LANGLANDS RECIPROCITY: L-FUNCTIONS, AUTOMORPHIC FORMS, AND DIOPHANTINE EQUATIONS

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Abstract. This chapter gives a description of the theory of reciprocity laws in algebraic number theory and its relationship to the theory of L-functions. It begins with a historical overview of the work of Euler, Dirichlet, Riemann, Dedekind, Hecke, and Tate. It then proceeds through the development of class field theory, before turning to the theory of Artin L-functions, and then of general motivic L-functions. It culminates in a description of Langlands's very general reciprocity conjecture, relating arbitrary motivic L-functions to the automorphic L-functions that he defined. The reciprocity conjecture is closely bound up with another fundamental conjecture of Langlands, namely his functoriality conjecture, and we give some indication of the relationship between the two. We conclude with a brief discussion of some recent results related to reciprocity for curves of genus one and two.

The goal of this chapter is to discuss the theory of ζ- and L-functions from the viewpoint of number theory, and to explain some of the relationships (both established and conjectured) with Langlands's theory of automorphic L-functions and functoriality. The key conjectural relationship, also due to Langlands, is that of *reciprocity*. The word *reciprocity* is a storied one in number theory, beginning with the celebrated theorem of quadratic reciprocity, discovered by Euler and Legendre, and famously proved by Gauss (several times over, in fact, and many more times since by others). The meaning of *reciprocity*, both in its precise technical usage and in its more general, intuitive sense, has evolved in a complicated way since it was coined (by Legendre) in the context of quadratic reciprocity in the 18th century. We hope to give the reader some sense of its modern meaning, but also to give the reader an indication of how this modern meaning evolved from the original meaning of Legendre.[1] To this end, we have taken a quasi-historical approach to our subject matter in the earlier parts of the chapter, although our historical review is by no means complete.

One of Langlands's earliest statements regarding automorphic L-functions and reciprocity appeared in his Yale lectures (delivered in 1967, published as [Lan71]), in which he wrote "Before beginning the substantial part of these lectures let me make, without committing myself, a further observation. The Euler products mentioned above are defined by means of the Hecke operators. Thus they are defined in an entirely different manner than those of Artin or Hasse-Weil. An assertion that an Euler product of the latter type is equal to one of those associated to an automorphic form is tantamount to a reciprocity law (for one equation in one variable in the case

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[1] Although it only alludes to the general framework laid out by Langlands at its very end, the article [Wym72] of Wyman, titled "What is a Reciprocity Law?", remains an excellent introduction to the subtle notion of a reciprocity law in number theory. We strongly recommend it.

of the Artin L-series and for several equations in several variables in the case of the Hasse-Weil L-series)." The title of Langlands's lecture series was "Euler products". For this reason too it seems appropriate that our discussion should begin with the work of Euler, and trace the thread of ideas leading from that work to the work of Langlands.

1. Introduction

We elaborate slightly on our goals in writing this chapter, before giving a brief overview of its contents.

1.1. **The Langlands program, and unification in number theory.** As will be clear to any reader, the view of history adopted in this note is not a neutral one. Rather, in addition to explaining some of the mathematics related to the Langlands program, I hope that my presentation of the historical development will convince the reader of the essential unity of number theory as a discipline over time, despite the changes in language, in technique, and in apparent focus. In fact, more than just unity, I hope to illustrate to the reader that the trend in number theory is towards one of *unification.* Apparently disparate phenomena become related to one another over time, and the theories used to explain them become more overarching. The Langlands program currently stands as one these overarching theories. Its apparent abstraction, and air of inaccessibility to the novice, are manifestations of the grand scope of the ideas and problems (old and new) which it encompasses. I hope that this note will help some non-expert readers overcome some of the difficulties of appreciation that necessarily attend such a broad and general theory.

1.2. **Problems in number theory.** To counterbalance the abstract sentiments of the preceding paragraph, it is good to recall that number theory has always been, and continues to be, driven by attempts to solve concrete problems. One of the beauties of the subject is that these problems seem almost inevitably to lead to theoretical and structural considerations of the deepest nature, whether or not the necessity of such considerations was apparent in the original question.

As a historical example, one might mention the law of quadratic reciprocity itself, which as we already noted lies at the root of all the mathematics we will be discussing. Actually, the point is illustrated even if one looks just at the early history of the quadratic reciprocity law: in trying to prove this law, Legendre[2] was led to state (nowadays we might say "conjecture") the theorem (proved later by Dirichlet) on the infinitude of primes in arithmetic progressions. Conversely, as we will recall in more detail below, Dirichlet's proof of that result itself relies on quadratic reciprocity (which had been proved in the meantime by Gauss, along different lines to the one pursued by Legendre) — and so these two seemingly distinct threads of research are intertwined (one example of unity in number theory). As more recent examples, one can think of the modularity conjecture for elliptic curves, or the Sato–Tate conjecture, originally formulated on their own terms, but now seen as instances of the general conjectural framework of Langlands reciprocity.

[2] In fact Euler stated this result as a theorem, at least for progressions of the form $1+mN$ [Eulb]; this particular case is indeed known to admit a fairly direct proof. In his introductory discussion of the problem [Dir37], Dirichlet treats the statement as a generally observed phenomenon, and doesn't attribute it to any particular mathematician. He does discuss Legendre's attempts to prove it, though (and does not mention Euler in this context).

We hope that the occasional examples we have included, as well as the various proof sketches that we have given, will enable the reader to see some of the concrete mathematics underlying the generalities to be discussed.

1.3. **Overview.** The first half of our chapter discusses (what are now called) *abelian* L-functions, beginning at the beginning, with the work of Euler, Riemann, and Dirichlet, and moving on to discuss the contributions of Dedekind and Hecke. This initial discussion culminates in a brief presentation of Tate's thesis. One important thread woven through this history is the notion of *Euler product*. Originally a discovery of Euler, beautiful but seemingly contingent, by the time one comes to Tate's adèlic (re)formulation, it is built into the underlying structure of the entire theory.

Reciprocity was discovered in this abelian setting; from our perspective (i.e. in the context of studying L-functions) it arises first when one attempts to compare Dirichlet's formula for the value $L(1,\chi)$, when χ is a quadratic character, with Dedekind's general class number formula. The point is that the former L-function is "automorphic", while (in the context of this particular comparison) the latter should be thought of as "motivic". A reciprocity law (in this particular case, quadratic reciprocity) is required for the comparison to be effected.

The consideration of abelian reciprocity laws culminates in the general statements of class field theory. We briefly recall these general statements, emphasizing the formulations in terms of L-functions, and illustrating them with the special cases of quadratic and cyclotomic extensions of the rational numbers. In anticipation of our dicussion of compatible families of ℓ-adic Galois representations, we also explain how compatible families of characters can be constructed from algebraic Hecke characters.

We then turn to describing motivic L-functions in general. Historically, these L-functions were first discovered somewhat implicitly, in conjunction with the investigations of abelian reciprocity: they arise (implicitly) in the factorization of $\zeta_K(s)$ into a product of Hecke L-functions for F (as in (32) below), when K/F is an abelian extension. It was Artin who singled out the factorization of $\zeta_K(s)$ into a product of L-functions for arbitrary extension K/F (see (43) below), where the L-functions now depend on (typically greater than one-dimensional) representations of (typically non-abelian) Galois groups, rather than on characters (i.e. one-dimensional representations) of ray class or idèle class groups.

We discuss Artin's L-functions, and then go on to discuss ζ-functions of finite type $\mathbb{Z}$-algebras (or, more generally, of finite type schemes over $\mathbb{Z}$). We give a brief overview of the theory in the case of varieties over finite fields, and then turn to the case of varieties over number fields, where we encounter the Hasse–Weil ζ-functions and their factorization into L-functions associated now to a *compatible family* of ℓ-adic Galois representations; these are the motivic L-functions — a vast generalization of Artin's L-functions.

Along the way, we also briefly describe the theory of local root numbers (or ϵ factors) in this non-abelian context. Although the problem of developing such a theory was considered by several mathematicians before him, it was Langlands who ultimately solved it [Lanc].

We conclude our discussion of motivic L-functions by introducing the conjectural Langlands group. This is a hypothetical object, whose existence Langlands proposed in [Lan79].[3] It mediates the relationship between automorphic forms and motives — that is, it mediates reciprocity. Although it is usually introduced first in the automorphic context, we introduce it in the motivic context, by analogy with the construction of compatible families of Galois characters from algebraic Hecke characters. To provide motivation (and hopefully not too much confusion), our discussion of the Langlands group is intertwined with an explanation of the Sato–Tate conjecture for elliptic curves; in particular, we explain its interpretation as a Čebotarev density-type theorem for the image of the Langlands group.

We then turn to our discussion of automorphic forms and Langlands reciprocity itself. We first briefly recall the basic notions related to automorphic forms, and also Hecke's work associating L-functions to modular forms. We then describe Langlands's construction of automorphic L-functions in general — and observe that the very definition is in terms of an Euler product! This already suggests (and already suggested to Langlands, as our quote above indicates) that these L-functions could be related to the L-functions attached to motives. The reciprocity conjecture, which we state next, formalizes this suggestion. The reciprocity conjecture is closely related to another conjecture of Langlands, his functoriality conjecture, and we briefly recall this relationship as well. Indeed, if understood in a sufficiently broad fashion, the latter conjecture subsumes the former.

Finally, we discuss some of the progress on the reciprocity conjecture, though with no pretense at completeness.

1.4. **A remark on terminology and notation.** There is no precise rule for distinguishing between the usage of the expressions "ζ-function" and "L-function". Roughly, one speaks of the ζ-function of a variety, and the L-function of a motive (so ζ-functions can be broken up into products of L-functions, analogously to the way that varieties can be decomposed into motives). In the automorphic context, one speaks primarily of L-functions (although Tate uses the notation ζ in his thesis). The reader unfamiliar with the nuances of contemporary usage shouldn't worry overly much about trying to discern any distinction between the two choices of terminology and notation. Of course, the attribution of proper names to ζ- and L-functions is an even more fraught enterprise. In the discussion that follows, I have tried to observe both tradition and contemporary practice as best I know how.

2. The early history of ζ- and L-functions — Euler, Riemann, and Dirichlet

The problems that Langlands is concerned with, in his theory of automorphic L-functions and his reciprocity conjecture, have their origins in the very beginnings of modern number theory, and I hope that it will be useful to the reader to review some of the history of this "early modern" number theory, if only to convince them of the essential continuity between the contemporary Langlands program and the number theory of previous eras.

2.1. **Euler.** The story of ζ- and L-functions begins with the study of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^k} = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots . \tag{1}$$

[3] Actually, we follow Kottwitz [Kot84] and Arthur [Art02] in modifying Langlands' construction.

A comparison with the improper integral $\int_1^\infty \frac{1}{x^k}$ shows that (1) converges if $k > 1$, while it diverges if $k = 1$; indeed,

$$\sum_{n=1}^{N} \frac{1}{n} \sim \log N. \tag{2}$$

The investigation of this series for various values of k seems to date back to the very beginning of the study of mathematical analysis in the 17th century. For example, the Basel Problem, posed in 1650, asked for the value of the sum

$$1 + 1/4 + 1/9 + \cdots$$

(i.e. the value of (1) when $k = 2$). Eighty-odd years later, this problem was solved by Euler [Eulc], who showed that the value was equal to $\frac{\pi^2}{6}$. His method of proof was to factor the function $\sin z$ as

$$\sin z = z \prod_{n=1}^{\infty} \big(1 - \frac{z}{\pi n}\big)\big(1 + \frac{z}{\pi n}\big) = z \prod_{n=1}^{\infty} \big(1 - \frac{z^2}{\pi^2 n^2}\big)$$

(via a consideration of its zeroes, and via the analogy with polynomials); expanding out the product and equating the resulting power series with the usual Taylor series for $\sin z$ yields the formula

$$1 + 1/4 + 1/9 + \cdots = \frac{\pi^2}{6} \tag{3}$$

(equate the coefficients of z^3 on either side of the equation) and, with some effort, analogous formulas for all higher even values of k. (As is well known, one obtains rational multiples of π^k.)

In fact, in the same paper, Euler considered other analogous factorizations derived in the same manner, beginning with the factorization

$$1 - \sin z = \prod_{n=0}^{\infty} \big(1 - \frac{z}{(4n+1)\pi/2}\big)^2 \prod_{n=1}^{\infty} \big(1 - \frac{z}{(4n-1)\pi/2}\big)^2,$$

which yields for example the formula

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} \tag{4}$$

(by identifying the coefficients of z on either side of the equation), due originally to Leibniz, as well as another (Euler's first) proof of (3) (by identifying the coefficients of z^2).

An even more consequential factorization, more directly related to the series (1), was discovered by Euler soon afterwards. Namely, Euler [Eulf] observed that the fundamental theorem of arithmetic (which states that each natural number admits a unique factorization into a product of prime powers) leads to the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^k} = \prod_{p \text{ prime}} \big(1 + \frac{1}{p^k} + \frac{1}{p^{2k}} + \frac{1}{p^{3k}} + \cdots\big) = \prod_{p \text{ prime}} \big(1 - \frac{1}{p^k}\big)^{-1}. \tag{5}$$

As an immediate consequence, by considering the case $k = 1$, Euler deduced that there must be infinitely many primes, since a minimal necessary condition for the product over p to diverge (as it must when we set $k = 1$, since the sum over n does) is that it involve an infinite number of factors. More quantitatively, one may take logarithms of both sides of this formula when $k = 1$; taking into account (2), one deduces that

$$\sum_{p \text{ prime } \leq N} \frac{1}{p} \sim \log\log N. \tag{6}$$

Euler obtains an analogous factorization of Leibniz's formula (4), to the effect that

$$\prod_{p \text{ prime } \equiv 1 \bmod 4} (1 - \frac{1}{p})^{-1} \prod_{p \text{ prime } \equiv -1 \bmod 4} (1 + \frac{1}{p})^{-1} = \frac{\pi}{4}. \tag{7}$$

Taking logarithms, one finds that

$$\sum_{\substack{p \text{ prime } \equiv 1 \bmod 4 \\ p \leq N}} \frac{1}{p} \quad - \quad \sum_{\substack{p \text{ prime } \equiv -1 \bmod 4 \\ p \leq N}} \frac{1}{p} \tag{8}$$

converges as $N \to \infty$. Comparing this with (6), one finds that each of the sums appearing in (8), considered individually, diverges as $N \to \infty$, and thus concludes that there are infinitely many primes congruent to each of $\pm 1 \bmod 4$.[4]

In his paper [Eula], Euler attempted to evaluate (1) at odd values of k. He was unsuccessful at this, but instead obtained generalizations of the formula (4), to the effect that

$$1 - \frac{1}{3^k} + \frac{1}{5^k} - \cdots = \text{a rational multiple of } \pi^k \tag{9}$$

when k is an odd natural number. In the subsequent paper [Eule], Euler studied the series (1), as well as the series appearing in (9), at *negative* integral values of k. Of course these latter series diverge, but Euler assigned them values via a regularization technique ("Abelian summation"): he considered the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{t^n}{n^k}$ and $\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)^k}$ as functions of t (functions which can be explicitly determined using formulas for geometric series and their derivatives, if $k \leq 0$), and then let $t \to 1$.[5] Euler then found explicit formulas relating the resulting values of (1), as well as the series appearing in (9), at k and at $1 - k$, and he conjectured that these formulas hold for arbitrary values of k. (And he rigorously verified his conjecture at $k = 1/2$ for the series (1), and verified it numerically at additional values of k.)

2.2. **Riemann.** Riemann [Rie] took up the investigation of the series (1) where Euler left off. He introduced the notation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots , \tag{10}$$

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[4] Much of Euler's results on what are now called *Euler products* are also discussed in Chapter 15 of his text [Euld]. The study of the particular difference appearing in (8) is the subject of his paper [Eulb]. This method of argument was vastly generalized by Dirichlet, as we recall below.

[5] Note that $1 - 1/2^k + 1/3^k - 1/4^k + \cdots = \big(1 - 2^{1-k}\big)\big(1 + 1/2^k + 1/3^k + 1/4^k + \cdots\big)$; thus one may regard this series as being obtained from the series (1) via a modification of the factor at $p = 2$ in the product formula (5). For a modern explication of Euler's method, and an indication of its relationship to more contemporary ideas in number theory, the reader can consult e.g. the first part of the paper [Kat75] of Katz.

and considered $\zeta(s)$ as a function of a complex variable, initially for $\Re s > 1$ (where the series (10) converges absolutely), but then for all s, after effecting a meromorphic continuation. Indeed, he showed that $\zeta(s)$ is holomorphic on $\mathbb{C} \setminus \{1\}$, and has a simple pole at $s = 1$, with residue 1.[6] He also established the famous functional equation for $\zeta(s)$, namely that

$$\pi^{-\frac{s}{2}}\Gamma\big(\frac{s}{2}\big)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\big(\frac{1-s}{2}\big)\zeta(1-s). \tag{11}$$

This recovered the results of Euler for integral values of s already mentioned, and amounts to Euler's conjectural generalization of those results (for arbitrary complex values of s!).

Riemann gave two proofs of the analytic continuation and functional equation of $\zeta(s)$. His second proof established a connection between $\zeta(s)$ and the Jacobi theta function, beginning an intimate relationship between the theories of L-functions and of automorphic forms which continues to this day.

Jacobi's theta function is defined by the formula

$$\vartheta(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} = 1 + 2\sum_{n=1}^{\infty} e^{\pi i n^2 \tau} = 1 + 2q + 2q^4 + 2q^9 + \cdots,$$

where $q = e^{\pi i \tau}$ and $\Im\tau > 0$ (equivalently, $|q| < 1$). An application of the Poisson summation formula to the Gaussian e^{-x^2} (which is essentially its own Fourier transform) yields the functional equation

$$\vartheta\big(\frac{-1}{\tau}\big) = \sqrt{\frac{\tau}{i}}\,\vartheta(\tau). \tag{12}$$

Combining this with the integral formula

$$\pi^{-\frac{s}{2}}\Gamma\big(\frac{s}{2}\big)\zeta(s) = \int_0^{i\infty} \frac{\vartheta(\tau)-1}{2}\big(\frac{\tau}{i}\big)^{s/2}\frac{d\tau}{\tau} \tag{13}$$

yields both the analytic continuation and functional equation.[7]

We can rewrite the formula (5) as

$$\zeta(s) = \prod_{p \text{ prime}} (1-p^{-s})^{-1} \quad (\Re s > 1); \tag{14}$$

this is now referred to as the *Euler product* formula for $\zeta(s)$. Riemann famously employed (14) to study the distribution of the prime numbers (building on Euler's application of it to reprove the infinitude of the set of primes). His ideas were further developed by Hadamard and de la Vallée Poussin to prove the Prime Number Theorem (that the number of primes $\leq x$ is asymptotic to $x/\log x$; note that this can be regarded as a strengthening of Euler's asymptotic (6), since it immediately implies that result). A fundamental conjecture of Riemann is that the zeroes of $\zeta(s)$ (other than those at the negative even integers, which were already discovered

[6] If one considers the function $\eta(s) = (1-2^{1-s})\zeta(s) = 1 - 1/2^s + 1/3^s - 1/4^s + \cdots$, then $\eta(s)$ is holomorphic on the entire complex plane — the zero at $s = 1$ of the factor $1 - 2^{1-s}$ cancels out the simple pole of $\zeta(s)$. The determination of the residue at $s = 1$ of $\zeta(s)$ is then seen to be equivalent to another formula of classical analysis, namely that $1 - 1/2 + 1/3 - 1/4 + \cdots = \log 2$.

[7] The functional equation (12) lets us rewrite the integral appearing in (13) in the form $\frac{1}{s(s-1)} + \int_i^{i\infty} \frac{\vartheta(\tau)-1}{2}\big((\frac{\tau}{i})^{\frac{s}{2}} + (\frac{\tau}{i})^{\frac{1-s}{2}}\big)\frac{d\tau}{\tau}$, from which the meromorphic continuation and functional equation both immediately follow.

by Euler) all lie on the line $\Re s = \frac{1}{2}$ — this is the famous *Riemann Hypothesis*. The crux of both Hadamard and de la Vallée Poussin's proofs of the Prime Number Theorem is a weaker result in the direction of the Riemann Hypothesis, namely that $\zeta(s)$ is zero-free on the line $\Re s = 1$.

We remark that the two properties of $\zeta(s)$ that we have discussed — (i) its analytic continuation and functional equation (which are mediated via its relationship to the automorphic form $\vartheta(\tau)$), and (ii) its Euler product — are *completely unrelated* in Riemann's treatment of them. This apparently technical point is in fact of a more than merely technical nature; it is crucial, and we will return to it below.

2.3. **Dirichlet.** Dirichlet introduced what are now called *Dirichlet L-functions* in his celebrated investigation of primes in arithmetic progressions [Dir37].

Following contemporary notation,[8] if $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ is a character (i.e. a homomorphism between the indicated multiplicative groups), then we write

$$L(s,\chi) = \sum_{\substack{n \geq 1 \\ n \text{ coprime to } N}} \frac{\chi(n)}{n^s} = \prod_{\substack{p \text{ prime} \\ p \nmid N}} \big(1 - \frac{\chi(p)}{p^s}\big)^{-1}, \tag{15}$$

where the validity of the Euler product follows from the multiplicative property of χ. Dirichlet cites Chapter 15 of Euler's text [Euld] as an inspiration for his consideration of such Euler products.

If χ is *not* primitive, i.e. if χ can be written as a composite $(\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/M\mathbb{Z})^\times \xrightarrow{\psi} \mathbb{C}^\times$ for some proper divisor M of N and some character ψ (more colloquially, if the *period* of χ is a proper divisor of N), then $L(s,\chi)$ and $L(s,\psi)$ coincide, up to a finite number of factors in their Euler product related to primes that divide N but not M. Thus we may, and typically do, restrict attention to *primitive* L-functions, i.e. those $L(s,\chi)$ for which χ is primitive.

Note that if $N = 1$, so that χ is trivial, then $L(s,\chi) = \zeta(s)$, which has a simple pole at $s = 1$. On the other hand, it is easily seen that $L(s,\chi)$ extends holomorphically over the entire complex plane if χ is non-trivial.[9] Dirichlet then proves that $L(1,\chi) \neq 0$ if χ is non-trivial. Passing to logarithms and taking into account the Euler product, he obtains the result that

$$\sum_{p \text{ prime}} \frac{\chi(p)}{p} \tag{16}$$

converges if χ is a non-trivial character. Combining this with (2) (the corresponding statement for the trivial χ), an application of finite Fourier theory shows that $\sum_{p \equiv a \bmod N} \dfrac{1}{p}$ diverges if $(a,N) = 1$, proving in particular that there are infinitely many primes in the arithmetic progression $a + mN$.

Dirichlet proves that $L(1,\chi) \neq 0$ (for non-trivial χ) by first considering the sum of the expressions (16), namely

$$\sum_{\substack{\chi \text{ a char.} \\ \text{mod } N}} \sum_{p \text{ prime}} \frac{\chi(p)}{p} = \sum_{p \text{ prime}} \frac{\sum_\chi \chi(p)}{p}. \tag{17}$$

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[8]This notation differs from that of Dirichlet, who wrote simply L, or L with a subscript if he wished to indicate a particular character.

[9]We mention, though, that Dirichlet only considers s as a real variable.

The orthogonality relations for characters imply that this series has non-negative terms, and hence if it diverges, it must diverge to $+\infty$. On the other hand, each expression (16) is either finite (if $L(1,\chi)$ is finite and non-zero), diverges to $+\infty$ (which happens precisely when χ is trivial), or diverges to $-\infty$ (if $L(1,\chi) = 0$). Making an analysis of the rates of convergence, one finds that at most one $L(1,\chi)$ can possibly equal zero (since the sum of at least two divergences to $-\infty$ would overwhelm the divergence to $+\infty$ that comes from the contribution of the trivial character, contradicting the positivity already noted). Since $L(1,\overline{\chi}) = \overline{L(1,\chi)}$, this shows that $L(1,\chi) \neq 0$ if χ is a genuinely complex character (i.e. if $\chi \neq \overline{\chi}$).

If χ is a real- (i.e. ± 1-) valued character (one also refers to such characters as *quadratic*), then Dirichlet's proof that $L(1,\chi) \neq 0$ is more involved, and is number-theoretic in nature — if χ is a primitive quadratic character mod N, then he shows [Dir] that $L(1,\chi)$ is a certain positive multiple of the class number of the field $\mathbb{Q}(\sqrt{\pm N})$ (here the sign is chosen equal to $\chi(-1)$).[10] Since this class number is non-zero, so is $L(1,\chi)$.

As a special case, when $N = 4$ and χ is the unique non-trivial character, he recovers Leibniz's formula (4); from Dirichlet's point-of-view, this formula can be interpreted as showing that $\mathbb{Q}(i)$ has class number 1.

3. Dedekind ζ-functions and Hecke L-functions

Over the course of the 19th century, algebraic number theory emerged as a topic of central importance, and many preceding preoccupations of number theorists (such as the theory of Diophantine equations, and the theory of quadratic forms) became to a large extent absorbed into the general apparatus of algebraic number theory. This is in particular true of the theory of L-functions.

Two of the principal contributors to these developments, both the development of algebraic number theory in general and the positioning of the theory of L-functions at the centre of it, were Kummer and Dedekind. Lack of space prohibits us from discussing Kummer's seminal contributions in any detail (though we refer the reader to the review [Maz77] for a wonderful overview), and we instead limit ourselves to recalling Dedekind's definition of the ζ-function of a number field, and the statement of his class number formula. This formula in a sense generalizes Dirichlet's formula for the special value $L(1,\chi)$ when χ is a quadratic character, but there is a fundamental point that has to be understood to make *precise* the sense in which this is true; indeed, as we already remarked in the introduction, comparing these two formulas is a first example of *reciprocity*, and we discuss it more fully in Section 4.1 below.

In this section, after briefly recalling Dedekind's contributions, we move onto the key (for our purposes) further developments of the analytic theory of L-functions in the first half of the 20th century — first those of Hecke (whose results generalized those of both Riemann and Dirichlet from the field $\mathbb{Q}$ to a general number field), and then those of Tate, as famously developed in his thesis.

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[10] Dirichlet actually employs the older language of quadratic forms, due to Legendre and Gauss. The transition to the language of algebraic number theory comes about in the work of Kummer and Dedekind; see e.g. this MathOverflow answer by Lemmermeyer [hl]. See also (24) below for the explicit form of Dirichlet's result.

3.1. **Dedekind.** If K is a number field (i.e. a finite extension of $\mathbb{Q}$), with ring of algebraic integers $\mathcal{O}_K$, then we may imitate the definition of the Riemann zeta function, and define a ζ-function attached to K, via the formula

$$\zeta_K(s) = \sum_{\mathfrak{a} \neq 0} \frac{1}{N(\mathfrak{a})^s};$$

here the sum is over all non-zero ideals $\mathfrak{a} \subseteq \mathcal{O}_K$, and $N(\mathfrak{a})$ denotes the norm of $\mathfrak{a}$, i.e. the order of the (finite) quotient ring $\mathcal{O}_K/\mathfrak{a}$. We call this the *Dedekind ζ-function* of K. If we take into account the unique factorization of ideals into products of prime ideals, along with the multiplicativity of the norm, we find that $\zeta_K(s)$ admits an Euler product

$$\zeta_K(s) = \prod_{\mathfrak{p}} (1 - \frac{1}{N(\mathfrak{p})^s})^{-1}; \tag{18}$$

here the product runs over non-zero prime ideals (equivalently, maximal ideals) of $\mathcal{O}_K$.

We remark that the series and Euler product converge, and hence these formulas make sense, when $\Re s > 1$, and indeed Dedekind regarded $\zeta_K(s)$ as a function on the halfline $s > 1$.[11]

Following Dirichlet's arguments for evaluating $L(1, \chi)$ for quadratic characters χ, Dedekind proved the famous *class number formula*

$$\lim_{s \to 1} (s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2} Rh}{w\sqrt{|D|}}. \tag{19}$$

where r_1 and r_2 denote the number of real and complex places of K respectively, h and R respectively denote the class number and regulator of K, w denotes the number of roots of unity contained in K, and $|D|$ is the absolute value of the discriminant of K.

