Draft: January 19, 2010

ORDINARY PARTS OF ADMISSIBLE REPRESENTATIONS OF p-ADIC REDUCTIVE GROUPS II. DERIVED FUNCTORS

MATTHEW EMERTON

Contents

1. Introduction

The goal of this paper, which is a sequel to [10], is to extend the functors of ordinary parts introduced in that paper to certain δ-functors. On the one hand, the δ-functors that we construct are easy to compute with in many situations; on the other hand, we expect that they coincide with the derived functors of the functors of ordinary parts. Since the functors of ordinary parts are right adjoint to the functors given by parabolic induction, we expect that the δ-functors introduced here will be useful in studying homological questions involving parabolically induced representations. As some evidence for this, we close the paper by applying them to make some Ext calculations in the category of admissible smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ over a finite field of characteristic p. These calculations play a role in the mod p and p-adic Langlands programme. (See [6] and [9].)

To describe our results more precisely, let p be a prime, and A a finite Artinian local $\mathbb{Z}_p$-algebra of residue characteristic p. Let G be a connected reductive p-adic group, and P a parabolic subgroup of G, admitting a Levi factorization $P = MN$. We define a δ-functor $H^\bullet \mathrm{Ord}_P$ on the category of admissible smooth G-representations over A, taking values in the category of admissible smooth M-representations over A, such that $H^0 \mathrm{Ord}_P \xrightarrow{\sim} \mathrm{Ord}_P$ (the functor of ordinary parts defined in [10, Def. 3.1.9]). The functors $H^\bullet \mathrm{Ord}_P$ are defined in terms of continuous N_0-cohomology, where N_0 is a compact open subgroup of N. This continuous cohomology is quite accessible to computation, and hence so are the functors $H^\bullet \mathrm{Ord}_P$. For example, one finds that the functors $H^i \mathrm{Ord}_P$ vanish for $i > \dim N$, and that $H^{\dim N} \mathrm{Ord}_P$ coincides (up to a twist) with the Jacquet functor of N-coinvariants. We conjecture that $H^\bullet \mathrm{Ord}_P$ is in fact a universal δ-functor, and thus coincides with the derived functors of Ord_P. In an appendix to this paper [11], joint with V. Paškūnas, we verify this conjecture in the case when $G = \mathrm{GL}_2(F)$, for any finite extension F of $\mathbb{Q}_p$.

The author was supported in part by NSF grants DMS-0401545 and DMS-0701315.

In the case when $G = \mathrm{GL}_2(\mathbb{Q}_p)$, P is a Borel subgroup of G, and $A = k$ is a field of characteristic p, we explicitly evaluate the functors $H^\bullet\mathrm{Ord}_P$ on all absolutely irreducible admissible smooth G-representations over k. As already mentioned, we are then able to apply the results of this calculation to compute certain Ext spaces in the category of admissible smooth G-representations over k. Some of the Ext computations that we make have also been made by Breuil and Paškūnas [4] and by Colmez [6], in both cases using different methods.

1.1. **Arrangement of the paper.** Section 2 of the paper is devoted to recalling some results about injective objects in certain categories of representations, as well as describing the relationship between certain Ext functors and continuous cohomology. In Section 3 we define the δ-functors $H^\bullet\mathrm{Ord}_P$, and develop some of their basic properties. In Section 4 we study the case when $G = \mathrm{GL}_2(\mathbb{Q}_p)$ and P is a Borel subgroup in detail. Some of the more technical aspects of the calculations, which are not directly related to the general ideas of the paper, have been placed in an appendix. A second, separate, appendix, written jointly with V. Paškūnas, establishes an important homological result which is applied in Section 4.

1.2. **Notation and terminology.** Throughout the paper, we fix a prime p. We let $\mathrm{Art}(\mathbb{Z}_p)$ denote the category of Artinian local $\mathbb{Z}_p$-algebras having finite residue field. All our rings of coefficients will be objects of $\mathrm{Art}(\mathbb{Z}_p)$.

We freely employ the terminology and notation that was introduced in the paper [10]. In particular, if V is a representation of a p-adic analytic group G over an object A of $\mathrm{Art}(\mathbb{Z}_p)$, then we let V_{sm} (resp. $V_{\mathrm{l.adm}}$) denote the G-subrepresentation of V consisting of the smooth (resp. locally admissible) vectors in V (see [10, Defs. 2.2.1, 2.2.15]).

1.3. **Acknowledgments.** I would like to thank Pierre Colmez, whose talk at the workshop on p-adic representations held in Montréal in September 2005 inspired me to develop the ideas of this paper, and to undertake the calculations of Section 4. I would also like to thank Adrian Iovita and Henri Darmon, as well as the Centre de Recherches Mathématiques at the University of Montréal, for organizing and hosting this wonderful workshop, during which many of the results presented here were first worked out. I thank the referee for their careful reading of the paper, which led to several corrections and clarifications of the text, and also Florian Herzig, who provided several useful corrections to and comments on an earlier version of the paper. Finally, I thank Vytas Paškūnas for his interest in this work, and for allowing [11] to be included as an appendix to this paper.

2. Injective and acyclic objects

Throughout this section, we let G denote a p-adic analytic group, and A an object of $\mathrm{Art}(\mathbb{Z}_p)$.

2.1. **Injective objects.** In this subsection we present some basic results regarding the existence and properties of injective objects in various categories of G-representations over A. These properties are largely standard, and presumably well-known. We recall them here, with proofs, for the sake of completeness.

2.1.1. **Proposition.** *Each of the full subcategories* $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ *and* $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$ *of* $\mathrm{Mod}_G(A)$ *has enough injectives.*

Proof. Certainly the category $\mathrm{Mod}_G(A)$ has enough injectives (since it coincides with the category of left modules over the ring $A[G]$). The functor $V \mapsto V_{\mathrm{sm}}$ is right adjoint to the natural embedding $\mathrm{Mod}_G^{\mathrm{sm}}(A) \hookrightarrow \mathrm{Mod}_G(A)$. Thus it takes injective objects in $\mathrm{Mod}_G(A)$ to injective objects in $\mathrm{Mod}_G^{\mathrm{sm}}(A)$. In particular, if V is an object of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, and $V \hookrightarrow I$ an $A[G]$-linear embedding of V into an injective object of $\mathrm{Mod}_G(A)$, then the induced embedding $V \hookrightarrow I_{\mathrm{sm}}$ is an embedding of V into an injective object of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$. The proof in the case of $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$ proceeds similarly, using the functor $V_{\mathrm{l.adm}}$. □

2.1.2. **Proposition.** *If I is an injective object of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, and H is an open subgroup of G, then I is also injective when regarded as an object of $\mathrm{Mod}_H^{\mathrm{sm}}(A)$.*

Proof. The forgetful functor $\mathrm{Mod}_G^{\mathrm{sm}}(A) \to \mathrm{Mod}_H^{\mathrm{sm}}(A)$ is right adjoint to the exact functor $V \mapsto A[G] \otimes_{A[H]} V$. (This functor is naturally isomorphic to compact induction, $V \mapsto c\text{−}\mathrm{Ind}_H^G V$, and so takes smooth H-representations to smooth G-representations.) Thus it takes injectives to injectives. □

2.1.3. **Proposition.** *If G is compact, then an inductive limit of injective objects of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ is again an injective object of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$.*

Proof. Let $\{I_i\}$ be a directed system of injective object of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, let $V \hookrightarrow W$ be an embedding of objects of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, and suppose given a G-equivariant map

$$V \to \varinjlim_i I_i. \tag{2.1.4}$$

We must show that it extends to a map $W \to \varinjlim_i I_i$. By the usual argument with Zorn's lemma, it suffices to show that if $w \in W$, and if we write $W_0 = A[G]w$, then (2.1.4) extends to the G-subrepresentation $V + W_0$ of W. It is evidently equivalent to show that the restriction to $V_0 := V \bigcap W_0$ of (2.1.4) extends to W_0. Since G is compact and W is smooth, we see that W_0, and thus V_0, is finitely generated over A, and in particular over the completed group ring $A[[G]]$. Since $A[[G]]$ is Noetherian [13], V_0 is in fact finitely presented over $A[[G]]$. Thus the restriction to V_0 of (2.1.4) lifts to a map $V_0 \to I_i$ for some (sufficiently large) value of i. (See the following lemma.) We may then extend this map to W_0, since I_i is injective. Composing such an extension with the natural map $I_i \to \varinjlim_i I_i$, we obtain a G-equivariant map $W_0 \to \varinjlim_i I_i$ which extends the restriction to V_0 of (2.1.4), as required. □

The following lemma is a standard piece of algebra, whose proof we recall for the benefit of the reader.

2.1.5. **Lemma.** *Let R be an associate ring with unit, and let M be a finitely presented left R-module. If $\{N_i\}$ is any directed system of left R-modules, then the natural map*

$$\varinjlim_i \mathrm{Hom}_R(M, N_i) \to \mathrm{Hom}_R(M, \varinjlim_i N_i) \tag{2.1.6}$$

is an isomorphism.

