $\rho$ to the conjugacy class of Frobenius elements at $\ell$ in $G_{\mathbb{Q}}$. We refer to [3, Conj. 3.2.1] and the subsequent remarks for a more precise statement and discussion.

We note in addition that to an algebraic automorphic eigenform we can associate a collection of integral "Hodge numbers" ([3, Rem. 3.2.3]; these are constructed using the Langlands parameter of the representation of $\mathbb{G}(\mathbb{R})$ generated by $f$, or alternatively, using the infinitesimal character of $f$), and that if $f$ and $\rho$ match in the above sense, then the Hodge numbers associated to $f$ should match with the Hodge–Tate weights of the de Rham representation $\rho_{|G_{\mathbb{Q}_p}}$.

Now the property of being algebraic is defined in terms of these "Hodge numbers" being integral [3] (the precise definition of algebraic then depending on exactly how the Hodge numbers are normalized [3]). It is then possible to enlarge the automorphic side of the conjectured correspondence by considering automorphic forms that are non-algebraic, i.e. whose associated "Hodge numbers" are not integral. It is also possible to enlarge the Galois side of the conjectured correspondence by considering automorphic forms that are non-de Rham. These representations have associated Hodge–Sen–Tate weights which are typically non-integral.

Unfortunately, it seems that when we enlarge the two sides of the correspondence in this manner, the enlarged sets are not related to each other. To understand why, note that on the automorphic side we are enlarging the set of allowed Hodge numbers in an archimedean manner, while on the Galois side, we are doing so in a $p$-adic manner, and there is no reason (that we know of) to think that these two kinds of enlargements should have any relationship to one another.

What we would like, in order to extend the Langlands correspondence so as to allow non-de Rham Galois representations to be included, is to introduce a notion of $p$-adic automorphic form which generalizes the notion of algebraic automorphic form, whose associated "Hodge numbers" (whatever they might be) are $p$-adic rather than archimedean. We believe that $p$-adically completed cohomology provides a partial (and slightly indirect) solution to the problem of defining such a notion of $p$-adic automorphic form.

We put ourselves in the context of Section 1.5. As noted there, the completed homology modules $\tilde{H}_n$ admit an action of a completed Hecke algebra $\mathbb{T}$.

Franke's theorem [11] describing the cohomology of congruence quotients in terms of automorphic forms shows that any system of Hecke eigenvalues appearing in some degree of cohomology (with $\mathbb{C}$-coefficients, or more generally with coefficients in some local system associated to an algebraic representation of $\mathbb{G}_{/\mathbb{C}}$) of $\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A})/A_\infty K_\infty G_r K^{\infty,p}$, is also a system of Hecke eigenvalues arising from some algebraic automorphic Hecke eigenform (more precisely, a $C$-algebraic eigenform in the sense of [3]) that is invariant under $K^{\infty,p}$. We call such automorphic Hecke eigenforms *cohomological Hecke eigenforms of tame level* $K^{\infty,p}$.

Using our fixed isomorphism $\imath : \mathbb{C} \to \mathbb{Q}_p$, we may identify cohomology with coefficients in $\mathbb{C}$ (or a local system associated to an algebraic representation of $\mathbb{G}_{/\mathbb{C}}$) with cohomology with coefficients in $\overline{\mathbb{Q}}_p$ (or a local system associated to an algebraic representation of $\mathbb{G}_{/\overline{\mathbb{Q}}_p}$). One can verify that the Hochschild–Serre-type spectral sequence of Theorem 1.1 (4) (and its generalization for local systems mentioned in Remark 1.2) is Hecke equivariant, and so we conclude that each cohomological Hecke eigenform of tame level $K^{\infty,p}$ gives rise to a point $\lambda \in \operatorname{Spec} \mathbb{T}(\overline{\mathbb{Q}}_p)$.

We expect that $\operatorname{Spec} \mathbb{T}$ will be topologically finitely generated as a $\mathbb{Z}_p$-algebra. The point of the preceding discussion is that every system of Hecke eigenvalues associated to a cohomological Hecke eigenform of tame level $K^{\infty,p}$ appears as a point of $\operatorname{Spec} \mathbb{T}(\overline{\mathbb{Q}}_p)$. Thus we regard $\operatorname{Spec} \mathbb{T}(\overline{\mathbb{Q}}_p)$ as a space parameterizing "$p$-adic cohomological systems of Hecke eigenvalues"; points of this space serve as a working model for the notion of (cohomological) $p$-adic automorphic Hecke eigenvalues.