Subsequently, Landau [Lan03] proved that $\zeta_K(s)$ has an analytic continuation to the left of $\Re s = 1$, holomorphic except for a simple pole at $s = 1$ (whose residue is then computed by (19)). He was then able to follow the method of Hadamard–de la Vallée-Pousin to establish the *Prime Ideal Theorem* (that the number of prime ideals of norm $\leq X$ is asymptotic to $X/\log X$), which is the number field analogue of the prime number theorem.[12]

It is natural to consider number field analogues of Dirichlet's theorem on primes in arithmetic progression as well. Already Dedekind suggested considering (what

[11] Dedekind developed the theory of $\zeta_K(s)$ — which he denotes $\Omega(s)$, in line with his notational choice of Ω for the number field that we are denoting K — in §184 of his Supplement XI to Dirichlet's lectures on number theory [Ded].

[12] We recall one particular consequence of this theorem. Suppose that K/F is a Galois extension of number fields. As is well-known, 100% (in the sense of density) of prime ideals of K are split completely over F. (A non-split prime $\mathfrak{q}$ of K lying over some prime $\mathfrak{p}$ of F has norm at least $N(\mathfrak{p})^2$, and so the contribution of these primes to $\lim_{s \to 1} \sum_{\mathfrak{q}} N(\mathfrak{q})^{-s}$ is actually convergent.) Since there are $[K : F]$ primes of K lying over a given prime of F that splits in K, we then deduce — by comparing the prime ideal theorems for F and for K — that the primes in F that split completely in K have density $1/[K : F]$ among all the primes in F. This proves the Čebotarev density theorem (see Subsection 5.3 below) for the trivial conjugacy class in $\mathrm{Gal}(K/F)$.

we would now call) L-functions

$$L(s,\chi) = \sum_{\mathfrak{a}\neq 0} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s},$$

where χ is a character of the class group of K [Ded, §184]. One might hope to use such L-functions to prove a result analogous to Dirichlet's theorem on primes in arithmetic progression, to the effect that each ideal class contains infinitely many prime ideals. This idea was taken up by Hecke.

3.2. **Hecke.** Hecke [Hec17, Hec20] proved the analytic continuation of and functional equation for Dedekind ζ-functions, and for a more general class of L-functions

$$L(s,\chi) = \sum_{\substack{\mathfrak{a} \\ (\mathfrak{a},\mathfrak{m})=1}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})}^{s},$$

where χ is (what is now called) a *Hecke character*, or equivalently a *Grössencharacter*, or (again equivalently, after an appropriate reformulation) an *idèle class character*, of conductor $\mathfrak{m}$.

If $\mathfrak{m}$ is a non-zero ideal in $\mathcal{O}_K$, and if S denotes the finite set of prime ideals dividing $\mathfrak{m}$, then a Hecke character of conductor[13] $\mathfrak{m}$ is a homomorphism of multiplicative groups

$$\chi : \{\text{fractional ideals coprime to } S\} \to \mathbb{C}^\times$$

for which there is some continuous character of multiplicative groups[14]

$$\theta : (\mathbb{R}\otimes_{\mathbb{Q}} K)^\times \to \mathbb{C}^\times$$

such that for any $\alpha \in K^\times$ with $\alpha \equiv 1 \bmod \mathfrak{m}$, we have $\chi\big((\alpha)\big) = \theta(\alpha)$ (where we think of α as lying in $(\mathbb{R}\otimes_{\mathbb{Q}} K)^\times$ via the natural embedding of $K^\times$ into the former group).

If we take θ to be the trivial character (so that we simply insist that $\chi\big((\alpha)\big) = 1$ if $\alpha \equiv 1 \bmod \mathfrak{m}$), then we obtain the notion of a *ray class character* of conductor $\mathfrak{m}$. These are precisely the Hecke characters of finite order.

The multiplicative property of a Hecke character χ ensures that Hecke's L-functions again admit an Euler product

$$L(s,\chi) = \prod_{\mathfrak{p}\notin S} \big(1 - \frac{\chi(\mathfrak{p})}{N(\mathfrak{p})^s}\big)^{-1}.$$

3.2.1. *Example.* If $K = \mathbb{Q}$, then conductors are either of the form N or $N\infty$ (where $N > 0$ is a natural number, and ∞ denotes the embedding $\mathbb{Q} \hookrightarrow \mathbb{R}$). Ray class characters of conductor N are precisely the *even* primitive characters $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, while those of conductor $N\infty$ are the *odd* primitive characters $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$. (We say a character is *even* or *odd* according to whether

[13] In fact, the notion of conductor should be broadened in the usual way: a conductor $\mathfrak{m}$ should consist of a non-zero ideal (the finite, or non-archimedean part of the conductor) together with any subset of the set of real embeddings of K (the infinite, or archimedean, part of the conductor). If $\sigma : K \hookrightarrow \mathbb{R}$ is a real embedding, and if $\alpha \in K^\times$, then the condition $\alpha \equiv 1 \bmod \sigma$ is interpreted as meaning that $\sigma(\alpha) > 0$ (i.e. that $\sigma(\alpha)$ and 1 lie in the same connected component of $\mathbb{R}^\times$).

[14] Note that $\mathbb{R}\otimes_{\mathbb{Q}} K$ is just a more functorial way of writing $\mathbb{R}^{r_1}\times\mathbb{C}^{r_2}$, and so we are simply considering characters $(\mathbb{R}^\times)^{r_1}\times(\mathbb{C}^\times)^{r_2} \to \mathbb{C}^\times$. The canonical embedding of K into $\mathbb{R}\otimes_{\mathbb{Q}} K$ (and hence of $K^\times$ into $(\mathbb{R}\otimes_{\mathbb{Q}} K)^\times$) is just a functorial way of describing the embedding of K into $\mathbb{R}^{r_1}\times\mathbb{C}^{r_2} \cong \mathbb{R}^{r_1+2r_2}$ that is traditionally considered in algebraic number theory.

$\chi(-1) = 1$ or -1.) Thus in this case, Hecke's ray class L-functions reduce to Dirichlet's L-functions.

The notion of Hecke character in this case (i.e. when $K = \mathbb{Q}$) is scarcely more general: any Hecke character has the form $(a) \mapsto \chi(a)|a|^{s_0}$ for some mod N character χ and some $s_0 \in \mathbb{C}$, in which case the corresponding L-function is simply $L(s - s_0, \chi)$ — i.e. a Dirichlet L-function with the variable shifted. In particular, the only *unitary* Hecke characters (i.e. those that land in the group of complex numbers of modulus 1) are products of the finite order characters that already appear in Dirichlet's theory with a unitary character $a \mapsto |a|^{iy}$ of $\mathbb{R}^\times$.

3.2.2. *Example.* If $K = \mathbb{Q}(i)$, then the ring of integers is $\mathbb{Z}[i]$, which is well-known to have trivial class group. The unique prime ideal above 2 is $(1+i)$; all other prime ideals (which we might call *odd*) have a unique generator π satisfying the congruence $\pi \equiv 1 \bmod (1+i)^3$. The character $\psi : (\pi) \to \pi$ (where the generator is chosen to satisfy the preceding congruence) is an infinite order Hecke character of conductor $(1+i)^3$. (The corresponding character θ is simply the identity character $z \mapsto z$ on $\mathbb{C}^\times$.) The character ψ is not unitary, but it is easy enough to modify it so as to make it so, e.g. by defining $\chi\big((\pi)\big) = \dfrac{\pi}{\sqrt{N(\pi)}}$.

(More generally, for any number field K, we always have the one-parameter family of Hecke characters of trivial conductor $\mathfrak{a} \mapsto |N(\mathfrak{a})|^s$. If ψ is any Hecke character, then $|\psi|$ belongs to this one-parameter family, and $\chi = \psi|\psi|^{-1/2}$ is unitary.)

Hecke proved the analytic properties of his L-functions by combining Riemann's arguments using theta functions with Dirichlet's and Dedekind's arguments (appearing in their proofs of the class number formula) involving sums over ideal classes. As in Riemann's case, the analytic aspects of his theory apparently bear no relation to the Euler product structure of the L-series. Rather than writing more about Hecke's arguments, though, we now turn to Tate's reworking of them, since this marked a decisive shift in the analytic theory of L-functions.

3.3. **Adèles and idèles.** In his thesis [Tat67], Tate gives a reformulation of the theory of Hecke L-functions in terms of adèles and idèles. We begin by briefly recalling the definitions of these latter objects.

As usual, we let $\widehat{\mathbb{Z}}$ denote the profinite completion of $\mathbb{Z}$, so

$$\widehat{\mathbb{Z}} = \varprojlim_N \mathbb{Z}/N\mathbb{Z} = \prod_p \mathbb{Z}_p,$$

the second equality holding by virtue of the Chinese Remainder Theorem. By its construction, $\widehat{\mathbb{Z}}$ is a profinite (and, in particular, a compact) commutative ring. We may extend scalars to $\mathbb{Q}$ to obtain a locally compact $\mathbb{Q}$-algebra [15] $\widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, which we also denote by $\mathbb{A}_{\mathbb{Q}}^\infty$, and refer to as the ring of *finite adèles* of $\mathbb{Q}$. We may also describe $\mathbb{A}_{\mathbb{Q}}^\infty$ as the *restricted product* $\prod_p{}' \mathbb{Q}_p$ of the various p-adic fields $\mathbb{Q}_p$: it consists of those tuples $(x_p) \in \prod_p \mathbb{Q}_p$ for which x_p is an integer for all but finitely

[15] Writing $\mathbb{Q} = \bigcup_{N>0} \frac{1}{N}\mathbb{Z}$, we find that $\mathbb{A}_{\mathbb{Q}}^\infty = \bigcup_{N>0} \frac{1}{N}\widehat{\mathbb{Z}}$ is simply a union of copies of $\widehat{\mathbb{Z}}$, in a manner analogous to that in which $\mathbb{Q}_p = \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Q} = \bigcup_{n\geq 0} \frac{1}{p^n}\mathbb{Z}_p$ is a union of copies of $\mathbb{Z}_p$.

many p. We then define the full ring of *adèles* of $\mathbb{Q}$ to be the product

$$\mathbb{A}_{\mathbb{Q}} = \mathbb{R} \times \mathbb{A}_{\mathbb{Q}}^{\infty};$$

it is again a locally compact $\mathbb{Q}$-algebra, and also admits a description as the restricted product $\prod_v{}' \mathbb{Q}_v$, where v now ranges over *all* places of $\mathbb{Q}$, including the infinite (archimedean) place.

If K is any number field, then we define

$$\mathbb{A}_K^{\infty} = \mathbb{A}_{\mathbb{Q}}^{\infty} \otimes_{\mathbb{Q}} K = \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} K = \prod_{\mathfrak{p}}{}' K_{\mathfrak{p}},$$

the final description here being the restricted product of the various non-archimedean completions $K_{\mathfrak{p}}$ of K (in the same sense as used above, i.e. consisting of those elements of the product for which all but finitely many of their components are integers). We define

$$\mathbb{A}_K = \mathbb{A}_{\mathbb{Q}} \otimes_{\mathbb{Q}} K = (\mathbb{R} \otimes_{\mathbb{Q}} K) \times \mathbb{A}_K^{\infty} = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \times \mathbb{A}_K^{\infty} = \prod_v{}' K_v,$$

where now the restricted product is over *all* places of K. We refer to these two K-algebras as the ring of *finite adèles* over K and the ring of *adèles* over K, respectively. They are both locally compact K-algebras. (As a $\mathbb{Q}$-vector space, i.e. forgetting the K-algebra structure, $\mathbb{A}_K$ is just a product of $[K : \mathbb{Q}]$ copies of $\mathbb{A}_{\mathbb{Q}}$.)

If G is any affine algebraic group over K, then we may consider its group $G(\mathbb{A}_K)$ of $\mathbb{A}_K$-valued points; the hypothesized Zariski closed embedding of G into some d-dimensional affine space over K induces an embedding of $G(\mathbb{A}_K)$ as a closed subset of $\mathbb{A}_K^d$, giving $G(\mathbb{A}_K)$ the structure of a locally compact group. For now, we apply this to the group

$$\mathbb{G}_m = \mathrm{GL}_1 = \{(x, y) \in \mathbb{A}^2 \mid xy = 1\}.$$

As a group, $\mathrm{GL}_1(\mathbb{A}_K)$ is simply the group of units $\mathbb{A}_K^{\times}$; the algebraic group viewpoint is important for endowing it with the correct topology. We refer to the abelian group $\mathbb{A}_K^{\times}$, endowed with the locally compact topology just discussed (so we identify it with the closed subset $\{(x, y) \in \mathbb{A}_K^2 \mid xy = 1\}$ of $\mathbb{A}_K^2$) as the group of idèles of K. We may also write $\mathbb{A}_K^{\times}$ as the restricted product $\prod_v{}' K_v^{\times}$ consisting of elements in the product which are integral units at all but finitely many finite places.

More important for our purposes is not the idèle group of K itself, but the idèle class group of K. Before defining this, we note that the natural (diagonal) embedding $\mathbb{Q} \hookrightarrow \mathbb{A}_{\mathbb{Q}}$ has discrete image.[16] Extending scalars we obtain the discrete embedding $K \hookrightarrow \mathbb{A}_K$, and consequently a discrete embedding $K^{\times} \hookrightarrow \mathbb{A}_K^{\times}$. The *idèle class group* of K is defined to be the quotient[17] $K^{\times} \backslash \mathbb{A}_K^{\times}$. We are quotienting a locally compact abelian group by a discrete subgroup; the result is again a locally compact abelian group.

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[16]This embedding is an analogue of the more classical discrete embedding $\mathbb{Z} \hookrightarrow \mathbb{R}$, and one of the technical jobs that the adèles perform is to mimic this latter embedding, while nevertheless allowing one to work with structures over $\mathbb{Q}$ — such as $\mathbb{Q}$-vector spaces, or groups like $\mathrm{GL}_n(\mathbb{Q})$, rather than having to contend with more general abelian groups, or groups like $\mathrm{GL}_n(\mathbb{Z})$, whose technical properties are typically not as pleasant.

[17]It is traditional at this point, if disorienting for the uninitiated, to write the group by which we are taking the quotient on the left.

In fact, the group $K^\times \setminus \mathbb{A}_K^\times$ is very close to being compact. Namely, the product formula for global fields shows that the surjective homomorphism

$$| \, | : \mathbb{A}_K^\times = \prod_v{}' K_v^\times \to \mathbb{R}_{>0}^\times$$

defined by $(x_v) \mapsto \prod_v |x_v|_v$ (where $| \, |_v$ is an appropriately normalized absolute value on $K_v^\times$) contains $K^\times$ in its kernel. If we let $\mathbb{A}_K^1$ denote this kernel, then $K^\times \setminus \mathbb{A}_K^1$ *is* compact,[18] and we have the short exact sequence

$$1 \to K^\times \setminus \mathbb{A}_K^1 \to K^\times \setminus \mathbb{A}_K^\times \to \mathbb{R}_{>0}^\times \to 1.$$

The first key reinterpretation of Hecke's theory in terms of these objects is that a Hecke character is *the same thing as* a continuous character

$$\chi : K^\times \setminus \mathbb{A}_K^\times \to \mathbb{C}^\times.$$

Indeed, composing χ with the inclusion $K_v^\times \to \mathbb{A}_K^\times \to K^\times \setminus \mathbb{A}_K^\times$ yields a continuous character $\chi_v : K_v^\times \to \mathbb{C}^\times$, and we may factor χ as

$$\chi = \prod_v \chi_v \tag{20}$$

The continuity of χ ensures that for all v outside a finite set S (which we now take to include the infinite places) the character χ_v is *unramified*, in that it factors through the quotient $\mathbb{Z} = K_v^\times / \mathcal{O}_{K_v}^\times$ — i.e. $\chi_v(x)$ depends only on the v-adic valuation of x. We now define a character[19] as an idèle class character via the discussion of

$$\tilde{\chi} : \{\text{fractional ideals coprime to } S\} \to \mathbb{C}^\times$$

via

$$\mathfrak{a} = \prod_{\mathfrak{p} \notin S} \mathfrak{p}^{n_\mathfrak{p}} \mapsto \prod_{\mathfrak{p} \notin S} \chi_\mathfrak{p}(\pi_\mathfrak{p}^{-1})^{n_\mathfrak{p}},$$

where $\pi_\mathfrak{p}$ is some choice of uniformizer in $K_\mathfrak{p}$ (it doesn't matter which). If we let θ denote

$$\prod_{v \text{ infinite}} \chi_v : (\mathbb{R} \otimes_\mathbb{Q} K)^\times \to \mathbb{C}^\times,$$

then the continuity of χ will ensure that $\tilde{\chi}$ satisfies the requirements to be a Hecke character with respect to the character θ and *some* conductor $\mathfrak{m}$ involving just the places in S. Note in particular that although the definition of $\tilde{\chi}$ only involves the χ_v for $v \notin S$, we may recover χ, and in particular the remaining χ_v for $v \in S$, from $\tilde{\chi}$, using the fact that χ is continuous and is defined on the quotient $K^\times \setminus \mathbb{A}_K^\times$.

The finite order idèle class characters correspond to Hecke's ray class characters. These necessarily factor through a finite quotient of $K^\times \setminus \mathbb{A}_K^\times$. In fact, the group of connected components $\pi_0(K^\times \setminus \mathbb{A}_K^\times)$ of $K^\times \setminus \mathbb{A}_K^\times$ is naturally a profinite group, and may be described as the inverse limit, over all conductors $\mathfrak{m}$, of the ray class groups of conductor $\mathfrak{m}$. The connected component of the identity in $K^\times \setminus \mathbb{A}_K^\times$ is the product of a copy of $\mathbb{R}_{>0}^\times$ with the connected component of the identity in $K^\times \setminus \mathbb{A}_K^1$,

[18]This is a reformulation of two key theorems of elementary algebraic number theory, namely the finiteness of the class group and Dirichlet's unit theorem.

[19] The exponent -1 on $\pi_\mathfrak{p}$ is chosen to ensure a compatibly between various conventions and constructions; e.g. if we identify a character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ with an idèle class character via the discussion of Example 3.3.1, then the character $\tilde{\chi}$ coincides with the ray class character associated to χ via the discussion of Example 3.2.1.

this latter group being a connected compact group. It is trivial in the simplest case of $K = \mathbb{Q}$, but is otherwise non-trivial.

3.3.1. *Example.* One easily computes that

$$\mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times = \mathbb{R}_{>0}^\times \times \widehat{\mathbb{Z}}^\times.$$

This recovers the description of Hecke characters for $\mathbb{Q}$ given in (3.2.1) above.

3.3.2. *Example.* It is also easy to compute that

$$\mathbb{Q}(i)^\times \setminus \mathbb{A}_{\mathbb{Q}(i)}^\times = \mathbb{C}^\times \times \{x \in \widehat{\mathbb{Z}[i]}^\times \mid x \equiv 1 \bmod (1+i)^3\}.$$

(More precisely, there is an obvious map from the right-hand side to the left, which is an isomorphism.) In these terms, the Hecke character ψ of (3.2.2) may be described simply as the projection onto the $\mathbb{C}^\times$ factor.

3.3.3. *Example.* Recalling that $\mathbb{Q}(\sqrt{2})$ has class number 1, that $1+\sqrt{2}$ is a fundamental unit of $\mathbb{Q}(\sqrt{2})$, and that $N(1+\sqrt{2}) = -1$, one finds that

$$\mathbb{Q}(\sqrt{2})^\times \setminus \mathbb{A}_{\mathbb{Q}(\sqrt{2})}^\times = \langle (1+\sqrt{2})^2 \rangle \setminus \big(\mathbb{R}_{>0}^\times \times \mathbb{R}_{>0}^\times \times \widehat{\mathbb{Z}[\sqrt{2}]}^\times\big).$$

In particular, the connected component of the identity is equal to

$$\varprojlim_n \Big(\langle (1+\sqrt{2})^{2n} \rangle \setminus \big(\mathbb{R}_{>0}^\times \times \mathbb{R}_{>0}^\times\big)\Big),$$

which is the product of $\mathbb{R}_{>0}^\times$ with a *solenoid* (the inverse limit of an inverse system of circles with respect to non-trivial covering maps).

3.4. **Tate.** In his thesis [Tat67], Tate reproved Hecke's results on L-functions of Hecke characters, using the formalism of the idèles and idèle class characters. As in Riemann's and Hecke's arguments, he uses an integral transform to obtain the analytic continuation and functional equation. However, the integral is now taken over the group of idèles $\mathbb{A}_K^\times$ (from this point of view, the integral in Riemann's formula (13) should be viewed as taking place over the purely archimedean group $\mathbb{R}_{>0}^\times$), and the application of Poisson summation that underlies the proof of the functional equation is now performed with respect to the discrete group $K \subset \mathbb{A}_K$. The fact that the integral is taken over $\mathbb{A}_K^\times$ has a crucial consequence: the Euler product for the Hecke L-functions appears naturally, as a part of the same sequence of arguments that proves the analytic continuation and functional equation!

In a little more detail:[20] Tate considers a *unitary* idèle class character χ, and then considers integrals of the form

$$\zeta(f, \chi|\ |^s)) = \int_{\mathbb{A}_K^\times} f(x)\chi(x)|x|^s d^\times x,$$

where f is a function on $\mathbb{A}_K$ which decays suitably at infinity (i.e. belongs to an appropriately defined Schwarz space of functions on $\mathbb{A}_K$), and $d^\times x$ is a suitably

[20]We sketch the contents of Tate's thesis here, since it played, and continues to play, such a key role in the development of the the theory of automorphic L-functions. For more details, we encourage the interested reader to study the thesis itself, as well as Weil's important Seminaire Bourbaki [Wei95], which gives a distribution-theoretic interpretation of Tate's arguments; suitable extensions of the ideas in Weil's lecture are basic to the contemporary analysis of integral formulas in the theory of automorphic forms. There are also many excellent commentaries on Tate's thesis available; among these, we particularly recommend the article [Kud03] of Kudla.

normalized Haar measure on the idèle group $\mathbb{A}_K^\times$, obtained as a product of Haar measures $d^\times x_v$ on each of the multiplicative groups $K_v^\times$.

If the function f factors as a product $f = \otimes_v f_v$ of (appropriately Schwarz) functions f_v on each K_v, then Tate's ζ-integral admits a factorization

$$\zeta(f,\chi|\ |^s) = \int_{\mathbb{A}_K^\times} f(x)\chi(x)|x|^s d^\times x \\ = \prod_v \int_{K_v^\times} f_v\chi_v(x_v)|x_v|_v^s d^\times x_v = \prod_v \zeta_v(f_v,\chi_v|\ |_v^s), \tag{21}$$

where the local ζ-integrals are defined according to the indicated formula. Directly evaluating these local ζ-integrals, and then (via the factorization (21)) analyzing the global ζ-integrals, Tate finds that the local ζ-integrals converge for $\Re s > 0$ — this directly corresponds to the fact that the geometric series

$$\sum_{n=0}^{\infty} \frac{1}{p^{ns}} = \big(1 - \frac{1}{p^s}\big)^{-1}$$

defining the Euler factors in (5) converge when $\Re s > 0$ — and that the global ζ-integrals converge for $\Re s > 1$ — directly corresponding to the fact the Euler product (5) itself converges[21] precisely when $\Re s > 1$.

Tate establishes two crucial results. The first, local, result, concerns the ratio

$$\frac{\zeta_v(f_v,\chi_v|\ |_v^s)}{\zeta_v(\hat{f}_v,\overline{\chi}_v|\ |^{1-s})}, \tag{22}$$

where $\hat{f}_v$ denotes the additive Fourier transform of f_v (with respect to some chosen additive character ψ_v); note that both integrals are defined when $0 < \Re s < 1$, and so this ratio makes sense (provided that the denominator does not vanish). What Tate shows is that the ratio (22) is in fact independent of the choice of the function f_v, and analytically continues to a meromorphic function of s. This is proved by an elementary manipulation of integrals.

The second, global, result, states that $\zeta(f,\chi|\ |^s)$ has a meromorphic continuation to the entire complex plane, and satisfies a functional equation

$$\zeta(f,\chi|\ |^s) = \zeta(\hat{f},\overline{\chi}|\ |^{1-s});$$

here $\hat{f}$ denotes the additive Fourier transform of f, computed with respect to an additive character of $\mathbb{A}_K$ which is chosen to be trivial on K. The proof of this result follows Riemann's proof of (11) via the integral transform of a theta series; in particular, as already remarked, Poissson summation (but now with respect to the discrete subgroup K of $\mathbb{A}_K$) plays a key role.

By choosing $f = \otimes_v f_v$ to be (essentially) Fourier self-dual (so at an infinite place, f_v will be an appropriate Gaussian — and so we connect back to Riemann's proof via Jacobi's ϑ — while at a finite place, f_v will be the characteristic function of $\mathcal{O}_K$), we find that $\zeta(f,\chi|\ |^s)$ is essentially Hecke's L-function $L(s,\chi)$. In fact, we essentially obtain the *completed* L-function $\Lambda(s,\chi)$ — i.e. with the appropriate

[21] It is important that χ be unitary for these bounds on the region of convergence to be correct. As we already saw in (3.2.1), and as is clear from the formulas defining the ζ-integrals, if we multiply χ by some power $||^{s_0}$ so as to make it possibly non-unitary, then this amounts to shifting the variable s in the ζ-integral by s_0, and correspondingly translating the regions of convergence by $\Re s_0$.

Γ factors, powers of π, etc., included — and (via (21)) we obtain it directly in its *Euler product form*! We say "essentially", because at the primes dividing the conductor of χ the local ζ-integrals are a little more finicky (they involve Gauss sum-like expressions); and we also have factors at the infinite places. When we put everything together, we obtain Hecke's functional equation[22]

$$\Lambda(1-s,\overline{\chi}) = \epsilon(\chi)\Lambda(s,\chi),$$

where the quantity $\epsilon(\chi)$ which mediates the functional equation is itself written as an "Euler product"

$$\epsilon(\chi) = \prod_{v \in S} \epsilon_v(\chi, \psi_v); \tag{23}$$

here S denotes the set of ramified primes for χ, together with the infinite places, and the local factors ϵ_v arise from an explicit evaluation of the ratios of (22) at the places $v \in S$.[23] (As the notation indicates, the local ϵ factors depend on the additive character ψ_v used to compute the local additive Fourier transforms — although in an easily understood way — whereas the global ϵ factor does not. Indeed, since the functional equation is what it is, regardless of how we normalize the Fourier transforms used to establish it, the global ϵ factor cannot depend on the choice of ψ; more concretely, one sees that this is so via an application of the global product formula.)

If we take χ to be the trivial character, Tate's ζ-integral essentially computes the Dedekind ζ-function $\zeta_K(s)$. In this optic, the class number formula (19) emerges quite directly from a computation of the volume of the compact group $K^\times \setminus \mathbb{A}_K^1$.

3.5. **Applications to density.** Given the analytic continuation of his L-series, Hecke was able to generalize Dirichlet's arguments (and indeed the subsequent improvements on them by Landau) to prove the infinitude (and indeed, the appropriate density) of prime ideals in a given ray class (by working with $L(s,\chi)$ for χ running over all characters of the appropriate ray class group), and even more general density theorems (by using $L(s,\chi)$ for infinite order χ), such as results on the density of Gaussian primes lying in a given angular sector of the complex plane. (One uses Fourier theory on $\mathbb{C}^\times$ to approximate the indicator function of the sector in terms of characters θ on $\mathbb{C}^\times$, and then uses a version of Dirichlet's argument, but applied to Hecke characters χ that extend the relevant θ.)

