An introduction to the Riemann-Hilbert Correspondence for Unit F-Crystals.

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In memory of Bernie Dwork

Introduction

Let X be a smooth scheme over $\mathbb{C}$, and $(\mathcal{M}, \nabla)$ an integrable differential equation on X. That is, $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module of finite rank, equipped with an integrable connection $\nabla : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^1_X$. The sheaf of local solutions of this differential equation forms a local system $\mathcal{L}$ of $\mathbb{C}$-vector spaces on X. One can recover $(\mathcal{M}, \nabla)$ as $\mathcal{L} \otimes_{\mathbb{C}} \mathcal{O}_X$.

The Riemann-Hilbert correspondence extends the correspondence $(\mathcal{M}, \nabla) \leftrightarrow \mathcal{L}$ to the situation where the differential equation $(\mathcal{M}, \nabla)$ may have singular points. The desirability of such an extension may be seen in geometry. If $f : Y \longrightarrow X$ is a smooth proper map of smooth $\mathbb{C}$-schemes, then for any non-negative integer i, the relative De Rham cohomology $R^i f_* \Omega^\bullet_{Y/X}$ carries a connection called the Gauss-Manin connection, with sheaf of solutions naturally isomorphic to the higher direct image sheaf $R^i f_* \mathbb{C}$. If f is not smooth, or not proper, however, then the sheaves $R^i f_* \mathbb{C}$ are not local systems in general; they are merely constructible sheaves. One would like to construct a "singular differential equation" on X, of which $R^i f_* \mathbb{C}$ is the sheaf of solutions.

The notion of singular differential equations is introduced in a precise way through the category of regular holonomic $\mathcal{D}$-modules. The solutions of a regular holonomic $\mathcal{D}$-module are indeed a constructible sheaf. To obtain an equivalence of categories, however, one must pass to derived categories. Making this transition, one finds that the derived solutions functor induces an equivalence of triangulated categories between the derived category of regular holonomic $\mathcal{D}$-modules and the derived category of constructible sheaves. Furthermore, this equivalence respects Grothendieck's six operations $f^!, f_!, f^*, f_*, \underline{Rhom}, \overset{\mathbb{L}}{\otimes}$. (In particular, if $f : Y \to X$ is a map of smooth $\mathbb{C}$-schemes, not necessarily smooth or projective, then the complex $Rf_* \mathbb{C}$ is naturally isomorphic to the derived solutions of a complex of regular holonomic $\mathcal{D}$-modules, usually denoted $f_+ \mathcal{O}_Y$.) This result, known as the "Riemann-Hilbert correspondence", was first proved by Kashiwara and by Mebkhout (See [Bo] for an algebraic proof.)

The purpose of this paper is to describe some results whose proof is the subject of the papers [EK1], [EK2], and which provide a p-adic analogue of the Riemann-Hilbert correspondence. Our starting point is a theorem of Nick Katz [Ka, Prop. 4.1.1], which provides the analogue of the correspondence between differential equations and local systems.

Theorem 0.1. (Katz) Let k be a perfect field of characteristic p. If X is a smooth scheme over $W_n(k)$, the ring of Witt vectors of length n, and if F_X is a lift to X of the Frobenius on $W_n(k)$, then there is an equivalence of categories between the category of étale sheaves of locally free $\mathbb{Z}/p^n$ modules E of finite rank, and the category of locally free $\mathcal{O}_X$ modules $\mathcal{E}$ of finite rank, equipped with an $\mathcal{O}_X$ linear isomorphism $F_X^*\mathcal{E}\tilde{\longrightarrow}\mathcal{E}$. The equivalence is realized by associating $\mathcal{E} = E \otimes_{W_n(k)} \mathcal{O}_X$ to E.

The sheaves $\mathcal{E}$ appearing in Katz's theorem are the analogues of differential equations. The analogue of $\mathscr{D}$-modules turn out to be sheaves over certain rings $\mathcal{O}_{F,X}$ (when $n = 1$) and $\mathscr{D}_{F,X}$ (in general). The first three sections of this paper outline the Riemann-Hilbert correspondence for unit $\mathcal{O}_{F,X}$-modules proved in [EK 1]. We assume that $n = 1$, so that X is a smooth k-scheme. As in the case of differential equations, we obtain not just an (anti-)equivalence of triangulated categories, but also its compatibility with certain cohomological functors.

A consequence of the Riemann-Hilbert correspondence over $\mathbb{C}$ is that one obtains a distinguished class of complexes of constructible $\mathbb{C}$-vector spaces, which correspond to a single regular holonomic $\mathscr{D}$-module. These are called "perverse sheaves." They form an abelian category, and have a description in term of the so called "middle extension". In §4, we complete the results of [EK, §11] regarding the structure of "perverse sheaves" in our context, and show that they have a similar description in terms of middle extension.

In §5 we sketch the applications to L-functions that are the subject of [EK, §12]. Finally in §6 we explain the results of [EK 2] on the Riemann-Hilbert correspondence for a smooth $W_n(k)$-scheme. This relies heavily on the theory of arithmetic $\mathscr{D}$-modules developed by Berthelot [Be].

To keep the paper a reasonable length, we have had to omit many details, and refinements. However we have tried to give indications of proofs where possible, and have included some typical calculations.

The study of unit root F-crystals and in particular their degeneration and L-functions was pioneered by Bernard Dwork. We dedicate this paper to his memory.

1. $\mathcal{O}_{F,X}$-modules

1.1. Let k be a perfect field of characteristic p. For any k-algebra R, we denote by $R[F]$ the non-commutative polynomial ring in a formal variable F, which satisfies the relation $Fa = a^pF$ for $a \in R$. For any k-scheme X, we denote by $\mathcal{O}_{F,X}$ the quasi-coherent sheaf of non-commutative rings which over on an affine open $\operatorname{Spec} R$ takes the value $R[F]$. As an $(\mathcal{O}_X, \mathcal{O}_X)$-bimodule $\mathcal{O}_{F,X}$ is isomorphic to $\oplus_{i=0}^{\infty} F^{i*}\mathcal{O}_X$, where F denotes the absolute Frobenius on X. Using this description, one sees that to give an $\mathcal{O}_{F,X}$-module $\mathcal{M}$ is the same as giving the underlying $\mathcal{O}_X$-module $\mathcal{M}$ together with the map

$$\phi_{\mathcal{M}} : F^*\mathcal{M} = F^*\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{M} \tag{1.1.1}$$

1.2. An $\mathcal{O}_{F,X}$-module $\mathcal{M}$ is called a *unit* $\mathcal{O}_{F,X}$-module if $\mathcal{M}$ is quasi-coherent as an $\mathcal{O}_X$-module, and if the map $\phi_{\mathcal{M}}$ is an isomorphism. There is a general technique for constructing unit $\mathcal{O}_{F,X}$-modules, due to Lyubeznik [Lyu, 1.9]. Consider a quasi-coherent $\mathcal{O}_X$-module M, and a map $\beta : M \to F^*M$. Set $\mathcal{M} = \varinjlim F^{i*}M$ where the transition maps are given by $F^{i*}\beta$. Then $\mathcal{M}$ has a natural structure of a unit $\mathcal{O}_{F,X}$-module, obtained by taking $\phi_{\mathcal{M}}$ to be the limit of the maps

$$F^*(F^{i*}M) \xrightarrow{\sim} F^{i+1*}M$$

Conversely, any unit $\mathcal{O}_{F,X}$-module can be obtained in this way, since we may take $M = \mathcal{M}$ and $\beta = \phi_{\mathcal{M}}^{-1}$. We say that the morphism β generates $\mathcal{M}$.

For our purposes the category of unit $\mathcal{O}_{F,X}$ which are, locally on X, finitely generated over $\mathcal{O}_{F,X}$ will be very important. They are the analogue, in our present situation, of holonomic $\mathscr{D}$-modules. Over smooth schemes, one has the following characterisation of such modules [EK 1, 6.1.3]

Proposition 1.2.1. Let X be a smooth k-scheme, and $\mathcal{M}$ an $\mathcal{O}_{F,X}$-module. Then $\mathcal{M}$ is a locally finitely generated unit $\mathcal{O}_{F,X}$-module if and only if there exists an injective generator $\beta : M \to F^*M$ for $\mathcal{M}$, with M a coherent $\mathcal{O}_X$-module.

In the situation of (1.2.1) an injective generator $\beta : M \to F^*M$ with M coherent is called a "root" of $\mathcal{M}$. From now on we will assume that X is a smooth k-scheme. We abbreviate the phrase "locally finitely generated unit" to "lfgu". As an application of (1.2.1), we have the following structural result, which often enables one to make excision arguments.

Corollary 1.2.2. Let $\mathcal{M}$ be an lfgu $\mathcal{O}_{F,X}$-module. There exists a dense open $U \subset X$, such that $\mathcal{M}|_U$ is a finite, locally free $\mathcal{O}_X$-module.

Proof. Let $\beta : M \to F^*M$ be a root of M. Then we may choose U so that M is locally free over U and β is an isomorphism. □

The following result shows that the subset $U \subset X$ can be taken to be the biggest open subset over which $\mathcal{M}$ is coherent.

Proposition 1.2.3. Let $\mathcal{M}$ be a unit $\mathcal{O}_{F,X}$-module, which is coherent as an $\mathcal{O}_X$-module. Then $\mathcal{M}$ is locally free over $\mathcal{O}_X$.

Proof. One proof may be found in [EK 1, 6.9.3]. Another, slicker, proof is as follows: We may work locally, and assume that $X = \operatorname{Spec} R$ for some integral domain R. The isomorphism $\phi_{\mathcal{M}}$ implies that the Fitting ideals of $\mathcal{M}$ and $F^*\mathcal{M}$ are equal. But if I^k is the k-th Fitting ideal of $\mathcal{M}$ then the k-th Fitting ideal of $F^*\mathcal{M}$ is $F(I^k)R$. This implies that I^k is either 0 or R. If k_0 is the smallest integer such that $I^{k_0} = R$, then $\mathcal{M}$ is locally free of rank k_0. □

A coherent unit $\mathcal{O}_{F,X}$-module is called an F-crystal.

1.3. Suppose that $\mathcal{M}$ is an $\mathcal{O}_{F,X}$ module. Then we have the $\mathcal{O}_X$-linear map

$$-\phi_{\mathcal{M}} \oplus \mathrm{id} : F^*\mathcal{M} \to \mathcal{M} \oplus F^*\mathcal{M} \subset \oplus_{i=0}^{\infty} F^{*i}\mathcal{M} = \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} \mathcal{M}$$

Extending this map $\mathcal{O}_{F,X}$-linearly one gets a map $\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} F^*\mathcal{M} \to \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} \mathcal{M}$ and it is not hard to check that this map sits in an exact sequence

$$0 \to \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} F^*\mathcal{M} \to \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{M} \to 0 \qquad (1.3.1)$$

Thus (1.3.1) provides a canonical resolution of any $\mathcal{O}_{F,X}$-module, by $\mathcal{O}_{F,X}$ modules of the form $\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} N$. This resolution is an important tool in cohomological calculations. As a first simple application we have (cf. [EK 1, 1.8.4])

Lemma 1.3.2. Let $\mathcal{M}$ be an $\mathcal{O}_{F,X}$-module. Then $\mathcal{M}$ has Tor-dimension at most $\dim X + 1$.