Proof. Fix a finite presentation $R^s \to R^r \to M \to 0$ of M. Applying $\mathrm{Hom}_R(-, N_i)$ to this exact sequence, and taking into account the natural isomorphism

$$\mathrm{Hom}_R(R, N) \xrightarrow{\sim} N, \tag{2.1.7}$$

which holds for any R-module N, we obtain an exact sequence

$$0 \to \mathrm{Hom}_R(M, N_i) \to N_i^r \to N_i^s.$$

Passing to the inductive limit over all N_i yields an exact sequence

$$0 \to \varinjlim_i \mathrm{Hom}_R(M, N_i) \to \varinjlim_i N_i^r \to \varinjlim_i N_i^s.$$

Applying $\mathrm{Hom}_R(-, \varinjlim_i N_i)$ to the presentation of M, and again taking into account the natural isomorphism (2.1.7), we obtain an exact sequence

$$0 \to \mathrm{Hom}_R(M, \varinjlim_i N_i) \to (\varinjlim_i N_i)^r \to (\varinjlim_i N_i)^s.$$

The preceding two exact sequences fit into the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \varinjlim_i \mathrm{Hom}_R(M, N_i) & \longrightarrow & \varinjlim_i N_i^r & \longrightarrow & \varinjlim_i N_i^s \\ & \downarrow{\scriptstyle (2.1.6)} & & \downarrow & & \downarrow \\ 0 \longrightarrow & \mathrm{Hom}_R(M, \varinjlim_i N_i) & \longrightarrow & (\varinjlim_i N_i)^r & \longrightarrow & (\varinjlim_i N_i)^s, \end{array}$$

in which the second and third arrows are clearly isomorphisms. Thus so is the first, as claimed. □

2.1.8. **Remark.** If the transition maps in the directed system $\{N_i\}$ are injective, then the map (2.1.6) will be an isomorphism provided that M is finitely generated. One can easily strengthen Lemma 2.1.5 to show that (2.1.6) is an isomorphism for every directed system $\{N_i\}$ if and only if M is a finitely presented R-module.

2.1.9. **Proposition.** *If G is compact, then the category $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ has enough injectives, and injective objects in this category are also injective in the category $\mathrm{Mod}_G^{\mathrm{sm}}(A)$.*

Proof. If V is an admissible smooth G-representation, then its Pontrjagin dual V^* is a finitely generated $A[[G]]$-module. Fixing a surjection $A[[G]]^r \to V^*$, and dualizing, we obtain an embedding $V \hookrightarrow \mathcal{C}^{\mathrm{sm}}(G, A^*)^r$, where A^* is the Pontrjagin dual of A, thought of as an A-module, and $\mathcal{C}^{\mathrm{sm}}(G, A^*)$ denotes the space of smooth A^*-valued functions on G (regarded as a smooth admissible G-representation over A via the right regular G-action). Since $A[[G]]^r$ is free of finite rank as an $A[[G]]$-module, one easily verifies that $\mathcal{C}^{\mathrm{sm}}(G, A^*)$ is injective in the category $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, and so also in the category $\mathrm{Mod}_G^{\mathrm{adm}}(A)$. Thus any object of $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ embeds into an injective object of $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ that is also injective in $\mathrm{Mod}_G^{\mathrm{sm}}(A)$. In particular, $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ has enough injectives. The following lemma then implies that *every* injective object of $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ is injective in $\mathrm{Mod}_G^{\mathrm{sm}}(A)$. □

2.1.10. **Lemma.** *Let $\mathcal{C}$ and $\mathcal{C}'$ be abelian categories, and let $F : \mathcal{C} \to \mathcal{C}'$ be an additive functor. Suppose that every object X of $\mathcal{C}$ admits a monomorphism $X \hookrightarrow J$ for some J such that $F(J)$ is an injective object of $\mathcal{C}'$. Then if I is any injective object of $\mathcal{C}$, the object $F(I)$ is injective in $\mathcal{C}'$.*

Proof. Let I be an injective object of $\mathcal{C}$, and choose a monomorphism $I \hookrightarrow J$, where $F(J)$ is injective in $\mathcal{C}'$. The injectivity of I in $\mathcal{C}$ implies that this monomorphism splits, and thus that I is a direct summand of J. Consequently, $F(I)$ is a direct summand of $F(J)$, and hence is also injective in $\mathcal{C}'$. □

2.1.11. **Proposition.** *If H is a compact subgroup of G, then any injective object of* $\mathrm{Mod}^{\mathrm{sm}}_G(A)$ *is also injective as an object of* $\mathrm{Mod}^{\mathrm{sm}}_H(A)$.

Proof. Since H is compact, we may find a compact open subgroup G' of G containing H. Appealing to Proposition 2.1.2, we see that we may replace G by G', and thus assume that G is also compact. We do so for the remainder of the proof.

Since G is profinite, we may find a continuous section to the quotient map $G \to G/H$, and thus find an isomorphism of right homogeneous H-spaces $G \xrightarrow{\sim} (G/H) \times H$ (where the H-action on the target is defined via its action on the second factor). Thus we obtain an H-equivariant isomorphism

$$\mathcal{C}^{\mathrm{sm}}(G, A^*) \xrightarrow{\sim} \mathcal{C}^{\mathrm{sm}}(G/H, A) \otimes_A \mathcal{C}^{\mathrm{sm}}(H, A^*),$$

where A^* denotes the Pontrjagin dual of A, though of as an A-module. As we observed in the proof of Proposition 2.1.9, $\mathcal{C}^{\mathrm{sm}}(H, A^*)$ is injective in $\mathrm{Mod}^{\mathrm{sm}}_H(A)$, and thus so is $\mathcal{C}^{\mathrm{sm}}(G/H, A) \otimes_A \mathcal{C}^{\mathrm{sm}}(H, A^*)$, by Proposition 2.1.3. Thus $\mathcal{C}^{\mathrm{sm}}(G, A^*)$ is injective in $\mathrm{Mod}^{\mathrm{sm}}_H(A)$. The proof of the preceding proposition shows that $\mathcal{C}^{\mathrm{sm}}(G, A^*)$ in injective in $\mathrm{Mod}^{\mathrm{sm}}_G(A)$. It follows from Proposition 2.1.3 that a direct sum of any number of copies of $\mathcal{C}^{\mathrm{sm}}(G, A^*)$ is again injective in both $\mathrm{Mod}^{\mathrm{sm}}_G(A)$ and $\mathrm{Mod}^{\mathrm{sm}}_H(A)$.

As noted in the proof of the preceding proposition, any admissible smooth representation admits an embedding into $\mathcal{C}^{\mathrm{sm}}(G, A^*)^r$ for some $r \geq 0$. Since any object of $\mathrm{Mod}^{\mathrm{sm}}_G(A)$ is an inductive limit of admissible smooth representations (G is compact), any object of $\mathrm{Mod}^{\mathrm{sm}}_G(A)$ admits an embedding into direct sum of (a possibly infinite number of) copies of $\mathcal{C}^{\mathrm{sm}}(G, A^*)$. Thus any object of $\mathrm{Mod}^{\mathrm{sm}}_G(A)$ admits an embedding into an injective object of $\mathrm{Mod}^{\mathrm{sm}}_G(A)$ that is also injective in $\mathrm{Mod}^{\mathrm{sm}}_H(A)$. It follows that any injective object of $\mathrm{Mod}^{\mathrm{sm}}_G(A)$ is also injective as an object of $\mathrm{Mod}^{\mathrm{sm}}_H(A)$, by Lemma 2.1.10. ☐

2.2. **Continuous cohomology as derived functors.** Since, by Proposition 2.1.1, the category $\mathrm{Mod}^{\mathrm{sm}}_G(A)$ has enough injectives, we may define the derived functors of the functor of G-invariants on $\mathrm{Mod}^{\mathrm{sm}}_G(A)$. Of course, these derived functors coincide with the Ext functors $\mathrm{Ext}^{\bullet}_G(\underline{1}, \text{–})$ (computed in $\mathrm{Mod}^{\mathrm{sm}}_G(A)$, of course), where $\underline{1}$ denotes A equipped with the trivial G-action.

If V is an object of $\mathrm{Mod}^{\mathrm{sm}}_G(A)$, then we let $H^i(G, V)$ denote the cohomology of G computed using continuous cochains taking values in V, where V is equipped with its discrete topology. The functors $H^i(G, \text{–})$ form a δ-functor on $\mathrm{Mod}^{\mathrm{sm}}_G(A)$. (Since any surjection of smooth G-representations certainly admits a set-theoretic splitting, which is automatically continuous for the discrete topologies on its source and target, one sees that a short exact sequence in $\mathrm{Mod}^{\mathrm{sm}}_G(A)$ gives rise to a corresponding short exact sequence of continuous cochain complexes, and hence to a long exact sequence of continuous cohomology modules.) Of course for any object V of $\mathrm{Mod}^{\mathrm{sm}}_G(A)$, we have the equalities $H^0(G, V) = \mathrm{Hom}_G(\underline{1}, V) = V^G$ (the space of G-invariants), and so the universality of derived functors as δ-functors gives rise to a natural transformation of δ-functors

$$\mathrm{Ext}^{\bullet}_G(\underline{1}, \text{–}) \to H^{\bullet}(G, \text{–}). \tag{2.2.1}$$

2.2.2. **Proposition.** *The natural transformation*

$$\mathrm{Ext}^1_G(\underline{1}, \text{–}) \to H^1(G, \text{–}) \tag{2.2.3}$$

is in fact a natural equivalence.