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[22] In the case of the Riemann, or more generally Dedekind, ζ-function, which in Hecke's optic is the case of the trivial character, one has the simplest possible functional equation: if $\xi_K(s)$ denotes the appropriately completed form of $\zeta_K(s)$, then $\xi_K(1-s) = \xi_K(s)$. For quadratic (equivalently, real-valued unitary) characters, the functional equation again takes a simple form, namely

$$\Lambda(1-s,\chi) = \pm\Lambda(s,\chi)$$

(a factor of ± 1 is the only possibility which is consistent with the involutive nature of the symmetry $s \mapsto 1-s$). But for genuinely complex characters, the functional equation relates $\Lambda(s,\chi)$ and $\Lambda(1-s,\overline{\chi})$; since χ and $\overline{\chi}$ are now distinct, the constant in the functional equation is less constrained; *a priori* it is merely a complex number of absolute value 1.

[23] It is more correct to include the variable s in the global and local ϵ factors as well; the factor $\big(|D|N(\mathfrak{m})\big)^{s/2}$ which appears in the classical completed L-function is then incorporated into the ϵ factors, and written as the product of its various local contributions. The ϵ factors in our formula are obtained by evaluating at the point of symmetry, i.e. at $s = 1/2$. When we evaluate the ϵ factors at a specific value of s in this way, it might be better to refer to them as local and global *root numbers*.

3.6. **Summary.** If one unpacks Tate's proof of Hecke's functional equation in sufficient detail, it does not differ so much in its technical details from Hecke's original proof. (As one example, the step in Hecke's argument, going back to Dirichlet's original analysis of his L-functions, in which one decomposes the L-function into a sum of separate series, each indexed by ideals lying in a given ideal class, is related to the proof of the compactness of $\mathbb{A}_K^1/K^\times$, in which (as we noted above) the finiteness of the class group plays a key role.) But the difference in presentation is enormous! Tate's integral formulas lead directly to the presentation of the L-functions as an Euler product, his local calculations explicate the structure of the global root number by writing it as a product of local factors, and the class number formula falls out as another manifestation of the basic structure of $\mathbb{A}_K^1/K^\times$.

We close this discussion by a recapitulation of the role of the adèles in this argument, and, more generally, in contemporary number theory. Like many mathematical innovations, the use of the adèles at first seems to be a merely technical device, which one becomes used to over time; and which one might eventually regard as natural and even intuitive, just through habit alone. There is certainly an aspect of this with regard to the use of the adèles, and I believe that it reflects the attitude of many, perhaps even most, students of number theory upon encountering them for the first time. And the adèles do have many advantages which appear purely technical at first; e.g. as we already indicated (in footnote 16 above), one such advantage is that we get to replace the ring $\mathbb{Z}$ with the field $\mathbb{Q}$. It seems, then, worthwhile to reflect on what fundamental number theoretic ideas and intuitions might be being captured in the concept of the adèles, beyond their purely technical advantages. We conclude with a short series of remarks in this spirit:

The product structure of the adèles, a *structural* product in the sense of modern algebra, reflects in a subtle but important way the *literal, arithmetic* product that appears in the statement of the fundamental theorem of arithmetic (that any natural number factors uniquely as a product of prime powers). The *restricted* nature of the product reflects the boundedness of the denominator of any rational number. And even though $\mathbb{Q}$ is a field, and so in some sense has no intrinsic arithmetic (unlike the ring $\mathbb{Z}$), the primes are still present and making their influence felt, once we consider the embedding $\mathbb{Q} \hookrightarrow \mathbb{A}$. One might summarize these points by saying that the embedding $\mathbb{Q} \hookrightarrow \mathbb{A}$ "externalizes" the traditional arithmetic concepts of prime number, unique factorization, and boundedness of denominators, and reinterprets them in a framework which more comfortably accommodates the mores of modern algebra, which require one to manipulate rings and fields themselves, rather than their individual elements.[24]

[24]One might regard the passage from elements of sets to sets themselves, in one's view of what is an "atomic" mathematical object, as a first step in the process of *categorification*, in which the passage from sets or other mathematical objects to the categories of which they are constituents, then to higher categories, and so on, make up the subsequent steps. The transition from Hecke's viewpoint to that of Tate's thesis can then be regarded as just one step in the ongoing evolution of number theory in general and the Langlands program in particular. Nevertheless, however much the ideas and problems apparently transform, the connection to the concrete mathematics, not just of Hecke, but of Euler, Riemann, and Dirichlet, persists, since the inner nature of the number theory remains the same.

4. Class field theory

We now begin our discussion of *reciprocity*, by treating the so-called *abelian* case, which is encapsulated in the statements of class field theory. The history of class field theory is such a vast topic that we are unable to do it any justice here.[25] It has its origins in a multitude of topics from 19th century algebraic number theory, such as Gauss's theory of quadratic forms, the theory of quadratic and higher reciprocity laws (studied first by Gauss, then by Eisenstein, Jacobi, and Kummer, among others), much of the theory of cyclotomic fields that Kummer developed in his study of higher reciprocity, the theory of complex multiplication elliptic curves as studied by Kronecker (among others); and also the relationship between algebraic number theory and L-functions brought to light by Dirichlet's proof of his theorem on primes in arithmetic progressions. This last topic will be the starting point for our own discussion of the theory.

4.1. **Comparing Dirichlet and Dedekind.** As is well-known, and as we described above, the crux of Dirichlet's proof of his theorem on primes in arithmetic progression is the non-vanishing of the special values $L(1,\chi)$ when χ is a non-trivial character mod N, and the difficult case of this result occurs when χ is a primitive quadratic (i.e. real-valued, i.e. ±1-valued) character.

There is a bijection between non-trivial primitive quadratic characters and quadratic extensions K of $\mathbb{Q}$, given by associating to χ the field $K=\mathbb{Q}\big(\sqrt{\chi(-1)N}\big)$. What Dirichlet shows is that

$$L(1,\chi)=\begin{cases}\dfrac{2Rh}{\sqrt{|D|}} \text{ if } \chi(-1)=1\\[2ex] \dfrac{2\pi h}{w\sqrt{|D|}} \text{ if } \chi(-1)=-1,\end{cases} \tag{24}$$

where h, etc. are the usual invariants attached to K, as in our discussion of (19). Indeed, a moment's consideration shows that the right hand expression in this formula *is* the value on the right hand side of the class number formula (19) (for the particular quadratic field K at hand). Clearly, there is a relationship between the two formulas!

Let us consider $\zeta_K(s)$. To reduce the number of symbols on the page, and remembering that $\chi(-1)$ is simply a sign, we write $\pm N$ in place of $\chi(-1)N$ (the particular choice of sign of course being dictated by the value $\chi(-1)$). The field $K=\mathbb{Q}(\sqrt{\pm N})$ is Galois over $\mathbb{Q}$, with Galois group of order 2; we identify its Galois group with $\{\pm1\}$. For each unramified prime in K, i.e. for each $p\nmid N$, there is a Frobenius element $\sigma(p)\in\{\pm1\}$ which determines the splitting behaviour of p in K: it equals $+1$ (respectively -1) if p splits completely (respectively remains inert) in K. If p is odd, then this splitting behaviour depends just on the question of whether or not $\pm N$ is a square mod$p$, and so we find that

$$\sigma(p)=\Big(\frac{\pm N}{p}\Big)$$

(the right-hand side denoting the Legendre symbol). We can now compute $\zeta_K(s)$ via its Euler product (18); if p is ramified, i.e. if $p|N$, then there is one prime of K

[25]For the reader interested in the history, we refer the reader to the survey of Hasse [Has67].

above p, having norm p; if p is unramified and split there are two such; and if p is unramified and inert then there is one prime above p of norm p^2. The corresponding factors in (18) multiply out to either

$$\big(1-\frac{1}{p^s}\big)^{-1},\quad \big(1-\frac{1}{p^s}\big)^{-2},\quad \text{or } \big(1-\frac{1}{p^{2s}}\big)^{-1}=\big(1-\frac{1}{p^s}\big)^{-1}\big(1+\frac{1}{p^s}\big)^{-1},$$

depending on which case we are in. The latter two cases admit the uniform description

$$\big(1-\frac{1}{p^s}\big)^{-1}\big(1-\frac{\sigma(p)}{p^s}\big)^{-1},$$

and so we obtain the formula

$$\zeta_K(s)=\prod_{p \text{ prime}}\big(1-\frac{1}{p^s}\big)^{-1}\times\prod_{\substack{p \text{ prime}\\ p\nmid N}}\big(1-\frac{\sigma(p)}{p^s}\big)^{-1} = \zeta_{\mathbb{Q}}(s)\times\prod_{\substack{p \text{ prime}\\ p\nmid N}}\big(1-\frac{\sigma(p)}{p^s}\big)^{-1}, \tag{25}$$

where we have used (5) to rewrite the first of the two products over primes. If we recall that $\zeta_{\mathbb{Q}}(s)$ has a simple pole with residue 1 at $s=1$ (the specialization of (19) to the case of the number field $\mathbb{Q}$), then we may rewrite the class number formula for K in the form

$$\lim_{s\to 1}\prod_{\substack{p \text{ prime}\\ p\nmid N}}\big(1-\frac{\sigma(p)}{p^s}\big)^{-1}=\begin{cases}\dfrac{2Rh}{\sqrt{|D|}} & \text{if } \chi(-1)=1\\[2ex] \dfrac{2\pi h}{w\sqrt{|D|}} & \text{if } \chi(-1)=-1.\end{cases} \tag{26}$$

Most readers who have made it through to this point no doubt know where we are heading, and those few who don't can surely guess: the key to Dirichlet's formula (24) is to identify the Euler product appearing in (26) with $L(s,\chi)$. But, taking into account the Euler product formula of (15), this amounts to verifying an identity of Euler products

$$\prod_{\substack{p \text{ prime}\\ p\nmid N}}\big(1-\frac{\chi(p)}{p^s}\big)^{-1}=\prod_{\substack{p \text{ prime}\\ p\nmid N}}\big(1-\frac{\sigma(p)}{p^s}\big)^{-1}, \tag{27}$$

which in turn amounts to verifying an identity of individual Euler factors, which itself comes down to establishing the formula[26]

$$\chi(p)=\sigma(p)=\big(\frac{\pm N}{p}\big); \tag{28}$$

but if we recall that χ is a primitive quadratic character modulo N, then this formula reduces to the law of quadratic reciprocity (together with its supplementary laws, if N is even). We sketch a proof of this law in Subsection 4.3 below.

This is the crux of reciprocity: a Diophantine quantity, such as $\sigma(p)$, which we may think of as being defined by the condition

$$1+\sigma(p)= \text{ the number of solutions to the equation } x^2=\pm N \bmod p,$$

[26] For odd primes; we ignore the details related to the special case $p=2$ when N is odd.

is related to an "automorphic" quantity, in this case the character χ. ("Automorphic" because it can be interpreted as an idèle class character, i.e. as a character of $\mathrm{GL}_1(\mathbb{Q}) \setminus \mathrm{GL}_1(\mathbb{A}_{\mathbb{Q}})$.)

And so we obtain an interpretation of the notion of reciprocity which goes well beyond the evident "switching p and N" meaning which is at the origin of its use in the term "quadratic reciprocity": reciprocity is an identification of one kind of L-function, the *automorphic* kind with which we already have some familiarity, with a quite different kind of L-function, defined by Euler products such as the one appearing in (26) above, which are defined not by mod N characters or the like, but in terms of Frobenius elements of Galois groups!

In the subsequent sections we turn to a detailed discussion of these new kinds of L-functions (one might reasonably call them "Diophantine", but in fact the traditional label is "motivic"); but before doing that, we explain how the preceding example can be generalized; this is the content of class field theory.

4.2. **Class field theory (I).** If we take into account the identification of Euler products (27), then we may rewrite the factorization (25) in the simple form

$$\zeta_K(s) = \zeta_{\mathbb{Q}}(s) L(s, \chi). \tag{29}$$

As we have seen, it is this factorization that yields Dirichlet's class number formula (24).[27] Now the Dedekind ζ-function $\zeta_K(s)$ may be thought of as *either* automorphic or motivic, from the perspective of the field K (for any number field K): it is the Hecke L-function for the trivial character on $K^\times \setminus \mathbb{A}_K^\times$ (and so automorphic) but it admits the Euler product

$$\prod_{\mathfrak{p}} \left(1 - \frac{\mathbf{1}}{N(\mathfrak{p})^s}\right)^{-1},$$

and the coefficient 1 that we have emphasized (in order to make the analogy with the coefficients $\sigma(p)$ considered above) counts the number of solutions to the (extremely trivial!) equation $x = 0$ in the residue field at $\mathfrak{p}$. (Just to be clear, this equation has 1 (!) solution in any field.) This makes $\zeta_K(s)$ motivic.

However, from the perspective of $\mathbb{Q}$, the L-function $\zeta_K(s)$ is inexorably *motivic* — as we already saw in (25), its Euler factors (when we index them by rational primes p, rather than by the primes $\mathfrak{p}$ of K) relate to counting the number of solutions to a quadratic equation mod p. What (29) shows is that $\zeta_K(s)$ is nevertheless also automorphic from the perspective of $\mathbb{Q}$: it can be written as a product of Hecke L-functions *for characters of* $\mathbb{Q} \setminus \mathbb{A}_{\mathbb{Q}}^\times$.

It is natural to ask: for which other extensions K of $\mathbb{Q}$ does $\zeta_K(s)$ have this property? More generally, we can ask: for which extensions K/F of number fields

[27] We note that Dirichlet's theorem $L(1, \chi) \neq 0$ follows immediately from this factorization, since we know that each of the two ζ factors have a simple pole at $s = 1$. It is also interesting to compare the functional equation for $L(s, \chi)$ that one obtains from this factorization of $\zeta_K(s)$ with the functional equation that one obtains directly from Hecke's theory. Indeed, since each of $\zeta_{\mathbb{Q}}(s)$ and $\zeta_K(s)$ have a sign of $+1$ in their functional equations, we find that also $\Lambda(s, \chi) = \Lambda(1 - s, \chi)$ for the quadratic character χ; i.e. the global root number for $L(s, \chi)$ is also equal to $+1$. If one compares this with the formula (23) and computes the product of the local root numbers explicitly, one recovers Gauss's celebrated theorem on the sign of the quadratic Gauss sum (another example of unity and unification!). Indeed, the theory of ϵ factors is a modern descendent of that classical line of enquiry by Gauss. See Subsection 5.2 below for a discussion of one aspect of this theory.

can $\zeta_K(s)$ be written as a product of Hecke L-functions with respect to idèle class characters of F? Class field theory answers these questions.

4.2.1. **Theorem** (Main theorem of class field theory, in terms of L-functions)**.** *If K/F is an extension of number fields, then $\zeta_K(s)$ may be written as a product of Hecke L-functions with respect to idèle class characters of F if and only if K/F is Galois with* abelian *Galois group. Furthermore, the L-functions that appear in the factorization are ray class L-functions, and any ray class L-function over F appears in such a factorization (for an appropriately chosen abelian extension K of F).*

We elaborate on the statement of this theorem. Suppose to begin with that K/F is an abelian Galois extension, with Galois group G. We begin by generalizing the factorization (25). If $\chi : G \to \mathbb{C}^\times$ is *any* character, let $H_\chi \subseteq G$ denote its kernel, and let E_χ denote the subfield of K fixed by H_χ (so, by Galois theory, χ induces an embedding $\mathrm{Gal}(E_\chi/F) \hookrightarrow \mathbb{C}^\times$). Let S_χ denote the set of primes in F which ramify in E_χ. Given a prime $\mathfrak{p} \notin S_\chi$, we have a Frobenius element[28] $\mathrm{Frob}_\mathfrak{p} \in \mathrm{Gal}(E_\chi/F)$, which is taken, via χ, to an element $\chi(\mathrm{Frob}_\mathfrak{p}) \in \mathbb{C}^\times$. It is now an exercise, generalizing the computations leading to (25), to show that

$$\zeta_K(s) = \prod_{\chi} \prod_{\substack{\mathfrak{p} \text{ a prime of } F \\ \mathfrak{p} \notin S_\chi}} \big(1 - \frac{\chi(\mathrm{Frob}_\mathfrak{p})}{N(\mathfrak{p})^s}\big)^{-1}. \tag{30}$$

For example, the factor corresponding to the trivial character is precisely $\zeta_F(s)$. If K/F is quadratic, then there is a unique non-trivial character $\chi : G \xrightarrow{\sim} \{\pm 1\}$, so (30) has just two factors. If we further assume that $F = \mathbb{Q}$, then (30) reduces to (25). (The quantity denoted there by $\sigma(p)$ coincides with the quantity $\chi(\mathrm{Frob}_p)$ of the present discussion.)

The first statement in Theorem 4.2.1 is then proved by establishing a generalization of the equality (27) — one proves that each of the various Euler products appearing in (30), labelled by the various characters χ of the Galois group, is in fact a Hecke L-function for some ray class character of F. In fact, one proves a more structural statement — one shows that, in an appropriate sense, the character χ *is* a ray class character, in such a way that $\chi(\mathrm{Frob}_\mathfrak{p})$ just becomes $\chi(\mathfrak{p})$. The Euler products in (30) will then literally be Euler products for ray class L-functions, and the factorization of $\zeta_K(s)$ as a product of such L-functions will be established.

4.3. **Cyclotomic fields.** We begin by explaining a special case of the preceding discussion. Let $N > 0$ be a natural number, and set $K = \mathbb{Q}(\zeta_N)$. Then the irreducibility of the Nth cyclotomic polynomial implies that there is a canonical isomorphism $\mathrm{Gal}(K/\mathbb{Q}) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^\times$, given as follows: an element $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ is identified with the Galois automorphism which maps ζ_N to ζ_N^a. If $p \nmid N$, so that K is unramified above p, then one sees that Frob_p corresponds to the the residue

[28]Typically, the Frobenius elements $\mathrm{Frob}_\mathfrak{q} \in \mathrm{Gal}(E/F)$, for a Galois extension of number fields, are associated to unramified primes $\mathfrak{q}$ of the extension field E. Any two $\mathfrak{q}$ lying over a given unramified prime $\mathfrak{p}$ of F are conjugate by some element of $\mathrm{Gal}(E/F)$, implying that the corresponding Frobenius elements in $\mathrm{Gal}(E/F)$ are conjugate (in the sense of group theory); we write $\mathrm{Frob}_\mathfrak{p}$ to denote this conjugacy class (or, sometimes, some choice of element in it). If $\mathrm{Gal}(E/F)$ is *abelian* — which is the case for $E = E_\chi$ — then conjugacy classes in $\mathrm{Gal}(E/F)$ are singletons, so that $\mathrm{Frob}_\mathfrak{p}$ is indeed a well-defined element.

class of p itself. The characters of $\mathrm{Gal}(K/\mathbb{Q})$ are thus identified with the characters of $(\mathbb{Z}/N\mathbb{Z})^\times$.

A character χ has conductor M (i.e. is a primitive character mod M, for some divisor M of N) if and only if E_χ is contained in $\mathbb{Q}(\zeta_M)$, but in no smaller cyclotomic subfield of K. One then sees that E_χ is ramified at precisely the primes dividing M, and so, identifying the Galois character χ with the corresponding character of $(\mathbb{Z}/M\mathbb{Z})^\times$, we find that

$$\prod_{p \notin S_\chi} \big(1 - \frac{\chi(\mathrm{Frob}_p)}{p^s}\big)^{-1} = \prod_{p \nmid M} \big(1 - \frac{\chi(p)}{p^s}\big)^{-1} = L(s, \chi).$$

Combining this identification of Euler products with (30), we find that

$$\zeta_K(s) = \prod_{\chi \text{ a char. mod } N} L(s, \chi),$$

establishing the claim of Theorem 4.2.1 for the cyclotomic extensions $K = \mathbb{Q}(\zeta_N)$ of $\mathbb{Q}$. Additionally, since Dirichlet's L-functions are precisely the ray class L-functions for $\mathbb{Q}$, we have established the second ("Furthermore, ...") statement of Theorem 4.2.1 in the case $F = \mathbb{Q}$.

Theorem 4.2.1 in general, for $F = \mathbb{Q}$, then comes down to the following celebrated theorem of Kronecker and Weber.

4.3.1. **Theorem.** *Any abelian Galois extension $K/\mathbb{Q}$ is contained in $\mathbb{Q}(\zeta_N)$ for some N.*

Let us sketch a proof of this result in the case of *quadratic* extensions $K/\mathbb{Q}$. Firstly, we note that if $\chi : \mathrm{Gal}\big(\mathbb{Q}(\zeta_N)/\mathbb{Q}\big) = (\mathbb{Z}/N\mathbb{Z})^\times \to \{\pm 1\}$ is a primitive quadratic character, then E_χ is a quadratic subextension of $\mathbb{Q}(\zeta_N)$ ramified precisely at the primes dividing N. Furthermore, it is real or imaginary depending on whether $\chi(-1) = 1$ or -1 (since $-1 \in (\mathbb{Z}/N\mathbb{Z})^\times$ corresponds to complex conjugation in $\mathrm{Gal}\big(\mathbb{Q}(\zeta_N)/\mathbb{Q}\big)$). From the explicit classification of quadratic extensions of $\mathbb{Q}$, this information uniquely determines E_χ, and indeed shows that

$$E_\chi = \mathbb{Q}\big(\sqrt{\chi(-1)N}\big). \tag{31}$$

As already noted, as χ runs over all primitive quadratic characters, the fields $\mathbb{Q}\big(\sqrt{\chi(-1)N}\big)$ run over all quadratic extensions of $\mathbb{Q}$. Thus we find that, indeed, every quadratic extension of $\mathbb{Q}$ is contained in some $\mathbb{Q}(\zeta_N)$.[29] Furthermore, from (31), and the fact that $\mathrm{Frob}_p \in \mathrm{Gal}\big(\mathbb{Q}(\zeta_N)/\mathbb{Q}\big)$ corresponds to $p \in (\mathbb{Z}/N\mathbb{Z})^\times$, we find that $\mathrm{Frob}_p \in \mathrm{Gal}\Big(\mathbb{Q}\big(\sqrt{\chi(-1)N}\big)/\mathbb{Q}\Big) = \{\pm 1\}$ is equal to $\chi(p)$. This establishes the law of quadratic reciprocity (28), and illustrates how the apparently more abstract analysis of the Galois theory of abelian extensions can be used to deduce concrete reciprocity laws.

4.3.2. *Remark.* Certainly $\mathbb{Q}$ admits other abelian extensions than quadratic ones. For example, since there are infinitely many primes $p \equiv 1 \bmod 3$, there are infinitely many extensions $\mathbb{Q}(\zeta_p)$ which contain a degree 3 subextension. One might wonder, then, why there is not a more classical reciprocity law which relates to (say) the

[29] A more traditional proof of this result uses quadratic Gauss sums to express $\sqrt{\chi(-1)N}$ explicitly as an element of $\mathbb{Q}(\zeta_N)$.

degree 3 case of Theorem 4.3.1 in the same way that the quadratic case is related to quadratic reciprocity.

The reason is that there is no simple family of Diophantine equations that gives rise to a corresponding family of degree 3 abelian extensions of $\mathbb{Q}$ in the same way that the family of equations $X^2 = D$ gives rise to the family of quadratic extensions. Indeed, while we can write down the family of cubic equations $x^3 + ax + b = 0$, such a cubic will typically have Galois group S_3 rather than C_3. In order to have abelian Galois group, we need to impose the additional condition that $-4a^3 - 27b^2$ be a square. Thus parameterizing such cubics amounts to finding rational points on the surface $4a^3 + 27b^2 + c^2 = 0$ — which is not a particularly simple parameter space. In practice, the simplest way to describe abelian extensions of $\mathbb{Q}$ of degree > 2 is as subfields of appropriate cyclotomic fields.

4.4. **Class field theory (II).** In its more Galois-theoretic formulation, class field theory provides an analogue of Theorem 4.3.1 for general number fields. More precisely, if F is any number field, and $\mathfrak{m}$ is any conductor in F, then one introduces the notion of a *ray class field* $K_{\mathfrak{m}}$ of F. This is a finite Galois extension of F satisfying the following properties:

(1) $\mathrm{Gal}(K_{\mathfrak{m}}/F)$ is isomorphic to the ray class group of K of conductor $\mathfrak{m}$;
(2) $K_{\mathfrak{m}}$ is unramified[30] at places of F not dividing $\mathfrak{m}$. Further, if $\mathfrak{p} \nmid \mathfrak{m}$, then the Frobenius element $\mathrm{Frob}_{\mathfrak{p}} \in \mathrm{Gal}(K_{\mathfrak{m}}/F)$ is identified with the class of $\mathfrak{p}$ in the ray class group (for an appropriate choice of the isomorphism in (1), which is then pinned down uniquely by this requirement).

4.4.1. *Example.* The cyclotomic field $\mathbb{Q}(\zeta_N)$ is the ray class field of $\mathbb{Q}$ of conductor $N\infty$.

One then proves the following theorem.

4.4.2. **Theorem** (Main theorem of class field theory, in terms of abelian extensions)**.** *For any choice of conductor* $\mathfrak{m}$ *in* F*, a corresponding ray class field* $K_{\mathfrak{m}}$ *exists* (*and is unique up to isomorphism*)*. Further, if* K/F *is an abelian Galois extension, then* K *is contained in some ray class field of* F*.*

If we take into account (4.4.1), then this result generalizes Theorem 4.3.1 in an evident sense. We also see (applying (30)) that

$$\zeta_{K_{\mathfrak{m}}}(s) = \prod_{\chi} L(s, \chi),$$

where the product is taken over all ray class characters of conductor dividing $\mathfrak{m}$. More generally, if K/F is a subextension of $K_{\mathfrak{m}}$, and if we write $H = \mathrm{Gal}(K_{\mathfrak{m}}/K) \subseteq \mathrm{Gal}(K_{\mathfrak{m}}/F)$, so that H is a subgroup of the ray class group of conductor $\mathfrak{m}$, then

$$\zeta_K(s) = \prod_{\chi \text{ which are trivial on } H} L(s, \chi). \tag{32}$$

Thus Theorem 4.2.1 follows from Theorem 4.4.2.

[30] Here one has to extend the notion of ramification to the infinite places: if a real place of F lifts to a complex place of $K_{\mathfrak{m}}$, we say it is ramified; otherwise we say it is unramified. Complex places are always unramified.

As the conductor $\mathfrak{m}$ varies, the ray class fields $K_{\mathfrak{m}}$ form a tower of field extensions (if $\mathfrak{m}|\mathfrak{m}'$ then $K_{\mathfrak{m}} \subseteq K_{\mathfrak{m}'}$), and so we may form their union. Theorem 4.4.2 shows that this is precisely the maximal abelian Galois extension F^{ab} of F. Thus

$$\mathrm{Gal}(F^{\mathrm{ab}}/F) \quad = \quad \varprojlim_{\mathfrak{m}} \mathrm{Gal}(K_{\mathfrak{m}}/F) \quad = \quad \varprojlim_{\mathfrak{m}} (\text{ray class group of conductor } \mathfrak{m}).$$

Recalling that this last inverse limit is precisely the group of connected components of the idèle class group, we obtain the following reformulation of the main theorem of class field theory.

4.4.3. **Theorem** (Main theorem of class field theory, in terms of idèles)**.** *There is a canonical isomorphism* $\mathrm{Gal}(F^{\mathrm{ab}}/F) \xrightarrow{\sim} \pi_0(F^\times \setminus \mathbb{A}_F^\times)$.

4.4.4. *Attributions.* Class field theory was first proved in general by Takagi [Tak20], although the characterization of ray class fields that he used was different. Class fields were first introduced by Weber, who defined them in terms of the splitting behaviour of primes in $K_{\mathfrak{m}}/F$ (but could not prove their existence in general).[31] In his work Takagi begins with a different definition, related to Weber's as well as to norms;[32] we refer the reader to Hasse's excellent historical survey [Has67] for more details on Takagi's point-of-view (and on all of the history that we are summarizing here). In our formulation they are characterized by the description of Frobenius elements in the Galois group (the splitting behaviour of primes then being a consequence of this description).