Proof. The assertion is that for any right $\mathcal{O}_{F,X}$-module $\mathcal{N}$, $\mathrm{Tor}^i_{\mathcal{O}_{F,X}}(\mathcal{M}, \mathcal{N})$ vanishes for $i > d + 1$. These Tor groups may be computed by taking a resolution of $\mathcal{M}$ by locally free $\mathcal{O}_{F,X}$ modules [EK, 1.6.2]. To see that they vanish, it suffices, using the resolution (1.3.1), to check that they vanish for $i > d$ when we replace $\mathcal{M}$ by $\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} \mathcal{M}$. However, this follows because $\mathcal{M}$ has Tor-dimension at most d as an $\mathcal{O}_X$-module (Serre's theorem), together with the fact that $\mathcal{O}_{F,X}$ is flat as a right $\mathcal{O}_X$-module. □

1.3.3. When $\mathcal{M}$ is a unit module, we may identify $\mathcal{M}$ with $F^*\mathcal{M}$ via the isomorphism $\phi_{\mathcal{M}}$, and replace $F^*\mathcal{M}$ by $\mathcal{M}$ in (1.3.1). Then the map $-\phi_{\mathcal{M}} \oplus \mathrm{id}$ becomes $-\mathrm{id} \oplus \beta : \mathcal{M} \to \mathcal{M} \oplus F^*\mathcal{M}$ where $\beta = \phi_{\mathcal{M}}^{-1}$. There is a useful generalisation of this construction: Let $\beta : M \to F^*M$ be any generator of $\mathcal{M}$. Then we have a map $-\mathrm{id} \oplus \beta : M \to M \oplus F^*M$, which induces a map $\mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} M \to \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} M$ as above, sitting in a short exact sequence

$$0 \to \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} M \to \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} M \to \mathcal{M} \to 0 \qquad (1.3.4)$$

In particular, if $\mathcal{M}$ is an lfgu $\mathcal{O}_{F,X}$-module, then M can be taken to be coherent.

2. Cohomological Operations

2.1. Let $\bullet$ be one of $+$, $-$, b or $\emptyset$. Denote by $D^\bullet(\mathcal{O}_{F,X})$ the derived category of complexes of $\mathcal{O}_{F,X}$ modules with the indicated boundedness condition. We denote by $D^\bullet_{qc}(X, \mathcal{O}_{F,X})$ (resp. $D^\bullet_u(\mathcal{O}_{F,X})$, resp. $D^\bullet_{lfgu}(\mathcal{O}_{F,X})$) the full subcategory of $D^\bullet(\mathcal{O}_{F,X})$ consisting of complexes with quasi-coherent (resp. unit, resp. lfgu) cohomology sheaves. If is easy to see that $D^\bullet_{qc}(\mathcal{O}_{F,X})$ is a triangulated subcategory of $D^\bullet(\mathcal{O}_{F,X})$. For $D^\bullet_u(\mathcal{O}_{F,X})$ and $D^\bullet_{lfgu}(\mathcal{O}_{F,X})$ this follows from the following result [EK 1, 5.2, 6.2.3]

Lemma 2.1.1. Let

$$\mathcal{M}_1 \to \mathcal{M}_2 \to \mathcal{M}_3 \to \mathcal{M}_4 \to \mathcal{M}_5$$

be an exact sequence of $\mathcal{O}_{F,X}$-modules. If each of $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_4$, and $\mathcal{M}_5$ is a unit (resp. lfgu) $\mathcal{O}_{F,X}$-module, then $\mathcal{M}_3$ is a unit (resp. lfgu) $\mathcal{O}_{F,X}$-module.

2.2. Let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{O}_{F,X}$-modules. Then we have a natural map

$$F^*(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}) \xrightarrow{\sim} F^*\mathcal{M} \otimes_{\mathcal{O}_X} F^*\mathcal{N} \xrightarrow{\phi_{\mathcal{M}} \otimes \phi_{\mathcal{N}}} \mathcal{M} \otimes \mathcal{N}$$

Thus $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ has a natural structure of $\mathcal{O}_{F,X}$-module.

We already remarked in the proof of (1.3.2) above, that any $\mathcal{O}_{F,X}$ module has a resolution by locally free $\mathcal{O}_{F,X}$-modules, and that the terms of such a resolution are flat over $\mathcal{O}_X$. Thus we can define the derived functor

$$\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \colon D^-(\mathcal{O}_{F,X}) \times D^-(\mathcal{O}_{F,X}) \to D^-(\mathcal{O}_{F,X})$$

and (1.3.2) implies (in particular) that this restricts to

$$\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \colon D^b(\mathcal{O}_{F,X}) \times D^b(\mathcal{O}_{F,X}) \to D^b(\mathcal{O}_{F,X})$$

This bifunctor respects the subcategories D_{qc}, D_u and D_{lfgu} [EK 1, 1.9.4, 5.5.2, 6.4.1]

Proposition 2.2.1. Let $*$ be one of qc, u or $lfgu$. Then $\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}$ restricts to

$$\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \colon D^-_*(\mathcal{O}_{F,X}) \times D^-_*(\mathcal{O}_{F,X}) \to D^-_*(\mathcal{O}_{F,X})$$

2.3. Let $f : Y \to X$ be a morphism of smooth k-schemes. There is a map of rings $f^{-1}(\mathcal{O}_{F,X}) \to \mathcal{O}_{F,Y}$ induced by the natural map $f^{-1}\mathcal{O}_X \to \mathcal{O}_Y$. Using this map we regard $\mathcal{O}_{F,Y}$ as a right $f^{-1}\mathcal{O}_{F,X}$-module, and it has a natural structure as a left $\mathcal{O}_{F,Y}$-module We denote by $\mathcal{O}_{F,Y\to X}$ the ring $\mathcal{O}_{F,Y}$ considered as a $(\mathcal{O}_{F,Y}, f^{-1}\mathcal{O}_{F,X})$-bimodule.

We denote by d_X the function which to a point $x \in X$ assigns the dimension of the connected component containing x. We set $d_{Y/X} = d_Y - d_X \circ f$

For $\mathcal{M}$ in $D^-(\mathcal{O}_{F,X})$, we set $f^!\mathcal{M}^\bullet = \mathcal{O}_{F,Y\to X} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathcal{O}_{F,X}} f^{-1}\mathcal{M}^\bullet[d_{Y/X}]$. In fact (1.3.2) can be used to show that $f^!$ is even well defined for $\mathcal{M}^\bullet$ in $D(\mathcal{O}_{F,X})$. It also implies that $f^!$ restricts to a functor

$$f^! : D^b(\mathcal{O}_{F,X}) \to D^b(\mathcal{O}_{F,Y})$$

We have [EK 1, 2.3.2, 5.8, 6.7]

Proposition 2.3.1. Let $*$ be one of qc, u or $lfgu$. Then $f^!$ induces a functor

$$f^! : D^-_*(\mathcal{O}_{F,X}) \to D^-_*(\mathcal{O}_{F,Y})$$

Sketch of proof. The case $* = qc$ follows from the fact that on underlying complexes of $\mathcal{O}_X$-modules, $f^!$ is given by $\mathcal{O}_Y \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} -$ followed by a shift. For the case $* = u$, or $lfgu$ one reduces to the case of a single module $\mathcal{M}$, and X affine. If $\beta : M \to F^*M$ is an injective generator for M, we may take a resolution $P^\bullet$ of M by finite free $\mathcal{O}_X$-modules, and lift β to a map $\beta' : P^\bullet \to F^*P^\bullet$. If $\mathcal{P}^\bullet$ denotes the complex of unit $\mathcal{O}_{F,X}$ modules generated by β', then one easily checks that $\mathcal{P}^\bullet$ is a resolution of $\mathcal{M}$ and that, up to a shift, $f^!\mathcal{P}^\bullet$ is the complex generated by $f^*\beta$. In particular, $f^!\mathcal{M}$ has unit cohomology sheaves. If in addition, $\mathcal{M}$ is locally finitely generated, then (1.2.1) implies that we may take M, and hence the terms of $P^\bullet$, to be coherent. In this case $f^!\mathcal{P}^\bullet$ is a complex of lfgu $\mathcal{O}_{F,X}$-modules, and in particular has cohomology sheaves with are lfgu $\mathcal{O}_{F,Y}$-modules □

We record two more properties of the functor $f^!$ [EK 1, 2.4, 2.5]

Proposition 2.3.2.

1. If $g : Z \to Y$ and $f : Y \to X$ are maps of smooth k-schemes, then there exists a natural isomorphism $(fg)^! \xrightarrow{\sim} g^! f^!$
2. If $\mathcal{M}^\bullet$ and $\mathcal{N}^\bullet$ are in $D^-(\mathcal{O}_{F,X})$ then there exists a canonical isomorphism

$$f^!(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} \mathcal{N}^\bullet) \xrightarrow{\sim} f^!\mathcal{M}^\bullet \otimes_{\mathcal{O}_Y} f^!\mathcal{N}^\bullet[-d_{Y/X}]$$

2.4. Let $f : Y \to X$ be a map of smooth k-schemes. We denote by $\omega_{Y/X}$ the relative dualising sheaf. This is isomorphic to $\omega_Y \otimes_{\mathcal{O}_Y} f^*\omega_X^{-1}$, which is a line bundle, since X and Y are smooth. The $(f^{-1}\mathcal{O}_{F,X}, \mathcal{O}_Y)$-bimodule $f^{-1}\mathcal{O}_{F,X} \otimes_{f^{-1}\mathcal{O}_X} \omega_{Y/X}$ can be given the structure of a $(f^{-1}\mathcal{O}_{F,X}, \mathcal{O}_{F,Y})$-bimodule [EK 1, 3.3.1]. The construction involves the formalism of duality theory for quasi-coherent sheaves, and especially the Cartier operator. Rather than giving it in generality, we describe the bimodule structure explicitly in two special cases.

In the first example, we take $\mathcal{O}_Y = \mathcal{O}_X/a$, where a cuts out the smooth divisor Y of X. In this case we have an exact sequence

$$0 \to \mathcal{O}_Y \cdot da \to f^*\Omega^1_X \to \Omega^1_Y \to 0$$

so that $\omega_{Y/X} = \mathcal{O}_Y \cdot (da)^{-1}$. The required right $\mathcal{O}_{F,Y}$-module structure is then given by [EK 1, 3.3.4]

$$bF^n(da)^{-1} \overset{\cdot F}{\mapsto} ba^{p^n(p-1)}F^{n+1}(da)^{-1}$$

where b is a section of $\mathcal{O}_X$.