Proof. For any object V of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, the A-module $\mathrm{Ext}^1_G(\underline{1}, V)$ classifies extensions of $\underline{1}$ by V in the category $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, while it is well-known, and easily verified, that for any object V of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, the A-module $H^1(G, V)$ classifies extensions of $\underline{1}$ by V in the category of Hausdorff topological G-modules over A. The map (2.2.3) is then the evident one, which maps an extension in $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ to the corresponding extension in the category of topological modules (each member of the exact sequence being endowed with its discrete topology). This map is easily checked to be injective. (Alternatively, this is a general fact from the theory of δ-functors.) Furthermore, any Hausdorff topological extension is easily verified to be smooth, and to necessarily be equipped with its discrete topology. Thus (2.2.3) is also surjective. This proves the proposition. $\square$

2.2.4. **Corollary.** *The natural transformation* $\mathrm{Ext}^2_G(\underline{1}, V) \to H^2(G, V)$ *is injective for any object* V *of* $\mathrm{Mod}_G^{\mathrm{sm}}(A)$.

Proof. It is a general fact from the theory of δ-functors (and is easily verified via a dimension-shifting argument) that if the map (2.2.1) is an isomorphism in degrees $i \leq n$, then it is injective in degree $i = n+1$. The lemma follows from this together with the preceding proposition. $\square$

Nothing is known (to the author at least) regarding the nature of the map (2.2.1) in degrees > 2 in general. The difficulty is that if V is an object of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, then the natural way to efface the completed cohomology of V is via the map

$$V \to \mathcal{C}^{\mathrm{sm}}(G, V), \tag{2.2.5}$$

where the target is the space of continuous (or equivalently, locally constant) V-valued functions on G, equipped with its compact-open topology, and the map sends an element v of V to the corresponding orbit function $g \mapsto gv$; see e.g. [12, 17]. If G is not compact, then the topological G-module $\mathcal{C}^{\mathrm{sm}}(G, V)$ is neither discrete nor smooth as a G-representation; thus it does not provide an effacement of the completed cohomology of V in the category $\mathrm{Mod}_G^{\mathrm{sm}}(A)$. On the other hand, if G is compact, then $\mathcal{C}^{\mathrm{sm}}(G, V)$ *is* a discrete, smooth G-representation, and so we obtain the following well-known result.

2.2.6. **Proposition.** *If G is compact, then the natural transformation* (2.2.1) *is a natural equivalence of δ-functors.*

Proof. The proof is precisely the one sketched above: namely, the reader may easily check that if V is an object of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, regarded as a discrete topological G-representation, then $\mathcal{C}^{\mathrm{sm}}(G, V)$ is discrete and smooth when equipped with its compact-open topology. (The smoothness follows from the fact that since G is compact and V is discrete, a map from G to V is continuous if and only if it is locally constant if and only if it is uniformly locally constant.) The map (2.2.5) then effaces the higher continuous cohomology of V [12, §2]. Thus $H^i(G, –)$ is an effaceable δ-functor, and hence universal; the proposition follows. $\square$

As a point of terminology, we say that an object V of $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ is G-acyclic if $H^i(G, V)$ vanishes for $i > 0$.

3. The δ-functor $H^\bullet \mathrm{Ord}_P$

Throughout this section, we let G denote (the group of $\mathbb{Q}_p$-valued points of) a connected reductive linear algebraic group over $\mathbb{Q}_p$, and let P denote a parabolic subgroup of G. We also fix a Levi factorization $P = MN$, and let $\overline{P} = M\overline{N}$ denote an opposite parabolic to P, chosen so that $P \bigcap \overline{P} = M$.

Our goal in this section is to define a certain δ-functor $H^\bullet\mathrm{Ord}_P$ from the category of (locally) admissible G-representations to the category of (locally) admissible M-representations, for which $H^0\mathrm{Ord}_P$ is naturally isomorphic to the functor Ord_P. We conjecture that $H^\bullet\mathrm{Ord}_P$ is the universal δ-functor extending Ord_P (and thus coincides with the derived functor of Ord_P on the category $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$), although we cannot prove this. Nevertheless, $H^\bullet\mathrm{Ord}_P$ is quite computable in some situations (as we will see in the following section), and provides a useful tool for investigating homological questions related to the functor Ord_P.

3.1. **Some properties of Z_M.** We let Z_M denote the centre of the Levi factor M of P. The group Z_M, together with certain of its submonoids, plays an important role in the definition of the functor Ord_P. In this subsection we establish some structural results about Z_M and some of its submonoids which we will need to apply later.

Let Z_0 denote the maximal compact subgroup of Z_M, and let $Z_G \subset Z_M$ denote the centre of G. Let Z_M^0 denote (the group of $\mathbb{Q}_p$-points of) the maximal connected subgroup of (the algebraic group underlying) Z_M, so that Z_M^0 is (the group of $\mathbb{Q}_p$-points of) a torus. Let S denote (the group of $\mathbb{Q}_p$-points of) the maximal split torus in Z_M^0, and write $S_0 := S \bigcap Z_0$ to denote the maximal compact subgroup of S, and $S_G := S \bigcap Z_G$ to denote the maximal split torus in Z_G. If Δ denotes the set of positive simple restricted roots appearing in the action of S on the Lie algebra of N, then there is a natural embedding

$$S/S_0S_G \hookrightarrow \mathbb{Z}^\Delta, \tag{3.1.1}$$

with finite cokernel, given by $z \mapsto \big(\mathrm{ord}_p(\alpha(z))\big)_{\alpha\in\Delta}$. (Here ord_p denotes the p-adic ordinal on $\mathbb{Q}_p^\times$.) Since Z_M contains Z_M^0 with finite index, while Z_M^0 (like any torus) is isogenous to the product of a split torus and a compact torus, we see that the natural embedding $S/S_0S_G \hookrightarrow Z_M/Z_0Z_G$ has finite cokernel. Thus (noting that Z_M/Z_0Z_G is a finitely generated torsion-free abelian group) the embedding (3.1.1) extends to an embedding

$$Z_M/Z_0Z_G \hookrightarrow (\frac{1}{d}\mathbb{Z})^\Delta, \tag{3.1.2}$$

for an appropriately chosen $d \geq 1$. We again denote this embedding

$$z \mapsto \big(\mathrm{ord}_p(\alpha(z))\big)_{\alpha\in\Delta}.$$

3.1.3. **Lemma.** *Let N_0 be any compact open subgroup of N.*

1. *If $z \in Z_M$ has the property that $zN_0z^{-1} \subset N_0$, then $\mathrm{ord}_p(\alpha(z)) \geq 0$ for all $\alpha \in \Delta$.*
2. *An element $z \in Z_M$ has the property that the conjugates $z^iN_0z^{-i}$ form a basis of neighbourhoods of the identity of N if and only if $\mathrm{ord}_p(\alpha(z)) > 0$ for all $\alpha \in \Delta$.*
3. *There exists $C \geq 0$ such that if $z \in Z_M$ and $\mathrm{ord}_p(\alpha(z)) \geq C$ for all $\alpha \in \Delta$, then $zN_0z^{-1} \subset N_0$.*
4. *The intersection $\bigcap_{\substack{z \text{ s.t. all} \\ \mathrm{ord}_p(\alpha(z))\leq 0}} zN_0z^{-1}$ (where, as indicated, the intersection is taken over all $z \in Z_M$ such that $\mathrm{ord}_p(\alpha(z)) \leq 0$ for all $\alpha \in \Delta$) is an open subgroup of N_0.*

Proof. Decompose $\mathfrak{n}$ as a sum of restricted root spaces $\mathfrak{n} = \oplus_{\alpha\in\Delta}\mathfrak{n}_\alpha$. Choose a bounded $\mathbb{Z}_p$-lattice $\mathfrak{n}_1 \subset \mathfrak{n}$, write $(\mathfrak{n}_1)_\alpha := \mathfrak{n}_1 \cap \mathfrak{n}_\alpha$ for each $\alpha \in \Delta$, and set $\mathfrak{n}_2 := \oplus_{\alpha\in\Delta}(\mathfrak{n}_0)_\alpha$; then $\mathfrak{n}_2$ is again a bounded $\mathbb{Z}_p$-lattice in $\mathfrak{n}$. For any $z \in Z_M$, it is clear that the collection $\{\mathrm{Ad}_{z^j}(\mathfrak{n}_2)\}_{j\geq 0}$ forms a uniformly bounded collection of subsets of $\mathfrak{n}$ (resp. a neighbourhood basis of 0 in $\mathfrak{n}$) if and only if $z \in Z_M$ satisfies $\mathrm{ord}_p(\alpha(z)) \geq 0$ (resp. > 0) for all $\alpha \in \Delta$, while $\bigcap_{j\geq 0}\mathrm{Ad}_{z^j}(\mathfrak{n}_2)$ is open if and only if $\mathrm{ord}_p(\alpha(z)) \leq 0$ for all $\alpha \in \Delta$. If Ω is any compact open subset of $\mathfrak{n}$, then we may find integers $i \leq j$ such that $p^i\Omega \subset \mathfrak{n}_2 \subset p^j\Omega$. Thus we similarly conclude that the collection $\{\mathrm{Ad}_{z^j}(\Omega)\}_{j\geq 0}$ forms a uniformly bounded collection of subsets of $\mathfrak{n}$ (resp. a neighbourhood basis of 0 in $\mathfrak{n}$) if and only if $z \in Z_M$ satisfies $\mathrm{ord}_p(\alpha(z)) \geq 0$ (resp. > 0) for all $\alpha \in \Delta$, and that $\bigcap_{j\geq 0}\mathrm{Ad}_{z^j}(\Omega)$ is open if and only if $\mathrm{ord}_p(\alpha(z)) \leq 0$ for all $\alpha \in \Delta$.