The emphasis on Frobenius elements is due to Artin [Art27], who reformulated the theory in terms of his famous reciprocity law. Although none of our various formulations of the main theorems of class field theory state Artin's reciprocity law explicitly, it follows from condition (2) in the definition of the ray class field $K_{\mathfrak{m}}$, together with the statement that any abelian extension of a given ground field F is contained in a ray class field.

The idelic reformulation is due to Chevalley [Che40].

4.5. **Algebraic Hecke characters.** Since the ray class characters of a number field K are precisely the finite order idèle class characters, they may also be regarded as the characters of $\pi_0(K^\times \setminus \mathbb{A}_K^\times)$. Theorem 4.4.3 then gives a very direct sense to the notion that characters of Galois groups are the same as ray class characters.

What is less obvious from Theorem 4.4.3 is that certain *infinite order* idèle class characters are also related to characters of Galois groups, in a manner which we now explain.

Firstly, among all the characters

$$\theta : (\mathbb{R} \otimes_{\mathbb{Q}} K)^\times = (\mathbb{R}^\times)^{r_1} \times (\mathbb{C}^\times)^{r_2} \to \mathbb{C}^\times,$$

[31] Suppose that K is an extension of F in which the primes that split completely are precisely the primes that are trivial in a given ray class group. Then the result described in footnote 12, together with Hecke's density result discussed in Subsection 3.5, shows that $[K : F]$ necessarily equals the order of the ray class group. This argument sketch may give the reader a sense of how one can even begin to relate the splitting behaviour in a field to other of its properties.

[32] The role of norms from an extension K down to F is fundamental in the study of class field theory, although since it is not so relevant for our purposes, we have omitted any discussion of it here.

let us single out those that are *algebraic*, in the sense that they are given by a formula (using obvious notation)

$$(x_1, \ldots, x_{r_1}, z_1, \ldots, z_{r_2}) \mapsto x_1^{n_1} \ldots x_{r_1}^{n_{r_1}} z_1^{p_1} \overline{z}_1^{q_1} \cdots z_{r_2}^{p_{r_2}} \overline{z}_{r_2}^{q_{r_2}},$$

for some integers $n_1, \ldots, n_{r_2}, p_1, q_1, \ldots, p_{r_2}, q_{r_2}$. We then say that an idèle class character χ is *algebraic*[33] if the associated character θ is algebraic in the above sense.

Now let ℓ be a prime,[34] and choose an isomorphism[35] $\imath : \mathbb{C} \xrightarrow{\sim} \overline{\mathbb{Q}}_\ell$.

Let ψ be an algebraic Hecke character, with associated algebraic character $\theta : (\mathbb{R}^\times \otimes_{\mathbb{Q}} K)^\times \to \mathbb{C}^\times$. Note that $(\mathbb{C} \otimes_{\mathbb{Q}} K)^\times = (\mathbb{C}^\times)^{r_1} \times (\mathbb{C}^\times \times \mathbb{C}^\times)^{r_2}$, and so θ extends to a character

$$(\mathbb{C} \otimes_{\mathbb{Q}} K)^\times \to \mathbb{C}^\times$$

via the formula (again using obvious notation)

$$(x_1, \ldots, x_{r_1}, z_1, w_1, \ldots, z_{r_2}, w_{r_2}) \mapsto x_1^{n_1} \ldots x_{r_1}^{n_{r_1}} z_1^{p_1} w_1^{q_1} \cdots z_{r_2}^{p_{r_2}} w_{r_2}^{q_{r_2}}.$$

We continue to denote this extension via θ.

The isomorphism $\imath$ allows us to regard θ as a character

$$(\overline{\mathbb{Q}}_\ell \otimes_{\mathbb{Q}} K)^\times \to \overline{\mathbb{Q}}_\ell^\times.$$

Composing this with the embedding $(\mathbb{Q}_\ell \otimes_{\mathbb{Q}} K)^\times \hookrightarrow (\overline{\mathbb{Q}}_\ell \otimes_{\mathbb{Q}} K)^\times$, and remembering that $\mathbb{Q}_\ell \otimes_{\mathbb{Q}} K = \prod_{\lambda | \ell} K_\lambda$, we obtain a character

$$\theta_\ell : \prod_{\lambda | \ell} K_\lambda^\times \to \overline{\mathbb{Q}}_\ell.$$

Furthermore, this character is again *continuous*.[36]

Now define a character $\mathbb{A}_K^\times \to \overline{\mathbb{Q}}_\ell^\times$ via the formula

$$x \mapsto \imath\big(\psi(x)\theta(x)^{-1}\big)\theta_\ell(x); \tag{33}$$

here $\theta(x)$ is computed by projecting x to its archimedean components, while $\theta_\ell(x)$ is computed by projecting x to its components at primes $\lambda|\ell$. This a continuous character, which by its definition and the construction of θ_ℓ is trivial on $K^\times$. Furthermore, since, in the formula (33), we have shifted the "infinite order" aspects contributed by θ from the archimedean primes to the ℓ-adic primes of K, we see that (33) is also trivial on the connected part of $\mathbb{A}_K^\times$ (as indeed it must be, since

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[33] In the original terminology of Weil [Wei56b], such a character is said to be of *type* (A_0).

[34] The use of ℓ, rather than p, in this context is traditional.

[35] Such an isomorphism exists, by the axiom of choice. Its precise nature is not terribly important, and making a choice of $\imath$ is largely an expedient. Its main purpose is to allow us to match the collection of embeddings $K \hookrightarrow \mathbb{C}$ with the collection of embeddings $\imath : K \hookrightarrow \overline{\mathbb{Q}}_\ell$, and to do this it suffices to identify the algebraic closures of $\mathbb{Q}$ inside $\mathbb{C}$ with the algebraic closure of $\mathbb{Q}$ inside $\overline{\mathbb{Q}}_\ell$. Such a choice of identification *is* provided by $\imath$, but is a considerably tamer piece of data than $\imath$ itself.

[36] It is here that we use the assumption that θ is algebraic. Indeed, the isomorphism $\imath$ is certainly *not* continuous, since $\mathbb{C}$ and $\overline{\mathbb{Q}}_\ell$ have utterly different topological natures. But the character θ is given simply by raising to various integer powers, and such homomorphism are continuous on *any* topological abelian group.

it is continuous and takes values in the totally disconnected group $\overline{\mathbb{Q}}_\ell^\times$). Thus (33) defines a continuous character[37]

$$\widetilde{\psi}_\ell : \pi_0(K^\times \setminus \mathbb{A}_K^\times) \to \overline{\mathbb{Q}}_\ell^\times,$$

which by Theorem 4.4.3 we may equally well regard as a continuous character

$$\widetilde{\psi}_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \overline{\mathbb{Q}}_\ell^\times. \tag{34}$$

Thus the algebraic Hecke character ψ gives rise to a *family*[38] of ℓ-adic characters (34). If ψ has conductor $\mathfrak{m}$, then one sees from the construction that $\widetilde{\psi}_\ell$ is unramified at primes $\mathfrak{p} \nmid \mathfrak{m}\ell$, and for such a prime $\mathfrak{p}$, one computes[39]

$$\widetilde{\psi}_\ell(\mathrm{Frob}_\mathfrak{p}) = \psi_\mathfrak{p}(\pi_\mathfrak{p}^{-1}), \tag{35}$$

where $\psi_\mathfrak{p}$ denotes the local factor of ψ at $\mathfrak{p}$, and $\pi_\mathfrak{p}$ denotes some uniformizer at $\mathfrak{p}$ (the indicated expression being independent of this choice). This value is *independent* of ℓ (it depends just on the original character ψ), and gives meaning to the idea that the $\widetilde{\psi}_\ell$ are a *compatible family* of ℓ-adic Galois characters.

4.5.1. *Remark.* If ψ is in fact a ray class character, then θ is trivial, the character ψ takes values in $\overline{\mathbb{Q}}^\times$, and $\widetilde{\psi}_\ell$ coincides with ψ (just using $\imath$ to relocate $\overline{\mathbb{Q}}$ from being a subfield of $\mathbb{C}$ to being a subfield of $\overline{\mathbb{Q}}_\ell$). So in this case the family $\widetilde{\psi}_\ell$ is really just the single character $\widetilde{\psi}$, reinterpreted as a finite order Galois character via Theorem 4.4.3.

If, however, θ is non-trivial, then the $\widetilde{\psi}_\ell$ are truly distinct characters as ℓ varies (for example, $\widetilde{\psi}_\ell$ will be infinitely ramified above ℓ, but not above any other rational prime), and the sense in which they are related is a subtle one, mediated by the compatibility condition (35).

The notion of compatible families of ℓ-adic Galois representations was introduced by Taniyama [Tan57], and further developed by Serre [Ser68]. It has evolved into a fundamental notion in number theory, and as we will see in what follows, it is the notion that underlies the definition of *motivic* L-functions. The preceding construction allows us to produce compatible families of ℓ-adic characters whose associated L-function is just the Hecke L-function $L(\psi, s)$; thus these are compatible families for which reciprocity is known to hold.

4.5.2. *Example.* The absolute value character $\psi = |\ | : \mathbb{Q}^\times \setminus \mathbb{A}_\mathbb{Q}^\times \to \mathbb{R}_{>0}^\times$ is algebraic, and the associated family of ℓ-adic characters is just the family of ℓ-adic cyclotomic characters $\chi_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^\times$, defined by $\sigma(\zeta) = \zeta^{\chi_\ell(\sigma)}$ for any Galois automorphism σ and any ℓ-power root of unity ζ.

4.6. **Brauer groups and Weil groups.** Another aspect of class field theory that we have so far not mentioned is the role of the theory of Brauer groups (over both local and global fields). The reader can consult [Has67] for some indication of the historical importance of Brauer groups in the development of the subject. From the modern cohomological perspective, the construction of the fundamental class in

[37]We use the notation $\widetilde{\psi}_\ell$ just to avoid conflicting with our already-established convention, in the particular case when $K = \mathbb{Q}$, of writing ψ_ℓ for the local component of ψ at ℓ.

[38]"Family", because we may vary the prime ℓ.

[39]Whether or not one includes the power -1 in the following formula depends on the normalization of the isomorphism in Theorem 4.4.3; it is adapted to our (admittedly implicit) choice of normalization. See also footnote 19.

$H^2\big(\mathrm{Gal}(K/F), K^\times\backslash\mathbb{A}_K^\times\big)$ is one of the pivotal steps in the entire development of the theory. We may interpret this class as an extension of $\mathrm{Gal}(K/F)$ by $K^\times\backslash\mathbb{A}_K^\times$; we call this extension the *Weil group* $W_{K/F}$. (See [Wei51] for Weil's original construction, and [Tat79] for a presentation along more modern lines.)

Just as $K^\times\backslash\mathbb{A}_K^\times$ relates to $\mathrm{Gal}(K^{\mathrm{ab}}/K)$, the group $W_{K/F}$ also relates to a Galois-theoretic object. Namely, if we let $\mathrm{Gal}(\overline{\mathbb{Q}}/K)^c$ denote the closure of the commutator subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$, so that $\mathrm{Gal}(K^{\mathrm{ab}}/K) = \mathrm{Gal}(\overline{\mathbb{Q}}/K)/\,\mathrm{Gal}(\overline{\mathbb{Q}}/K)^c$, then we have a canonical morphism of short exact sequences of groups

(36)
$$\begin{array}{ccccccccc} 1 & \longrightarrow & K^\times\backslash\mathbb{A}_K^\times & \longrightarrow & W_{K/F} & \longrightarrow & \mathrm{Gal}(K/F) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathrm{Gal}(K^{\mathrm{ab}}/K) & \longrightarrow & \mathrm{Gal}(\overline{\mathbb{Q}}/F)/\,\mathrm{Gal}(\overline{\mathbb{Q}}/K)^c & \longrightarrow & \mathrm{Gal}(K/F) & \longrightarrow & 1 \end{array}$$

where the left vertical arrow is induced by (the inverse to) the isomorphism of Theorem 4.4.3.

If ψ denotes an algebraic Hecke character on $K^\times\backslash\mathbb{A}_K^\times$, then we may induce ψ to obtain a representation $\rho : W_{K/F} \to \mathrm{GL}_n(\mathbb{C})$, if $n = [K:F]$. Correspondingly, we may induce each of the ℓ-adic characters $\widetilde{\psi}_\ell$ of $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ to obtain a representation $\widetilde{\rho}_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ (which is trivial on $\mathrm{Gal}(\overline{\mathbb{Q}}/K)^c$). The $\widetilde{\rho}_\ell$ give an example of a compatible family of n-dimensional ℓ-adic Galois representations.

We finish this discussion by recalling how to construct the *absolute* Weil group of F. First, we recall that if $E \subseteq K$ with E again Galois over F, then there is a natural surjection $W_{K/F} \to W_{E/F}$, inducing the natural surjection on Galois groups, and the norm map on idèle class groups. In particular, taking $E = F$, and noting that by construction $W_{F/F} = F^\times\backslash\mathbb{A}_F^\times$, we obtain a surjection $W_{K/F} \to F^\times\backslash\mathbb{A}_F^\times$ that identifies the target with the abelianization of its source. We then find that the kernel $W_{K/F}^1$ of

$$W_{K/F} \to W_{K/F}^{\mathrm{ab}} = F^\times\backslash\mathbb{A}_F^\times \xrightarrow{|\,|} \mathbb{R}_{>0}^\times$$

sits in a short exact sequence

$$1 \to K^\times\backslash\mathbb{A}_K^1 \to W_{K/F}^1 \to \mathrm{Gal}(K/F) \to 1,$$

and since $K^\times\backslash\mathbb{A}_K^1$ is compact, so is $W_{K/F}^1$. Thus $W_{K/F}$ is an extension of $\mathbb{R}_{>0}^\times$ by a compact group.

Now, just as we may take an inverse limit to form the profinite group $\mathrm{Gal}(\overline{\mathbb{Q}}/F) = \varprojlim_K \mathrm{Gal}(K/F)$, we may form an inverse limit $W_F = \varprojlim_K W_{K/F}$. This is a locally compact group (in fact, it is an extension of $\mathbb{R}_{>0}^\times$ by $\varprojlim_K W_{K/F}^1$, and the latter group is compact, being an inverse limit of compact groups), which admits a continuous morphism $W_F \to \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. The kernel of this morphism is *connected*; it is an inverse limit, under the norm maps, of the connected parts of all the various $K^\times\backslash\mathbb{A}_K^\times$, and so is an extension of $\mathbb{R}^\times$ by a very complicated solenoid. (See (3.3.3).)

5. Motivic L-functions

Our discussion up to this point essentially completes the story of reciprocity in the *abelian* case (in summary form, to be sure!). As far as chronology is concerned, it takes us up to more-or-less the middle of the 20th century.

We must now turn to the discussion of *non-abelian* reciprocity, which might also be called *non-abelian* class field theory. The search for such a theory began already in the first half of the 20th century, but even the conjectural statements and frameworks remained entirely elusive outside of certain special cases, and it was only with Langlands' letter to Weil [Lana], and his elaborations [Lan70] and [Lan71] on this letter, that a general framework for non-abelian reciprocity was proposed. There is no doubt that this is indeed the correct framework within which to consider the problem, although much remains to be proved.

To begin with, we will describe non-abelian L-functions on the motivic side, since it is these L-functions which were the first to be discovered in the non-abelian context. Indeed, while these L-functions exist in abundance, the absolutely fundamental difficulty in making even a conjectural statement of a general non-abelian reciprocity law, prior to Langlands' letter to Weil, was the seeming lack of any kind of L-functions on the automorphic side that remotely resembled the creatures littering the zoo of non-abelian motivic L-functions.

5.1. **Artin.** Suppose that K/F is a finite Galois extension of number fields, and write $G = \mathrm{Gal}(K/F)$. Artin [Art31] was the first to observe that even when G is non-abelian, the Dedekind ζ-function $\zeta_K(s)$ admits a factorization analogous to (30).

Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be a representation. If $\mathfrak{p}$ is a prime of F unramified in K, then the conjugacy class in G of $\mathrm{Frob}_\mathfrak{p}$ is well-defined, and so $\rho(\mathrm{Frob}_\mathfrak{p})$ is well-defined as a matrix up to conjugation. In particular, its characteristic polynomial is well-defined. Actually, we are more interested in its reciprocal characteristic polynomial $\det\bigl(\mathrm{Id}_{n\times n} - T\rho(\mathrm{Frob}_\mathfrak{p})\bigr)$, which is a degree n polynomial in T with constant term 1. We use this latter polynomial to construct an Euler factor

$$\det\bigl(\mathrm{Id}_{n\times n} - N(\mathfrak{p})^{-s}\rho(\mathrm{Frob}_\mathfrak{p})\bigr)^{-1}; \tag{37}$$

this expression is now the reciprocal of a degree n polynomial in the quantity $N(\mathfrak{p})^{-s}$. In fact, if $\mathfrak{p}$ is unramified in the splitting field of ρ (i.e. the fixed field of the kernel of ρ) then this same definition makes sense. Note that if ρ is 1-dimensional, i.e. just a character χ, then we recover the definition of the Euler factors appearing in the Euler product labelled by χ that appears in the factorization (30).

If $\mathfrak{p}$ is ramified in the splitting field of ρ, then the definition of the Euler factor $\mathfrak{p}$ is a little more involved. It will be of degree $< n$, but (since $n > 1$ if ρ is not a character) will typically not be trivial (whereas in the abelian contexts we've considered up till now, the Euler factors at ramified primes have always been trivial.) To define it, we let V denote the vector space $\mathbb{C}^n$ on which G acts via ρ. We also make a choice $I_\mathfrak{p} \subset D_\mathfrak{p} \subset G$ of inertia and decomposition group at $\mathfrak{p}$ (again well-defined up to conjugation). If $\mathrm{Frob}_\mathfrak{p} \in D_\mathfrak{p}$ denotes a choice of element lifting the Frobenius element of $D_\mathfrak{p}/I_\mathfrak{p}$, then the reciprocal characteristic polynomial of $\rho(\mathrm{Frob}_\mathfrak{p})$ acting on the invariant subspace $V^{I_\mathfrak{p}}$ is well-defined independent of choices, and we form the Euler factor

$$\det\bigl(\mathrm{Id}_{V^{I_\mathfrak{p}}} - N(\mathfrak{p})^{-s}\rho(\mathrm{Frob}_\mathfrak{p})_{|V^{I_\mathfrak{p}}}\bigr)^{-1}. \tag{38}$$

Of course, if $\mathfrak{p}$ is unramified in K (or, more generally, in the splitting field of ρ), then $V^{I_\mathfrak{p}} = V = \mathbb{C}^n$, and so the two Euler factors (38) and (37) coincide. In general, the quantity in (38) is the reciprocal of a polynomial in $N(\mathfrak{p})^{-s}$ of degree equal to $\dim V^{I_\mathfrak{p}}$.

We now define the Artin L-function of ρ as the product over all $\mathfrak{p}$ of these Euler factors, i.e.

$$L(s,\rho) = \prod_{\mathfrak{p}} \det\bigl(\mathrm{Id}_{V^{I_\mathfrak{p}}} - N(\mathfrak{p})^{-s}\rho(\mathrm{Frob}_\mathfrak{p})_{|V^{I_\mathfrak{p}}}\bigr)^{-1}. \tag{39}$$

These L-functions are perhaps the first to appear in the literature that are truly *defined* as an Euler product, rather than as a Dirichlet series which is subsequently factored into a product over primes. The usual comparisons with $\zeta(s)$ show that the product defining $L(s,\rho)$ converges to a holomorphic function on the half-plane $\Re s > 1$; and this all one can say about them to begin with.

In the case of a character $\chi : G \to \mathbb{C}^\times$, class field theory allows us to identify χ with a ray class character, so that $L(s,\chi)$ (as defined above) coincides with the Hecke L-function $L(s,\chi)$ (as the formula (35) shows). In particular, in this case, $L(s,\chi)$ has an analytic continuation and functional equation.

One aspect of defining L-functions for general representations of G, rather than just 1-dimensional representations, is that the set of all representations of G has more structure: for example, we can take direct sums of representations, or induce representations from subgroups. (Note that both of these processes tend to output representations of dimension > 1, and so were not visible in the context of the L-functions of characters that we've studied up till now.)

One rather easily verifies the following properties of Artin L-functions:

$$L(s,\rho_1 \oplus \rho_2) = L(s,\rho_1)L(s,\rho_2), \tag{40}$$

and, if $H \subset G$ is a subgroup and θ is a representation of H, then

$$L(s, \mathrm{Ind}_H^G \theta) = L(s,\theta). \tag{41}$$

This second formula looks a bit paradoxical at first, since the dimensions of the two representations $\mathrm{Ind}_H^G \theta$ and θ itself are different (assuming H is a proper subgroup of G) — the dimension of the former is $[G : H]$ times the dimension of the latter, and so the Euler factors in the Euler product (at least all the unramified primes) are of different degrees! But the Euler products are taken over different collections of primes! The representation $\mathrm{Ind}_H^G \theta$ is a representation of G, and so $L(s, \mathrm{Ind}_H^G)$ is defined as an Euler product over primes of F. On the other hand, θ is a representation of $H = \mathrm{Gal}(K/E)$, if E denotes the fixed field of H, and so $L(s,\theta)$ is defined as an Euler product over the primes of E. Now $[E : F] = [G : H]$, and so degree d Euler factors indexed by primes in E, if we write them in terms of primes of F, will indeed become of degree $[G : H]d$. Thus we see that (41) is related to, indeed formalizes and generalizes, the computations we made to derive the factorization (25).

Indeed, let us apply these formulas in the case when $H = \{1\} \subset G$, with θ being the (necessarily) trivial character of H. Then the fixed field of H is just K itself, and the Artin L-function of the trivial character equals the Hecke L-function of the trivial character equals $\zeta_K(s)$. On the other hand, the induction of the trivial character from $\{1\}$ to G is just the regular representation $\mathbb{C}[G]$ of G, and we have the well-known decomposition

$$\mathbb{C}[G] = \bigoplus_{\rho} \rho^{\oplus \dim \rho},$$

where ρ runs over (a set of isomorphism class representatives of) all the irreducible representations of G. Thus, applying first (41) and then (40), we find that

$$\zeta_K(s) = \prod_{\rho} L(s,\rho)^{\dim \rho}, \tag{42}$$

where the product is taken over the all the irreducible representations ρ of G.

If G is abelian, then the irreducible representations of G are exactly the various characters of G, and so (42) reduces to (30), which we now recognize as a factorization of $\zeta_K(s)$ into a product of Artin L-functions. Class field theory then allows us to reinterpret this as expressing $\zeta_K(s)$ (for the abelian extension K of F) as a product of Hecke L-functions for F. The factorization (29), essentially due to Dirichlet, gives one of the simplest illustrations of this phenomenon, and already has far-reaching consequences (being the crux of Dirichlet's proof of this theorem, in so far as it leads to the non-vanishing of $L(1,\chi)$).

What are the number-theoretic consequences of the more general factorizations (42)? Is there a non-abelian class field theory which gives us an alternative interpretation of the L-functions $L(s,\rho)$ that appear in it? The answers to these questions (some known, most still conjectural) are provided by Langlands's notions of reciprocity and functoriality, to be discussed in the following section.

For now, we state the following celebrated conjecture of Artin.

5.1.1. **Conjecture** (Artin). *If ρ is an irreducible representation of G distinct from the trivial character, then $L(\rho,s)$ admits a holomorphic continuation to the complex plane and satisfies a functional equation relating* (*a suitably completed form of*) *$L(\rho,1-s)$ and* (*a suitably completed form of*) *$L(\rho^\vee,s)$.* (*Here $\rho^\vee$ denotes the contragredient to ρ.*)

5.1.2. *Example.* We explain one concrete consequence of Artin's conjecture. If E/F is an arbitrary finite extension, let K denote the Galois closure of E, and write $H = \mathrm{Gal}(K/E) \subseteq \mathrm{Gal}(K/F) = G$. If θ denotes the trivial character of H, then $\mathrm{Ind}_H^G \theta$ is a transitive permutation representation, and so contains exactly one copy of the trivial character of G as direct summand. Thus we find that

$$\zeta_E(s) = L(s,\theta) = L(s, \mathrm{Ind}_H^G \theta) = \zeta_F(s) \prod_{\rho} L(\rho,s), \tag{43}$$

where ρ ranges over the collection of non-trivial irreducible direct summands of $\mathrm{Ind}_H^G \theta$. Thus, if Artin's conjecture is true, we find that $\zeta_E(s)/\zeta_F(s)$ is an entire function, or (more expressively), that $\zeta_F(s)$ *divides* $\zeta_E(s)$.

One approach to studying Artin's conjecture, initiated by Artin himself and further pursued by Brauer, is to attempt to express representations of G in terms of inductions of characters (i.e. 1-dimensional representations) of subgroups of G. This becomes a group-theoretic problem, via the study of which Brauer was led to prove [Bra47a] the following result.

5.1.3. **Theorem** (Brauer). *If ρ is a non-trivial irreducible representation of $G = \mathrm{Gal}(K/F)$, then $L(\rho,s)$ is a product of integral powers of Hecke L-functions* (*attached to Hecke characters of various fields intermediate between F and K*). *In* *particular, $L(\rho, s)$ has a* meromorphic *continuation to the entire complex plane, and satisfies a functional equation of the expected form.*

Artin had already proved the corresponding statement with rational rather than integer powers, thus obtaining an *a priori* multi-valued meromorphic continuation. The integral powers appearing in Brauer's theorem can be negative in general, and so it does not suffice to prove the *holomorphic* continuation that Artin conjectured. For particular (not necessarily irreducible) ρ, variants are possible; for example, in some contexts one can obtain powers that are positive rational numbers. Combined with Brauer's general theorem, this suffices to prove that $L(s, \rho)$ is entire. E.g. in this way Brauer proved [Bra47b] the divisibility discussed in (5.1.2) when E/F is already Galois.

5.2. **Epsilon factors.** There is a subtle but important point that arises with regard to the functional equation for $L(s, \rho)$ that comes out of Brauer's theorem. It indeed has the "expected" form (as formulated by Artin — it is a natural generalization of the form of the functional equation in the abelian case, involving Γ-factors for the archimedean places, and a power of the discriminant of K and also of the *conductor* of ρ; the notion of conductor having been formulated by Artin with exactly this purpose in mind). But the root number in the equation comes as a single quantity; it does not appear as a product of local factors as abelian root numbers do (via (23)).

It is natural to hope that such a factorization of the non-abelian root numbers should exist, especially if one anticipates that Artin L-functions, and Artin's conjecture, are to be explained by a non-abelian class field theory. Versions of this problem were studied by Hasse [Has54] and by Dwork [Dwo56], among others; and Dwork succeeded in constructing a local root number "up to a sign", in an appropriate sense.

The difficulties in studying this problem are two-fold. Firstly, there is the the problem of characterizing the local ϵ factors. One imagines that they should satisfy analogues of the properties (40) and (41); given this, one can follow Brauer's proof of Theorem 5.1.3 and show that these properties allow one to uniquely compute the local root numbers, assuming they exist. The second, and seemingly more serious, difficulty is that the product appearing in Brauer's theorem is not at all unique, and so correspondingly, the local root numbers whose computation we have just described are not at all obviously well-defined. In fact, if one adapts (41) in the most naive manner, it turns out the local root numbers can *only* be well-defined up to a sign, and this explains the limitation on Dwork's result.

The problem was subsequently taken up by Langlands, who realized that the key to finding the correct construction of the local root numbers was to adapt (41) in a more subtle manner.[40] With the correct characterization of the local ϵ factors in hand, Langlands proceeded, in his manuscript [Lanc], to show that they are well-defined. Again, this required dealing with the non-uniqueness in Brauer's theorem; the necessary calculations are famously elaborate, and make [Lanc] very long. By employing Langlands's characterization, Deligne soon after found a more indirect, but shorter, local-global argument that establishes the well-definedness [Del73].