Next suppose that $Y \to X$ is smooth of relative dimension 1. Then locally, Y admits an étale map to the X-scheme, $\mathbb{G}_m \times X$. If t denotes the canonical co-ordinate on $\mathbb{G}_m$, then this induces an invertible section t of $\mathcal{O}_Y$, and $\omega_{Y/X}$ is generated by dt/t. The required right $\mathcal{O}_{F,Y}$-module structure is then given by

$$aF^n t^i dt/t \overset{\cdot F}{\mapsto} aF^{n+1} t^{i/p} dt/t$$

where a is a section of $f^{-1}\mathcal{O}_X$, and $t^{i/p} = 0$ if $p \nmid i$.

We denote the $(f^{-1}\mathcal{O}_{F,X}, \mathcal{O}_{F,Y})$-bimodule $f^{-1}\mathcal{O}_{F,X} \otimes_{f^{-1}\mathcal{O}_X} \omega_{Y/X}$ by $\mathcal{O}_{F,X\leftarrow Y}$. If $g : Z \to Y$ is another morphism of smooth k-schemes, then the natural isomorphism $\omega_{Z/X} \xrightarrow{\sim} \omega_{Z/Y} \otimes_{\mathcal{O}_Z} g^*\omega_{Y/X}$ induces an isomorphism of $((fg)^{-1}\mathcal{O}_{F,X}, \mathcal{O}_Z)$-bimodules

$$\mathcal{O}_{F,X\leftarrow Z} \xrightarrow{\sim} g^{-1}\mathcal{O}_{F,X\leftarrow Y} \otimes_{g^{-1}\mathcal{O}_{F,Y}} \mathcal{O}_{F,Y\leftarrow Z} \tag{2.4.1}$$

In fact this map is an isomorphism of $(f^{-1}\mathcal{O}_{F,X}, \mathcal{O}_{F,Z})$-bimodules [EK 1, 3.3.3]. The calculation of the general right $\mathcal{O}_{F,Y}$-module structure on $\mathcal{O}_{F,X\leftarrow Y}$ can be reduced to the two special cases considered above, by locally factoring any map $Y \to X$ as a composite of a closed embedding, and smooth curve fibrations, and using the isomorphism (2.4.1).

We define a functor

$$f_+ : D^-(\mathcal{O}_{F,Y}) \to D^-(\mathcal{O}_{F,X})$$

by

$$f_+\mathcal{M}^\bullet = Rf_*(\mathcal{O}_{F,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F,Y}} \mathcal{M}^\bullet)$$

By (1.3.2), and the fact that Rf_* has finite cohomological dimension, f_+ is well defined (in fact we could have even worked with unbounded complexes). Moreover, it induces a functor

$$f_+ : D^b(\mathcal{O}_{F,Y}) \to D^b(\mathcal{O}_{F,X})$$

As for $f^!$, we have the following result [EK 1, 3.5.3, 5.8, 6.8.4]

Proposition 2.4.2. Let $*$ be one of qc, u, or $lfgu$. Then f_+ restrict to a functor

$$f_+ : D^-_*(\mathcal{O}_{F,Y}) \to D^-_*(\mathcal{O}_{F,X})$$

We record some other properties of f_+

Proposition 2.4.3. If $f : Y \to X$ and $g : Z \to Y$ are morphisms of smooth k-schemes, then

1. There exists a canonical isomorphism $(fg)_+ \xrightarrow{\sim} g_+ f_+$.
2. If $\mathcal{M}^\bullet$ is in $D^-(\mathcal{O}_{F,Y})$ and $\mathcal{N}^\bullet$ is in $D^-_{qc}(\mathcal{O}_{F,X})$ then we have the projection formula

$$f_+(\mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_Y} f^!\mathcal{N}^\bullet) \xrightarrow{\sim} \mathcal{M}^\bullet[d_{Y/X}] \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} f_+ f^!\mathcal{N}^\bullet$$

3. If $\mathcal{M}^\bullet$ is in $D^-(\mathcal{O}_{F,X})$ and $\mathcal{N}^\bullet$ is in $D^b(\mathcal{O}_{F,Y})$, and f is an open immersion, then there is a canonical isomorphism

$$Rf_*\underline{Rhom}_{\mathcal{O}_{F,Y}}(f^!\mathcal{M}^\bullet, \mathcal{N}^\bullet) \xrightarrow{\sim} \underline{Rhom}_{\mathcal{O}_{F,X}}(\mathcal{M}^\bullet, f_+\mathcal{N}^\bullet)$$

4. If $\mathcal{M}^\bullet$ is in $D^-_{qc}(\mathcal{O}_{F,Y})$ and $\mathcal{N}^\bullet$ is in $D^b_{qc}(\mathcal{O}_{F,X})$, and f is a proper map, then there is a canonical isomorphism

$$\underline{Rhom}_{\mathcal{O}_{F,X}}(f_+\mathcal{M}^\bullet, \mathcal{N}^\bullet) \xrightarrow{\sim} Rf_*\underline{Rhom}_{\mathcal{O}_{F,Y}}(\mathcal{M}^\bullet, f^!\mathcal{N}^\bullet)$$

Proof. (1) is a formal consequence of the isomorphism (2.4.1). (2) and (3) are formal consequences of respectively, the projection formula and the adjointness between Rf_* and f^{-1} for Zariski sheaves [EK 1, 4.2, 4.3]. The proof of (4) is one of most delicate points in the whole theory, and uses Grothendieck duality, expressed on the level of residual complexes [Ha, Ch VI]. A key point is the construction of the trace map, explained in the following lemma [EK 1, 4.4.9]. □

Lemma 2.4.4. Let $f : Y \to X$ be a proper map of smooth k-schemes. Then the complex of $\mathcal{O}_{F,X}$-modules $f_+\mathcal{O}_{F,Y}$ has a natural structure of an object in the derived category of $(\mathcal{O}_{F,X}, \mathcal{O}_{F,X})$-bimodules, and there exists a map

$$tr_{F,f} : f_+\mathcal{O}_{F,Y}[d_{Y/X}] \to \mathcal{O}_{F,X}$$

in this derived category, which on the level of complexes of $\mathcal{O}_{F,X}$-modules becomes the map

$$f_+\mathcal{O}_{F,Y}[d_{Y/X}] = \mathcal{O}_{F,X} \otimes_{\mathcal{O}_X} Rf_*\omega_{Y/X}[d_{Y/X}] \overset{1\otimes tr_f}{\to} \mathcal{O}_{F,X}$$

where $tr_f : Rf_*\omega_{Y/X}[d_{Y/X}] \to \mathcal{O}_X$ is the trace map of Grothendieck-Serre duality.

2.5. In this section we explain the analogue of Kashiwara's theorem for unit modules. The following result is proved in [EK 1, 5.10.1, 5.10.3]

Theorem 2.5.1. Let $f : Y \to X$ be a closed immersion of smooth k schemes.

1. If $\mathcal{M}$ is a unit $\mathcal{O}_{F,X}$-module supported on Y, then $\underline{H}^0(f^!\mathcal{M}) \xrightarrow{\sim} f^!\mathcal{M}$.

2. If $\mathcal{M}$ is a unit $\mathcal{O}_{F,Y}$-module then $\underline{H}^0(f_+\mathcal{M}) \xrightarrow{\sim} f_+\mathcal{M}$.

3. The functors $\underline{H}^0(f^!\mathcal{M})$ and $\underline{H}^0(f_+\mathcal{M})$ induce an equivalence between the category of unit $\mathcal{O}_{F,Y}$ modules, and the category of unit $\mathcal{O}_{F,X}$-modules supported on Y.

4. If $\mathcal{M}$ is any unit $\mathcal{O}_{F,X}$, then adjunction map $f_+f^!\mathcal{M} \to \mathcal{M}$ induced by the adjointness of (2.4.3)(4) induces an inclusion $f_+\underline{H}^0(f^!\mathcal{M}) \subset \mathcal{M}$ which identifies $f_+\underline{H}^0(f^!)\mathcal{M}$ with the subsheaf $\Gamma_Y(\mathcal{M})$ of $\mathcal{M}$, consisting of sections supported on Y.

We also have an analogous result for the derived category [EK 1, 5.11.3, 5.11.5].To explain it, for any closed subset $Y \subset X$, denote by $D^-_{u,Y}(\mathcal{O}_{F,X})$ the full (triangulated) subcategory of $D^-_u(\mathcal{O}_{F,X})$ consisting of complexes whose cohomology sheaves are supported on Y.

Corollary 2.5.2. Let $f : Y \to X$ be a closed immersion of smooth k-schemes.

1. The functor
$$f_+ : D^-_u(\mathcal{O}_{F,Y}) \to D^-_u(\mathcal{O}_{F,X})$$
induces an equivalence of $D^-_u(\mathcal{O}_{F,Y})$ with $D^-_{u,Y}(\mathcal{O}_{F,X})$. The functor $f^!$ provides a quasi-inverse.

2. If $\mathcal{M}^\bullet$ is in $D^b_u(\mathcal{O}_{F,X})$, then $R\Gamma_Y(\mathcal{M}^\bullet)$ has a natural structure of an object of $D_{u,Y}(\mathcal{O}_{F,X})$, and there is an isomorphism
$$R\Gamma_Y(\mathcal{M}^\bullet) \xrightarrow{\sim} f_+f^!\mathcal{M}^\bullet)$$

The following result is often used in [EK 1] to make arguments by induction on the support of $\mathcal{M}^\bullet$ [EK 1, 5.12.1]

Corollary 2.5.3. Let $\mathcal{M}^\bullet$ be in $D^-_u(\mathcal{O}_{F,X})$, and let Y be the support of the cohomology sheaves of $\mathcal{M}$. Then for any smooth open subscheme $U \subset Y$, if $f : U \to X$ denotes the natural map, then there is a natural map $\mathcal{M}^\bullet \to f_+f^!\mathcal{M}^\bullet$ whose cone is supported on $Y \backslash U$.

Proof. Let $W = X \setminus (Y \setminus U)$. Then U is closed in W and W is open in X. Let $i : U \to W$ and $j : W \to X$ denote the corresponding immersions, so that $f = ji$. Then by (2.4.3)(3), there is the natural morphism of adjunction $\mathcal{M}^\bullet \to j_+j^!\mathcal{M}^\bullet$. Since $\mathcal{M}^\bullet$ is supported on Y, we see that $j^!\mathcal{M}^\bullet$ is supported on $Y \cap W = U$. Thus by (2.5.2) we see that
$$j^!\mathcal{M}^\bullet \xrightarrow{\sim} i_+i^!j^!\mathcal{M}^\bullet \tag{2.5.4}$$
Thus we get maps
$$\mathcal{M}^\bullet \to j_+j^!\mathcal{M}^\bullet \overset{\sim}{\to} j_+i_+i^!j^!\mathcal{M}^\bullet \xrightarrow{\sim} f_+f^!\mathcal{M}^\bullet$$
Applying $j^!$ to the composite of these maps recovers the isomorphism (2.5.4), so this composite map has cone supported on $X \setminus W = Z \setminus U$. □

3. The Riemann-Hilbert correspondence

3.1. We now move from working in the Zariski site, to the étale site. If X is a smooth k-scheme, we denote by $\pi_X : X_{ét} \to X_{Zar}$ the natural morphism of sites. We set $\mathcal{O}_{F,X_{ét}} = \pi^*_X\mathcal{O}_{F,X}$. We call a $\mathcal{O}_{F,X_{ét}}$-module unit (resp. lfgu) if it has this property when restricted to the Zariski site of an étale covering of X. Recall that the exact functors π_X^* and π_X^* induce quasi-inverse equivalences of categories between quasi-coherent sheaves on $X_{ét}$ and quasi-coherent sheaves on X_{Zar}. Using this one sees easily that every unit (resp. lfgu) $\mathcal{O}_{F,X_{ét}}$-module $\mathcal{M}_{ét}$ has the form $\mathcal{M}_{ét} = \pi_X^*\mathcal{M}$ for some unit (resp. lfgu) $\mathcal{O}_{F,X}$-module.