Recall that exponentiation provides a canonical identification of $\mathfrak{n}$ with N. This identification is Z_M-equivariant, where Z_M acts on $\mathfrak{n}$ via the adjoint map, and on N via conjugation. From the discussion of the preceding paragraph, we conclude that for any compact open subset Ψ of N, the collection $\{z^j\Psi z^{-j}\}_{j\geq 0}$ forms a uniformly bounded collection of subsets of N (resp. a neighbourhood basis of the identity in N) if and only if $z \in Z_M$ satisfies $\mathrm{ord}_p(\alpha(z)) \geq 0$ (resp. > 0) for all $\alpha \in \Delta$, and that $\bigcap_{j\geq 0} z^j\Psi z^{-j}$ is open if and only if $\mathrm{ord}_p(\alpha(z)) \leq 0$ for all $\alpha \in \Delta$. The lemma follows upon taking Ψ to be N_0. □

This lemma has an evident analogue for subgroups of $\overline{N}$, which we now state, and which is proved in an identical fashion.

3.1.4. **Lemma.** *Let $\overline{N}_0$ be any compact open subgroup of $\overline{N}$.*

(1) *If $z \in Z_M$ has the property that $z\overline{N}_0 z^{-1} \subset \overline{N}_0$, then $\mathrm{ord}_p(\alpha(z)) \leq 0$ for all $\alpha \in \Delta$.*
(2) *An element $z \in Z_M$ has the property that the conjugates $z^i\overline{N}_0 z^{-i}$ form a basis of neighbourhoods of the identity of $\overline{N}$ if and only if $\mathrm{ord}_p(\alpha(z)) < 0$ for all $\alpha \in \Delta$.*
(3) *There exists $C \leq 0$ such that if $z \in Z_M$ and $\mathrm{ord}_p(\alpha(z)) \leq C$ for all $\alpha \in \Delta$, then $z\overline{N}_0 z^{-1} \subset \overline{N}_0$.*
(4) *The intersection $\bigcap_{\substack{z \text{ s.t. all} \\ \mathrm{ord}_p(\alpha(z))\geq 0}} z\overline{N}_0 z^{-1}$ (where, as indicated, the intersection is taken over all $z \in Z_M^+$ such that $\mathrm{ord}_p(\alpha(z)) \geq 0$ for all $\alpha \in \Delta$) is an open subgroup of $\overline{N}_0$.*

3.1.5. **Lemma.** *If N_0 (resp. $\overline{N}_0$) is a compact open subgroup of N (resp. $\overline{N}$), and if we write $Z_M^+ := \{z \in Z_M \mid zN_0z \subset N_0\}$, then we may choose $z \in Z_M^+$ so that:*

(1) *Z_M^+ together with z^{-1} generate Z_M as a monoid.*
(2) *$z^{-1}\overline{N}_0 z \subset \overline{N}_0$.*
(3) *The descending sequence of subgroups $z^{-i}\overline{N}_0 z^i$ forms a neighbourhood basis of the identity in $\overline{N}$.*

Proof. If $z \in Z_M$ is chosen so that $\mathrm{ord}_p(\alpha(z))$ is sufficiently large for all $\alpha \in \Delta$, then it follows from Lemmas 3.1.3 (3) and 3.1.4 (2), (3) that $zN_0z^{-1} \subset N_0$, that $z^{-1}\overline{N}_0 z \subset \overline{N}_0$, and that the descending sequence of subgroups $z^{-i}\overline{N}_0 z^i$ forms a neighbourhood basis of the identity in $\overline{N}$. If z' is any element of Z_M, then $\mathrm{ord}_p(\alpha(z^i z'))$ can be made arbitrarily large for all $\alpha \in \Delta$ by taking i large enough, and so, again by Lemma 3.1.3 (3), we see that $z^i z' \in Z_M^+$ if i is large enough. Thus

$z' \in z^{-i} Z_M^+$ if i is large enough, and so z^{-1} and Z_M^+ together generate Z_M as a monoid. $\square$

3.2. **A review of the functor** Ord_P**.** Recall from [10, §3.1] that in order to define Ord_P, we must first choose an open subgroup P_0 of P. We fix such a choice, and following [10, §3.1], we write $M_0 := M \bigcap P_0$, $N_0 := N \bigcap P_0$, and $M^+ := \{m \in M \mid m N_0 m^{-1} \subset N_0\}$. As in the preceding subsection, we let Z_M denote the centre of M, and we write $Z_M^+ := Z_M \bigcap M^+$.

If V is any object of $\mathrm{Mod}_P^{\mathrm{sm}}(A)$, then there is a natural action (the "Hecke action") of M^+ on V^{N_0} [10, §3.1]. We then recall from [10, Def. 3.1.9] that

$$\mathrm{Ord}_P(V) := \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], V^{N_0})_{Z_M-\mathrm{fin}}.$$

The remainder of this subsection is devoted to showing that, in some situations, we may compute $\mathrm{Ord}_P(V)$ using a single (well-chosen) element $z \in Z_M^+$, rather than the entire monoid Z_M^+. Fix an element such that Z_M^+ together with z^{-1} generate Z_M as a monoid, as we may, by Lemma 3.1.5. Let C denote the cyclic subgroup of Z_M generated by z, and write $C^+ = Z_M^+ \bigcap C$ (so C^+ is the cyclic monoid generated by z). We regard V^{N_0} as an $A[C^+]$-module via the inclusion $C^+ \subset Z_M^+$.

We adapt the terminology and notation of [10, Def. 2.3.1] in an evident way to the subgroup C of Z_M; namely, we say that an element in a C-representation V is locally C-finite if the $A[C]$-module that it generates is finitely generated over A, and we write $V_{C-\mathrm{fin}}$ to denote the submodule of V consisting of locally C-finite vectors.

3.2.1. **Lemma.** *Let $\mathcal{X}$ denote the full subcategory of the category of $A[Z_M^+]$-modules consisting of modules which may be written as the union of a collection of finitely generated Z_M^+-invariant A-modules.*

(1) *For any object X of $\mathcal{X}$, the composite*

$$\mathrm{Hom}_{A[C^+]}(A[C], X)_{C-\mathrm{fin}} \to X \to A[C] \otimes_{A[C^+]} X,$$

where the first arrow is given by evaluation at $1 \in A[C]$, and the second arrow is the natural one, is an isomorphism, as is the composite

$$\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], X)_{Z_M-\mathrm{fin}} \to X \to A[Z_M] \otimes_{A[Z_M^+]} X,$$

defined in the same manner.

(2) *For any object X of $\mathcal{X}$, the restriction map*

$$\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], X)_{Z_M-\mathrm{fin}} \to \mathrm{Hom}_{A[C^+]}(A[C], X)_{C-\mathrm{fin}}$$

is an isomorphism.

(3) *The functor $X \mapsto \mathrm{Hom}_{A[Z_M^+]}(A[Z_M], X)_{Z_M-\mathrm{fin}}$ is exact on the category $\mathcal{X}$.*

Proof. We begin by proving the first claim of (1). If X is an object of $\mathcal{X}$, then by assumption we may write it as the union of finitely generated A-modules that are Z_M^+-invariant, and so in particular C^+-invariant. The functor $\mathrm{Hom}_{A[C^+]}(A[C], –)$ commutes with the formation of inductive limits, by (an evident analogue for C^+ and C of) [10, Lem. 3.2.2], as does the functor $A[C] \otimes_{A[C^+]} –$. Thus, in proving (1), we may assume that X is finitely generated over A.