We then expect that the conjectural correspondence (1.5) (or, more precisely, its restriction to the cohomological Hecke eigenforms) should extend to identify $\mathbb{T}$ with a space of $p$-adic Galois representations. Just slightly more precisely, we write the complete ring $\mathbb{T}$ as a product $\mathbb{T} = \prod_{\mathfrak{m}} \mathbb{T}_{\mathfrak{m}}$ of its local factors. Attached to each $\mathfrak{m}$, and to a choice of an embedding $\mathbb{T}/\mathfrak{m} \hookrightarrow \overline{\mathbb{F}}_p$, we expect that there should be an associated Galois representation $\rho_{\mathfrak{m}} : G_{\mathbb{Q}} \to \mathbb{G}^L(\overline{\mathbb{F}}_p)$. (This is quite possibly an oversimplification, as the discussion of [3] shows,[2] or perhaps more accurately, it is an idealization of the actual situation; however, it will serve well enough for our present purpose, which is only to explain our motivations in the most general way.) Associated to $\rho_{\mathfrak{m}}$ is a deformation ring $R_{\rho_{\mathfrak{m}}}$ parameterizing deformations of $\rho_{\mathfrak{m}}$. (Here we should impose ramification conditions away from $p$ expressing the idea that the prime-to-$p$ conductor of the deformations considered is bounded by $K^{\infty,p}$ — this is a relatively minor point. The key point is that we impose no ramification conditions at $p$. Since we are only discussing heuristics here, we don't try to make our discussion more precise; in particular, we don't attempt to address the precise meaning of $R_{\rho_{\mathfrak{m}}}$ in the case when $\rho_{\mathfrak{m}}$ is "reducible" (i.e. factors through a parabolic of $\mathbb{G}^L$).) What we then expect is that there will be an isomorphism $\mathbb{T}_{\mathfrak{m}} \xrightarrow{\sim} R_{\rho_{\mathfrak{m}}}$.

Now one can compute the expected dimension of $R_{\rho_{\mathfrak{m}}}$ via the global Euler characteristic formula [15], and what one finds is that expected

[2] To be just slightly less oblique, we are (in the terminology of [3]) considering $C$-algebraic, rather than $L$-algebraic, automorphic forms, and the discussion of [3], especially Section 5, shows that it may be necessary to replace $G$ by a $z$-extension in order to get a correspondence between $C$-algebraic automorphic eigenforms and $p$-adic Galois representations.

dimension of $R_{\rho_{\mathfrak{m}}}$ is $1 + \dim \mathbb{B} - l_0$, where $\mathbb{B}$ is a Borel subgroup of $\mathbb{G}$. Thus we expect that each $\mathbb{T}_{\mathfrak{m}}$, and so $\mathbb{T}$ itself, should have Krull dimension $1 + \dim \mathbb{B} - l_0$.

Now consider $\tilde{H}_{q_0}$ as $\mathbb{T}$-module. We in fact hope that it will be faithful (in other words, there are no systems of eigenvalues appearing in $\tilde{H}_n$ for $n \neq q_0$ which don't also appear in $\tilde{H}_{q_0}$), the reason being that $q_0$ is the first dimension in which tempered representations of $G_\infty$ can contribute cohomology [2], and so is the first "interesting dimension of cohomology". Since it is the first dimension which knows about the interesting automorphic forms, we also expect that it knows about all the interesting Galois representations, and we hope that all the Galois representations are in the Zariski closure of the interesting ones.

Assuming this faithfulness, one can then naively view $\tilde{H}_{q_0}$ as a kind of bundle over $\operatorname{Spec} \mathbb{T}$, and one might then guess the following dimension formula:

$$\begin{aligned}&\text{Iwasawa dim. of } \tilde{H}_{q_0}\\ &\qquad \stackrel{?}{=} \text{Krull dim. of } \operatorname{Spec} \mathbb{T} + \text{Iwasawa dim. of the fibres.} \qquad (1.6)\end{aligned}$$

What do we expect the Iwasawa dimension of a fibre to be? The fibre of $\tilde{H}_{q_0}$ over a point $\lambda \in \operatorname{Spec} \mathbb{T}(\overline{\mathbb{Q}}_p)$ associated to a cohomological Hecke eigenform is at least morally dual to the $\lambda$-eigenspace in cohomology, and the rough analogy between Iwasawa dimensions and Gelfand–Kirillov dimensions of smooth representations (see Theorem 1.18 and Remark 1.19) then suggests that the dimension of this fibre might be equal to the Gelfand–Kirillov dimension of the $\lambda$-eigenspace in cohomology, which in turn will be equal to $\dim \mathbb{G}/\mathbb{B}$ in the generic situation. (An irreducible admissible smooth representation of $\mathbb{G}(\mathbb{Q}_p)$ has Gelfand–Kirillov dimension at most equal to $\dim \mathbb{G}/\mathbb{B}$, with equality precisely when it is generic, i.e. when it admits a Whittaker model.) Combining this final heuristic with (1.6) and our computation of the expected dimension of $\mathbb{T}$, we are left with the following guess:

$$\text{Iwasawa dim. of } \tilde{H}_{q_0} \stackrel{?}{=} 1 + \dim \mathbb{B} - l_0 + \dim \mathbb{G}/\mathbb{B} = 1 + \dim \mathbb{G} - l_0.$$

Equivalently, we guess that the codimension of $\tilde{H}_{q_0}$ is equal to $l_0$, which is Conjecture 1.5 (2). The preceding discussion provides one of the main motivations for this conjecture.

Since we expect that the systems of eigenvalues appearing in $\tilde{H}_n$ for $n < q_0$ will be quite special, we expect that the support of $\tilde{H}_n$ in $\operatorname{Spec} \mathbb{T}$ will have positive codimension. This helps to motivate Conjecture 1.5 (1). As noted in Remark 1.7, if we believe parts (1) and (2) of Conjecture 1.5, we are naturally led to believe the entire conjecture.

We close by remarking that a large part of the framework described here has been worked out in detail in the particular case of $\mathbb{G} = \mathrm{GL}_2$ [9].

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