[40]We don't go into the details here, but the key point is that local ϵ factors are literally induction invariant only for the induction of virtual representations of degree 0.

5.3. **Čebotarev.** We recall an important consequence of Brauer's Theorem 5.1.3. Namely, an appropriate generalization of Hadamard and de la Vallée Poussin's work shows that Hecke L-functions are zero-free on the closed half-plane $\Re s \geq 1$. Thus the ratio of Hecke L-functions appearing in Brauer's theorem is holomorphic in the same half-plane. This is enough to deduce an analogue of the prime number theorem, which in this context proves that as $\mathfrak{p}$ ranges over the primes in F (excluding the finitely many primes that ramify in K), the Frobenius elements $\mathrm{Frob}_{\mathfrak{p}}$ are equidistributed in $\mathrm{Gal}(K/F)$. More precisely, since $\mathrm{Frob}_{\mathfrak{p}}$ is only well-defined up to conjugacy, the asymptotic proportion of $\mathfrak{p}$ for which $\mathrm{Frob}_{\mathfrak{p}}$ lies in a given conjugacy class is proportional to the size of the class; in particular, there are an infinitude of $\mathfrak{p}$ for which $\mathrm{Frob}_{\mathfrak{p}}$ lies in a given conjugacy class. Note that if K/F is abelian, and thus contained in a ray class field, this reduces to Hecke's theorem regarding primes lying in a given ray class. (And the proof becomes Hecke's proof.)

This result was first proved by Čebotarev [Tsc26], using a different argument, building on earlier partial results of Frobenius (see e.g. the result mentioned in footnote 12 above for the simplest such precursor). Čebotarev's argument was the inspiration for Artin's proof of his reciprocity law.

5.4. **Motivic ζ-functions.** If we recall the definition of the Dedekind ζ-function of a number field K as a Dirichlet series $\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s}$, where $\mathfrak{a}$ runs over all non-zero ideals of $\mathcal{O}_K$, an obvious generalization suggests itself. Namely, if A is any ring (commutative, and with a unit), one could attempt to imitate this definition so as to obtain a ζ-function attached to A, by taking a sum over all *cofinite* ideals in A (i.e. ideals $\mathfrak{a}$ for which $A/\mathfrak{a}$ is finite; the point being that, in the number field case, $N(\mathfrak{a}) = |\mathcal{O}_K/\mathfrak{a}|$). However, there is a serious defect with this definition: ignoring for now the question of convergence of such a series (which would be related to bounding the number of ideals $\mathfrak{a}$ in A with $|A/\mathfrak{a}| = n$, as n grows), we see that, since there is no unique factorization of ideals into prime ideals in a general ring,[41] there is no reason in general for a ζ-function, so defined, to have an Euler product; and hence there is no reason (at least for our purposes) to imagine that this proposed definition is useful!

Indeed, we immediately abandon the momentary proposal of the preceding paragraph. Instead, we will insist that ζ-function of A (for suitable A) have an Euler product, and we will enforce this by definition! That is, we will imitate not the description of $\zeta_K(s)$ as a series, but its description as an Euler product. Any cofinite prime ideal in A is certainly maximal, and the natural condition to place on A (which will also lets us control the counting problem related to convergence) is that it be finitely generated as a $\mathbb{Z}$-algebra. Indeed, the Nullstellensatz then implies that any maximal ideal of A is cofinite, and so we define

$$\zeta_A(s) = \prod_{\mathfrak{p} \text{ maximal in } A} \left(1 - \frac{1}{|A/\mathfrak{p}|^s}\right)^{-1}. \tag{44}$$

[41]Essentially the only interesting and general context in which we have unique factorization of cofinite ideals into cofinite prime ideals is when A is a product of finitely many Dedekind domains which are of finite type over $\mathbb{Z}$. Working on each Dedekind domain factor separately, and so assuming that A *is* a Dedekind domain, we then find that either $A = \mathcal{O}_K[1/a]$ for some number field K and some $a \in K$, or that $\mathrm{Spec}\, A$ is a smooth curve over a finite field; we will discuss this latter case in the following subsection.

Any cofinite ideal of A has a positive residue characteristic, and so (assuming some half-plane of convergence) we may rearrange the product in (44) to write

$$\zeta_A(s) = \prod_{p \text{ prime}} \prod_{\substack{\mathfrak{p} \text{ max. in } A \\ \text{of res. char. } p}} \big(1 - \frac{1}{|A/\mathfrak{p}|^s}\big)^{-1}. \tag{45}$$

Each Euler factor in the product labelled by the prime number p is of the form $\big(1 - \frac{1}{p^{fs}}\big)^{-1}$, if $A/\mathfrak{p}$ is a finite field of order p^f. The difference between this general setting and the case when $A = \mathcal{O}_K$ is that there might be infinitely many such factors for each prime p!

There is an important geometric perspective on this construction: maximal ideals in A are the same as closed points in $\operatorname{Spec} A$. In this optic, we can generalize the preceding construction to any finite type scheme X over $\operatorname{Spec} \mathbb{Z}$. If X is such a scheme, write $\kappa(x)$ for the residue field of a point $x \in X$; then we can define[42]

$$\zeta_X(s) = \prod_{x \in X \text{ closed}} \big(1 - \frac{1}{|\kappa(x)|^s}\big)^{-1}. \tag{46}$$

Since the underlying topological space of X, and the residue fields at its closed points, are insensitive to any nilpotent elements in the structure sheaf of X, we may certainly assume that X is reduced when we analyze $\zeta_X(s)$. (In the affine case when $X = \operatorname{Spec} A$, this amounts to quotienting out A by its nilradical, and so supposing that A contains no non-zero nilpotents.) Also, if we write $X = Y \coprod Z$ as the disjoint union of two subschemes, then clearly

$$\zeta_X(s) = \zeta_Y(s)\zeta_Z(s). \tag{47}$$

Since any reduced scheme of finite type over $\operatorname{Spec} \mathbb{Z}$ may be stratified into a disjoint union of integral affine locally closed subschemes, in analyzing some of the coarser properties of $\zeta_X(s)$, e.g. its convergence properties, we may assume that $X = \operatorname{Spec} A$ with A an integral domain.

Momentarily suppose, then, that A is an integral domain of finite type over $\mathbb{Z}$. Either A is torsion free, in which case A contains maximal ideals of residue characteristic p for all but finitely many primes p, or A is of characteristic p for some fixed prime p. In the second case, a rough count of closed points on $\operatorname{Spec} A$ (e.g. via a Noether normalization to compare $\operatorname{Spec} A$ with an affine space of the same dimension) shows that $\zeta_A(s)$ converges if $\Re s > \dim A$. In the first case, we find that $\operatorname{Spec} A/p$ is either empty (if p is invertible in A, which happens only for a finite number of p) or of dimension equal to $\dim A - 1$. Inputting the estimates from our analysis of the second (i.e. positive characteristic) case into the product over (all but finitely many) p, we again find that $\zeta_A(s)$ converges if $\Re s > \dim A$. Working backwards through the stratification argument, we then find, for any scheme X of finite type over $\mathbb{Z}$, that $\zeta_X(s)$ converges if $\Re s > \dim X$.

[42] The author first learnt this definition from the note [Ser65] of Serre. The author's impression is that this definition developed via a complex and somewhat organic process over several decades, beginning with the function field analogue of Dedekind ζ-functions of quadratic field in Artin's thesis [Art24], progressing through the work of several others, especially Hasse and Weil, until by the 1960s, this emerged as the evidently natural definition. We strongly recommend [Ser65] to the reader, as well as Tate's article [Tat65], which immediately follows it in the same volume; our discussion here owes much to both papers.

Note that the preceding argument provides an important geometric interpretation of the first product (the product over p) in (45). If we consider the more general case of a finite type $\mathbb{Z}$-scheme X, then we may break up the product in (46) into a product over primes p of a product over all closed points x of residue characteristic p. Since these latter points coincide with the closed points in the base-change $X_{/\mathbb{F}_p}$, we find that

$$\zeta_X(s) = \prod_p \prod_{\substack{x \in X_{/\mathbb{F}_p} \\ \text{closed}}} \big(1 - \frac{1}{|\kappa(x)|^s}\big)^{-1} = \prod_p \zeta_{X_{/\mathbb{F}_p}}(s). \tag{48}$$

This suggests that in order to analyze $\zeta_X(s)$ for general X, we should first consider the case when X lies over a particular finite field, say $\mathbb{F}_p$.

5.4.1. *Remark.* How do these considerations connect to number theory? Well, a finite type $\mathbb{Z}$-algebra A is just a quotient

$$A = \mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$$

for some variables $x_1, \ldots, x_n$, and some polynomials f_i (with integer coefficients) in these variables. Maximal ideals of A, or equivalently, closed points of $X = \operatorname{Spec} A$, correspond (more or less — we will see the precise relationship soon) to solutions to the equations

$$f_1 = \cdots = f_n = 0 \tag{49}$$

over some finite field $\mathbb{F}_q$. Maximal ideals of fixed residue characteristic p then correspond to such solutions with q taken to run through the powers of some fixed p. So $\zeta_{X_{/\mathbb{F}_p}}(s)$ encodes information about solutions to the system of simultaneous Diophantine equations (49) modulo the given prime p, while $\zeta_X(s)$ then packages this information together as the prime p varies.

5.5. **ζ-functions for varieties over finite fields.** We very briefly recall the theory of ζ-functions of varieties over finite fields. This is an enormous topic in its own right, which is the subject of Weil's celebrated series of conjectures [Wei49] — now theorems of Dwork, Grothendieck (and his collaborators), and Deligne.

Suppose that X is of finite type over $\mathbb{F}_p$. We first recall the precise relationship between $\zeta_X(s)$ and counting solutions to Diophantine equations. If X is cut out by some equations, then giving an $\mathbb{F}_{p^n}$-valued solution to the equations cutting out X is the same as giving a morphism $\operatorname{Spec} \mathbb{F}_{p^n} \to X$, i.e. an $\mathbb{F}_{p^n}$-valued point of X — as usual, we denote this set of solutions, or equivalently, morphisms, by $X(\mathbb{F}_{p^n})$. Such a morphism has a closed image, say $x \in X$, and so determines, and is determined by, an embedding $k(x) \hookrightarrow \mathbb{F}_{p^n}$. Such an embedding exists if and only if $k(x) = \mathbb{F}_{p^m}$ for some $m \mid n$, and the number of such embeddings is then equal to m (the order of the automorphism group of $\mathbb{F}_{p^m}$). From these remarks, we compute that

$$\log \zeta_X(s) = \sum_{n=1}^{\infty} \frac{|X(\mathbb{F}_{p^n})|}{np^{ns}}. \tag{50}$$

5.5.1. *Example.* If $X = \mathbb{A}^d_{/\mathbb{F}_p}$ (affine d-space over $\mathbb{F}_p$), then (50) shows that

$$\zeta_{\mathbb{A}^d_{/\mathbb{F}_p}}(s) = \frac{1}{(1 - p^{d-s})}.$$

If $X = \mathbb{P}^d_{/\mathbb{F}_p}$ (projective d-space of $\mathbb{F}_p$), then writing $\mathbb{P}^d_{/\mathbb{F}_1}$ as a union of $\mathbb{A}^d_{/\mathbb{F}_p}$ and a copy of $\mathbb{P}^{d-1}_{/\mathbb{F}_p}$ at infinity in the usual way, taking into account (47), and arguing inductively, we find that

$$\zeta_{\mathbb{P}^d_{/\mathbb{F}_p}}(s) = \frac{1}{(1-p^{-s})(1-p^{1-s})\cdots(1-p^{d-s})}.$$

As we noted in footnote 41 above, if A is the ring of regular functions on a smooth affine curve over $\mathbb{F}_p$, then A is a Dedekind domain, and in this case the Euler product defining ζ_A can be expanded as a sum over non-zero ideals in A, and so it is in this case that ζ_A is most similar to a Dedekind ζ-function. In fact, it is more natural to consider $\zeta_X(s)$ for X a *complete* curve over $\mathbb{F}_p$ (this is analogous to forming the completed ζ-function by adding in the Γ-factors, etc., in the number field case). In the number field case, the completed ζ-function conjecturally has all its zeroes along the line $\Re s = 1/2$ (the Γ-factors absorb the trivial zeroes), and Artin [Art24] proposed the analogous conjecture for a complete curve X.

The case of elliptic curves was proved by Hasse [Has36] and the general case was proved by Weil.[43] In fact, Hasse (in the elliptic curve case), and Weil (for general curves), proved a much more precise statement, which we illustrate by recalling the elliptic curve case.

5.5.2. *Example.* If E is an elliptic curve over $\mathbb{F}_p$, then there are two algebraic numbers α and β both of absolute value $\sqrt{p}$, satisfying $\alpha\beta = p$, such that

$$|E(\mathbb{F}_{p^n})| = 1 + p^n - \alpha^n - \beta^n,$$

for any n. Thus

$$\zeta_E(s) = \frac{(1-\alpha p^{-s})(1-\beta p^{-s})}{(1-p^{-s})(1-p^{1-s})}.$$

Note that the Riemann Hypothesis for $\zeta_E(s)$ is encapsulated in the statement about the absolute values of α and β.

By examining the precise shape of $\zeta_X(s)$ in a range of examples for which it could be computed (such as each of the previous examples, the case of general curves, and the case of diagonal hypersurfaces — this last case being discussed in a very readable form in [Wei49]), Weil was led to make his eponymous conjectures on $\zeta_X(s)$ for a general smooth projective variety X over $\mathbb{F}_p$. These conjectures suggested the existence of a robust cohomology theory of such varieties, compatible in a suitable sense with the usual (that is, singular) cohomology of (the spaces consisting of the complex points of) varieties over $\mathbb{C}$. Such a cohomology theory was then constructed, in the form of Grothendieck's ℓ-adic cohomology, and, using it as a tool, Weil's conjectures were proved.[44] Rather than recapitulating Weil's well-known conjectures here,[45] we proceed straight to the upshot, by recalling the description of $\zeta_X(s)$ that comes out of the ℓ-adic theory.

[43]The precise history of and citations for Weil's multiple proofs are involved, and rather than giving them here, we refer the reader to Milne's recent excellent survey [Mil16] for a thorough discussion and references.

[44]The *rationality* part of the conjectures was in fact proved first by Dwork [Dwo60], using p-adic techniques. It was reproved by Grothendieck [AGV72] using ℓ-adic cohomology, as was the functional equation of $\zeta_X(s)$. Finally, the general Riemann Hypothesis was proved by Deligne [Del74].

[45]We again refer the reader to [Mil16] for more details. Appendix C of Hartshorne's textbook [Har77] also gives a short readable summary.

If X is a variety over a separably closed field k, then we can define its ℓ-adic cohomology groups $H^i(X, \mathbb{Q}_\ell)$ for any prime ℓ different from the characteristic of k (i.e. if we are in positive characteristic p, we require $\ell \neq p$). When $k = \mathbb{C}$, we recover singular cohomology with $\mathbb{Q}_\ell$ coefficients. If X is defined over a not-necessarily separably closed field k, and we let $\overline{X}$ denote the base-change of X to some separably closed field Ω containing k, then we may form the cohomology groups $H^i(\overline{X}, \mathbb{Q}_\ell)$. These are in fact independent of the choice of Ω, and so it is no loss of generality to take[46] $\Omega = \overline{k}$, a separable closure of k. The action of $\mathrm{Gal}(\overline{k}/k)$ on $\overline{X}$ then induces, by an appropriate functoriality, a continuous action of $\mathrm{Gal}(\overline{k}/k)$ on $H^i(\overline{X}, \mathbb{Q}_\ell)$.

Now if $k = \mathbb{F}_p$, then Frob_p generates $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$, and so if X is a variety over $\mathbb{Q}$, we obtain an action of Frob_p on each $H^i(\overline{X}, \mathbb{Q}_\ell)$. Now the $\mathbb{F}_{p^n}$ points of X are precisely the Frob_p^n-fixed points of $\overline{X}$, and so if X is projective, we may compute the fixed points of Frob_p^n via the ℓ-adic form of the Lefschetz fixed point formula.[47] The upshot is that[48]

$$\zeta_X(s) = \prod_{i=1}^{2d} \det\bigl((\mathrm{Id} - p^{-s}\mathrm{Frob}_p^{-1}) \text{ acting on } H^i(\overline{X}, \mathbb{Q}_\ell)\bigr)^{(-1)^{i-1}}; \tag{51}$$

here d denotes the dimension of X (or, equivalently, of $\overline{X}$, so that the cohomology lies in degrees between 0 and $2d$). Each factor in this product is a polynomial in p^{-s} (raised to a power ± 1), of degree equal to the dimension of the corresponding cohomology group. Thus this expression exhibits, for general X, the structure of $\zeta_X(s)$ that can be observed in the simple examples described above. In particular, from this formula one immediately obtains the meromorphic continuation of $\zeta_X(s)$.

If X is smooth as well as projective, then the cohomology of $\overline{X}$ satisfies Poincaré duality, and this gives rise to a functional equation for $\zeta_X(s)$. Finally, the *Riemann Hypothesis* in this context (i.e. for X smooth and projective over $\mathbb{F}_p$) states that the eigenvalues of Frob_p^{-1} on H^i are algebraic numbers all of whose conjugates have absolute value $p^{i/2}$. As Serre explained in his letter [Ser60] to Weil, this also admits a cohomological "explanation", by analogy with the Hodge theory of smooth projective complex varieties (the particular focus of the analogy being the definiteness of the polarization pairings on primitive cohomology that comes out

[46]The point of allowing more general Ω is that if, e.g., X is a variety over $\mathbb{Q}$, then we find that the cohomology of X base-changed to $\overline{\mathbb{Q}}$, which inherits a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action, coincides with the cohomology of X base-changed to $\mathbb{C}$, which can then be interpreted as singular cohomology.

[47]If X is not necessarily projective, then this still works, provided that we use compactly supported ℓ-adic cohomology. We also remark that Weil proposed this "Lefschetz formula" interpretation of his conjectures in his ICM address [Wei56a].

[48]The reason that Frob_p^{-1}, rather than Frob_p, appears in the formula is slightly technical: in order to apply the Lefschetz formula, we have to work with a morphism of varieties, *not* a Galois automorphism. Thus we work with the Frobenius endomorphism $F_{\overline{X}}$ of $\overline{X}$, defined by raising the coordinates of the a point to the pth power. If we compose this endomorphism with the Galois automorphism Frob_p of $\overline{X}$, we obtain the *absolute Frobenius* of $\overline{X}$ (defined simply by the map $f \mapsto f^p$ on the structure sheaf $\mathcal{O}_{\overline{X}}$). This latter morphism induces the identity on the underlying topological space of $\overline{X}$, and consequently acts via the identity on the cohomology of $\overline{X}$. Thus the traces of powers Frob_p^{-1} on cohomology coincide with the traces of powers of $F_{\overline{X}}$, and so it is Frob_p^{-1} that appears in the Lefschetz formula for $\zeta_X(s)$. Motivated by this relationship between Frob_p^{-1} and $F_{\overline{X}}$, the Galois element Frob_p^{-1} is referred to as the *geometric* Frobenius automorphism, in contrast to the *arithmetic* Frobenius Frob_p itself.

of the Hodge–Riemann bilinear relations). Grothendieck proposed an approach to proving the Riemann Hypothesis in the present context via a series of so-called "standard conjectures" [Kle94], which would allow one to make the same kind of definiteness arguments as in the context of complex varieties. (Such arguments were already employed by Hasse and Weil in their proofs of the Riemann Hypothesis in the case when X is a curve.) Unfortunately, these conjectures remain unproved in general, and Deligne [Del74] found a different approach to proving the Riemann Hypothesis.[49]

We close this discussion by pointing out that if X is smooth and projective, then the various factors

$$\det\big((\mathrm{Id} - p^{-s}\mathrm{Frob}_p^{-1}) \text{ acting on } H^i(\overline{X}, \mathbb{Q}_\ell)\big) \tag{52}$$

in (51) are mutually coprime as i varies, since the Riemann Hypothesis ensures that they have no roots in common. This implies that the factorization of $\zeta_X(s)$ given by the right-hand side of (51) is *intrinsic* to $\zeta_X(s)$, and can be constructed independently of the choice of $\ell \neq p$. In particular, the various determinants (52) have integer coefficients (as polynomials in p^{-s}), and are *independent* of the choice of ℓ!

5.6. **Hasse–Weil ζ- and L-functions.** Let us return now to the case of a finite type scheme over $\mathbb{Z}$, which we now assume to be flat over $\mathbb{Z}$, so that it does not lie in any particular residue characteristic p; the reader should have in mind an integral model of a variety over $\mathbb{Q}$ or some other number field. If we combine (48) with (51), we obtain the formula (in which d denotes the relative dimension of X over $\mathbb{Z}$)

$$\zeta_X(s) = \prod_p \prod_{i=1}^{2d} \det\big((\mathrm{Id} - p^{-s}\mathrm{Frob}_p^{-1}) \text{ acting on } H^i(\overline{X_{/\mathbb{F}_p}}, \mathbb{Q}_\ell)\big)^{(-1)^{i-1}}, \tag{53}$$

which expresses $\zeta_X(s)$ as an Euler product, where the Euler factors are now determined by the action of (geometric) Frobenius on the ℓ-adic cohomology of the various base-changes $X_{/\mathbb{F}_p}$. But this looks very similar[50] to Artin's definition (39) of his L-functions! It is also slightly inconsistent: for any particular p, we are free to choose any $\ell \neq p$ to compute the Euler factor at p; but since ℓ *is* a prime, and so appears as one of the indices in the product over p, there is no single choice of ℓ which works for every factor in the product (53)!

To further develop the observations just made, we now change notation. Namely, we now let X be a $\mathbb{Q}$-scheme, assumed to be smooth and projective, and we let $\mathcal{X}$ be a model of X over $\mathbb{Z}$, i.e. a projective flat $\mathbb{Z}$-scheme whose fibre over $\operatorname{Spec}\mathbb{Q}$ equals X. (We can obtain $\mathcal{X}$ just by suitably clearing denominators in some system of equations defining X; and $\mathcal{X}$ will now be the $\mathbb{Z}$-scheme that was previously denoted by X.) There is a finite set S of primes (the set of primes of *bad reduction*) such that $\mathcal{X}_{/\mathbb{F}_p}$ is smooth if $p \notin S$.

Now write $\overline{X}$ to denote the base-change of X to $\overline{\mathbb{Q}}$. If we choose a prime ℓ, then we may form the ℓ-adic cohomology $H^i(\overline{X}, \mathbb{Q}_\ell)$, with its natural $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action;

[49]There is still a certain "definiteness", or "positivity", that goes into Deligne's proof, but it is more elementary — it is, *grosso modo*, just the fact that the point counts that are encoded by $\zeta_X(s)$ are necessarily non-negative.

[50]The use of geometric Frobenius rather than arithmetic Frobenius can be thought of as simply a certain choice of normalization.

we are in characteristic zero, so no primes ℓ are excluded. General principles of ℓ-adic cohomology ("local acyclicity") imply that if $p \notin S$, and if $\ell \neq p$, then

$$H^i(\overline{X}, \mathbb{Q}_\ell) \cong H^i(\overline{\mathcal{X}_{/\mathbb{F}_p}}, \mathbb{Q}_\ell) \tag{54}$$

(where, as above, $\overline{\mathcal{X}_{/\mathbb{F}_p}}$ denotes the base-change of $\mathcal{X}_{/\mathbb{F}_p}$, the reduction mod p of $\mathcal{X}$, to $\overline{\mathbb{F}}_p$.) To make this identification canonical, we should choose a place of $\overline{\mathbb{Q}}$ above p; or equivalently, fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, up to the action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on the target; or, again equivalently, fix a choice of decomposition group D_p at p in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$; or, yet again equivalently, fix a choice of surjection from the integral closure of $\mathbb{Z}$ in $\overline{\mathbb{Q}}$ onto $\overline{\mathbb{F}}_p$. With this choice made, the isomorphism (54) is functorial, and so is equivariant for the action of the chosen decomposition group D_p, which acts on the left-hand side via restricting the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action to D_p, and acts on the right hand side via the surjection $D_p \to D_p/I_p = \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$, and the action of $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ on the right-hand side.

In particular, we find that the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on $H^i(\overline{X}, \mathbb{Q}_\ell)$ is unramified at $p \notin S \cup \{\ell\}$, and (for such p) we have

$$\begin{aligned} &\det\bigl((\mathrm{Id} - p^{-s}\mathrm{Frob}_p^{-1}) \text{ acting on } H^i(\overline{X}, \mathbb{Q}_\ell)\bigr)^{(-1)^{i-1}} \\ &\qquad = \det\bigl((\mathrm{Id} - p^{-s}\mathrm{Frob}_p^{-1}) \text{ acting on } H^i(\overline{\mathcal{X}_{/\mathbb{F}_p}}, \mathbb{Q}_\ell)\bigr)^{(-1)^{i-1}}. \end{aligned} \tag{55}$$

Finally, the "independence of ℓ" noted above shows that the determinant (55) is independent of ℓ, as long as we assume that $p \notin S$ and $\ell \neq p$.

The conclusion is that (for each i) we have a family of ℓ-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, namely the representations $H^i(\overline{X}, \mathbb{Q}_\ell)$, each of which is unramified outside $S \cup \{\ell\}$ (where S is fixed independently of ℓ), and for which the characteristic polynomial of Frobenius at p, and hence the Euler factor (55), is independent of the choice of $\ell \neq p$, for all $p \notin S$. We say that the $H^i(\overline{X}, \mathbb{Q}_\ell)$ form a *compatible family*, and we define the associated (partial) *Hasse–Weil* ζ-function via Artin's formula[51]

$$\zeta_X^S(s) = \prod_{p \notin S} \prod_{i=0}^{2d} \det\bigl((\mathrm{Id} - p^{-s}\mathrm{Frob}_p^{-1}) \text{ acting on } H^i(\overline{X}, \mathbb{Q}_\ell)\bigr)^{(-1)^{i-1}}, \tag{56}$$

where it is understood that to compute the factor at a given prime p, we choose any $\ell \neq p$. The superscript S indicates that we are forming an imprimitive ζ-function, in which we have omitted Euler factors at the finitely many bad primes in S. (We will return to this point below.)

The formula defining $\zeta_X^S(s)$ then suggests that we should follow Artin in defining an L-function for each of the individual compatible families of representations $H^i(\overline{X}, \mathbb{Q}_\ell)$, and indeed we can do just that, setting

$$L^S\bigl(s, H^i(X)\bigr) = \prod_{p \notin S} \det\bigl((\mathrm{Id} - p^{-s}\mathrm{Frob}_p^{-1}) \text{ acting on } H^i(\overline{X}, \mathbb{Q}_\ell)\bigr)^{-1}$$

with the same convention as above, namely that in computing the pth factor we choose any $\ell \neq p$. We then have

$$\zeta_X^S(s) = \prod_{i=0}^{2d} L^S\bigl(s, H^i(X)\bigr)^{(-1)^i}.$$

[51] Up to working with geometric rather than arithmetic Frobenius.

The Riemann Hypothesis gives an improvement on the half-plane of convergence for the individual L-functions $L^S\big(s, H^i(X)\big)$; namely, this L-function converges for $\Re s > 1 + \frac{i}{2}$.

How should we eliminate the imprimitivity of the L-functions $L^S\big(s, H^i(X)\big)$, or the ζ-function $\zeta_X^S(s)$? I.e., what Euler factors should we include for the missing primes $p \in S$? One would like to use Artin's recipe (forming the reciprocal characteristic polynomial of geometric Frobenius on the I_p-invariants). Unfortunately, the problem now arises as to whether or not this is independent of ℓ. This problem is related to the following one: our definition of $\zeta_X^S(s)$, and of the $L^S\big(s, H^i(X)\big)$, no longer uses the integral model $\mathcal{X}$! For example, it could be that for some primes in S, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $H^i(\overline{X}, \mathbb{Q}_\ell)$ happens to be unramified for some choice of ℓ; does this imply the same for other choices of ℓ? Even if it does, are the resulting characteristic polynomials of Frobenius independent of ℓ? One way this could happen is if X admitted a different integral model $\mathcal{X}'$ which *did* have good reduction at S. Then the answers to these questions would be "yes". But in general the answers to these questions aren't known (although the answers *are* still expected to be "yes").