Let B denote the image of $A[C^+]$ in $\mathrm{End}_A(X)$; then B is a finite A-algebra, and the $A[C^+]$-structure on X allows us to regard it as a B-module. We may write B as a product of local factors $B = \prod_{\mathfrak{m}} B_{\mathfrak{m}}$, where $\mathfrak{m}$ runs over the finite set of maximal ideals of B. We say that a maximal ideal $\mathfrak{m}$ is ordinary if the image of z in B does not lie in $\mathfrak{m}$, and write

$$B_{\mathrm{ord}} := \prod_{\mathfrak{m} \text{ ordinary}} B_{\mathfrak{m}}, \qquad B_{\mathrm{nil}} := \prod_{\mathfrak{m} \text{ not ordinary}} B_{\mathfrak{m}},$$

so that $B = B_{\mathrm{ord}} \times B_{\mathrm{nil}}$. We also write $X_{\mathrm{ord}} := B_{\mathrm{ord}} \otimes_B X$ and $X_{\mathrm{nil}} := B_{\mathrm{nil}} \otimes_B X$, so that we have the C^+-invariant decomposition $X = X_{\mathrm{ord}} \times X_{\mathrm{nil}}$. The element z acts invertibly on X_{ord}, while some sufficiently high power of z annihilates X_{nil}. Given this, the reader may easily verify that there are isomorphisms

$$\begin{aligned}\mathrm{Hom}_{A[C^+]}(A[C], X)_{C-\mathrm{fin}} = \mathrm{Hom}_{A[C^+]}(A[C], X) &\xrightarrow{\sim} \mathrm{Hom}_{A[C^+]}(A[C], X_{\mathrm{ord}}) \\ &\xrightarrow{\sim} X_{\mathrm{ord}} \xrightarrow{\sim} A[C] \otimes_{A[C^+]} X_{\mathrm{ord}} \xrightarrow{\sim} A[C] \otimes_{A[C^+]} X.\end{aligned}$$

(Note that the equality is obvious, since X is finitely generated over A.) Thus the first claim of (1) is proved.

We turn to proving (2), and thus we again suppose that X is an arbitrary object of $\mathcal{X}$. Since Z_M^+ and C generate Z_M as a monoid, the natural map $A[Z_M^+] \otimes_{A[C^+]} A[C] \to A[Z_M]$ is an isomorphism, which in turn implies that the restriction map

$$\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], X) \to \mathrm{Hom}_{A[C^+]}(A[C], X) \tag{3.2.2}$$

is an isomorphism, and also that the natural map

$$A[C] \otimes_{A[C^+]} X \to A[Z_M] \otimes_{A[Z_M^+]} X \tag{3.2.3}$$

is an isomorphism. By [10, Lem 3.1.6], together with an evident variant with C^+ and C in place of Z_M^+ and Z_M, we see that an element ϕ in the source (resp. the target) of (3.2.2) is locally Z_M-finite (resp. locally C-finite) if and only if its image is finitely generated as an A-module. Since $Z_M = Z_M^+ C$, and since X is a union of finitely generated Z_M^+-invariant A-modules, we conclude that the isomorphism (3.2.2) restricts to an isomorphism

$$\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], X)_{Z_M-\mathrm{fin}} \xrightarrow{\sim} \mathrm{Hom}_{A[C^+]}(A[C], X)_{C-\mathrm{fin}},$$

proving (2).

We may now deduce the second claim of (1). Indeed, it follows from the first claim of (1), together with the isomorphism of (2) and the isomorphism (3.2.3). Claim (3) then follows as well, since the functor $A[Z_M] \otimes_{A[Z_M^+]} -$ is exact (as $A[Z_M]$ is a localization of $A[Z_M^+]$). □

3.3. **The definition of $H^\bullet \mathrm{Ord}_P$.** Fix P_0 as in the preceding subsection, and define N_0, M_0, M^+, etc., as in that subsection. Let V be an object of $\mathrm{Mod}_P^{\mathrm{sm}}(A)$, and let $V \hookrightarrow I^\bullet$ be an injective resolution of V in the category $\mathrm{Mod}_P^{\mathrm{sm}}(A)$. Such a resolution exists, by Proposition 2.1.1, and Proposition 2.1.11 show that such a resolution is also an N_0-acyclic resolution of V (since N_0 is a compact subgroup of P). Thus $(I^\bullet)^{N_0}$ computes the N_0-cohomology $H^\bullet(N_0, V)$. The complex $(I^\bullet)^{N_0}$ is naturally a complex of $A[M^+]$-modules (via the Hecke M^+-action), and thus each cohomology module $H^\bullet(N_0, V)$ is also naturally an $A[M^+]$-module.

3.3.1. **Definition.** For any object V of $\mathrm{Mod}_P^{\mathrm{sm}}(A)$, we define $H^\bullet \mathrm{Ord}_P(V) := \mathrm{Hom}_{A[Z_M^+]}\big(A[Z_M], H^\bullet(N_0, V)\big)_{Z_M-\mathrm{fin}}$.

Note that we may compute the M^+-action on $H^\bullet(N_0, V)$, and hence compute $H^\bullet \mathrm{Ord}_P(V)$, using any resolution of V in the category $\mathrm{Mod}_P^{\mathrm{sm}}(A)$ by objects that are acyclic as N_0-representations.

Our first goal is to show that the computation of $H^\bullet \mathrm{Ord}_P$ is independent of the choice of P_0. To this end, suppose that P_0' is another compact open subgroup of P, giving rise to N_0', $(M^+)'$, and $(Z_M^+)'$ in analogy to N_0, M^+, and Z_M^+. Write $Y' := M^+ \bigcap (M^+)'$ and $Y := Z_M^+ \bigcap (Z_M^+)'$; then Y generates Z_M as a group [8, Prop. 3.3.2 (i)], and Y' and Z_M generate M [8, Prop. 3.3.2 (ii)].

Choose $z \in Z_M$ such that $z N_0' z^{-1} \subset N_0$. If V is an object of $\mathrm{Mod}_P^{\mathrm{sm}}(A)$, and $I^\bullet$ is an injective resolution of V, then [10, Lem. 3.1.11] gives rise to a Y'-equivariant map

$$h_{N_0, N_0', z} : (I^\bullet)^{N_0'} \to (I^\bullet)^{N_0}.$$

Since Y generates Z_M as a group, the inclusions

$$\mathrm{Hom}_{A[(Z_M^+)']}\big(A[Z_M], H^\bullet(N_0', V)\big) \subset \mathrm{Hom}_{A[Y]}\big(A[Z_M], H^\bullet(N_0', V)\big)$$

and

$$\mathrm{Hom}_{A[Z_M^+]}\big(A[Z_M], H^\bullet(N_0, V)\big) \subset \mathrm{Hom}_{A[Y]}\big(A[Z_M], H^\bullet(N_0, V)\big)$$

are equalities. The Y'-equivariant map $h_{N_0, N_0, z}$ induces an $M = Y' Z_M$-equivariant map

$$\mathrm{Hom}_{A[Y]}\big(A[Z_M], H^\bullet(N_0', V)\big) \to \mathrm{Hom}_{A[Y]}\big(A[Z_M], H^\bullet(N_0, V)\big),$$

and hence an M-equivariant map

$$\mathrm{Hom}_{A[(Z_M^+)']}\big(A[Z_M], H^\bullet(N_0', V)\big) \to \mathrm{Hom}_{A[Z_M^+]}\big(A[Z_M], H^\bullet(N_0, V)\big),$$

which we denote by $H^\bullet \imath_{N_0, N_0', z}$. Now define

$$H^\bullet \imath_{N_0, N_0'} := z^{-1} H^\bullet \imath_{N_0, N_0', z} : \mathrm{Hom}_{A[(Z_M^+)']}\big(A[Z_M], H^\bullet(N_0', V)\big) \to \mathrm{Hom}_{A[Z_M^+]}\big(A[Z_M], H^\bullet(N_0, V)\big).$$

3.3.2. **Proposition.** (1) *The map $H^\bullet \imath_{N_0, N_0'}$ is well-defined independently of the choice of z.*
(2) *The map $H^\bullet \imath_{N_0, N_0'}$ is an M-equivariant isomorphism.*
(3) *If P_0'' is another choice of compact open subgroup of P, giving rise to N_0'', etc., then $H^\bullet \imath_{N_0, N_0'} H^\bullet \imath_{N_0', N_0''} = H^\bullet \imath_{N_0, N_0''}$.*

Proof. This is proved in an identical manner to [10, Prop. 3.1.12]. □

The preceding result shows that $H^\bullet \mathrm{Ord}_P$ is defined up to canonical isomorphism independently of the choice of P_0. It is also independent, up to a canonical isomorphism, of the choice of Levi factor M of P. Indeed, if M' is another choice of Levi factor of P, let $H^\bullet \mathrm{Ord}_P'$ denote the analogue of $H^\bullet \mathrm{Ord}_P$, computed using M' in place of M. There is a uniquely determined element $n \in N$ such that $n M n^{-1} = M'$.

Let V be an object of $\mathrm{Mod}_P^{\mathrm{sm}}(A)$, with injective resolution $V \hookrightarrow I^\bullet$. We will compute $H^\bullet \mathrm{Ord}_P'$ using $P_0' := n P_0 n^{-1}$. Write $N_0' := N \bigcap P_0' = n N_0 n^{-1}$. We find that multiplication by n induces an isomorphism $(I^\bullet)^{N_0} \xrightarrow{\sim} (I^\bullet)^{N_0'}$, and hence an isomorphism $H^\bullet(N_0, V) \xrightarrow{\sim} H^\bullet(N_0', V)$. One immediately verifies that this isomorphism in turn induces an isomorphism $H^\bullet \mathrm{Ord}_P(V) \xrightarrow{\sim} H^\bullet \mathrm{Ord}_P'(V)$. (See the discussion at the end of [10, §3.1].)