For $\bullet$ equal to one of $\emptyset$, $+$, $-$, or b, and $*$ one of $\emptyset$, qc, u, or $lfgu$ we denote by $D_*^\bullet(\mathcal{O}_{F,X_{ét}})$ the derived category of $\mathcal{O}_{F,X_{ét}}$-modules with the indicated boundedness condition, and whose cohomology sheaves satisfy the condition $*$. If $*$ is one of qc, u or $lfgu$, then π_X^* induces an equivalence of triangulated categories between this category and $D_*^\bullet(\mathcal{O}_{F,X})$ [EK 1, 7.3.2]. This essentially follows from descent, and a theorem of Bernstein which implies that any complex in $D^b(\mathcal{O}_{F,X_{ét}})$ is quasi-isomorphic to a complex of quasi-coherent $\mathcal{O}_{F,X}$-modules.

For a map of smooth k-schemes $f : Y \to X$, we can define the bimodules $\mathcal{O}_{F,Y\to X}$ and $\mathcal{O}_{F,X\leftarrow Y}$, and the functors $\otimes_{\mathcal{O}_{X_{ét}}}$, $f^!$ and f_+. When restricted to bounded complexes with quasi-coherent cohomology, these satisfy

$$\pi_X^*(- \otimes_{\mathcal{O}_X} -) \xrightarrow{\sim} \pi_X^*(-) \otimes_{\mathcal{O}_{X_{ét}}} \pi_X^*(-), \quad \pi_Y^* f^! \xrightarrow{\sim} f^! \pi_X^* \text{ and } \pi_X^* f_+ \xrightarrow{\sim} \pi_Y^* f_+$$

3.2. For $\bullet$ as above, we denote by $D^\bullet(X_{ét}, \mathbb{Z}/pZ)$ the derived category of étale sheaves of $\mathbb{Z}/p\mathbb{Z}$-modules on X with the indicated boundedness condition. We denote by $D_c^\bullet(X_{ét}, \mathbb{Z}/p\mathbb{Z})$ the full triangulated subcategory of $D^\bullet(X_{ét})$ consisting of complexes with constructible cohomology sheaves.

Now define the functor

$$\mathrm{Sol}_{ét}(-) = \underline{Rhom}_{\mathcal{O}_{F,X_{ét}}}(-, \mathcal{O}_{X_{ét}})[d_X] : D^-_{lfgu}(\mathcal{O}_{F,X_{ét}}) \to D^+(X_{ét}, \mathbb{Z}/p\mathbb{Z})$$

This functor has the following properties [EK 1, 9.3, 9.7.1, 9.8, 9.9]

Proposition 3.2.1. Let $f : Y \to X$ be a morphism of smooth k-schemes.

1. There is a natural isomorphism $f^{-1}\,\mathrm{Sol}_{ét} \xrightarrow{\sim} \mathrm{Sol}_{ét}\, f^!$
2. If f is quasi-projective, there is a natural isomorphism $\mathrm{Sol}_{ét}\, f_+ \xrightarrow{\sim} f_!\,\mathrm{Sol}_{ét}$.
3. There is a natural isomorphism
$$\mathrm{Sol}_{ét}(-) \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p\mathbb{Z}} \mathrm{Sol}_{ét}(-) \xrightarrow{\sim} \mathrm{Sol}_{ét}(- \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{X_{ét}}} -)[d_X]$$
4. $\mathrm{Sol}_{ét}$ restricts to a functor
$$\mathrm{Sol}_{ét} : D^b_{lfgu}(\mathcal{O}_{F,X_{ét}}) \to D^b_c(X_{ét}, \mathbb{Z}/p\mathbb{Z})$$

Sketch of proof. Each of these claims is non-trivial. For (1), the required natural transformation is deduced from the map

$$\underline{Rhom}_{\mathcal{O}_{F,X_{ét}}}(\mathcal{M}^\bullet, \mathcal{O}_{X_{ét}}) \to \underline{Rhom}_{\mathcal{O}_{F,Y_{ét}}}(f^!\mathcal{M}^\bullet, f^!\mathcal{O}_{X_{ét}})$$

and the remark that $f^!\mathcal{O}_{X_{ét}} = \mathcal{O}_{Y_{ét}}[d_{Y/X}]$.

We will explain how to check that this map is an isomorphism in the special case that Y is a smooth divisor, cut out by a section a of $\mathcal{O}_X$, and $\mathcal{M}^\bullet$ is a single lfgu $\mathcal{O}_{F,X}$-module, with a generator of the form $\mu : \mathcal{O}^n_{X_{ét}} \to \mathcal{O}^n_{X_{ét}}$. The general case can be reduced to this one using various formal arguments, and the technique explained in the proof of (2.3.1).

Using the resolution (1.3.4) to compute $\mathrm{Sol}_{ét}$, one sees that what we have to show is that the natural map

$$f^{-1}(\mathcal{O}^n_{X_{ét}} \stackrel{1-\mu F}{\to} \mathcal{O}^n_{X_{ét}}) \to (\mathcal{O}^n_{Y_{ét}} \stackrel{1-\mu F}{\to} \mathcal{O}^n_{Y_{ét}}) \tag{3.2.2}$$

is a quasi-isomorphism. Both sides of (3.2.2) are surjective in the étale topology (because the differential of $1-\mu F$ is the identity), so we have to show (3.2.2) induces an isomorphism on kernels.

Suppose that a section (ax_i) of $a\mathcal{O}^n_{X_{ét}}$ satisfies $(1-\mu F)(ax_i) = 0$. Then $(ax_i) = \mu F(ax_i) = a^p \mu F(x_i)$, and

$$\begin{aligned}(x_i) = a^{p-1}\mu F(x_i) &= a^{p-1}\mu F(a^{p-1}\mu F(x_i)) \\ &= a^{(p+1)(p-1)}\mu F \mu F(a^{p-1}\mu F(x_i)) = \dots . \end{aligned} \tag{3.2.3}$$

Continuing, we find that the x_i are divisible by arbitrarily high powers of a, and so vanish. This shows that (3.2.2) induces an injective map on kernels. To see the surjectivity, suppose that $(\bar{x}_i)$ are sections of $(\mathcal{O}_{X_{ét}}/a)^n$ which satisfy $(1 - \mu F^r)(\bar{x}_i) = 0$. Let (x_i) be a section of $\mathcal{O}^n_{X_{ét}}$ lifting $(\bar{x}_i)$. Then $(1-\mu F)(x_i) = (ax'_i)$ for some section (x'_i) of $\mathcal{O}_{X_{ét}}$. Now étale locally, we may find a section (y_i) of $\mathcal{O}^n_{X_{ét}}$ solving $(1 - a^{p-1}\mu F)(y_i) = (x'_i)$. Then $x_i - ay_i = \bar{x}_i$ modulo a, and

$$(1-\mu F)(x_i - ay_i) = (ax'_i - ax'_i) = 0$$

For the proof of (2) one considers separately the case of a proper map, and of an open immersion. The general isomorphism is then constructed by factoring a quasi-projective map as a composite of such maps (one has to work to check that the result is independent of the factorisation). The case of a proper map can be deduced from the adjointness in (2.4.3)(4). For the case of f, an open immersion, we have a natural isomorphism $f^{-1}\,\mathrm{Sol}_{ét}(f_+\mathcal{M}^\bullet) \xrightarrow{\sim} \mathrm{Sol}_{ét}(\mathcal{M}^\bullet)$, because the definition of $\mathrm{Sol}_{ét}$ is local on X. By adjointness of $f_!$ and f^{-1}, this gives a map $\mathrm{Sol}_{ét}(f_+\mathcal{M}^\bullet) \xrightarrow{\sim} f_!\,\mathrm{Sol}_{ét}(\mathcal{M}^\bullet)$. To show this is an isomorphism one has to show that the stalks of $\mathrm{Sol}_{ét}(f_+\mathcal{M}^\bullet)$ have vanishing cohomology at points of $X\backslash Y$. This is done (after some reductions) by an explicit calculation, similar to the one we explained in the proof of (1).

The proofs of (3) and (4) proceed by induction on the support of (the cohomology sheaves of) $\mathcal{M}^\bullet$, using (1.2.2) and (2.5.3). This argument uses (1) and (2). Note that when $\mathcal{M}$ is an F-crystal (i.e. a unit module, which is finite flat over $\mathcal{O}_{X_{ét}}$) of rank n, then computing $\mathrm{Sol}_{ét}(\mathcal{M})$ by using the resolution (1.3.4), with the root $\mu = \phi^{-1}_{\mathcal{M}}$, one sees that

$$\mathrm{Sol}_{ét}(\mathcal{M}) = (\mathcal{M} \stackrel{1-\mu F}{\to} \mathcal{M})[d_X]$$

Then Katz's theorem, mentioned in the introduction, implies that $\mathrm{Sol}_{ét}(\mathcal{M}) = \mathcal{L}[d_X]$ where $\mathcal{L}$ is a locally constant étale sheaf of $\mathbb{Z}/p\mathbb{Z}$-modules of rank n. □

3.3. We define a functor

$$\mathrm{M}_{\acute{e}t} = \underline{Rhom}_{\mathbb{Z}/p\mathbb{Z}}(-, \mathcal{O}_{X_{\acute{e}t}})[d_X] : D^b_c(X_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z}) \to D^+(\mathcal{O}_{F,X_{\acute{e}t}})$$

We have the following result [EK 1, 10.4]

Proposition 3.3.1. The functor $\mathrm{M}_{\acute{e}t}$ induces a functor

$$\mathrm{M}_{\acute{e}t} : D^b_c(X_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z}) \to D^b_{lfgu}(\mathcal{O}_{F,X_{\acute{e}t}})$$

The proof of (3.3.1) uses excision and induction on the dimension of the support of (the cohomology sheaves) of a complex $\mathcal{L}^\bullet$ in $D^b_c(X_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z})$, together with the fact that if $\mathcal{L}$ is an étale local system, then $\mathrm{M}_{\acute{e}t}(\mathcal{L})$ is easily seen to be an F-crystal. As in the proof of (3.2.1), in order to carry out this excision argument, one first has to prove that in certain special cases, $\mathrm{M}_{\acute{e}t}$ exchanges f_+ and $f_!$, and $f^!$ and f^{-1}.