The preceding question is closely related to the problem of finding "good" (e.g. semistable) integral models of smooth projective schemes over $\mathbb{Q}$, or over $\mathbb{Q}_p$ (since the ramification behaviour at p can be investigated locally at p). And this is related to problems such as resolution of singularities in positive characteristic, or in mixed characteristic (i.e. over $\mathbb{Z}_p$) — problems which also remain open. For example, the isomorphism (54) typically won't hold at primes of bad reduction, and so the question arises as to what the true relationship between the two sides is in that case; the theory of nearby and vanishing cycles is aimed at answering this question. But even for varieties over $\mathbb{F}_p$, if they are singular then we don't know independence of ℓ for their ℓ-adic cohomology in general, and without knowing this it doesn't seem reasonable to imagine we could prove an independence of ℓ result at a prime of bad reduction for a general smooth projective variety over $\mathbb{Q}$.

All that being said, one *does expect* that the compatible families of ℓ-adic representations $H^i(\overline{X}, \mathbb{Q}_\ell)$ *are* independent of $\ell \neq p$ locally at every prime p in a suitable sense, and in particular, that if we apply Artin's formula, we obtain a well-defined L-function

$$L\big(s, H^i(X)\big) = \prod_p \det\big((\mathrm{Id} - p^{-s}\mathrm{Frob}_p^{-1}) \text{ acting on } H^i(\overline{X}, \mathbb{Q}_\ell)^{I_p}\big)^{-1},$$

as well as a ζ-function

$$\zeta_X(s) = \prod_{i=0}^{2d} L\big(s, H^i(X)\big)^{(-1)^i}.$$

Independence of ℓ is known in many interesting cases, including the case when X is a curve, or an abelian variety. We refer to Serre's lecture [Ser70] for a more detailed discussion, as well as for a discussion of how to complete these functions at the archimedean place by adding an appropriate Γ-factor.

Once we have added the appropriate factors, we conjecture that $\zeta_X(s)$ will have an analytic continuation (this doesn't actually depend on the manner in which $\zeta_X(s)$ is completed at the $p \in S$ or at ∞, since the Euler factors that we add are themselves meromorphic functions) and a functional equation (and having a functional equation is very much dependent on completing in the correct way!).

More precisely, the functional equation will relate $\zeta_X(s)$ to $\zeta_X(d+1-s)$; in terms of the individual L-factors, it will relate $L\big(s, H^i(X)\big)$ to $L\big(1+d-s, H^{2d-i}\big)$; if one takes into account the cup product pairing of H^i and H^{2d-i} into H^{2d}, and the Galois-action on H^{2d}, this in fact becomes a functional equation relating $L\big(s, H^i(X)\big)$ to $L\big(1-s, H^i(X)^\vee\big)$, just as in the formalism of Artin L-functions. And just as is the case for Artin's L-functions, we anticipate that the global root number will be a product of local root numbers, which can be determined using the formalism of Langlands–Deligne discussed above.

5.7. **Motives.** When we write $L\big(s, H^i(X)\big)$, what "is" $H^i(X)$? For now, it is just a symbol standing for the compatible family of ℓ-adic $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representations

$$\{H^i\big(\overline{X}, \mathbb{Q}_\ell\big)\}.$$

Grothendieck's theory of motives is intended to supply a more substantive answer to this question.

Grothendieck's proposal is based on the fact that although ℓ-adic cohomology is a functor on the category of varieties (e.g. over $\mathbb{Q}$, or $\mathbb{F}_p$), it is functorial with respect to a larger class of morphisms then just the morphisms of varieties. Namely, because cohomology is valued in linear objects (vector spaces), multi-valued functions between varieties can induce single valued functions between their cohomology groups (by simply adding up the multiple values!). Now multi-valued functions between varieties are actually very natural objects to consider — they are just the correspondences. (The reader might think of the Hecke correspondences on modular curves, or if $f: C \to C'$ is a surjective morphism of curves of degree > 1, one can think of the "reflected" graph of f in $C \times C' \xrightarrow{\sim} C' \times C$ as providing a multivalued morphism $f^{-1}: C' \to C$.) So cohomology extends to a functor on the category of varieties, with suitable correspondences as morphisms. But we can also do spectral theory of operators on cohomology groups — decomposing them into eigenspaces. So we should also allow ourselves to decompose varieties into "eigenspaces" for the action of correspondences. Roughly speaking, this gives us the category of motives. But many technicalities arise — for example, two coresspondences that can be algebraically deformed, one into the other, will induce the same morphism on cohomology, and so we might want to identify them. Very quickly, in trying to make a precise definition and investigate its properties, one gets into thorny questions of equivalence relations on cycles. Grothendieck's standard conjectures [Kle94] are intended in part to resolve some of these questions, and thus to define a category — the category of so-called *Grothendieck motives*. The objects $H^i(X)$ are then supposed to be motives. (How would we cut them out as eigenspaces of correspondences? We saw the answer above, at least over $\mathbb{F}_p$, in the course of proving independence of ℓ: the characteristic polynomials of Frobenius on the different H^i are coprime polynomials, and so we can use the Frobenius endomorphism of X to cut X up into the various $H^i(X)$.)

Let us return momentarily to the context of Artin L-functions. If K is a number field, then $\operatorname{Spec} K$ is a zero-dimensional smooth $\mathbb{Q}$-scheme. If $F \subset K$ is a subfield for which K/F is Galois, then $\mathrm{Gal}(K/F)$ acts as automorphism of $\operatorname{Spec} K$, and elements in the group ring of $\mathrm{Gal}(K/F)$ induces correspondences on $\operatorname{Spec} K$. "Cutting up" $\operatorname{Spec} K$ using these correspondences amounts to decomposing $\operatorname{Spec} K$ in the category of motives. Thus the Galois representations appearing in the theory of Artin L-functions are examples of motives (0-dimensional motives). Note that there was no ℓ in that story; this is because H^0 (the only degree of cohomology involved in the 0-dimensional context) can be perfectly well defined with $\mathbb{Q}$-coefficients, so "independence of ℓ" is elementary for H^0.

Note, though, that not all $\mathrm{Gal}(K/F)$-representations will necessarily be defined over $\mathbb{Q}$. In the theory of Artin L-functions we used representations on $\mathbb{C}$-vector spaces, but since $\mathrm{Gal}(K/F)$ is a finite group, we could just as well have considered representations on $\overline{\mathbb{Q}}$-vector spaces. For higher degree cohomology, when we have to allow ℓ to vary, we should take cohomology with $\overline{\mathbb{Q}}_\ell$-coefficients, so as to allow finer motivic decompositions than can be obtained with $\mathbb{Q}_\ell$-coefficients. (One way to express this passage from $\mathbb{Q}_\ell$ to $\overline{\mathbb{Q}}_\ell$-coefficients is to say that we are allowing motives "with coefficients".)

Whatever a motive M is, then, it should have cohomology (however we cut up our original scheme X, we can correspondingly cut up its cohomology into the appropriate eigenspaces), and this cohomology will have a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action (as long as we are using correspondences defined over $\mathbb{Q}$), or at least a $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-action (if we use correspondences defined over some extension F of $\mathbb{Q}$, as we did above when we decomposed $\mathrm{Spec}\, K$ into its various irreducible Artin representations). We anticipate that, as ℓ varies, the cohomology of M will give compatible families of Galois representations.[52] Each motive will then have an associated L-function $L(s, M)$, which we again anticipate will admit an analytic continuation and functional equation (the latter relating $L(s, M)$ to $L(1-s, M^\vee)$ for a suitably understood dual[53] motive $M^\vee$, with a root number determined by the Langlands–Deligne theory).

We have written "whatever a motive is" above, because Grothendieck's standard conjectures remain unproved, and so the theory of motives is not complete. There are other standard conjectures relating cohomology and cycles (not in the sense of being on Grothendieck's list, but in the usual meaning of "standard"), such as the Hodge conjecture and the Tate conjecture [Tat65], which would also have to be solved to give anything like a complete theory of motives. (For example, one conjectures[54] that if X is smooth and projective over $\mathbb{Q}$, then the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on each $H^i(\overline{X}, \mathbb{Q}_\ell)$ is semisimple, and that furthermore each simple constituent is cut out by a suitable correspondence, i.e. is *motivic*. This would make the analogy between our general context and the original context of Artin L-functions particularly tight and compelling; but this semisimplicity conjecture is proved in very few cases.)

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[52]Meaning that there is a number field E independent of ℓ such that, for $\ell \neq p$, the characteristic polynomial of Frob_p has coefficients in E (thought of as lying inside $\overline{\mathbb{Q}}_\ell$), and is independent of ℓ.

[53]To obtain duals, there is an additional step that we should perform in the construction of the category of motives, namely "invert the Lefschetz motive", i.e. the motive $H^2(\mathbb{P}^1)$. If we do this, then in fact $H^{2d}(X)$ becomes an invertible motive for any smooth projective geometrically connected X of dimension d, and so the cup product pairings on cohomology can be used to construct dual motives.

[54]This conjecture is usually included under the general umbrella of "the Tate conjecture", although in [Tat65], Tate attributes it to Grothendieck and Serre. For a discussion of this conjecture and its relation to other conjectures regarding the Galois action on ℓ-adic cohomology, one can consult [Tat94].

5.8. **Compatible families of Galois representations.** Number theorists often circumvent the difficulties of rigorously defining/constructing motives by working directly with compatible families of ℓ-adic representations. Since this notion was introduced by Taniyama [Tan57], it has evolved into a fundamental concept, which provides a practical and flexible replacement for the notion of motive. Being inherently linear, Galois representations are easier to manipulate than varieties!

One technical difficulty in working with compatible families, such as the families $\{H^i(\overline{X}, \mathbb{Q}_\ell)\}$ considered above, is ensuring compatibility at *all* primes p (i.e. including the bad ones). Although, as already noted, arithmetic geometry is not currently advanced enough to prove this directly in all cases of interest, in practice, there are many compatible families for which one *does* have control at all primes, and hence for which one can define complete L-functions unambiguously. As we already mentioned, the families $\{H^1(\overline{X}, \mathbb{Q}_\ell\}$ for a smooth projective curve X over a number field give examples of such families.

In Subsection 4.6 we gave examples of compatible families of ℓ-adic representations $\{\widetilde{\rho}_\ell\}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ (for some number field F), constructed from algebraic Hecke characters ψ on some extension K of F. These particular compatible families are in fact motivic. Indeed, the compatible families of characters $\{\widetilde{\psi}_\ell\}$ from Subsection 4.5 and 4.6 are associated to motives over K that can be constructed from abelian varieties with complex multiplication. (This goes back to Taniyama [Tan57] and [Ser68].) The group-theoretic induction that constructs $\{\widetilde{\rho}_\ell\}$ from $\{\widetilde{\psi}_\ell\}$ then corresponds in the world of varieties and motives to restriction of scalars from K to F. Thus the L-functions $L(s, \{\widetilde{\rho}_\ell\})$ are motivic L-functions. In this case, the induction formalism for L-functions (41), together with (35), shows that

$$L(s, \{\widetilde{\rho}_\ell\}) = L(s, \psi), \tag{57}$$

and so the L-functions $L(s, \{\widetilde{\rho}_\ell\})$ are also automorphic[55] (in this particular instance they are Hecke L-functions). In particular, these L-functions *do* admit analytic continuations and satisfy functional equations.

5.9. **The Sato–Tate conjecture, unitary normalization, and the Langlands group.** We begin by considering a specific, historically important, class of examples: elliptic curves E over $\mathbb{Q}$.

Suppose that E has good reduction outside the finite set S of primes; we abuse notation by writing simply $E_{/\mathbb{F}_p}$ for the reduction modulo $p \notin S$. For each $p \notin S$, we see from (5.5.2) that

$$\zeta_{E_{/\mathbb{F}_p}}(s) = \frac{(1-\alpha_p p^{-s})(1-\beta_p p^{-s})}{(1-p^{-s})(1-p^{1-s})},$$

where $|\alpha_p| = |\beta_p| = \sqrt{p}$, and $\alpha_p \beta_p = p$. The numerator arises from H^1, and the factors in the denominator from H^0 and H^2. We deduce that

$$L\big(s, H^0(E)\big) = \zeta(s) \quad \text{ and } L\big(s, H^2(E)\big) = \zeta(s-1)$$

(strictly speaking we haven't computed the missing Euler factors at $p \in S$, but completing to the Riemann ζ-function and its translate in this way are the only

[55] Automorphic *over* K, since we obtain Hecke L-functions for K. The theory of *automorphic induction*, briefly recalled in Subsection 7.3 below, shows that they are also automorphic over F.

possibilities compatible with a functional equation), while

$$L^S\big(s, H^1(E)\big) = \prod_{p \notin S} \frac{1}{(1-\alpha_p p^{-s})(1-\beta_p p^{-s})} = \prod_{p \notin S} \frac{1}{1 - a_p p^{-s} + p^{1-2s}},$$

where

$$a_p = \alpha_p + \beta_p = 1 + p - |E_{/\mathbb{F}_p}(\mathbb{F}_p)|.$$

The condition $|\alpha_p| = |\beta_p| = \sqrt{p}$ implies (indeed, is equivalent to) the condition

$$a_p \in [-2\sqrt{p}, 2\sqrt{p}]; \tag{58}$$

and both conditions are equivalent to the characteristic polynomial of Frob_p^{-1} on $H^1(\overline{E}, \mathbb{Q}_\ell)$ — which equals $X^2 - a_p X + p$ — having non-positive discriminant.[56]

Our next immediate goal is to discuss the *Sato–Tate* conjecture, which is a conjecture describing the distribution of the quantities a_p as p varies. But we also intend to explain how it relates to a more far-reaching idea, that of the *Langlands group*.

To begin with, a consideration of (58) suggests that we rescale the a_p, and consider (for each $p \notin S$) the quantity

$$A_p = \frac{a_p}{\sqrt{p}} \in [-2, 2];$$

the quantities A_p now lie in the same interval, as p varies, and so it makes sense to discuss their distribution. It turns out to be convenient to write $A_p = 2\cos\theta_p$, where $\theta_p \in [0, \pi]$. (Then, up to switching the labelling, we have $\alpha_p = \sqrt{p}e^{i\theta_p}$ and $\beta_p = \sqrt{p}e^{-i\theta_p}$.)

If E admits CM by an imaginary quadratic field K, then it is well-known (and easily seen) that $a_p = 0$ if p is inert in K (which happens half the time, by Dirichlet), so that $\theta_p = \pi/2$ for all such p; while for the p that are split in K, one finds that θ_p is distributed uniformly throughout $[0, \pi]$.

If E does not admit CM, then the Sato–Tate conjecture[57] states that θ_p is distributed over $[0, \pi]$ according to the measure $\dfrac{2}{\pi}\sin^2\theta d\theta$.

Before we try to explain the meaning of these results, we begin by describing a more structural interpretation of the above rescaling by $p^{-1/2}$. If we let $\{\chi_\ell\}$ denote the compatible family of cyclotomic characters (see example (4.5.2)), then $\chi(\mathrm{Frob}_p^{-1}) = p^{-1}$. Suppose for a moment that we could form a compatible family $\{\chi_\ell^{1/2}\}$, with the property that $\chi_\ell^{1/2}(\mathrm{Frob}_p^{-1}) = p^{-1/2}$ for each $p \neq \ell$. Then we could form the twisted compatible family $H^1(E) \otimes \chi^{1/2}$, and for a given $p \notin S$ (working with $\ell \neq p$) we would find that the characteristic polynomial of Frob_p^{-1} on this twist is

$$X^2 - A_p X + 1. \tag{59}$$

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[56]The ℓ-adic H^1 of E is dual to the ℓ-adic Tate module of E, and so this polynomial is also the characteristic polynomial of Frob_p on the ℓ-adic Tate module of E; which is how it is often described in discussions of elliptic curves.

[57] Now a theorem for elliptic curves over any totally real of CM number field. Over totally real fields, the proof is due to a large number of people, including Clozel–Harris–Taylor [CHT08], Harris–Shepherd-Barron–Taylor [HSBT10], Taylor [Tay08], Shin [Shi11], Clozel–Harris–Labesse–Ngô [CHLN11], and Barnet-Lamb–Geraghty–Harris–Taylor [BLGHT11]. The recent extension [ACC+18] to the case of CM fields is discussed in Subsection 7.3 below.

Now unfortunately, the characters χ_ℓ don't admit square roots. (For example, complex conjugation is an element $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ of order 2 for which $\chi_\ell(c) = -1$.) On the other hand, the compatible family $\{\chi_\ell\}$ arises from the absolute value character $|\ |$ on $\mathbb{Q}^\times \setminus \mathbb{A}_{\mathbb{Q}}^\times$, and this character evidently admits a square root, namely $|\ |^{1/2}$.

This prompts the question: is there a process via which we can "unravel" the various two-dimensional ℓ-adic representations $H^1(\overline{E}, \mathbb{Q}_\ell)$, so as to obtain a single two-dimensional representation, over the complex numbers, of some group — in the way that we can "undo" the passage from $|\ |$ to $\{\chi_\ell\}$, at the cost of passing from $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}) = \pi_0(\mathbb{Q}^\times \setminus \mathbb{A}_{\mathbb{Q}}^\times)$ to the group $\mathbb{Q}^\times \setminus \mathbb{A}_{\mathbb{Q}}^\times$ itself? If so, then we could twist this unravelled representation by $|\ |^{1/2}$, and obtain a representation which (we might hope) would give a conceptual sense to the rescaled characteristic polynomials (59).

5.9.1. *Example.* Consider the elliptic curve E cut out by the Weierstrass equation

$$y^2 = x^3 - x.$$

This is an elliptic curve with complex multiplication by $\mathbb{Z}[i]$, defined over $\mathbb{Q}(i)$, via the formula

$$[i](x, y) = (-x, iy).$$

This action makes each $H^1(\overline{E}, \mathbb{Q}_\ell)$ free of rank one over $\mathbb{Q}_\ell \otimes_{\mathbb{Q}} \mathbb{Q}(i)$, so that the $\mathrm{Gal}\big(\overline{\mathbb{Q}}/\mathbb{Q}(i)\big)$-action on $H^1(\overline{E}, \mathbb{Q}_\ell)$ (which commutes with the $\mathbb{Q}_\ell \otimes_{\mathbb{Q}} \mathbb{Q}(i)$-action) is abelian. In fact, this action is given by the character $\widetilde{\psi}_\ell$, where $\widetilde{\psi}_\ell$ is constructed from the (algebraic!) Hecke character ψ which is *inverse* to the character of examples (3.2.2) and (3.3.2). It is then easy to see that the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action on $H^1(\overline{E}, \mathbb{Q}_\ell)$ is given by the two-dimensional representation $\widetilde{\rho}_\ell$ obtained by inducing $\widetilde{\psi}_\ell$.

In this case, an "unravelling" of the desired type is possible: we have the Weil-group representation $\rho : W_{\mathbb{Q}(i)/\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ obtained by inducing ψ, as in Subsection 4.6, which we can then twist to form

$$\rho \otimes |\ |^{1/2} = \mathrm{Ind}_{\mathbb{Q}(i)^\times \setminus \mathbb{A}_{\mathbb{Q}(i)}^\times}^{W_{\mathbb{Q}(i)/\mathbb{Q}}} (\psi \otimes |\ |^{1/2}).$$

Thus $\rho \otimes |\ |^{1/2}$ is also obtained by the induction construction of Subsection 4.6, but now applied to $\psi \otimes |\ |^{1/2}$, which is the unitarization of ψ (see (3.2.2)). Thus $\rho \otimes |\ |^{1/2}$ is a *unitary* representation of $W_{\mathbb{Q}(i)/\mathbb{Q}}$, and we see that the polynomials (59) are the characteristic polynomials of geometric Frobenius elements[58] in $W_{\mathbb{Q}(i)/\mathbb{Q}}$. Thus they are the characteristic polynomials of *unitary* 2×2-matrices — which explains why their roots have the form $e^{\pm i\theta_p}$!

The result about the distribution of the A_p is then seen to be a result about the distribution of the images $\rho(\mathrm{Frob}_p^{-1})$, and so is a certain kind of generalization of Čebotarev density.

On the other hand, if E is *not* a CM elliptic curve, then Serre showed in [Ser68] that the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $H^1(\overline{E}, \mathbb{Q}_\ell)$ is irreducible, and remains so after restriction to any open subgroup $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Thus ρ_ℓ is *not* induced

[58] We don't recall the details of the construction of these elements right now; it is part of the local-global compatibility theory for Weil groups, which is discussed in the following subsection.

from a character, and so we cannot describe the compatible family $\{\rho_\ell\}$ in terms of a Weil group representation ρ.

However, we have the following conjecture. (We remind the reader that we introduced the absolute Weil group W_F of F in Subsection 4.6 above.)

5.9.2. **Conjecture** (Existence of the Langlands group). *For any number field F, there is a locally compact topological group L_F, the* Langlands group *of F, admitting a surjection $L_F \to W_F$ with compact kernel and for which the induced morphism $L_F^{\mathrm{ab}} \to W_F^{\mathrm{ab}} = F^\times \setminus \mathbb{A}_F^\times$ is an isomorphism, and such that compatible families of n-dimensional ℓ-adic Galois representations $\{\rho_\ell\}$ arising from motives arise* (*in a suitable sense*) *from certain* (*"algebraic"*) *complex representations $L_F \to \mathrm{GL}_n(\mathbb{C})$.*

This conjecture is somewhat informal as stated, but we will make it more precise in the following subsection. For now, we just remark that the notion of "algebraic", for representations of L_F, should be analogous to the corresponding notion for Hecke characters, and that the mechanism which produces the compatible family $\{\rho_\ell\}$ from ρ should be (at least formally) analogous to the mechanism which produces the compatible family $\{\widetilde{\psi}_\ell\}$ from an algebraic Hecke character ψ.

If we admit this conjecture, then we find, for any elliptic curve, that the family of representations $H^1(\overline{E}, \mathbb{Q}_\ell)$ arises from a representation $L_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$. If we let ρ denote the twist of this latter representation by $|\ |^{1/2}$ (this twist now makes sense!), then ρ in fact factors through $SU(2) \subset \mathrm{GL}_2(\mathbb{C})$. Indeed its determinant is trivial, as each characteristic polynomial (59) has constant term 1, and since L_F is compact outside of its $\mathbb{R}_{>0}^\times$ part coming from W_F, we see that the image is compact with trivial determinant, and so can be factored through $SU(2)$ after an appropriate choice of basis. Now the image is a compact subgroup, and there aren't many possibilities for it: either a copy of $U(1)$; the normalizer of a copy of $U(1)$ in $SU(2)$ (which is a non-split extension of a group of order 2 by $U(1)$, with the non-trivial element in the group of order 2 acting on $U(1)$ via inversion); or $SU(2)$ itself. The first possibility can't occur, since the representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $H^1(\overline{E}, \mathbb{Q}_\ell)$ aren't abelian, and so one of the latter two possibilities must occur.

In the second case, restricting to an index two open subgroup of $L_{\mathbb{Q}}$, i.e. to L_K for some quadratic extension of K, yields an abelian representation. This is evidently the CM case. Otherwise, we must be in the third case. And now we see the meaning of the results on the distributions of the θ_p: what they are saying is that $\rho(\mathrm{Frob}_p^{-1})$ is equidistributed in the image of ρ, as p ranges over all primes $\notin S$. Just as in the case of the Čebotarev's theorem, we must remember that Frob_p^{-1} is only defined up to conjugacy, so we have to understand this equidistribution in the framework of conjugacy classes.

The normalizer of $U(1)$ has two connected components. The non-identity component is a single conjugacy class. The other component is just $U(1)$ itself, and the conjugacy classes are just the pairs $\{z, \overline{z}\}$ for $z \in U(1)$, which can be parameterized by the elements $z = e^{i\theta}$, with $\theta \in [0, \pi]$. Thus, when E has CM, we see that half the $\rho(\mathrm{Frob}_p^{-1})$ land in the non-identity component — these are the half of the primes for which $a_p = A_p = 0$. And the other half of the primes satisfy $\mathrm{Frob}_p^{-1} \in U(1)$, and θ_p is equidistributed in $[0, \pi]$.

The conjugacy classes in $SU(2)$ are again parameterized by their pairs of eigenvalues $\{z, \overline{z}\}$, and hence by $z = e^{i\theta}$ with $\theta \in [0, \pi]$. But the uniform (i.e. Haar) measure on $SU(2)$ doesn't induce the uniform measure on $[0, \pi]$. Rather, it induces the Sato–Tate measure $\frac{2}{\pi}\sin^2\theta d\theta$. And the Sato–Tate conjecture affirms that if E does not have CM, then the θ_p are indeed equidistributed according to this measure.

How does one *prove* this equidistribution? It is a density theorem, and the proof proceeds like the proof of all such theorems, via an appropriate Fourier analysis facilitated by a result on the non-vanishing of appropriate L-functions. Consider first the example of (5.9.1). For primes p that are split in $\mathbb{Q}(i)$, say $p = \pi\overline{\pi}$, with $\pi \equiv 1 \bmod (1+i)^3$, the corresponding conjugacy class in $U(1)$ is simply $(\pi/\sqrt{p}, \overline{\pi}/\sqrt{p})$ (see the definition of ψ in Example (3.2.2)), and so the claimed equidistribution amounts to the equidistribution of Gaussian primes in angular sectors that was already mentioned in Subsection 3.5.

What about the non-CM case? Then we have to perform a Fourier analysis of the conjugacy classes in $SU(2)$. To expand the indicator function of a conjugacy class in terms of traces of representations, we have to use not just the character of the standard two-dimensional representation of $SU(2)$, but the characters of *all* of its irreducible representations. These are obtained by taking the symmetric powers $\mathrm{Sym}^n\rho$ of the two-dimensional representation ρ. Now $\mathrm{Sym}^n\rho$ is a obtained as a direct summand of $\rho^{\otimes n}$ by writing it as the image of a certain symmetrization projector, and since ρ arises from $H^1(E)$ by a twist, the Künneth theorem assures us that $\rho^{\otimes n}$, and hence also $\mathrm{Sym}^n\rho$, will be cut out of (some twist of) $H^n(E^n)$ by some correspondence. Thus $\mathrm{Sym}^n\rho$ is motivic. So what we need to know, to prove the Sato–Tate conjecture, is that each of the motivic L-functions $L(s, \mathrm{Sym}^n\rho)$ has an analytic continuation up to $\Re s \geq 1$, with no zeroes or poles in this region.

Now, although the representation ρ is only hypothetical, its L-function is not. Indeed, since hypothetically $\rho = H^1(E)\otimes |\ |^{1/2}$, we see that (again, hypothetically)[59] $\mathrm{Sym}^n\rho = \mathrm{Sym}^n H^1(E)\otimes |\ |^{n/2}$. Thus

$$L(s, \mathrm{Sym}^n\rho) = L\big(s, \mathrm{Sym}^n H^1(E)\otimes |\ |^{n/2}\big) = L\big(s+\frac{n}{2}, \mathrm{Sym}^n H^1(E)\big),$$

and this equation gives the left-hand L-function a meaning, even if ρ is only hypothetical. Furthermore, the Riemann Hypothesis ensures that the Euler product defining $L\big(s, \mathrm{Sym}^n H^1(E)\big)$ converges if $\Re s > 1+\frac{n}{2}$, and so we see that the Euler product defining $L(s, \mathrm{Sym}^n\rho)$ converges if $\Re s > 1$ — so that L-functions of the unitary Langlands group representation $\mathrm{Sym}^n\rho$ behave formally just like the L-functions of unitary Weil group representations, or unitary Hecke characters: the Euler products converge if $\Re s > 1$ — and our problem is to analytically continue them to the line $\Re s = 1$.