Suppose now that $0 \to V_1 \to V_2 \to V_3 \to 0$ is a short exact sequence of objects in $\mathrm{Mod}_P^{\mathrm{sm}}(A)$. Applying the δ-functor $H^\bullet(N_0, –)$ we obtain a long exact sequence

$$(3.3.3)\quad \begin{aligned} 0 \to H^0(N_0, V_1) \to H^0(N_0, V_2) \to H^0(N_0, V_3) \to \cdots \\ \to H^i(N_0, V_1) \to H^i(N_0, V_2) \to H^i(N_0, V_3) \\ \to H^{i+1}(N_0, V_1) \to H^{i+1}(N_0, V_2) \to H^{i+1}(N_0, V_3) \to \cdots . \end{aligned}$$

Applying the functor $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], –)_{Z_M-\mathrm{fin}}$ to this long exact sequence, we obtain the complex

$$(3.3.4)\quad \begin{aligned} 0 \to \mathrm{Ord}_P(V_1) \to \mathrm{Ord}_P(V_2) \to \mathrm{Ord}_P(V_3) \to \cdots \\ \to H^i\mathrm{Ord}_P(V_1) \to H^i\mathrm{Ord}_P(V_2) \to H^i\mathrm{Ord}_P(V_3) \\ \to H^{i+1}\mathrm{Ord}_P(V_1) \to H^{i+1}\mathrm{Ord}_P(V_2) \to H^{i+1}\mathrm{Ord}_P(V_3) \to \cdots . \end{aligned}$$

Since $\mathrm{Hom}_{A[Z_M^+]}(A[Z_M], –)_{Z_M-\mathrm{fin}}$ is not an exact functor, we are not able to conclude that (3.3.4) is exact in general.

3.4. **The main theorem.** The goal of this subsection is to prove Theorem 3.4.7 below, which establishes the key properties of $H^\bullet(N_0, V)$ and $H^\bullet\mathrm{Ord}_P(V)$ in the case when V is an admissible (or more generally, locally admissible) G-representation. In particular, it implies as a corollary that (3.3.4) *is* exact when the V_i are locally admissible G-representations. (See Corollary 3.4.8 below.)

Before proving Theorem 3.4.7, we make some preliminary constructions and prove some necessary lemmas. We fix a compact open subgroup H_0 of G, which admits an Iwahori decomposition with respect to P and $\overline{P}$, i.e., writing $M_0 := H_0 \bigcap M$, $N_0 := H_0 \bigcap N$, and $\overline{N}_0 := H_0 \bigcap \overline{N}$, such that the product map $\overline{N}_0 \times M_0 \times N_0 \to H_0$ is a bijection. We will take the open subgroup P_0 with respect to which we compute $H^\bullet\mathrm{Ord}_P$ to be $P_0 := H_0 \bigcap P = M_0 N_0$.

Next, we apply Lemma 3.1.5 to choose an element $z \in Z_M^+$ such that Z_M is generated as a monoid by Z_M^+ together with z^{-1}. We furthermore choose z (as Lemma 3.1.5 allows us to) such that $z^{-1}\overline{N}_0 z^1 \subset \overline{N}_0$, and such that the groups $z^{-i}\overline{N}_0 z^i \subset \overline{N}_0$, which then form a descending sequence of compact open subgroups of $\overline{N}$, form a neighbourhood basis of the identity in $\overline{N}$.

3.4.1. **Lemma.** *There exists a descending sequence of compact open subgroups $\{H_i\}_{i\geq 1}$ of H_0, each admitting an Iwahori decomposition $\overline{N}_i \times M_i \times N_0 \xrightarrow{\sim} H_i$, so that $\overline{N}_i = z^{-i}\overline{N}_0 z^i$, and so that the descending sequence $\{M_i\}_{i\geq 1}$ forms a basis of neighbourhoods of the identity in M.*

Proof. Write $\overline{N}_i := z^{-i}\overline{N}_0 z^i$. One sees, using the fact that $z \in Z_M^+$ (so that $z^{-i}N_0 z^i \supset N_0$, while $z^{-i}M_0 z^i = M_0$), that $z^{-i}H_0 z^i \bigcap H_0 = \overline{N}_i M_0 N_0$ is a compact open subgroup of G which again admits an Iwahori decomposition. Let H_i' denote the closed subgroup of $z^{-i}H_0 z^i \bigcap H_0$ generated by $\overline{N}_i$ and N_0. If $h \in H_i'$, then, applying the Iwahori decomposition of $z^{-i}H_0 z^i \bigcap H_0$, we may write $h = \overline{n}mn$ with $\overline{n} \in \overline{N}_i$, $m \in M_0$, and $n \in N_0$. Thus $m = \overline{n}^{-1}hn^{-1} \in H_i'$, and so H_i' admits the Iwahori decomposition $H_i' = \overline{N}_i M_i' N_0$, where $M_i' := H_i' \bigcap M$ is a closed subgroup of M_0.

Fix a descending sequence of open subgroups $\{M_j\}_{j\geq 1}$ of M_0, chosen to form a neighbourhood basis of the identity in M. For a fixed value of i, let $j(i)$ denote the largest value of j such that $M_i' \subset M_j$. Since the groups $\overline{N}_i$ form a neighbourhood basis of the identity in $\overline{N}$, we see that the subgroups M'_i become arbitrarily small as $i \to \infty$, and so $j(i) \to \infty$ as $i \to \infty$.

If we write $H_i := \overline{N}_i M_{j(i)} N_0$, then one sees (using the fact that $M_{j(i)} \subset M_0$, so that $M_{j(i)}$ normalizes both $\overline{N}_i$ and N_0, together with the fact that $M'_i \subset M_{j(i)}$) that H_i is closed under multiplication. Since it is also an open subset of the profinite subgroup H_0, it is necessarily an open subgroup of H_0, which by construction admits an Iwahori decomposition of the form required by the lemma (once we relabel $M_{j(i)}$ as M_i). □

Since $\{\overline{N}_i\}$ (resp. M_i) forms a neighbourhood basis of the identity in $\overline{N}$ (resp. M), we find that

$$\bigcap_{i \geq 0} H_i = N_0. \tag{3.4.2}$$

3.4.3. **Lemma.** *For any smooth G-representation V, there is a natural isomorphism $\varinjlim_i H^\bullet(H_i, V) \xrightarrow{\sim} H^\bullet(N_0, V)$. Furthermore, if U_i denotes the image of $H^\bullet(H_i, V)$ in $H^\bullet(N_0, V)$, then there exists $j \geq 0$ such $A[Z_M^+]U_i \subset U_{i+j}$ for all $i \geq 0$. (Here we are regarding $H^\bullet(N_0, V)$ as an $A[Z_M^+]$-module via the Hecke Z_M^+-action.)*

Proof. Let $V \hookrightarrow I^\bullet$ be an injective resolution of V in the category $\mathrm{Mod}_G^{\mathrm{sm}}(A)$. It follows from Propositions 2.1.2 that $I^\bullet$ is an injective resolution of V in $\mathrm{Mod}_{H_i}^{\mathrm{sm}}(A)$, for each $i \geq 0$. Proposition 2.1.11 shows that we may use $I^\bullet$ to compute $H^\bullet(N_0, V)$. Taking into account (3.4.2), we see that there is a natural isomorphism

$$\varinjlim_i H^\bullet\big((I^\bullet)^{H_i}\big) \xrightarrow{\sim} H^\bullet\big((I^\bullet)^{N_0}\big),$$

and consequently, a natural isomorphism

$$\varinjlim_i H^\bullet(H_i, V) \xrightarrow{\sim} H^\bullet(N_0, V).$$

This proves the first claim of the lemma.

It follows from Lemmas 3.1.3 (1) and 3.1.4 (4) that $\bigcap_{z' \in Z_M^+} z' \,\overline{N}_0 (z')^{-1}$ is an open subgroup of $\overline{N}_0$, and thus contains $\overline{N}_j$ for some sufficiently large value of j. We then have that $\overline{N}_{i+j} \subset z' \,\overline{N}_i (z')^{-1}$ for every $z' \in Z_M^+$ and every $i \geq 0$, and so from [10, Lem 3.3.2] (applied with z_0 taken to be an arbitrary element z' of Z_M^+, I_0 taken to be $z^{-j} H_i z^j \bigcap H_i = \overline{N}_{i+j} M_i N_0$, and I_1 taken to be H_i), we conclude that $A[Z_M^+](I^\bullet)^{H_i} \subset (I^\bullet)^{(H_i \bigcap z^{-j} H_i z^j)} \subset (I^\bullet)^{H_{i+j}}$ for all $i \geq 0$. This in turn implies that $A[Z_M^+]U_i \subset U_{i+j}$, completing the proof of the lemma. □

3.4.4. **Lemma.** *If V in admissible smooth G-representation, then the cohomology modules $H^\bullet(H_i, V)$ are finitely generated A-modules, for any $i \geq 0$.*

Proof. Let $V \hookrightarrow J^\bullet$ be an injective resolution of V in the category $\mathrm{Mod}_{H_0}^{\mathrm{adm}}(A)$. It follows from Propositions 2.1.2 and 2.1.9 that $J^\bullet$ is an injective resolution of V in $\mathrm{Mod}_{H_i}^{\mathrm{sm}}(A)$, for each $i \geq 1$, and so there are natural isomorphisms

$$H^\bullet(H_i, V) \xrightarrow{\sim} H^\bullet\big((J^\bullet)^{H_i}\big).$$

Since $J^\bullet$ is an admissible H_i-representation, the complex $(J^\bullet)^{H_i}$ is a complex of finitely generated A-modules. Its cohomology modules are thus also finitely generated A-modules. The lemma follows. □

As in Subsection 3.2, let $C^+ \subset Z_M^+$ (resp. $C \subset Z_M$) denote the submonoid (resp. subgroup) of Z_M^+ (resp. Z_M) generated by z. If V is any smooth G-representation, then the Hecke Z_M^+-action on V^{N_0} restricts to an action of C^+ on V^{N_0}. If $i \geq 0$, then it follows from the fact that $z^{-1}\overline{N}_i z \subset \overline{N}_i$, together with [10, Lem. 3.3.2] (applied with $z = z_0$ and $I_0 = I_1 = H_i$) that V^{H_i} is invariant under the Hecke operator $h_{N_0,z}$, for each $i \geq 0$. Thus V^{H_i} is a C^+-submodule of V^{N_0}.