3.4. Now using (3.2.1)(4) and (3.3.1) we can define functors

$$\mathrm{Sol} = \mathrm{Sol}_{\acute{e}t} \circ \pi_X^* : D^b_{lfgu}(\mathcal{O}_{F,X}) \to D^b_c(X_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z})$$

and

$$\mathrm{M} = R\pi_{X*} \circ \mathrm{M}_{\acute{e}t} : D^b_c(X_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z}) \to D^b_{lfgu}(\mathcal{O}_{F,X})$$

The natural map

$$\mathcal{M}^\bullet \to \underline{Rhom}_{\mathbb{Z}/p\mathbb{Z}}(\underline{Rhom}_{\mathcal{O}_{F,X_{\acute{e}t}}}(\mathcal{M}^\bullet, \mathcal{O}_{X_{\acute{e}t}}), \mathcal{O}_{X_{\acute{e}t}})$$

induces a natural transformation $\zeta : \mathrm{id} \to \mathrm{M} \circ \mathrm{Sol}$. Similarly, we get a natural transformation $\eta : \mathrm{id} \to \mathrm{Sol} \circ \mathrm{M}$. We have the following theorem [EK 1, 11.3]

Theorem 3.4.1. The natural transformations ζ and η are isomorphisms. In particular M and Sol induce quasi-inverse anti-equivalences of triangulated categories. They exchange the functors $\overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p\mathbb{Z}}$ and $\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}$ (up to a shift), and f^{-1} and $f^!$. If f is quasi-projective they exchange $f_!$ and f_+.

Sketch of proof. That Sol takes $f^!$, f_+ and $\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}$ into f^{-1} $f_!$ and $\overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p\mathbb{Z}}$ follows from (3.2.1). The analogous claim for M follows once we know that ζ and η are isomorphisms. This is established by an excision argument. □

4. Perverse Sheaves

4.1. The canonical t-structure on $D^b_{lfgu}(\mathcal{O}_{F,X})$ induces via the anti-equivalence of categories in (3.4.1), a t-structure on $D^b_c(X_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z})$. To describe this exotic t-structure explicitly we recall the notion of a perversity function.

Let X be a k-scheme. For $x \in X$, denote by $i_x : x \to X$ the natural inclusion. Recall that the "middle perversity" function on X is defined by $\mathbf{p}(x) = -\dim \overline{\{x\}}$. Define full sub-categories ${}^{\mathbf{p}}D^{\le 0}$ and ${}^{\mathbf{p}}D^{\ge 0}$ of $D^b_c(X_{\acute{e}t}, \Lambda)$ by the conditions:

$\mathcal{F}^\bullet$ is in ${}^{\mathbf{p}}D^{\le 0}$ if and only if for all $x \in X$, $\underline{H}^i(i_x^{-1}\mathcal{F}^\bullet) = 0$ for $i > \mathbf{p}(x)$.

$\mathcal{F}^\bullet$ is in ${}^{\mathbf{p}}D^{\ge 0}$ if and only if for all $x \in X$, $\underline{H}^i(i_x^!\mathcal{F}^\bullet) = 0$ for $i < \mathbf{p}(x)$.

Gabber has shown that the subcategories ${}^{\mathbf{p}}D^{\le 0}$ and ${}^{\mathbf{p}}D^{\ge 0}$ underlie a (necessarily unique) t-structure on $D^b_c(X_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z})$ [Ga]. (In fact, Gabber establishes this more generally for perversity functions $\mathbf{p}$ having the property that $\mathbf{p}(x) \le \mathbf{p}(y) + 1$ whenever x is an immediate specialisation of y.) In [EK 1, 11.5] we show that this t-structure is equal to the one induced on $D^b_c(X_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z})$ by the anti-equivalence of categories in (3.4.1). In particular, the analysis there gives an independent proof that the categories ${}^{\mathbf{p}}D^{\le 0}$ and ${}^{\mathbf{p}}D^{\ge 0}$ underlie a t-structure on $D^b_c(X_{\acute{e}t}, \mathbb{Z}/p\mathbb{Z})$.

In this section we want to supplement the results of *loc. cit.* by proving a structure theorem for complexes in the heart of this t-structure – "perverse sheaves" – and showing that there is a reasonable theory of middle extension. These results were motivated by those of Blickle's thesis [Bl], which constructs the middle extension in certain cases.

4.2. The next result follows immediately from [Lyu, Thm. 3.2]

Proposition 4.2.1. Let X be a smooth k-scheme. The category of lfgu $\mathcal{O}_{F,X}$-modules is Artinian.

Corollary 4.2.2. Let $j : U \to X$ be a locally closed immersion of smooth k-schemes. If $\mathcal{M}$ is an lfgu $\mathcal{O}_{F,U}$-module, then the set of lfgu $\mathcal{O}_{F,X}$-submodules $\tilde{\mathcal{M}} \subset \underline{H}^0(j_+\mathcal{M})$ such that $j^!\tilde{\mathcal{M}} = \mathcal{M}$ has a smallest element.

Proof. If $\tilde{\mathcal{M}}_1$ and $\tilde{\mathcal{M}}_2$ are two such submodules, then we have an exact sequence

$$0 \to \tilde{\mathcal{M}}_1 \cap \tilde{\mathcal{M}}_2 \to \underline{H}^0(j_+\mathcal{M}) \to \underline{H}^0(j_+\mathcal{M})/\tilde{\mathcal{M}}_1 \oplus \underline{H}^0(j_+\mathcal{M})/\tilde{\mathcal{M}}_2$$

By (2.1.1) the right most term is lfgu, so applying (2.1.1) again shows that $\tilde{\mathcal{M}}_1 \cap \tilde{\mathcal{M}}_2$ is an lfgu $\mathcal{O}_{F,X}$-module. Now the corollary follows from (4.2.1). □

We denote the minimal extension $\tilde{\mathcal{M}}$ in (4.2.2) by $j_{!+}\mathcal{M}$. We call an lfgu $\mathcal{O}_{F,X}$-module *simple* if it has no non-trivial, proper lfgu $\mathcal{O}_{F,X}$-submodules.

Corollary 4.2.3. Every simple lfgu $\mathcal{O}_{F,X}$-module on X is of the form $j_{!+}\mathcal{N}$ for a locally closed immersion $j : U \to X$, and some simple F-crystal $\mathcal{N}$ on U.

Proof. Suppose that $\mathcal{N}$ is a simple F-crystal on U. Let $\mathcal{M} \subset j_{!+}\mathcal{N}$ be a non-zero submodule. Let Y denote the closure of U in X, set $W = X\backslash(Y\backslash U)$, and let $h : W \to X$ and $i : U \to W$ denote the natural (open, resp. closed) immersions. First we claim that $\mathcal{M}$ is not supported on $Y\backslash U$. Indeed, the composite $\mathcal{M} \to \underline{H}^0(j_+\mathcal{N}) \to j_+\mathcal{N} \xrightarrow{\sim} h_+i_+\mathcal{N}$ (in which the isomorphism is provided by (2.4.3)(1)) corresponds, by the adjointness of (2.4.3)(3), to a map $h^!\mathcal{M} \to i_+\mathcal{N}$. If $\mathcal{M}$ were supported on $Y\backslash U$, then $h^!\mathcal{M}$ would vanish, hence the map $\mathcal{M} \to h_+i_+\mathcal{N} = j_+\mathcal{N}$

would also vanish, and thus $\mathcal{M}$ would vanish, contradicting our assumption. Thus $j^!\mathcal{M}$ is a non-zero submodule of $\mathcal{N}$. Since $\mathcal{N}$ is simple, it follows that $j^!\mathcal{M} = \mathcal{N}$, which implies that $\mathcal{M} = j_{!+}\mathcal{N}$ as the latter module is minimal.

Conversely, suppose that $\mathcal{M}$ is a simple lfgu $\mathcal{O}_{F,X}$-module. Let Y be the support of $\mathcal{M}$ and choose a dense open $U \subset Y$ such that, if $j : U \to X$ denotes the natural map, then $j^!\mathcal{M}$ is an F-crystal. Now the map $\mathcal{M} \to j_+j^!\mathcal{M}$ is non-zero, as it becomes an isomorphism after applying $j^!$. Since $\mathcal{M}$ is simple, this map must be an injection, whence $j_{!+}j^!\mathcal{M} \subset \mathcal{M}$. Since $\mathcal{M}$ is simple, this inclusion must be an isomorphism. Finally if $\mathcal{N} \subset j^!\mathcal{M}$ is a non-zero subcrystal, then $j_{!+}\mathcal{N} \subset \mathcal{M}$, and this must be an isomorphism, as $\mathcal{M}$ is simple, so that $\mathcal{N} = j^!\mathcal{M}$. Thus, $j^!\mathcal{M}$ is simple. □

4.3. It remains to translate the results of the previous section into statements involving $D^b_c(X_{ét}, \mathbb{Z}/p\mathbb{Z})$. We call a complex $\mathcal{L}^\bullet$ in $D^b_c(X_{ét}, \mathbb{Z}/p\mathbb{Z})$ a *perverse sheaf* if it is of the form $\mathrm{Sol}_{ét}(\mathcal{M})$ for a single lfgu $\mathcal{O}_{F,X}$-module $\mathcal{M}$. We begin by showing that the construction on perverse sheaves, which corresponds to the functor $j_{!+}$, has a familiar description.

The conditions given in (4.1) to define ${}^{\mathbf{p}}D^{\leq 0}$ and ${}^{\mathbf{p}}D^{\geq 0}$ may be applied equally well to complexes of sheaves in $D^b(X, \mathbb{Z}/p\mathbb{Z})$, and it is shown in [Ga] that they determine a t-structure on this category. (Indeed, Gabber's strategy is to construct this t-structure first, and then to show that it restricts to a t-structure on $D^b_c(X, \mathbb{Z}/p\mathbb{Z})$, by showing that the perverse truncation of a complex with constructible cohomology sheaves again has constructible cohomology sheaves.) We denote the cohomology functors of this t-structure by ${}^{\mathbf{p}}\underline{H}^i$. If $j : U \to X$ is a locally closed immersion, and $\mathcal{L}^\bullet$ is a perverse sheaf in $D^b_c(U_{ét}, \mathbb{Z}/p\mathbb{Z})$, then we set

$$j_{!*}\mathcal{L}^\bullet = \mathrm{Im}({}^{\mathbf{p}}\underline{H}^0(j_!\mathcal{L}^\bullet) \to {}^{\mathbf{p}}\underline{H}^0(Rj_*\mathcal{L}^\bullet)).$$

Lemma 4.3.1. Let $j : U \to X$ be a locally closed immersion, and $\mathcal{M}$ an lfgu $\mathcal{O}_{F,U}$-module. If $\mathcal{L}^\bullet = \mathrm{Sol}(\mathcal{M})$ is the perverse sheaf corresponding to $\mathcal{M}$ then

1. $j_!\mathcal{L}^\bullet$ is in ${}^{\mathbf{p}}D^{\leq 0}$ and $Rj_*\mathcal{L}^\bullet$ is in ${}^{\mathbf{p}}D^{\geq 0}$
2. We have a canonical isomorphism $\mathrm{Sol}(j_{!+}\mathcal{M}) \xrightarrow{\sim} j_{!*}\mathcal{L}^\bullet$

Proof. To prove (1) let $i_x : x \to X$ be the inclusion of a point of X. We have to check that $\underline{H}^i(i_x^{-1}j_!\mathcal{L}^\bullet) = 0$ for $i > \mathbf{p}(x)$ and $\underline{H}^i(i_x^!Rj_*\mathcal{L}^\bullet) = 0$ for $i < \mathbf{p}(x)$. Let Y be the closure of U in X. If $x \notin Y\backslash U$ then these equalities follow from the analogous facts for the perverse sheaf $\mathcal{L}^\bullet$. If $x \in Y\backslash U$, then we have $i_x^{-1}j_!\mathcal{L}^\bullet = i_x^!Rj_*\mathcal{L}^\bullet = 0$, so these equalities are trivial.