For Artin L-functions, this was achieved by Brauer's Theorem 5.1.3. We will briefly recall in Subsection 7.3 below that Brauer's theorem plays a role in achieving the analogous result here as well.

We remark that in general, for any smooth projective $\mathbb{Q}$-scheme X (or, more generally, any motive), we may hypothetically form $H^i(X)\otimes |\ |^{i/2}$ as a representation of the Langlands group $L_{\mathbb{Q}}$, and the Riemann Hypothesis will ensure that this representation, if it existed, would be unitary. The Riemann Hypothesis also ensures that

$$L\big(s, H^i(X)\otimes |\ |^{i/2}\big) = L\big(s+\frac{i}{2}, H^i(X)\big)$$

[59]Now $H^1(E)$ just stands for the compatible family $\{H^1(\overline{E}, \mathbb{Q}_\ell)\}$, and we can simply form its symmetric power to obtain a compatible family $\mathrm{Sym}^n H^1(E)$, given by the representations $\{\mathrm{Sym}^n H^1(\overline{E}, \mathbb{Q}_\ell)\}$.

converges if $s > 1$. The left-hand L-function is sometimes said to be *unitarily* normalized. This can be thought of[60] as referring to the hypothetical unitary representation $H^i(X) \otimes |\ |^{i/2}$.

5.10. **The conjectural Langlands group — an elaboration.** We first recall the definition of the local Weil groups. They come in two flavours, archimedean and non-archimedean. The archimedean local Weil groups are the easiest to describe: $W_{\mathbb{C}} = \mathbb{C}^\times$, while $W_{\mathbb{R}}$ is the unique non-split extension of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ by $\mathbb{C}^\times$ (with $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ acting on $\mathbb{C}^\times$ via complex conjugation).[61]

In analogy with the short exact sequence (36) (and thinking of $W_{\mathbb{C}}$ as $W_{\mathbb{C}/\mathbb{C}}$ — since $\mathbb{C}$ has no non-trivial finite extensions — and of $W_{\mathbb{R}}$ as $W_{\mathbb{C}/\mathbb{R}}$ — since $\mathbb{C}$ is the algebraic closure of $\mathbb{R}$), we may place these Weil groups in short exact sequences

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow W_{\mathbb{C}} \longrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{C}) = 1 \tag{60}$$

and

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow W_{\mathbb{R}} \longrightarrow \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \longrightarrow 1\ . \tag{61}$$

Of course, $W_{\mathbb{C}}^{\mathrm{ab}} = W_{\mathbb{C}/\mathbb{C}} = W_{\mathbb{C}} = \mathbb{C}^\times$, while $W_{\mathbb{R}}^{\mathrm{ab}} = W_{\mathbb{R}/\mathbb{R}} = \mathbb{R}^\times$, the identification with $\mathbb{R}^\times$ being uniquely determined by the requirement that it induces the morphism $z \mapsto z\overline{z}$ when pulled back to $\mathbb{C}^\times \subset W_{\mathbb{R}}$.

If F_v is a non-archimedean local field (which we denote in this fashion in anticipation of the fact that it will arise by completing a global field F at some non-archimedean place v), with residue field $\mathbb{F}_q$, then there there is a short exact sequence of groups

$$1 \to I_v \to \mathrm{Gal}(\overline{F}_v/F_v) \to \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \to 1,$$

where I_v denotes the inertia subgroup of $\mathrm{Gal}(\overline{F}_v/F_v)$. The Galois theory of finite fields shows that the group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is isomorphic to $\widehat{\mathbb{Z}}$, with topological generator the Frobenius element $\mathrm{Frob}_v = \mathrm{Frob}_q$ (the automorphism of $\overline{\mathbb{F}}_q$ defined by $x \mapsto x^q$). Inside this Galois group we may consider the cyclic subgroup (just a copy of $\mathbb{Z}$) that is literally generated by Frob_v. Then the Weil group W_{F_v} is the preimage of this cyclic group in $\mathrm{Gal}(\overline{F}_v/F_v)$, so that W_{F_v} sits in a short exact sequence

$$1 \to I_v \to W_{F_v} \to \langle \mathrm{Frob}_v \rangle \to 1.$$

We recall that if K_v is a finite extension of F_v, then $\mathrm{Gal}(\overline{F}_v/K_v)^c$ (the closure of the commutator subgroup of $\mathrm{Gal}(\overline{F}_v/K_v)$) is contained in I_v, and so we may set $W_{K_v/F_v} = W_{F_v}/\,\mathrm{Gal}(\overline{F}_v/K_v)^c$. Local class field theory then yields a morphism of exact sequences analogous to (36)

(62)

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_v^\times & \longrightarrow & W_{K_v/F_v} & \longrightarrow & \mathrm{Gal}(K_v/F_v) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathrm{Gal}(K_v^{\mathrm{ab}}/K_v) & \longrightarrow & \mathrm{Gal}(\overline{F}_v/F_v)/\,\mathrm{Gal}(\overline{F}_v/K_v)^c & \longrightarrow & \mathrm{Gal}(K_v/F_v) & \longrightarrow & 1 \end{array}$$

[60]Although it might be historically more accurate to say that this name is derived from its meaning on the automorphic side of reciprocity, where the corresponding process amounts to twisting so as to obtain a unitary central character.

[61]If we regard $\mathbb{R}^\times_{>0}$ as a subgroup of scalar matrices in $\mathrm{GL}_2(\mathbb{C})$, then $\mathbb{C}^\times = \mathbb{R}^\times_{>0}U(1)$ embeds into $\mathbb{R}^\times_{>0}SU(2)$, and $W_{\mathbb{R}}$ can also be described as the normalizer of $\mathbb{C}^\times$ in $\mathbb{R}^\times_{>0}SU(2)$.

and of course $W_{F_v} = \varprojlim_{K_v} W_{K_v/F_v}$.

If F_v is a completion of the number field F at a place v, then there is a morphism[62] $W_{F_v} \to W_F$ compatible with the short exact sequences (36) and whichever choice of (60), (61), or (62) is appropriate to v.

We may also define local versions of the Langlands group. If v is archimedean, then $L_{F_v} = W_{F_v}$. If v is non-archimedean, then $L_{F_v} = W_{F_v} \times SU(2)$. (From the viewpoint of Conjecture 5.9.2, relating the hypothetical global Langlands group to compatible families of ℓ-adic Galois representations, the role of the $SU(2)$-factor is to account for the potentially infinite-order action of tame inertia on ℓ-adic representations.) In either case, we have a surjection $L_{F_v} \to W_{F_v}$.

We now elaborate on Conjecture 5.9.2. We conjecture the existence of a locally compact group L_F, the Langlands group of F, equipped with a surjection $L_F \to W_F$ whose kernel is compact and which induces an isomorphism

$$L_F^{\mathrm{ab}} \to W_F^{\mathrm{ab}} = W_{F/F} = F^\times \setminus \mathbb{A}_F^\times,$$

and also equipped with morphisms $L_{F_v} \to L_F$ for each place v that are compatible with the maps to and between local and global Weil groups. Granting this, if $\rho : L_F \to \mathrm{GL}_n(\mathbb{C})$ is a continuous representation, then we may pull-back ρ to a morphism $\rho_v : W_{F_v} = L_{F_v} \to \mathrm{GL}_n(\mathbb{C})$ for each archimedean place v of F. We say that ρ is *algebraic* if each of these pull-backs ρ_v becomes algebraic when restricted to the copy of $\mathbb{C}^\times \subset W_{F_v}$; i.e. if the induced morphism $\mathbb{C}^\times \to \mathrm{GL}_n(\mathbb{C})$ is a product of n algebraic characters of $\mathbb{C}^\times$, i.e. characters of the form $z \mapsto z^p \overline{z}^q$ for some $p, q \in \mathbb{Z}$.

We now conjecture that algebraic representations of L_F correspond to motivic compatible families of representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$. To make this precise, we follow Langlands [Lan79] in using the language of Tannakian categories.[63] We can consider the category of algebraic representations of L_F; this will be a Tannakian category, and we can form its Tannakian Galois group, a pro-algebraic group over $\mathbb{C}$, which we denote by L_F^{alg}. Assuming that Grothendieck's category of motives exists, we may also form its Tannakian group G_F^{mot}; this is a pro-algebraic group over $\mathbb{Q}$, or over $\overline{\mathbb{Q}}$ if we allow our motives to have coefficients. The existence of compatible families of Galois representations can be expressed in terms of the existence of morphisms $\mathrm{Gal}(\overline{\mathbb{Q}}/F) \to G_F^{\mathrm{mot}}(\overline{\mathbb{Q}}_\ell)$ (one for each ℓ and each choice of embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$) having certain properties. A more precise formulation of Conjecture 5.9.2, then, is that there is an isomorphism of pro-algebraic groups over $\mathbb{C}$

$$L_F^{\mathrm{alg}} \xrightarrow{\sim} (G_F^{\mathrm{mot}})_{/\mathbb{C}}.$$

This isomorphism should again satisfy various compatibilities, whose details we omit. (For example, it should be compatible with the construction of Subsection 4.5.)

This more precise form of Conjecture 5.9.2 is the top layer of a rather tall stack of conjectures, since its formulation depends on the existence of a category of motives satisfying a range of desirable, but so far unproved, properties. Nevertheless, in his work (e.g. [Lan79] and [LR87]) Langlands has consistently shown the value of substantively engaging with this circle of ideas, and of incorporating them into one's

[62] To determine this unambiguously, we should extend the embedding $F \hookrightarrow F_v$ to an embedding $\overline{\mathbb{Q}} = \overline{F} \hookrightarrow \overline{F}_v$.

[63] For abelian representations, this viewpoint was already introduced in [Ser68].

thinking. For this reason, we believe it is worthwhile, and in keeping with Langlands's own point-of-view, to place Conjecture 5.9.2 at the centre of our discussion of reciprocity.

6. Automorphic L-functions and reciprocity

6.1. **Automorphic representations.** Following the definition in Hecke [Hec27] (or Serre [Ser77] for a more contemporary reference), we may regard a modular form of weight k and level 1 as a function

$$f : \{\Lambda \mid \Lambda \subset \mathbb{C} \text{ a lattice } \} \to \mathbb{C}$$

which satisfies the condition $f(\lambda\Lambda) = \lambda^{-k} f(\Lambda)$ for any $\lambda \in \mathbb{C}^\times$, which is holomorphic as a function of those τ for which $\Im\tau > 0$, if we associate to τ the lattice $\mathbb{Z} + \mathbb{Z}\tau$, and which satisfies a moderate growth condition as $\tau \to i\infty$.

Now choosing a *based* lattice in $\mathbb{C} \cong \mathbb{R}^2$ is the same as choosing a basis of $\mathbb{R}^2$, i.e. an element of $\mathrm{GL}_2(\mathbb{R})$. Changing basis amounts to applying an element of $\mathrm{GL}_2(\mathbb{Z})$. So modular forms of weight k and level 1 are functions on the quotient

$$\mathrm{GL}_2(\mathbb{Z}) \setminus \mathrm{GL}_2(\mathbb{R}) \tag{63}$$

satisfying certain additional conditions. We now follow the adèlic philosophy of replacing $\mathbb{Z} \subseteq \mathbb{R}$ by $\mathbb{Q} \subseteq \mathbb{A}_{\mathbb{Q}}$, and observe that we can rewrite (63) as

$$\mathrm{GL}_2(\mathbb{Q}) \setminus \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) / \mathrm{GL}_2(\widehat{\mathbb{Z}}),$$

so that modular forms of weight k and level 1 are functions on the quotient $\mathrm{GL}_2(\mathbb{Q}) \setminus \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ satisfying certain conditions. Being invariant under the right translation action of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ is one of those conditions, and it is natural to omit it, thus incorporating modular forms of higher level into our picture. Relaxing the archimedean conditions (holomorphicity and the precise weight k condition) is also a sensible step; this incorporates Maass forms and non-holomorphic Eisenstein series into our picture.

Thus we are led to define the space of automorphic forms $\mathcal{A}\big(\mathrm{GL}_2(\mathbb{Q}) \setminus \mathrm{GL}_2(\mathbb{A})\big)$; it is the space of functions $f : \mathrm{GL}_2(\mathbb{Q}) \setminus \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ which are smooth in the archimedean variables and of moderate growth as these variables tend to ∞; invariant under some open subgroup of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^\infty)$; and which are generalized eigenvectors for the centre of the enveloping algebra of the Lie algebra $\mathfrak{gl}_2$ of $\mathrm{GL}_2(\mathbb{R})$ (so essentially we are asking them to be eigenvectors for the Casimir operator, which can be interpreted as the hyperbolic Laplacian in the classical picture). The space $\mathcal{A}\big(\mathrm{GL}_2(\mathbb{Q}) \setminus \mathrm{GL}_2(\mathbb{A})\big)$ is a complete locally convex topological vector space,[64] which becomes an *admissible* representation of $\mathrm{GL}_2(\mathbb{A})$ under the action by right translation. An automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ is then, by definition, an irreducible subquotient (in the topological sense, i.e. the quotient of one closed invariant subspace by another that it contains, which is irreducible in the topological sense of containing no proper closed invariant subspace) π of $\mathcal{A}\big(\mathrm{GL}_2(\mathbb{Q}) \setminus \mathrm{GL}_2(\mathbb{A})\big)$.

These notions extend directly to any connected reductive affine algebraic group G over a number field F — we may define the space $\mathcal{A}\big(G(F) \setminus G(\mathbb{A}_F)\big)$ of automorphic forms, which is an admissible representation of $G(\mathbb{A}_F)$ via right translation, and then define an *automorphic representation* π of $G(\mathbb{A}_F)$ to be a topologically

[64] We work with topological vector spaces so as to avoid having to take $O(2)$-finite vectors; the equivalence between this viewpoint and the $O(2)$-finite vector viewpoint follows from Harish-Chandra's theory of admissible representations of real reductive groups.

irreducible subquotient of this space. Since $G(\mathbb{A}_F)$ is the restricted direct product of the groups $G(F_v)$ (as v runs over all places of F), we may factor the irreducible representation π as a restricted tensor product (suitably defined)

$$\pi = \bigotimes{}' \pi_v, \tag{64}$$

where each π_v is an irreducible admissible representation of $G(F_v)$ [Fla79]. (This is the analogue in the non-abelian setting of the factorization (20) of an idèle class character.)

The space $\mathcal{A}\big(G(F) \setminus G(\mathbb{A}_F)\big)$ contains a subspace $\mathcal{A}^\circ\big(G(F) \setminus G(\mathbb{A}_F)\big)$ of cuspforms, whose elements are characterized by the vanishing of certain integrals that compute their "constant terms" at the boundary of $G(F) \setminus G(\mathbb{A}_F)$. The space $\mathcal{A}^\circ\big(G(F) \setminus G(\mathbb{A}_F)\big)$ has a positive definite $G(\mathbb{A}_F)$-equivariant inner product (the L^2-inner product, also called the Petersson inner product), and decomposes as a direct sum of irreducible representations. These are the *cuspidal* automorphic representations.

We refer the reader to [BJ79] for more details regarding these definitions.

Returning to the case of GL_2 over $\mathbb{Q}$ and classical modular forms: if f is a classical cuspform, then we have seen that we can interpret f as an automorphic form, i.e. as an element $f \in \mathcal{A}^0\big(\mathrm{GL}_2(\mathbb{Q}) \setminus \mathrm{GL}_2(\mathbb{A})\big)$, and we may then consider the $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$-subrepresentation that it generates. It is a theorem, essentially due to Atkin and Lehner[AL70], that f (thought of as an automorphic form) generates an irreducible subrepresentation (i.e. an automorphic representation) if and only if f (thought of as a modular form of some weight and level) is an eigenform for the Hecke operator T_p for all but finitely many primes p. Thus automorphic representations can be regarded as a representation-theoretic generalization of the notion of Hecke eigenform, and the list of the factors π_v occurring in the the decomposition (64) as a refinement of the classical data consisting of the list of Hecke eigenvalues of f.

In particular, returning to the general case, if π_v is a constituent of an automorphic form which is *unramified*, then it gives rise to a Frobenius–Hecke conjugacy class $c_v \in \widehat{G} \times \mathrm{Frob}_v^{-1}$ (as explained by Langlands in [Lan70], building non-trivially on Satake's work [Sat63]; see also Shahidi's chapter in this volume [Sha21], and the discussion of Section 6.3 below); this class is an analogue of the classical pth Hecke polynomial $X^2 - a_p X + \varepsilon(p) p^{k-1}$ associated to a modular Hecke eigenform. (Up to a possible issue of normalization, this polynomial is the characteristic polynomial of the Frobenius–Hecke conjugacy class that Langlands associates to the factor π_p of the automorphic representation π generated by f.)

6.2. **Hecke.** In his paper [Hec36], Hecke generalized Riemann's second proof of the analytic continuation and functional equation of $\zeta(s)$ to show that the Mellin transform $\Lambda(f, s)$ of a cusp form f of level one is the product of a Dirichlet series $L(f, s)$ with an appropriate Γ-factor, which admits an analytic continuation to an entire function and satisfies a functional equation. He also proved the converse: any entire function which satisfies a functional equation of the appropriate form, and which satisfies an appropriate growth condition, arises from a modular form of level one.

In the papers [Hec37], he introduced the operators which now bear his name, and proved that f is a Hecke eigenform if and only if $L(f, s)$ admits an Euler product — in which case the Euler factors are of the form $(1 - a_p p^{-s} + p^{k-2s})^{-1}$, where a_p is the pth Hecke eigenvalue, and k is the weight of f. Although he extended the theory of Hecke operators to higher level modular forms (and the theory, at least from a classical — i.e. non-adèlic — viewpoint was essentially completed by Atkin and Lehner [AL70]), the converse theorem for levels > 1 is not as obvious. It was eventually proved by Weil [Wei67]. Jacquet and Langlands [JL70] proved another form of the converse theorem, this time in the context of automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ for arbitrary number fields F. A key feature of their converse theorem is that the hypothesized functional equations must have the correct ϵ-factors. This provides one of the motivations for developing the theory discussed in Subsection 5.2.

We close this discussion by emphasizing that, just as in Riemann's work, Hecke's description of his L-functions associated to automorphic forms via an integral formula, and his analysis of their Euler product factorization, are completely independent of one another.

6.3. **Langlands.** We sketch Langlands's definition of automorphic L-functions, referring to Shahidi's chapter in this volume [Sha21] for more details.

Recall that if G is a connected reductive affine algebraic group over the number field F, then we may form first the dual group $\widehat{G}$, which is a group over $\mathbb{C}$ with dual root datum to that of G (the construction of $\widehat{G}$ depends only on G over $\overline{\mathbb{Q}}$), and then construct an action of the Galois group $\mathrm{Gal}(K/F)$ on $\widehat{G}$, so as to form the L-group

$$^{L}G = \widehat{G} \rtimes \mathrm{Gal}(K/F)$$

of G. The action of $\mathrm{Gal}(K/F)$ on $\widehat{G}$, and hence the L-group of G, only depends on the inner class of G over F.

Note that the L-group isn't entirely well-defined, since we are free to enlarge K, so it is better to speak of *an* L-group of G. This is actually an advantage in the theory. We are not even restricted to taking the semi-direct product with a Galois group. Any group that surjects onto $\mathrm{Gal}(K/F)$ (and hence can be made to act on $\widehat{G}$ through the original $\mathrm{Gal}(K/F)$-action) will do. Thus we can form the L-group in its "Weil form", as $^{L}G = \widehat{G} \rtimes W_{K/F}$, or $^{L}G = \widehat{G} \rtimes W_F$. If we grant ourselves the existence of the Langlands group, we could even form $^{L}G = \widehat{G} \rtimes L_F$.

In general, suppose that Γ is a group that surjects onto $\mathrm{Gal}(K/F)$, and that we form $^{L}G = \widehat{G} \rtimes \Gamma$. Suppose now that H is any group equipped with a homomorphism to Γ (and so also with a homomorphism to $\mathrm{Gal}(K/F)$, by composition). Then giving an *L-homomorphism* $H \to {}^{L}G = \widehat{G} \rtimes \Gamma$, i.e. a homomorphism which induces the given map upon projection to Γ, is the same as giving an *L-homomorphism* $H \to {}^{L}G = \widehat{G} \rtimes \mathrm{Gal}(K/F)$. (This just follows from description of $\widehat{G} \rtimes \Gamma$ as the fibre product $\big(\widehat{G} \rtimes \mathrm{Gal}(K/F)\big) \times_{\mathrm{Gal}(K/F)} \Gamma$.) Thus the notion of L-homomorphism is suitably compatible with the flexibility in the formation of the L-group of G.

If $\mathfrak{p}$ is a prime F, and $K_{\mathfrak{p}}$ is an extension of $F_{\mathfrak{p}}$ that splits G, then we may perform a similar local construction, obtaining a semi-direct product $\widehat{G} \rtimes \mathrm{Gal}(K_{\mathfrak{p}}/F_{\mathfrak{p}})$, and similar considerations apply. In particular, if G is unramified at $\mathfrak{p}$ (i.e. quasi-split and split over an unramified extension), then we may choose $K_{\mathfrak{p}}/F_{\mathfrak{p}}$ to be unramified, so that $\mathrm{Gal}(K_{\mathfrak{p}}/F_{\mathfrak{p}}) = \langle \mathrm{Frob}_{\mathfrak{p}} \rangle$. Then $\widehat{G}$-conjugacy classes of L-homomorphisms $\varphi : L_{F_{\mathfrak{p}}} \to {}^{L}G$ (for any choice of L-group; and remember that $L_{F_{\mathfrak{p}}} = W_{F_{\mathfrak{p}}} \times SU(2)$ is the Langlands group of $F_{\mathfrak{p}}$) correspond to $\widehat{G}$-conjugacy class of L-homomorphisms $L_{F_{\mathfrak{p}}} \to \widehat{G} \rtimes \langle \mathrm{Frob}_{\mathfrak{p}} \rangle$. We define one of the latter (and hence also one of the former) homomorphisms to be unramified if it is trivial on $I_v \times SU(2)$; giving an unramified φ then amounts to giving an element $c \in \widehat{G} \times \mathrm{Frob}_p^{-1}$, up to $\widehat{G}$-conjugacy, the *Frobenius–Hecke* conjugacy class associated to the unramified L-homomorphism φ. The class c and the unramified L-homomorphism φ determine one another.

We now consider a representation $\rho : {}^LG \to \mathrm{GL}_n$, required to be algebraic when restricted to $\widehat{G}$, and continuous on the Galois/Weil factor in ${}^LG$. Thus ρ will be unramified at all but finitely many primes $\mathfrak{p}$ of F. In other words, there is a finite set of primes S such that for $\mathfrak{p} \notin S$, when we restrict ρ to $\widehat{G} \rtimes W_{F_\mathfrak{p}}$ for $\mathfrak{p} \notin S$, the action of $W_{F_\mathfrak{p}}$ on $\widehat{G}$ factors through the quotient $W_{F_\mathfrak{p}}/I_\mathfrak{p} = \langle \mathrm{Frob}_\mathfrak{p} \rangle$, and the restriction of ρ to $\widehat{G} \rtimes W_{F_\mathfrak{p}}$ factors through $\widehat{G} \rtimes \langle \mathrm{Frob}_\mathfrak{p} \rangle$.

Suppose now that π is an automorphic representation of $G(\mathbb{A}_F)$. Then we may form the restricted tensor factorization $\pi = \bigotimes_v' \pi_v$. The group G is unramified at all but finitely many primes $\mathfrak{p}$, and for all but finitely many of these primes, the local factor $\pi_\mathfrak{p}$ will be unramified. As recalled in Subsection 6.1 above, for each such prime p, we obtain a corresponding Frobenius–Hecke conjugacy class, i.e. an element $c_\mathfrak{p} \in \widehat{G} \rtimes \mathrm{Frob}_\mathfrak{p}^{-1}$, well-defined up to $\widehat{G}$-conjugacy.

Now enlarge the finite set of places S so that it includes all the bad primes for ρ, all the bad primes for π (i.e. the primes for which $c_\mathfrak{p}$ is not defined), and also all the infinite places. Langlands then defines the (imprimitive) automorphic L-function associated to π and a representation $\rho : {}^LG \to \mathrm{GL}_n$ *as an Euler product*, via the formula

$$L^S(s, \pi, \rho) = \prod_{\mathfrak{p} \notin S} \frac{1}{\det\bigl(\mathrm{Id}_{n \times n} - N(\mathfrak{p})^{-s} \rho(c_\mathfrak{p})\bigr)} \tag{65}$$

Note the similarity to Artin's definition (39) of his L-functions!

The problem of completing the L-function (65) by adding Euler factors at the primes in S is a difficult one. We first recast the Frobenius–Hecke classes $c_\mathfrak{p}$ as unramified L-homomorphisms $\varphi_\mathfrak{p} : L_{F_\mathfrak{p}} \to {}^LG$, in the manner described above. We would then like to use the remaining $\pi_\mathfrak{p}$ to produce L-homomorphisms $\varphi_v : L_{F_v} \to {}^LG$ which are no longer necessarily unramified. The *local Langlands conjecture* for G states that this is possible. It is proved in a number of cases (we can vary G, and also, having fixed G, impose various ramification conditions on the π), and in particular is proved in the archimedean case for arbitrary G by Langlands himself [Lan89], and is proved completely for GL_n by Harris–Taylor and Henniart [HT01, Hen00]; but it remains open in general.

Langlands proves that the Euler product (65) converges in some right half-plane. There is one particular case in which we can say more. Let $G = \mathrm{GL}_n$, so that $\widehat{G} = \mathrm{GL}_n$ also, and take ρ to be the *standard representation*, i.e. simply the identity map of GL_n to itself; the resulting L-functions are referred to as the *standard* L-functions, and we denote them $L(s, \pi, \mathrm{standard})$. The work of Godement–Jacquet[GJ72] establishes the analytic continuation and functional equation for these L-functions by methods generalizing those of Tate's thesis. In general, Langlands conjectures that, after completing the Euler product (65) by adding factors at the missing places, the resulting function will admit an analytic continuation to the entire plane, and satisfy an appropriate functional equation. But in fact, he conjectures more: as we recall below, his *functoriality* conjecture states that an arbitrary automorphic L-function $L(s, \pi, \rho)$ should in fact be a standard L-function (for some other π, to be sure), whose analytic continuation and functional equation is then assured by [GJ72].

6.4. **The reciprocity conjecture.** If π is an automorphic representation of $G(\mathbb{A}_F)$, then we obtain the unramified L-homomorphisms $\varphi_{\mathfrak{p}} : W_{F_{\mathfrak{p}}} \to {}^LG$, for $\mathfrak{p} \notin S$, and, if we grant the existence of the local Langlands correspondence, we in fact obtain L-homomorphisms $\varphi_v : L_{F_v} \to {}^LG$ for every place v. We could now ask: is there an L-homomorphism $\varphi : L_F \to {}^LG$ which induces the various φ_v. The answer is "no" in general,[65] even if $G = \mathrm{GL}_n$. But we have the following conjecture:

6.4.1. **Conjecture** (Langlands reciprocity)**.** *There is a bijection between irreducible representations $\varphi : L_F \to \mathrm{GL}_n$ and cuspidal automorphic forms π on $\mathrm{GL}_n(\mathbb{A}_F)$, via which φ and π correspond if $\varphi_{|F_v} = \varphi_v$ (the local L-homomorphism associated to π_v via the local Langlands correspondence) for all places v.*

Why do we call this a reciprocity law? Because, according to Conjecture 5.9.2, each motive over F gives rise to a representation ρ of L_F. By the conjecture, each irreducible constituent of ρ then corresponds to a cuspidal automorphic representation π of some $\mathrm{GL}_n(\mathbb{A}_F)$. In particular, $L(s,\rho)$ (which will coincide with the L-function of our original motive) is a product of standard L-functions $L(s,\pi,\text{standard})$, and so has the anticipated analytic continuation and functional equation.