Now suppose that W is a smooth H_0-representation, and that we are given a pair of H_0-equivariant maps $\eta : V \to W$ and $\psi : W \to V$. (This situation will occur in the proof of Theorem 3.4.7 below.) The space of invariants W^{N_0} is a smooth M_0-representation, since M_0 normalizes N_0. We may define an action of C^+ on W^{N_0} by declaring z to act via the endomorphism $\eta h_{N_0,z}\psi$ of W^{N_0}. Since η and ψ are H_0-equivariant, we find that W^{H_i} is also C^+-invariant for each $i \geq 0$.

Since the C^+-action on W^{N_0} commutes with the M_0-action, the space

$$\mathrm{Hom}_{A[C^+]}(A[C], W^{N_0})_{C-\mathrm{fin}}$$

is equipped in a natural manner with an M_0-action, induced by the M_0-action on W^{N_0}. In this situation we have the following lemma.

3.4.5. **Lemma.** *If W is an admissible smooth H_0-representation, then the M_0-representation $\mathrm{Hom}_{A[C^+]}(A[C], W^{N_0})_{C-\mathrm{fin}}$ is smooth and admissible.*

Proof. As was already noted in the proof of Lemma 3.2.1, there is an evident analogue of [10, Lem. 3.1.6], with C and C^+ in place of Z_M and Z_M^+. Namely, if $\phi \in \mathrm{Hom}_{A[C^+]}(A[C], W^{N_0})$, then ϕ is locally C-finite if and only if $\mathrm{im}(\phi)$ is a finitely generated A-submodule of W^{N_0}. In particular, since the M_0-action on W^{N_0} is smooth, the M_0-action on $\mathrm{Hom}_{A[C^+]}(A[C], W^{N_0})_{C-\mathrm{fin}}$ is also smooth.

To prove admissibility, we must show that $\mathrm{Hom}_{A[C^+]}(A[C], W^{N_0})^{M_i}_{C-\mathrm{fin}}$, is finitely generated over A for each $i \geq 0$. Since the M_i-action on $\mathrm{Hom}_{A[C^+]}(A[C], W^{N_0})_{C-\mathrm{fin}}$ is defined by the M_i-action on W^{N_0}, we see that the natural embedding

$$\mathrm{Hom}_{A[C^+]}(A[C], W^{M_iN_0})_{C-\mathrm{fin}} \hookrightarrow \mathrm{Hom}_{A[C^+]}(A[C], W^{N_0})^{M_i}_{C-\mathrm{fin}}$$

is an isomorphism. Thus we must show that $\mathrm{Hom}_{A[C^+]}(A[C], W^{M_iN_0})_{C-\mathrm{fin}}$ is finitely generated over A, for $i \geq 0$.

Since W^{H_i} is a finitely generated A-module, for each $i \geq 1$, it follows from [10, Lem 3.1.5] (or rather, its analogue with C and C^+ in place of Z_M and Z_M^+) that $\mathrm{Hom}_{A[C^+]}(A[C], W^{H_i})_{C-\mathrm{fin}}$ is finitely generated over A, for each $i \geq 0$. Thus, it suffices to show that the natural embedding

$$\mathrm{Hom}_{A[C^+]}(A[C], W^{H_i})_{C-\mathrm{fin}} \hookrightarrow \mathrm{Hom}_{A[C^+]}(A[C], W^{M_iN_0})_{C-\mathrm{fin}} \tag{3.4.6}$$

is an isomorphism.

Let $\phi : A[C] \to W^{M_iN_0}$ be an element in the target of (3.4.6). Since ϕ is locally C-finite, by assumption, its image $\mathrm{im}(\phi)$ is finitely generated over A (as we noted above). Thus there is some value of $j \geq i$ such that $\mathrm{im}(\phi)$ is invariant under $\overline{N}_j := z^{-j}\overline{N}_0 z^j$, and thus under the subgroup $\overline{N}_j M_i N_0$ of H_0. It follows from [10, Lem 3.3.2] (taking $z_0 := z$, $I_0 := \overline{N}_{j-1}M_iN_0$, and $I_1 := \overline{N}_j M_i N_0$) that

$$h_{N_0,z}(V^{\overline{N}_j M_i N_0}) \subset V^{\overline{N}_{j-1} M_i N_0},$$

and thus that

$$z\, W^{\overline{N}_j M_i N_0} = \eta h_{N_0,z}\psi(W^{\overline{N}_j M_i N_0}) \subset W^{\overline{N}_{j-1} M_i N_0}.$$

Iterating this observation $k := j - i$ times, we find that

$$z^k\, W^{\overline{N}_j M_i N_0} = (\eta h_{N_0,z}\psi)^k (W^{\overline{N}_j M_i N_0}) \subset W^{\overline{N}_i M_i N_0} = W^{H_i}.$$

Since $\mathrm{im}(\phi) = z^k \,\mathrm{im}(\phi)$ (since $z^k \in A[C]^\times$), we find that $\mathrm{im}(\phi) \subset W^{H_i}$. This proves that (3.4.6) is an isomorphism, as required. □

We are now ready to state and prove the following key result.

3.4.7. **Theorem.** (1) *If V is a locally admissible smooth G-representation, then each of the M^+-representations $H^\bullet(N_0, V)$ is a union of Z_M^+-invariant finitely generated A-submodules.*

(2) *If V is a (locally) admissible smooth G-representation, then each of the smooth M-representations $H^\bullet \mathrm{Ord}_P(V)$ is (locally) admissible.*

Proof. Suppose first that V is an admissible smooth G-representation. Lemma 3.4.3 shows that $H^\bullet(N_0, V) = \bigcup_{i\geq 0} U_i$, where U_i denotes the image of the natural map $H^\bullet(H_i, V) \to H^\bullet(N_0, V)$. It also shows that $A[Z_M^+]U_i \subset U_{i+j}$, for all $i \geq 0$ and some $j \geq 0$. Lemma 3.4.4 shows that U_{i+j} is a finitely generated A-module (being the image of the finitely generated A-module $H^\bullet(H_{i+j}, V)$). Thus so is its submodule $A[Z_M^+]U_i$. Thus, we may write $H^\bullet(N_0, V) = \bigcup_{i\geq 0} A[Z_M^+]U_i$, exhibiting $H^\bullet(N_0, V)$ as a union of Z_M^+-invariant finitely generated A-submodules. This proves claim (1) for admissible V.

Continuing to assume that V is an object of $\mathrm{Mod}_G^{\mathrm{adm}}(A)$, let $V \hookrightarrow I^\bullet$ be an injective resolution of V in the category $\mathrm{Mod}_G^{\mathrm{sm}}(A)$, and let $V \hookrightarrow J^\bullet$ be an injective resolution of V in the category $\mathrm{Mod}_{H_0}^{\mathrm{adm}}(A)$. It follows from Propositions 2.1.2 and 2.1.9 that $I^\bullet$ and $J^\bullet$ are both injective resolutions of V in $\mathrm{Mod}_{H_i}^{\mathrm{sm}}(A)$, for each $i \geq 0$. Thus we may find an H_0-equivariant chain homotopy $\eta : I^\bullet \to J^\bullet$, as well as a homotopy inverse $\psi : J^\bullet \to I^\bullet$.

Writing C to denote the cyclic subgroup of Z_M generated by the element z, as above, the Hecke action of M^+ on $(I^\bullet)^{N_0}$ restricts to an action of C^+, which leaves $(I^\bullet)^{H_i}$ invariant, for each $i \geq 0$. We define a corresponding action of C^+ on $(J^\bullet)^{N_0}$ by having z act via the composite $\eta h_{N_0,z}\psi$. Note that we are in the precise situation introduced in the discussion preceding Lemma 3.4.5 (taking the V and W of that discussion to be $I^\bullet$ and $J^\bullet$ respectively).