Next we prove (2). Let $\mathcal{L}'^\bullet \subset {}^{\mathbf{p}}\underline{H}^0(j_!\mathcal{L}^\bullet)$ be a perverse subsheaf. Consider the composite map

$$\mathcal{L}'^\bullet \to {}^{\mathbf{p}}\underline{H}^0(j_!\mathcal{L}^\bullet) \to {}^{\mathbf{p}}\underline{H}^0(Rj_*\mathcal{L}^\bullet) \qquad (4.3.2)$$

If the map (4.3.2) vanishes then $\mathcal{L}'^\bullet$ is supported on $Y\backslash U$ because $j^{-1}j_{!*}\mathcal{L}^\bullet = \mathcal{L}^\bullet$. Conversely, if $\mathcal{L}'^\bullet$ is supported on $Y\backslash U$, then we claim that (4.3.2) vanishes. By (1) it suffices to check that the map obtained by composing (4.3.2) with

${}^{\mathrm{p}}\underline{H}^0(Rj_*\mathcal{L}^\bullet) \to Rj_*\mathcal{L}^\bullet$ vanishes. Since $\mathcal{L}'^\bullet$ is supported on $Y\backslash U$ this last map factors through a map

$$\mathcal{L}'^\bullet \to i_*i^!Rj_*\mathcal{L}^\bullet = 0,$$

where $i : Y\backslash U \to X$ denotes the inclusion.

It follows that $j_{!*}\mathcal{L}^\bullet$ is equal to the quotient of ${}^{\mathrm{p}}\underline{H}^0(j_!\mathcal{L}^\bullet)$ by its maximal perverse subsheaf supported on $Y\backslash U$. Now (2) follows from (3.4.1) and the definition of $j_{!+}\mathcal{M}$. □

Corollary 4.3.3. The category of perverse sheaves is Artinian. A perverse sheaf is irreducible if and only if it is of the form $j_{!*}\mathcal{L}[d_U]$, where $j : U \to X$ is a locally closed immersion, and $\mathcal{L}$ is an irreducible local system of $\mathbb{Z}/p\mathbb{Z}$-sheaves on U.

Proof. The first claim follows from (4.2.1). The second claim follows from (4.2.3) and (4.3.1)(2), keeping in mind that $\mathcal{M} \mapsto \underline{H}^0(\mathrm{Sol}(\mathcal{M})[-d_U])$ establishes an equivalence between the category of F-crystals on U, and the category of locally constant étale sheaves of finite $\mathbb{Z}/p\mathbb{Z}$-modules (cf. the proof of (3.2.1)). □

We remark that in the notes [Ga], Gabber gives another proof that the category of perverse sheaves is Artinian.

5. L-functions

5.1. In this section we will define L-functions for lfgu $\mathcal{O}_{F,X}$-modules and explain some of their properties. The theory explained in previous sections has its strongest applications to L-functions if it is first generalised to include "coefficients". Thus, in this section we let Λ be a Noetherian $\mathbb{F}_p$-algebra, and we set $\mathcal{O}_X^\Lambda = \Lambda \otimes_{\mathbb{F}_p} \mathcal{O}_X$ and $\mathcal{O}_{F,X}^\Lambda = \Lambda \otimes_{\mathbb{F}_p} \mathcal{O}_{F,X}$. We define $\mathcal{O}_{X_{\acute{e}t}}^\Lambda$ and $\mathcal{O}_{F,X_{\acute{e}t}}^\Lambda$ similarly.

We say that a $\mathcal{O}_{F,X}^\Lambda$ is "unit" (resp. quasi-coherent) if its underlying $\mathcal{O}_{F,X}$-module is unit (resp. quasi-coherent). We say that a $\mathcal{O}_{F,X}^\Lambda$-module is lfgu if it is unit, and if, locally on X, it is finitely generated over $\mathcal{O}_{F,X}^\Lambda$.

For $\bullet = \emptyset$, $-$, $+$, or b, and $*$ one of qc, u or $lfgu$ we define $D_*^\bullet(\mathcal{O}_{F,X}^\Lambda)$ to be the full (triangulated) subcategory of $D^\bullet(\mathcal{O}_{F,X}^\Lambda)$ consisting of complexes whose cohomology sheaves satisfy the condition $*$.

5.2. If $f : Y \to X$ is a morphism of smooth k-schemes, we can define functors

$$f^! : D^-(\mathcal{O}_{F,X}^\Lambda) \to D^-(\mathcal{O}_{F,Y}^\Lambda)$$

and

$$f_+ : D^-(\mathcal{O}_{F,X}^\Lambda) \to D^-(\mathcal{O}_{F,X}^\Lambda)$$

using the same definitions as in §2. These functors also preserve the conditions qc, u, and $lfgu$ (except for the last case, this is a formal consequence of the results cited in §2). For proofs of these results we refer to the same references in [EK 1] cited in §2. We also have a bifunctor

$$- \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X^\Lambda} - : D^-(\mathcal{O}_{F,X}^\Lambda) \times D^-(\mathcal{O}_{F,X}^\Lambda) \to D^-(\mathcal{O}_{F,X}^\Lambda)$$

but we will not need this in what follows.

Having introduced coefficients, we can also define functors corresponding to extension of scalars: Let $\Lambda \to \Lambda'$ be a morphism of Noetherian $\mathbb{F}_p$-algebras. Then we have a functor

$$\overset{\mathbb{L}}{\otimes}_\Lambda \Lambda' : D^-(\mathcal{O}_{F,X}^\Lambda) \to D^-(\mathcal{O}_{F,X}^{\Lambda'})$$

We have the following result

Proposition 5.2.1. If $*$ is one of qc, u or $lfgu$, then $\overset{\mathbb{L}}{\otimes}_\Lambda \Lambda'$ induces a functor

$$\overset{\mathbb{L}}{\otimes}_\Lambda \Lambda' : D_*^-(\mathcal{O}_{F,X}^\Lambda) \to D_*^-(\mathcal{O}_{F,X}^{\Lambda'})$$

Moreover, $- \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda'$ commutes with $f^!$ and f_+.

Proof. For the first claim see [EK 1, 1.13, 5.6, 6.5], and for the second [EK 1, 2.8, 3.10] □

5.3. If Λ is a finite ring we have a version of the Riemann-Hilbert correspondence. To explain it, we denote by $D^b_{lfgu}(\mathcal{O}_{F,X}^\Lambda)^\circ$ the full subcategory of $D^b_{lfgu}(\mathcal{O}_{F,X}^\Lambda)$ consisting of complexes which have finite Tor dimension when viewed as complexes of $\mathcal{O}_X^\Lambda$-modules. Note that if Λ is regular, then $\mathcal{O}_X^\Lambda$ is a sheaf of regular rings, and this finiteness condition is automatic.

We denote by $D^b_{ctf}(X_{ét}, \Lambda)$ the full subcategory of $D^b_c(X_{ét}, \Lambda)$ consisting of complexes of sheaves of Λ-modules which have finite Tor dimension.

We define functors

$$\mathrm{Sol}(-) = \underline{Rhom}_{\mathcal{O}_{F,X_{ét}}^\Lambda}(\pi_X^* -, \mathcal{O}_{X_{ét}}^\Lambda) : D^b_{lfgu}(\mathcal{O}_{F,X}^\Lambda)^\circ \to D(X_{ét}, \Lambda)$$

$$\mathrm{M}(-) = R\pi_{X*}\underline{Rhom}_\Lambda(-, \mathcal{O}_{X_{ét}}^\Lambda) : D^b_{ctf}(X_{ét}, \Lambda) \to D(\mathcal{O}_{F,X}^\Lambda)$$

The Riemann-Hilbert correspondence with "Λ-coefficients" then takes the form (with the same references in [EK 1] cited in §3)

Theorem 5.3.1. Suppose Λ is a finite ring. The functors Sol and M induce quasi-inverse anti-equivalences of triangulated categories between $D^b_{lfgu}(\mathcal{O}_{F,X}^\Lambda)^\circ$ and $D^b_{ctf}(X_{ét}, \Lambda)$. It exchanges the functors $\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X^\Lambda}$ and $\overset{\mathbb{L}}{\otimes}_\Lambda$ (up to a shift), and $f^!$ and f^{-1}. If f is quasi-projective, it exchanges f_+ and $f_!$. The equivalence is compatible with $- \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda'$ for any map of finite $\mathbb{F}_p$-algebras $\Lambda \to \Lambda'$.

5.4. We now begin our definition of L-functions. We assume from now on that k is a finite field. Suppose first that Λ is a product of finitely many fields, and that $X = x = \operatorname{Spec} k(x)$ where $k(x)$ is a finite extension of k.

Let $\mathcal{M}$ be a lfgu $\mathcal{O}_{F,x}^{\Lambda}$-module. Then $\mathcal{M}$ is actually finite as a $\Lambda \otimes_{\mathbb{F}_p} k'$-module. This follows from the Λ-coefficient version of (1.2.1). Alternatively, in this very simple situation, it follows from the fact that $\mathcal{M}$ is finitely generated over $\Lambda[F]$, and is also a module over the commutative ring $\Lambda[F, F^{-1}]$. Since Λ is a finite product of fields, $\mathcal{M}$ is finite projective over $\Lambda \otimes_{\mathbb{F}_p} k(x)$. Let $|k(x)| = p^s$ for some integer s. The structural isomorphism $\phi_{\mathcal{M}} : F^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}$ induces a map

$$\phi_{\mathcal{M},x}^{-1} : F^{s-1*}\phi_{\mathcal{M}}^{-1} \circ F^{s-2*}\phi_{\mathcal{M}}^{-1} \circ \cdots \circ \phi_{\mathcal{M}}^{-1} : \mathcal{M} \xrightarrow{\sim} F^{s*}\mathcal{M} \to \mathcal{M}$$

We define

$$L_u(x, \mathcal{M}) = \det_{\Lambda \otimes_{\mathbb{F}_p} k(x)} (1 - \phi_{\mathcal{M},x}^{-1} T^s | \mathcal{M})^{-1}$$

A priori $L_u(x, \mathcal{M})$ lies in $\Lambda \otimes_{\mathbb{F}_p} k(x)[\![T]\!]$ but one can show that it actually lies in $\Lambda[\![T]\!]$ [EK 1, 12.1.2].