The conjecture also has an implication in the reverse direction: if π is a cuspidal automorphic representation on $\mathrm{GL}_n(\mathbb{A}_F)$, whose factors at the infinite places are algebraic,[66] then the conjecture produces an irreducible representation ρ of L_F, which is algebraic in the sense of Conjecture 5.9.2, and so should correspond to a compatible family of ℓ-adic Galois representations coming from a motive over F.

6.4.2. *Reciprocity in terms of L-functions.* A key point, which we have already intimated, is that the reciprocity conjecture can be phrased in terms of L-functions, and hence stated (and studied!) independently of the problem of the existence of the Langlands group, at least in the algebraic case. Thus, suppose that π is an algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, conjecturally corresponding to an algebraic representation $\varphi : L_F \to \mathrm{GL}_n(\mathbb{C})$. This algebraic representation in turn corresponds to a motive M over F, and the conjectured relationship between π and φ can be rephrased as an equality

$$L(s,\pi,\text{standard}) = L(s,\varphi) = L(s,M). \tag{66}$$

Thus, phrased in terms of L-functions this way, we can study the conjecture in either direction: we can begin with an algebraic cuspidal automorphic representation π and try to construct a motive M, or at least a compatible family of ℓ-adic

[65]For $G = \mathrm{GL}_n$, this is due to *non-isobaric* automorphic representations, in the sense of Langlands's paper [Lan79]. For more general G, it is then related to the representations that Langlands (in the same paper) calls *anomolous*. Arthur's theory of non-tempered endoscopy [Art89] aims to explain automorphic representations that appear in $L^2\big(G(F)\setminus G(\mathbb{A}_F)\big)$, and which are anomolous in the sense of [Lan79], in terms of L-homomorphisms $L_F \times \mathrm{SL}_2 \to {}^LG$.

[66]Suppose that $\pi = \bigotimes' \pi_v$ is an automorphic representation of some $G(\mathbb{A}_F)$. At each archimedean place v, the factor π_v induces a morphism $\varphi_v : L_{F_v} \to {}^LG$ via the local Langlands correspondence [Lan89], which in turn restricts to a morphism $\mathbb{C}^\times \to \widehat{G}$. We can then ask that this morphism be algebraic, in the sense that it factor through a maximal torus $\widehat{T}$ of $\widehat{G}$ where it is then described by characters of the form $z \mapsto z^p \bar{z}^q$. If this condition holds, we say that π_∞ is *algebraic*. This corresponds to the notion of L-algebraic, in the terminology of [BG14].

Galois representations, satisfying (66). Conversely, we can begin with a (suitably irreducible) motive M and try to construct a cuspidal automorphic representation π satisfying (66).

6.4.3. *Functoriality.* [67] Langlands has made another fundamental conjecture related to L-groups and his L-functions, namely his *functoriality* conjecture. To describe it, we suppose that G and H are two connected reductive linear algebraic groups over F, and that $\rho : {}^L G \to {}^L H$ is an L-homomorphism of their L-groups. Suppose also that π is an automorphic representation of $G(\mathbb{A}_F)$, giving rise to the Frobenius–Hecke classes $c_\mathfrak{p}$ for good primes $\mathfrak{p}$. We may then consider the collection of classes $\rho(c_\mathfrak{p})$ in ${}^L H$, and Langlands conjectures that if H is furthermore quasi-split,[68] then there is an automorphic representation Π of $H(\mathbb{A}_F)$ whose associated Frobenius–Hecke classes are equal to the classes $r(c_\mathfrak{p})$ (for all but finitely many $\mathfrak{p}$). In short, automorphic representations are *functorial* in the L-group.

We can describe the conjectured relationship between π and Π in terms of their L-functions: if $r : {}^L H \to \mathrm{GL}_n$, then a direct consideration of the definitions shows that (taking S to be sufficiently large)

$$L^S(s, \pi, r \circ \rho) = L^S(s, \Pi, r). \tag{67}$$

6.4.4. *Example.* One example of functoriality is the case when $H = \mathrm{GL}_n$. Then ${}^L H$ is simply the direct product of GL_n with the Galois group, and so we may ignore the Galois factor, so that ρ is just a homomorphism ${}^L G \to \mathrm{GL}_n$. In this case, taking r to be the standard representation, the formula (67) reduces to the equality

$$L^S(s, \pi, \rho) = L^S(s, \Pi, \text{standard}). \tag{68}$$

Thus functoriality implies that *all* automorphic L-functions are *standard* L-functions, and so in particular have analytic continuation and satisfy a functional equation.

6.4.5. *Example.* Consider now the case when G is the trivial group! As Langlands has emphasized, although G is trivial, its L-group is not. If we take the Galois form for L-groups, then can write its L-group as $\mathrm{Gal}(K/F)$ for some finite extension K, and $\rho : \mathrm{Gal}(K/F) \to \mathrm{GL}_n$ is simply a representation of the kind considered in Subsection 5.1. There is only one automorphic representation of the trivial group, namely the trivial representation, and its Frobenius–Hecke classes are just the Frobenius conjugacy classes $\mathrm{Frob}_\mathfrak{p} \in \mathrm{Gal}(K/F)$. Now apply functoriality, in particular (68). The L-function of the trivial representation is precisely the Artin L-function of ρ. Thus functoriality predicts the existence of an automorphic representation Π of $\mathrm{GL}_n(\mathbb{A})$ for which

$$L(s, \rho) = L(s, \Pi, \text{standard})). \tag{69}$$

If ρ is irreducible then we anticipate that Π should be cuspidal, in which case Artin's Conjecture 5.1.1 would hold for ρ. In this way Artin's conjecture is situated in the much more general context of functoriality.

Of course, since the representation ρ is a particular example of a motive, the existence of Π satisfying (69) is also a special case of the reciprocity conjecture. In fact, assuming that the Langlands group L_F exists, we could use the L_F-form of the L-group. In this case, if G is the trivial group, then ${}^L G = L_F$, and giving

[67] For an elaboration on this topic, see Arthur's chapter in this volume [Art21].

[68] In (6.4.6) below we explain the significance of this assumption.

an L-homomorphism ρ as above amounts to giving a morphism $\rho : L_F \to \mathrm{GL}_n$. Applying functoriality in this context, we obtain an automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ for which $L(s, \rho) = L(s, \Pi, \text{standard})$. Thus functoriality, when interpreted in a sufficiently broad manner, implies the "motivic to automorphic" direction of reciprocity!

6.4.6. *Example.* If G is any connected linear reductive group over F, we can take H to be the quasi-split inner form of G, and let ρ be the equality ${}^L G = {}^L H$. We thus expect to have a functorial transfer of automorphic representations from $G(\mathbb{A}_F)$ to $H(\mathbb{A}_F)$. It is typically *not* surjective — automorphic representations for the quasi-split group H can't always be "descended" to the inner form H. This is why, in the statement of functoriality, we assume that H is quasi-split.

In the case when $H = \mathrm{GL}_2$, this transfer was established by Jacquet and Langlands in their book [JL70]. They did this via an application of the trace formula, and Langlands has emphasized the importance of the trace formula as a tool to study functoriality in general.

6.4.7. *Example.* There are many other examples of functoriality that we could mention in this context, and which will play a role in the discussion of the next section. Among them are base change, automorphic induction, symmetric power functoriality (corresponding to the representation $\mathrm{Sym}^n : \mathrm{GL}_2 = \widehat{\mathrm{GL}}_2 \to \mathrm{GL}_{n+1} = \widehat{\mathrm{GL}}_{n+1}$), and endoscopic functoriality. Another important example occurs when G is a classical (i.e. special orthogonal or symplectic) group. In this case $\widehat{G}$ is again classical, and we may consider $\rho : {}^L G \to \mathrm{GL}_n$ arising from the standard representation of $\widehat{G}$. This case of functoriality (subject to some splitness/quasi-splitness assumptions on G) has been proved by Arthur [Art13], as part of his theory of non-tempered endoscopy and stabilization of the trace formula [Art89].

7. Progress

The proof of class field theory — abelian reciprocity — can be thought of as consisting of two steps: (i) constructing the ray class fields, for which one has an explicit reciprocity law (i.e. their Galois groups, and the Frobenius elements inside them, are described in automorphic terms), and (ii) showing that any abelian extension is contained in a ray class field. In practice, in modern proofs of class field theory, this order of argumentation is not necessarily observed (e.g. one can prove Artin reciprocity for arbitrary abelian extensions before proving the existence theorem for ray class fields), but it provides a helpful framework for viewing the dominant contemporary approaches to studying reciprocity.

The analogue of ray class fields, then, will be certain varieties (or motives) that admit an explicit reciprocity law, i.e. for which the Galois action on their ℓ-adic cohomology can be described in automorphic terms. These are the *Shimura varieties.* Although not all motives over number fields can be constructed out of them, they play a fundamental role in the study of reciprocity.

7.1. **Shimura varieties.** These are a class of varieties attached to certain reductive groups over $\mathbb{Q}$, originally studied by Shimura, and subsequently by Langlands (who coined their name) and a host of other researchers. The simplest examples arise from the groups GL_2 over $\mathbb{Q}$, for which the associated Shimura varieties are modular curves, and the groups $\mathrm{Res}^E_{\mathbb{Q}} \mathbb{G}_m$ for imaginary quadratic extensions E of $\mathbb{Q}$ (which are forms of the group $\mathbb{R}^{\times}_{>0}U(1) \subset \mathrm{GL}_2(\mathbb{C})$), whose associated Shimura varieties are finite sets of "singular" j-invariants (i.e. j-invariants of CM elliptic curves).

The second example above is of course famously related to the class field theory of quadratic imaginary fields (Kronecker's *Jugendtraum*). The first example was also intensively studied throughout the 19th and 20th centuries. In the 1950's, it came to be realized (by Eichler [Eic54] and Shimura [Shi58] particularly, and by Taniyama as well) that the ζ-functions of modular curves admitted a description in terms of the L-functions of weight 2 cuspforms. Exploiting the universal family of elliptic curves over the modular curve (or, in an alternative formulation, allowing non-constant coefficients over the modular curves), Ihara [Iha67] found that higher weight cuspforms were also related to motivic ζ-functions.

Shimura introduced his eponymous varieties as generalizations of the modular curves, and (exploiting the 20th century outgrowths of Kronecker's Jugendtraum, i.e. class field theory and the theory of CM abelian varieties) he constructed their canonical models,[69] over an appropriate *reflex field*.[70] Langlands proposed the problem of studying their ζ-functions (with both constant and twisted coefficients), as a natural generalization of Kronecker's Jugendtraum [Lan76]. He further developed this suite of ideas in [Lan79] and [LR87].

There is an enormous amount of subsequent research on this problem, and it has proved to be one of the most stimulating directions of research in contemporary number theory and arithmetic geometry. (As just one example, Langlands's beginning investigations into this question led to his discovery of endoscopy.) We won't begin to attempt to describe the current state of the field. We do mention that Arthur has given a precise conjectural description of the ζ-function of the intersection cohomology of the minimal compactification of a Shimura variety in [Art89, §9]. These conjectures imply that these ζ-functions can be expressed in terms of automorphic L-functions. We also mention that these are typically not exhibited as standard L-functions, but as various L-functions attached to G and its endoscopic groups (if G is the group giving rise to the Shimura variety).[71]

We also mention Langlands's conjecture on conjugation of Shimura varieties [Lan79]. This conjecture proposes that the collection of Shimura varieties is stable under the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, even though any particular Shimura variety is defined only over its reflex field E (and so stable only by $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$). In fact, if we think of Shimura's results (the so-called *Shimura reciprocity law*) as describing the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ on H^0 of a particular Shimura variety, then Langlands's proposed extension describes the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ on the collection of H^0's taken over all Shimura varieties. Langlands' reciprocity law was proved (in most cases) in [DMOS82]. As far as I know, no one has posited an extension of Langlands's conjecture to the general H^i of Shimura varieties, although there should be such an extension, generalizing the conjecture of [Art89] for the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$ on the cohomology

[69]In many cases; the general case is treated in [Mil83].

[70]Shimura actually worked with what now called *connected* Shimura varieties, and showed that they are defined over ray class fields of the reflex field, the conductor of the field depending on the level. We follow Deligne's formulation of the theory [Del71b, Del79], and consider (disconnected, i.e. what are now called) Shimura varieties, which are defined over the reflex field, no matter what the level.

[71]The representation ρ of the L-group that appears in these L-functions is related to the datum originally used to define the Shimura variety. It first appeared in Langlands's letter to Lang [Lanb], and is also described in [Lan79].

of a particular Shimura variety. The paper [Tay12] establishes a particular result of this type.

7.2. **From automorphic representations to Galois representations.** Shimura varieties are endowed with large numbers of correspondences — "Hecke correspondences" — and so can be decomposed into motives according to the eigenvalues of the Hecke correspondences. Correspondingly, their ζ-functions will decompose into the product of the L-functions of these motives, which (following e.g. the general conjecture of [Art89]) one anticipates will be related to automorphic L-functions associated to G or to its endoscopic groups. Reversing the order of this discussion, one sees that if a particular automorphic representation "contributes" to the decomposition of the ζ-function, one can hope to find a motive which corresponds to this automorphic representation, in the sense that the L-function of the motive coincides with an L-function of the given automorphic representation.

The simplest case of this analysis is that of modular curves; in this case the upshot is the construction of a compatible family of ℓ-adic representations [Del71a] (and even a motive [Sch90]) associated to the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ generated by any cuspidal Hecke eigenform of weight $k \geq 2$. (The weight 2 case was treated first, by Shimura [Shi71].) This proves one direction of the reciprocity conjecture for such automorphic representations.

The preceding construction extends to the case of Hilbert modular forms, but in an indirect way: if F is totally real, then while $\mathrm{GL}_2(\mathbb{A}_F)$ gives rise to a Shimura variety (essentially a Hilbert modular variety), the reflex field is $\mathbb{Q}$, not F, and the cohomology of this Shimura variety won't produce the 2-dimensional representations that ought to be attached to cuspidal automorphic representations on $\mathrm{GL}_2(\mathbb{A}_F)$. (Rather, it produces their "tensor inductions" from $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$ to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.) Instead, one has to work with Shimura curves [Car86]; these are Shimura varieties attached to a non-split inner form of GL_2 over F, and so there are obstructions to functorially transferring automorphic representations from $\mathrm{GL}_2(\mathbb{A}_F)$ to these inner forms. The upshot is that one constructs compatible families of Galois representations for most, but not all, Hilbert modular eigenforms of weights $k_i \geq 2$.

In the case of modular eigenforms of weight $k = 1$, or Hilbert modular eigenforms with one or more of its weights $= 1$, there is no direct connection between these modular forms (or the automorphic representations that they generate) and the motives arising from modular curves or Shimura curves. One can still find associated families of ℓ-adic representations in these cases, but they are obtained by ℓ-adic limiting processes from the Galois representations associated to higher weight forms (using the theory of congruences of modular forms) [DS74, RT83, Oht84, Jar97]. In this case one works with a single ℓ at a time to construct the family; the compatibility comes at the end, by comparing the representations with varying ℓ to the fixed modular eigenform. There are no obvious motives giving rise to these compatible families. In the case of weight 1 modular forms, or parallel weight 1 Hilbert modular forms, one can prove *a posteriori* that the resulting Galois representations are of finite image, and hence *are* motivic. For partial weight 1 Hilbert modular forms, the problem of showing that these Galois representation are motivic remains open in general. The construction of compatible families associated to Hilbert modular forms of weights $k_i \geq 2$ that cannot be transferred to Shimura curves also proceeds by ℓ-adic limiting/congruence arguments. Again, the resulting compatible family is not evidently motivic. An alternative approach in this case is to find these compatible families in the cohomology of a higher dimensional Shimura variety via endoscopy [BR93]; this approach produces a motive, at least if one grants the Tate conjecture.

The problems of the preceding paragraphs compound if one tries to treat $\mathrm{GL}_n(\mathbb{A}_F)$ for $n > 2$. The group GL_n never gives rise to a Shimura variety if $n > 2$. If F is totally real or CM, one *does* have Shimura varieties associated to forms of $GU(n)$ defined over F. These groups are *outer* twists of GL_n. Roughly, the cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ which satisfy an appropriate self-duality arise as functorial transfers from automorphic representations on some form of $GU(n)$. One also has the happy fact that if the generalized unitary group under consideration has signature $(1, n-1)$ at one infinite place, and definite signature at all others, than the most significant of the L-functions appearing in the description of its ζ-function are in fact standard L-functions.

Applying appropriate extensions of all the methods described in the GL_2-case (cohomology of Shimura varieties, endoscopy, ℓ-adic limiting arguments), as well as a passage from dimension n to dimension $2n$ to impose self-duality in the non-essentially-self-dual case, leads to the following result.[72]

7.2.1. **Theorem.** *If F is totally real or CM, and if π is a cuspidal automorphic representation whose infinitesimal characters at all infinite places are regular, then there exists a compatible family of Galois representations $\{\varphi_\ell\}$ associated to π.*

In many of the cases when π is self-dual, these are known to be motivic (either unconditionally or subject to the Tate conjecture), since they are constructed in the cohomology of Shimura varieties. But not always, and not at all if π is not self-dual (because of the ℓ-adic limiting arguments that are required).

7.3. **From Galois representations to automorphic representations.** We now discuss the general problem of passing from a motive, or its associated compatible family of Galois representations, to an associated automorphic representation. It is helpful to consider this problem through the lens of various particular examples of 2-dimensional Galois representations.

To begin with, suppose that $K/\mathbb{Q}$ is a degree two extension, and that ψ is an algebraic Hecke character for K. The construction of Subsection 4.6 gives rise to a compatible family $\{\widetilde{\rho}_\ell\}$ of two-dimensional Galois representations, which, as we observed in Subsection 5.8, are motivic. As we also observed there, $L(s, \{\widetilde{\rho}_\ell\}) = L(s, \psi)$, and so in this case the associated L-function *does* satisfy the expected analytic continuation and functional equation. If K is imaginary, and in certain cases if K is real, one can then apply the Hecke–Weil converse theorem discussed in 6.2 to show that $L(s, \psi) = L(s, \pi, \text{standard})$ for some cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. In fact, using the relationship between quadratic fields and quadratic forms, one can also directly use arguments with θ-series (so, ultimately, the Poisson summation formula) to construct π (or, rather, to construct a modular eigenform f that generates π).[73] Maass [Maa49] made a similar construction in the remaining real quadratic cases (and introduced Maass forms to this end).

[72] An enormous number of authors are involved in the complete proof of this result. In the essentially self-dual case, we should mention [Clo91], [Kot92], [HT01], [Shi11], and [CH13]. In the not-necessarily self-dual case, the references are [HLTT16] and [Sch15].

[73] One can think of this as taking the θ-series arguments that go into proving the functional equation for $L(s, \psi)$, and using them directly to construct the required modular form.

There is an enormous generalization of the preceding construction, known as *automorphic induction*. If K/F is a cyclic Galois extension of degree n, if ψ is a Hecke character, and if ρ is the induction of ψ to $W_{K/F}$, then we may find an automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that $L(s,\rho) = L(s,\pi)$. In the case when ψ is algebraic, so that ρ gives rise to the motivic family $\{\widetilde{\rho}_\ell\}$ (generalizing the context of the preceding paragraph), we then have $L(s,\{\widetilde{\rho}_\ell\}) = L(s,\pi)$, and we have established reciprocity for the family $\{\widetilde{\rho}_\ell\}$.

When $n = 2$, the converse theorem of [JL70] can be used to prove this result. For general n, a different argument is required. One considers the problem in the optic of (6.4.3), by working with the Weil form of the L-group. Automorphic induction is then proved by an argument using the trace formula [AC89, Hen12].

Another 2-dimensional example comes from 2-dimensional Artin representations. If these have solvable image in $\mathrm{GL}_2(\mathbb{C})$, then by analyzing the structure of these representations (e.g. relating them to certain inductions), and using automorphic induction, as well as cases of *base change* functoriality, they can be shown to satisfy reciprocity — this is the celebrated Langlands–Tunnell theorem [Lan80, Tun81]. In particular, this establishes the Artin conjecture (Conjecture 5.1.1) for these representations. In fact, in this case, the Artin conjecture is essentially equivalent to reciprocity, by the Jacquet-Langlands converse theorem for GL_2.[74]

Yet another 2-dimensional example is given by $H^1(E)$ for an elliptic curve E over $\mathbb{Q}$. The conjecture on meromorphic continuation of $\zeta(s,E)$ is attributed to Hasse by Weil [Wei52], while the reciprocity conjecture for $H^1(E)$ seems to have been discovered by Shimura and Taniyama. Taking into account the converse theorems for GL_2, this reciprocity is seen to be equivalent to the holomorphic continuation and functional equation of $L\big(s,H^1(E)\big)$; and indeed, Weil proved his converse theorem in part to establish this equivalence. In particular, if E has CM, then we have seen that $H^1(E)$ arises from a Weil group representation, and so we are in the case discussed at the beginning of this subsection — the converse theorem indeed can be applied to deduce reciprocity for $H^1(E)$. For elliptic curves without CM, the compatible family $H^1(E)$ does not arise from a Weil group character, and there is no evident way to directly prove this analytic continuation and functional equation. Indeed, the only known proof of this analytic continuation and functional equation is *via* reciprocity.

In this case, reciprocity was famously proved by Wiles [Wil95] and Taylor–Wiles [TW95] for semistable E, and then by Breuil–Conrad–Diamond–Taylor [BCDT01] for general E. We will say the briefest amount about the methods. They are ℓ-adic in nature, exploiting the theory of congruences of modular forms. Very roughly, one finds an automorphic representation π' which is *known* to have an associated ℓ-adic representation, and for which this ℓ-adic representation, when reduced mod ℓ, is *isomorphic* to the mod ℓ reduction of $H^1(E)$ (so, concretely, to the Galois representation on the group of ℓ-division points on E). An *automorphy lifting theorem* (here "lifting" refers to lifting back from mod ℓ to the original ℓ-adic representation $H^1(\overline{E},\mathbb{Q}_\ell)$) then implies that $H^1(\overline{E},\mathbb{Q}_\ell)$ is itself "automorphic", i.e. the ℓ-adic representation associated to an automorphic representation π. Now we are done: $L(s,\pi) = L\big(s,H^1(E)\big)$ (since we can check this by working at a single ℓ).

[74]Tate once wryly noted that between the wars, both Artin and Hecke were in Hamburg, and both were studying the problem of analytic continuation of L-functions. Nevertheless, it took thirty-odd more years before Langlands made the connection between their respective viewpoints.

As Wiles remarks in the introduction to [Wil95], this method of argument does not need the entire compatible family of ℓ-adic representations, other than that some ℓ may be easier to argue with than others. Having flexibility in choosing which ℓ to work with turns out to be essential, though, because in implementing the method, one must exhibit an existing automorphic representation π' which will give rise to the desired automorphic representation π via automorphy lifting.

In the original argument of [Wil95], the choice of $\ell = 3$ is made; since $\mathrm{GL}_2(\mathbb{F}_3)$ is solvable, one can use the Langlands–Tunnell result to construct a π'. For certain E, the argument at $\ell = 3$ doesn't work, and one shifts to $\ell = 5$ instead. For this one, has to find a π' matching with the 5-torsion of E. To this end, one finds *another* elliptic curve E' with $E'[5] = E[5]$ (as $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules), but with better behaved 3-torsion, so that the Langlands–Tunnell result applies. The previous argument shows that E' satisfies reciprocity, so now one has a π' from which to construct the desired π via 5-adic automorphy lifting. The argument of [BCDT01] uses further switching between the primes 3 and 5. These switches are possible because there are an abundance of auxiliary E' available with prescribed 3 or 5-torsion, since the modular curves of level 3 and 5 are *rational* curves.

Once one tries to generalize this argument, the moduli schemes from which one is "sampling" one's auxiliary objects have essentially uncontrollable geometry, and there is no guarantee that they have rational points. This necessitates having to pass from $\mathbb{Q}$, or whatever ground field one is working over, to an extension field, in order to find the auxiliary automorphic representation π' that will effectuate the automorphy lifting. One also needs a source of "automatic automorphy", since an analogue of Langlands–Tunnell is typically not available. Automorphic induction provides the solution to this — mod ℓ Galois representations which are monomial automatically arise from automorphic representations π'.

We should also mention that the automorphy lifting arguments depend very much on the "automorphic to Galois" direction of reciprocity. And of course there are many other technical constraints which limit the precise arguments that can be made and, consequently, the scope of the results that can be proved. Nevertheless, much progress has been made in the last 25 years. Here are some sample recent results.

7.3.1. **Theorem** (Allen–Calegari–Caraiani–Gee–Helm–Le Hung–Newton–Scholze—Taylor–Thorne [ACC+18])**.** *The Sato–Tate conjecture holds for any elliptic curve over a CM field.*[75]

7.3.2. **Theorem** (Boxer–Calegari–Gee–Pilloni [BCGP18])**.** *If C is a genus two curve over a totally real field, then $\zeta_C(s)$ admits a meromorphic continuation to the entire complex plane, and satisfies the expected functional equation.*

Both results are about meromorphic continuation — the second evidently, and the first because, as we explained above, the proof of the Sato–Tate conjecture requires us to continue each $L\big(s, \mathrm{Sym}^n H^1(E)\big)$ to the line $s \geq 1 + \frac{n}{2}$, and to prove that it is zero-free along the edge of this region. Both results are established by proving a *potential* version of reciprocity; i.e. by constructing the relevant automorphic representation π that controls the given L-function, not over F, but over some (uncontrollably large) extension K of F. (As explained above, this is necessary to

[75]As indicated in footnote 57, the case of elliptic curves over totally real fields was proved earlier.

find the required auxiliary points on some moduli scheme.) A variant of Brauer's argument (Theorem 5.1.3) then shows that each $L(s, \mathrm{Sym}^m H^1(E))$, in the context of the Theorem 7.3.1, or $L(s, H^1(C))$, in the context of Theorem 7.3.2, has the required meromorphic continuation. We also remark that the the automorphy lifting argument used to prove Theorem 7.3.2 takes place in the context of automorphic representations of a (generalized) symplectic group, and the authors rely on Arthur's work on functoriality for classical groups [Art13] (mentioned in (6.4.7)) to compare these automorphic representations with automorphic representations on the group GL_4 — for example, in order to manipulate the associated Galois representations, and in order to apply base-change arguments such as those used in the Brauer-type argument for meromorphic continuation.

In his Shaw Prize address [Lan11], Langlands observes that the Sato–Tate conjecture, analogous as it is to Čebotarev's theorem, is closer to Brauer's Theorem 5.1.3 than to Artin's Conjecture 5.1.1; this is a valid point, and indeed, the potential reciprocity that is being applied here to deduce it is weaker than true reciprocity. On the other hand, from the point of view of current technique, reciprocity and potential reciprocity don't seem all that different; to establish one rather than the other, one just has to find a method for constructing the auxiliary automorphic representation π' without making an unwanted field extension. One recent instance of this is the paper [AKT19] of Allen–Khare–Thorne, in which the authors are able to improve on some of the results of [ACC+18] by finding a large class of elliptic curves over CM fields for which reciprocity holds genuinely, not just potentially. We also mention the paper [CCG20], which employs the arguments of [BCGP18] to give examples of abelian surfaces over $\mathbb{Q}$ for which reciprocity holds.

In the same address, Langlands also discusses the significance of establishing results such as Sato–Tate, which are consequences of functoriality, by using the technique of reciprocity instead. The same ten authors who proved Theorem 7.3.1 also established entirely new cases of the Ramanujan conjecture for cuspidal automorphic representations on $\mathrm{GL}_2(\mathbb{A}_F)$ (F a CM field). This is another application of reciprocity to establish a consequence of functoriality.

As we noted in (6.4.3), from a certain viewpoint reciprocity can be entirely subsumed into functoriality. In any event, the two problems seem to be just as, or even more, intertwined than ever before. Whether this is merely a contingent phenomenon, or the reflection of something deeper, remains to be seen.

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