More generally, for $\mathcal{M}^\bullet$ in $D_{lfgu}^b(\mathcal{O}_{F,x}^{\Lambda})$ we define

$$L_u(x, \mathcal{M}^\bullet) = \prod_i L_u(x, \underline{H}^i(\mathcal{M}^\bullet))^{(-1)^i}$$

Next if X is any smooth k-scheme, and $\mathcal{M}^\bullet$ is in $D_{lfgu}^b(\mathcal{O}_{F,X}^{\Lambda})$ we define

$$L_u(X, \mathcal{M}^\bullet) = \prod_{x \in |X|} L_u(x, i_x^! \mathcal{M}^\bullet)$$

where $|X|$ denotes the set of closed points in X, and $i_x : x \to X$ denotes the inclusion.

Finally, if Λ is any reduced ring, we denote by $Q(\Lambda)$ its total ring of fractions and for $\mathcal{M}^\bullet$ in $D_{lfgu}^b(\mathcal{O}_{F,X}^{\Lambda})$, we define

$$L_u(X, \mathcal{M}^\bullet) = L_u(X, \mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_{\Lambda} Q(\Lambda))$$

5.5. If Λ is finite we have the following result [EK 1, 12.3.1]

Proposition 5.5.1. Suppose Λ is a finite reduced ring (i.e a product of finite fields), let $\mathcal{M}^\bullet$ be in $D_{lfgu}^b(\mathcal{O}_{F,X}^{\Lambda})$ and set $\mathcal{L}^\bullet = \operatorname{Sol}(\mathcal{M}^\bullet)$. Then we have

$$L_u(X, \mathcal{M}^\bullet) = L_{\acute{e}t}(X, \mathcal{L}^\bullet) \tag{5.5.2}$$

where the right hand side is the L-function defined in [De, p. 116].

For the proof one reduces immediately to the case where X is a point, and the result is then a simple computation.

In fact the right hand side of (5.5.2) is defined for more general Λ, so one might ask if (5.5.2) holds for any (i.e not necessarily reduced) finite $\mathbb{F}_p$-algebra Λ. For this one first needs a definition of the left hand side. Such a definition is given in [EK 1, 12.1.5], and (5.5.2) then holds. However, we will not need this below.

One initial draw back of the definition of $L_u(X, \mathcal{M}^\bullet)$ we have given is that it is not *a priori* clear that $L_u(X, \mathcal{M}^\bullet)$ is compatible with change of rings, or that it lies in $\Lambda[\![T]\!]$. The key to overcoming these difficulties is the following result which proves this in a special case

Proposition 5.5.3. Let $\lambda : \Lambda \to \Lambda'$ be a map of reduced Noetherian $\mathbb{F}_p$-algebras, and suppose that Λ is regular. If $\mathcal{M}^\bullet$ is in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F,X})$ then $L_u(X, \mathcal{M}^\bullet)$ lies in $\Lambda[\![T]\!]$, and we have

$$L_u(X, \mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda') = \lambda(L_u(X, \mathcal{M}^\bullet))$$

The main point of the proof is that, because Λ has finite projective dimension, one can compute $L_u(X, \mathcal{M}^\bullet)$ without passing to $Q(\Lambda)$.

As a consequence of (5.5.3) one deduces [EK 1, 12.4.5, 12.4.6]

Theorem 5.5.4. Let $\lambda : \Lambda \to \Lambda'$ be a morphism of Noetherian, reduced $\mathbb{F}_p$-algebras, and $\mathcal{M}$ in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F,X})$. Then

1. If Λ is normal, then $L_u(X, \mathcal{M}^\bullet)$ is in $\Lambda[\![T]\!]$.
2. If Λ is of finite type over $\mathbb{F}_p$, $\mathcal{M}^\bullet$ is in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F,X})^\circ$, and $L_u(X, \mathcal{M}^\bullet)$ lies in $\Lambda[\![T]\!]$ then

$$L_u(X, \mathcal{M}^\bullet \overset{\mathbb{L}}{\otimes}_\Lambda \Lambda') = \lambda(L_u(X, \mathcal{M}^\bullet))$$

Sketch of proof. For (1), note that Λ is a product of normal domains, and it suffices to consider the case where Λ is a normal domain. Then for every height one prime $\mathfrak{p}$ of Λ we have that $L_u(X, \mathcal{M}^\bullet)$ is in $\Lambda_\mathfrak{p}[\![T]\!]$ by (5.5.3). On the other hand we have $\Lambda = \cap_\mathfrak{p} \Lambda_\mathfrak{p}$.

The proof of (2) is more involved, and uses de Jong's theorem on resolution of singularities to reduce to the case Λ regular, when one can apply (5.5.3). □

It is now easy to deduce the following result [EK 1, 12.4.3]

Theorem 5.5.5. Let $f : Y \to X$ be a morphism of smooth k-schemes, Λ a reduced Noetherian k-algebra, and $\mathcal{M}^\bullet$ in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F,X})$. Then we have

$$L_u(X, \mathcal{M}^\bullet) = L_u(X, f_+\mathcal{M}^\bullet) \tag{5.5.6}$$

Sketch of proof. To prove the theorem one first reduces by a limit argument to the case where Λ is of finite type over $\mathbb{F}_p$, and then to the case where Λ is smooth over $\mathbb{F}_p$. It suffices to show (5.5.6) holds after specialising both sides modulo every maximal ideal of Λ. Thus (5.5.3) implies that we may assume that Λ is a finite field. Then (5.5.6) follows from (5.3.1) and an analogous formula for complexes of constructible étale Λ-sheaves [De, 2.2, p. 116]. □

As a consequence of this theorem we have the following result which includes a conjecture of Goss proved by Taguchi-Wan [TW].

Corollary 5.5.7. Let $\mathcal{M}^\bullet$ be in $D^b_{lfgu}(\mathcal{O}^\Lambda_{F,X})$. Then $L_u(X,\mathcal{M}^\bullet)$ is a rational function.

6. $\mathscr{D}_{F,X}$-modules

6.1. We again let k be any perfect field. In this section we explain how to use arithmetic $\mathscr{D}$-modules [Be] to generalise the results of §1 - 3 to schemes which are smooth over $W_n(k)$ for some positive integer n. Before giving the formal definitions, we explain the basic idea.

Let X be a smooth $W_n(k)$ scheme. Suppose first that X admits a local system of co-ordinates $x_1,\ldots,x_d$, - i.e an étale map $X \to \mathbb{A}^d_{W_n(k)}$. Consider the ring $\mathscr{D}_{\mathbb{A}^d_{/\mathbb{Q}_p}}$ of differential operators on d-dimensional affine space $\mathbb{A}_{/\mathbb{Q}_p}$ over $\mathbb{Q}_p$. This is a "twisted polynomial ring":

$$D_{\mathbb{A}^d_{/\mathbb{Q}_p}} = \mathbb{Q}_p[x_1,\ldots,x_d,\partial_{x_1},\ldots,\partial_{x_d}],$$

with the commutation relations $[\partial_{x_i},x_j]=\delta_{ij}$. (Here δ_{ij} denotes the usual Kronecker delta.) For any multi-index $I=(i_1,\cdots,i_d)$, let $\partial^{[I]} = \frac{1}{i_1!\cdots i_d!}\partial^{i_1}_{x_1}\cdots\partial^{i_d}_{x_d}$. Then the $\mathbb{Z}_p$-submodule of $D_{\mathbb{A}^d_{/\mathbb{Q}_p}}$ generated by monomials in the elements x_i (as i ranges between 1 and d) and $\partial^{[I]}$ (as I ranges over all multi-indices) forms a $\mathbb{Z}_p$-subalgebra of $D_{\mathbb{A}^d_{/\mathbb{Q}_p}}$, which is in fact the ring of differential operators $D_{\mathbb{A}^d_{/\mathbb{Z}_p}}$ on d-dimensional affine space over $\mathbb{Z}_p$.

Now consider the composite map $h: X \to \mathbb{A}^d_{W_n(k)} \to \mathbb{A}^d_{W(k)} \to \mathbb{A}^d_{\mathbb{Z}_p}$ We set $\mathscr{D}_X = \mathcal{O}_X \otimes_{h^{-1}\mathcal{O}_{\mathbb{A}^d_{\mathbb{Z}_p}}} h^{-1}\mathscr{D}_{\mathbb{A}^d_{\mathbb{Z}_p}}$. This is the full ring of differential operators on X, and does not depend on the choice of co-ordinates.

Next let F be a lift of Frobenius to X. Such a lift can be constructed by choosing a lift of Frobenius to $\mathbb{A}^n_{W_n(k)}$ (e.g $x_i \mapsto x_i^p$), which then lifts uniquely to X by étaleness.

We denote by $\mathscr{D}_{F,X}$ the sheaf of rings obtained from $\mathscr{D}_X$ by adjoining an indeterminate F which satisfies the commutation relations

$$Fx_i = F(x_i)F, \text{ and } \partial_{x_i}F = \partial_{x_i}(F(x_i))F\partial_{x_i}.$$

where x is a section of $\mathcal{O}_X$. The key observation is that $\mathscr{D}_{F,X}$ does not depend on the choice of the Frobenius lift F : If F' is a second lift then, F' can be expressed by Taylor's formula

$$F' = \sum_I (F'(x_i) - F(x_i))^I F\partial^{[I]}$$

6.2. As a result of the independence of $\mathscr{D}_{F,X}$ of Frobenius lift, we can define a sheaf $\mathscr{D}_{F,X}$ for any smooth $W_n(k)$-scheme X, without assuming the existence of a global Frobenius lift.

The construction of $\mathscr{D}_{F,X}$ can be efficiently expressed in terms of Berthelot's interpretation of Frobenius descent [Be 2, §2.2]. Namely, Berthelot shows that there is always a functor F^* taking $\mathscr{D}_X$-modules to $\mathscr{D}_X$, and which locally on X realises pullback by Frobenius. Thus, we may set $\mathscr{D}_{F,X} = \oplus_{i\geq 0} F^{*i}\mathscr{D}_X$. By functoriality $F^{i*}\mathscr{D}_X$ is a $(\mathscr{D}_X, \mathscr{D}_X)$-bimodule, and the multiplication in $\mathscr{D}_{F,X}$ is defined via the isomorphism.

$$F^{i*}\mathscr{D}_X \otimes_{\mathscr{D}_X} F^{j*}\mathscr{D}_X \xrightarrow{\sim} F^{i+j*}\mathscr{D}_X$$

One checks [EK 2, 1.4.5] that this makes $\mathscr{D}_{F,X}$ into an associative sheaf of $\mathscr{D}_X$-algebras.

The following result result describes $\mathscr{D}_{F,X}$-modules, in terms of $\mathscr{D}_X$-modules, and is useful for transporting information from one theory to the other [EK 2, 1.4.7, 1.6.4]

Proposition 6.2.1.

1. To give a left $\mathscr{D}_{F,X}$-module is equivalent to giving a left $\mathscr{D}_X$-module $\mathcal{M}$, and a map of left $\mathscr{D}_X$-modules $\phi_{\mathcal{M}} : F^*\mathcal{M} \to \mathcal{M}$.
2. To give a right $\mathscr{D}_{F,X}$-module is equivalent to giving a right $\mathscr{D}_X$-module $\mathcal{M}$ and a map of right $\mathscr{D}_{F,X}$-modules $\mathcal{M} \to F^!\mathcal{M}$.

Here $F^!$ is the functor of coherent duality theory, and we have used that if $\mathcal{M}$ is a right $\mathscr{D}_X$=module, then $F^!\mathcal{M}$ has a canonical structure of right $\mathscr{D}_X$-module [Be 2, §2.4], [EK 2, 1.5.1].

6.3. We call a (left) $\mathscr{D}_{F,X}$-module $\mathcal{M}$ a unit $\mathscr{D}_{F,X}$-module if $\mathcal{M}$ is quasi-coherent as an $\mathcal{O}_X$-module, and the map $\phi_{\mathcal{M}}$ in (6.2.1)(1) is an isomorphism. We call $\mathcal{M}$ an lfgu $\mathscr{D}_{F,X}$-module if it is a unit module, and is locally finitely generated over $\mathscr{D}_{F,X}$. The connection between these notions and the corresponding ones for $\mathcal{O}_{F,X}$-modules is given by the following result [EK 2, 3.2.3]

Proposition 6.3.1. Let X be a smooth k-scheme.

1. For any unit $\mathcal{O}_{F,X}$-module $\mathcal{M}$ there is a natural isomorphism
$$\mathcal{M} \overset{1\otimes \mathrm{id}}{\to} \mathscr{D}_{F,X} \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_{F,X}} \mathcal{M}$$
In particular, $\mathcal{M}$ has a natural $\mathscr{D}_{F,X}$-module structure.
2. The functor $\mathcal{M} \mapsto \mathscr{D}_{F,X} \otimes_{\mathcal{O}_{F,X}} \mathcal{M}$ induces an equivalence between the category of unit $\mathcal{O}_{F,X}$-modules, and the category of unit $\mathscr{D}_{F,X}$-modules.
3. If $\mathcal{M}$ is a unit $\mathscr{D}_{F,X}$-module then $\mathcal{M}$ is locally finitely generated as a $\mathscr{D}_{F,X}$-module if and only if it is locally finitely generated as an $\mathcal{O}_{F,X}$-module.

6.4. As usual we let $\bullet$ denote one of $\emptyset$, $+$, $-$, or b, and $*$ one of $\emptyset$, qc, u or $lfgu$. We denote by $D^\bullet(\mathscr{D}_{F,X})$ the derived category of $\mathscr{D}_{F,X}$-modules with the indicated bounded condition. We denote by $D^\bullet_*(\mathscr{D}_{F,X})$ the full subcategory of $D^\bullet(\mathscr{D}_{F,X})$

consisting of complexes whose cohomology sheaves satisfy the condition indicated by $*$. As in §2, each of these subcategories is triangulated [EK 2, 3.1.2, 3.3.4].

In addition to these categories, we denote by $D^b_*(\mathscr{D}_{F,X})^\circ$ the full (triangulated) subcategory of $D^b_*(\mathscr{D}_{F,X})$ consisting of complexes which have finite Tor dimension as complexes of $\mathcal{O}_X$-modules.

As in §2, we have a bifunctor

$$- \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} - : D^-(\mathscr{D}_{F,X}) \times D^-(\mathscr{D}_{F,X}) \to D^-(\mathscr{D}_{F,X})$$

We also define, for a map of smooth $W_n(k)$-schemes $f : Y \to X$, functors $f^!$ and f_+. As in §2, the key point is to define the appropriate bimodules.

To construct $f^!$ we remark that for any $\mathscr{D}_{F,X}$-module $\mathcal{M}$, $f^*\mathcal{M}$ has a natural structure of $\mathscr{D}_{F,Y}$-module [EK 2, 2.2.2]. In fact, $f^*\mathcal{M}$ is natural a $\mathscr{D}_Y$-module [Be, §2.1], and we have a map

$$F^* f^* \mathcal{M} \xrightarrow{\sim} f^* F^* \mathcal{M} \overset{f^*\phi_{\mathcal{M}}}{\to} f^*\mathcal{M}$$

which makes $f^*\mathcal{M}$ a $\mathscr{D}_{F,Y}$-module by (5.2.1)(1).

In particular $\mathscr{D}_{F,Y\to X} := f^*\mathscr{D}_{F,X}$ is a $(\mathscr{D}_{F,Y}, f^{-1}\mathscr{D}_{F,X})$-bimodule by functoriality, and we define a functor

$$f^! : D^-(\mathscr{D}_{F,X}) \to D^-(\mathscr{D}_{F,Y})$$

by

$$f^!(\mathcal{M}^\bullet) = \mathscr{D}_{F,Y\to X} \overset{\mathbb{L}}{\otimes}_{f^{-1}\mathscr{D}_{F,X}} f^{-1}\mathcal{M}^\bullet[d_{Y/X}]$$

The definition of the $(f^{-1}\mathscr{D}_{F,X}, \mathscr{D}_{F,Y})$-bimodule $D_{F,X\leftarrow Y}$ used to define f_+ is more delicate [EK 2, 2.3.6]. Let us only mention that its underlying left $f^{-1}\mathscr{D}_X$-module is $f^*(\mathscr{D}_{F,X} \otimes_{\mathcal{O}_X} \omega_X^{-1}) \otimes_{\mathcal{O}_Y} \omega_Y$. The key points in constructing the right $\mathscr{D}_{F,Y}$-module structure is first to note that this module has a right $\mathscr{D}_Y$-module structure, by the same argument as in [Be, 2.4.1], and then use the fact that there is an isomorphism of $\mathscr{D}_Y$-modules $F^!\omega_Y \xrightarrow{\sim} \omega_Y$ (obtained from the compatibility of "$(-)^!$" with composition), together with (6.2.1)(2). We define the functor

$$f_+ : D^-(\mathscr{D}_{F,Y}) \to D^-(\mathscr{D}_{F,X})$$

by

$$f_+\mathcal{M}^\bullet = Rf_*(\mathscr{D}_{F,X\leftarrow Y} \overset{\mathbb{L}}{\otimes}_{D_{F,Y}} \mathcal{M}^\bullet)$$

We remark that on underlying $\mathscr{D}_Y$-modules this functor agrees with the one of [Be, §2.4].

Proposition 6.4.1. Let $\bullet$ be one of b or $-$, and $*$ one of $\emptyset$, qc, u, or $lfgu$. Also if $\bullet = b$, let $\times$ be one of $\emptyset$ or $\circ$, and $\emptyset$ otherwise. If $f : Y \to X$ is a map of smooth $W_n(k)$-schemes then the functors $f^!$ and f_+ restrict to functors

$$f^! : D^\bullet_*(\mathscr{D}_{F,X})^\times \to D^\bullet_*(\mathscr{D}_{F,Y})^\times.$$

and

$$f_+ : D^\bullet_*(\mathscr{D}_{F,Y})^\times \to D^\bullet_*(\mathscr{D}_{F,X})^\times.$$

The bifunctor $- \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} -$ restricts to a functor

$$- \overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} - : D^\bullet_*(\mathscr{D}_{F,X})^\times \times D^b_*(\mathscr{D}_{F,X})^\circ \to D^\bullet_*(\mathscr{D}_{F,X})^\times.$$

Proof. See [EK 2, 2.2.6, 2.3.13, 2.6, 3.5.1]. □

The functors $\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X} f^!$, and f_+ satisfy analogues of the properties listed in §2 for the case of $\mathcal{O}_{F,X}$-modules. In particular there is an analogue of Kashiwara's theorem, and f_+ is left adjoint to $f^!$ for proper maps. Rather than repeat the statements, we refer the reader to [EK 2, §2, 3].

6.5. Let X be a smooth $W_n(k)$-scheme. As in (3.1), we denote by $\pi_X : X_{ét} \to X_{Zar}$ the natural morphism of sites, and we set $\mathcal{O}_{X_{ét}} = \pi_X^*$, and $\mathscr{D}_{F,X_{ét}} = \pi_X^* \mathscr{D}_{F,X}$. We denote by $D^b_{ctf}(X_{ét}, \mathbb{Z}/p^n\mathbb{Z})$ the full (triangulated) subcategory of the category $D^b(X_{ét}, \mathbb{Z}/p^n\mathbb{Z})$, consisting of complexes which have constructible cohomology, and finite Tor dimension over $\mathbb{Z}/p^n\mathbb{Z}$.

The sheaf $\mathcal{O}_{X_{ét}}$ is naturally a sheaf of $\mathscr{D}_{F,X_{ét}}$-modules, and so we may define the functors

$$\mathrm{Sol}(-) = \underline{Rhom}_{\mathscr{D}_{F,X_{ét}}}(\pi_X^* -, \mathcal{O}_{X_{ét}}) : D^b_{lfgu}(\mathscr{D}_{F,X})^\circ \to D(X_{ét}, \mathbb{Z}/p^n\mathbb{Z})$$

and

$$\mathrm{M}(-) = R\pi_{X*}\underline{Rhom}_{\mathbb{Z}/p^n\mathbb{Z}}(-, \mathcal{O}_{X_{ét}}) : D^b_{ctf}(X_{ét}, \mathbb{Z}/p^n\mathbb{Z}) \to D(\mathscr{D}_{F,X})$$

The Riemann-Hilbert correspondence for unit $\mathscr{D}_{F,X}$-modules is the following result [EK 2, 4.2.8]

Theorem 6.5.1. The functors Sol and M induce quasi-inverse, anti-equivalences of triangulated categories between $D^b_{lfgu}(\mathscr{D}_{F,X})^\circ$ and $D^b_{ctf}(X_{ét}, \mathbb{Z}/p^n\mathbb{Z})$. They exchange the functors $\overset{\mathbb{L}}{\otimes}_{\mathbb{Z}/p^n\mathbb{Z}}$ and $\overset{\mathbb{L}}{\otimes}_{\mathcal{O}_X}$ (up to a shift) and f^{-1} and $f^!$. If f is quasi-projective, they exchange $f_!$ and f_+.

The proof of the theorem uses a dévissage technique to reduce to the case $n = 1$, when one can apply the results of [EK 1], thanks to (6.3.1).

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