It follows from (3.4.2) that $(J^\bullet)^{N_0}$ is the union of the C^+-invariant subcomplexes $(J^\bullet)^{H_i}$, each of which is a complex of finitely generated A-modules (since $J^\bullet$ is a complex of admissible smooth H_0-representations), and so Lemma 3.2.1 shows that the natural map

$$H^\bullet\Big(\mathrm{Hom}_{A[C^+]}\big(A[C], (J^\bullet)^{N_0}\big)_{C-\mathrm{fin}}\Big) \to \mathrm{Hom}_{A[C^+]}\Big(A[C], H^\bullet\big((J^\bullet)^{N_0}\big)\Big)_{C-\mathrm{fin}}$$

is an isomorphism. We also have the sequence of isomorphisms

$$\begin{aligned} H^\bullet \mathrm{Ord}_P(V) &:= \mathrm{Hom}_{A[Z_M^+]}\big(A[Z_M], H^\bullet(N_0, V)\big)_{Z_M-\mathrm{fin}} \\ &\xrightarrow{\sim} \mathrm{Hom}_{A[Z_M^+]}\Big(A[Z_M], H^\bullet\big((I^\bullet)^{N_0}\big)\Big)_{Z_M-\mathrm{fin}} \\ &\xrightarrow{\sim} \mathrm{Hom}_{A[C^+]}\Big(A[C], H^\bullet\big((I^\bullet)^{N_0}\big)\Big)_{C-\mathrm{fin}} \\ &\xrightarrow{\sim} \mathrm{Hom}_{A[C^+]}\Big(A[C], H^\bullet\big((J^\bullet)^{N_0}\big)\Big)_{C-\mathrm{fin}}, \end{aligned}$$

the first isomorphism following from Proposition 2.1.11 (as we noted above), the second being provided by part (2) of Lemma 3.2.1, and the third being provided by the chain homotopy η. Altogether, we obtain an M_0C-equivariant isomorphism

$$H^\bullet \mathrm{Ord}_P(V) \xrightarrow{\sim} H^\bullet\Big(\mathrm{Hom}_{A[C^+]}\big(A[C], (J^\bullet)^{N_0}\big)_{C-\mathrm{fin}}\Big).$$

Thus (since the category of admissible smooth M_0-representations is abelian [10, Prop. 2.2.13]), in order to prove claim (2) for V, it suffices to show that each member of the complex $\mathrm{Hom}_{A[C^+]}\big(A[C], (J^\bullet)^{N_0}\big)_{C-\mathrm{fin}}$ is an admissible M_0-representation. This follows from Lemma 3.4.5, and completes the proof of (2) for admissible V.

Suppose now that V is a locally admissible smooth G-representation. We may write V as an inductive limit, $V \xrightarrow{\sim} \varinjlim_j V_j$, where each V_j is an admissible smooth G-representation. We then obtain an M^+-equivariant isomorphism

$$\varinjlim_j H^\bullet(N_0, V_j) \xrightarrow{\sim} H^\bullet(N_0, V),$$

and hence (taking into account [10, Lem. 3.2.2]), an isomorphism

$$\varinjlim_j \mathrm{Hom}_{A[Z_M^+]}\big(A[Z_M], H^\bullet(N_0, V_j)\big)_{Z_M-\mathrm{fin}} \xrightarrow{\sim} \mathrm{Hom}_{A[Z_M^+]}\big(A[Z_M], H^\bullet(N_0, V)\big)_{Z_M-\mathrm{fin}}.$$

Claims (1) and (2) in the locally admissible case thus follow from the corresponding claims in the admissible case. □

3.4.8. **Corollary.** *$H^\bullet\mathrm{Ord}_P$ forms a δ-functor from the category $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ (resp. $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$) to the category $\mathrm{Mod}_M^{\mathrm{adm}}(A)$ (resp. $\mathrm{Mod}_M^{\mathrm{l.adm}}(A)$).*

Proof. We have already shown in part (2) of Theorem 3.4.7 that $H^\bullet\mathrm{Ord}_P$, when restricted to $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ (resp. $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$), takes values in $\mathrm{Mod}_M^{\mathrm{adm}}(A)$ (resp. $\mathrm{Mod}_M^{\mathrm{l.adm}}(A)$). Part (1) of Theorem 3.4.7, together with Lemma 3.2.1, shows that the complex (3.3.4) is exact, when $0 \to V_1 \to V_2 \to V_3$ is a short exact sequence in $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$. The corollary follows. □

3.5. **Some properties of $H^\bullet(N_0, –)$.** As its title indicates, in this subsection we prove some properties of the cohomology functors $H^\bullet(N_0, –)$, which we will need in our subsequent study of the functor $H^\bullet\mathrm{Ord}_P$.

It will be convenient (in order to facilitate the arguments to come) to assume for the duration of this subsection that N is an arbitrary connected unipotent linear algebraic group over $\mathbb{Q}_p$, of dimension d, say, and that N_0 is a compact open subgroup of N.

If $\mathfrak{n}$ denotes the Lie algebra of N, then the exponential map induces an isomorphism $\mathfrak{n} \xrightarrow{\sim} N$. Under this isomorphism, the group operation on N is obtained from the Lie bracket on $\mathfrak{n}$, together with the Cambpell–Hausdorff formula. We let $\mathfrak{n}_0$ denote the preimage of N_0 under this isomorphism.

3.5.1. **Definition.** We say that N_0 is a standard compact open subgroup of N, or simply standard, if $\mathfrak{n}_0$ is a $\mathbb{Z}_p$-Lie subalgebra of $\mathfrak{n}$.

3.5.2. **Lemma.** *The identity of N admits a neighbourhood basis of standard compact open subgroups.*

Proof. Let $\mathfrak{n}_0$ be any bounded $\mathbb{Z}_p$-lattice in $\mathfrak{n}$ (i.e. $\mathfrak{n}_0$ is a finitely generated $\mathbb{Z}_p$-submodule of $\mathfrak{n}$ which spans $\mathfrak{n}$ over $\mathbb{Q}_p$). Since $\mathfrak{n}_0$ is finitely generated over $\mathbb{Z}_p$, and spans $\mathfrak{n}$ over $\mathbb{Q}_p$, one sees that

$$[\mathfrak{n}_0, \mathfrak{n}_0] \subset p^{-m}\mathfrak{n}_0$$

for $m \geq 0$. Hence, for any integer n, there is an inclusion

$$[p^n\mathfrak{n}_0, p^n\mathfrak{n}_0] \subset p^{n-m} \cdot p^n\mathfrak{n}_0,$$

and thus the $\mathbb{Z}_p$-submodules $p^n\mathfrak{n}_0$, for $n \geq m$, form a basis of neighbourhoods of 0 in $\mathfrak{n}$ consisting of $\mathbb{Z}_p$-Lie subalgebras of $\mathfrak{n}$. Since every Lie bracket in $p^n\mathfrak{n}_0$ is in fact divisible by p^{n-m}, we see that if n is sufficiently large (depending on the dimension d of N) then $p^n\mathfrak{n}_0$ will exponentiate to a standard compact open subgroup of N (i.e. applying the Cambpell–Hausdorff formula to multiply elements of $\exp(p^n\mathfrak{n}_0)$ will not introduce any denominators). This proves the lemma. $\square$

Since N is unipotent, it admits a surjective homomorphism $N \to \mathbb{Q}_p$. The kernel N' of this homomorphism is again unipotent. The image N_0'' of N_0 under this homomorphism is a compact open subgroup of $\mathbb{Q}_p$, and hence is isomorphic to $\mathbb{Z}_p$. Thus if we write $N_0' := N_0 \bigcap N$, then we have a short exact sequence

$$0 \to N_0' \to N_0 \to N_0'' \to 0, \tag{3.5.3}$$

where N_0'' is one-dimensional and N_0' is again a compact open subgroup of a unipotent group. Since any compact open subgroup of $\mathbb{Q}_p$ is standard, it is clear that if N_0 is standard, then so is N_0'.

3.5.4. **Lemma.** *If V is any smooth N_0-representation over A, then $H^\bullet(N_0, V)$ vanishes in degrees greater than d.*

Proof. If $d = 1$, and if we let γ denote a topologically generator of N_0, then it is well-known that $H^\bullet(N_0, V)$ can be computed by the complex

$$V \xrightarrow{\gamma-1} V. \tag{3.5.5}$$

Thus the lemma holds in this case. The general case then follows by applying the Hochschild–Serre spectral sequence to the short exact sequence (3.5.3), and using induction on d. $\square$

If L is any finite rank free $\mathbb{Z}_p$-module, then we write $\det L$ to denote the top non-vanishing exterior power of L (so $\det L = \bigwedge^d L$ if L has rank d), and $L^\vee$ to denote the $\mathbb{Z}_p$-dual of L, i.e. $L^\vee := \operatorname{Hom}_{\mathbb{Z}_p}(L, \mathbb{Z}_p)$.

3.5.6. **Proposition.** *If N_0 is standard, corresponding to the Lie subalgebra $\mathfrak{n}_0$ of $\mathfrak{n}$, then for V any smooth N_0-representation over A, there is a natural isomorphism*

$$H^d(N_0, V) \xrightarrow{\sim} V_{N_0} \otimes_{\mathbb{Z}_p} \det \mathfrak{n}_0^\vee.$$

Proof. Suppose first that N_0 is one-dimensional, topologically generated by an element γ. In this case the Lie bracket on $\mathfrak{n}_0$ is trivial, and the exponential map $\mathfrak{n}_0 \xrightarrow{\sim} N_0$ is in fact an isomorphism of groups (with the source being given the group structure underlying its structure as a $\mathbb{Z}_p$-module). The choice of γ induces isomorphisms $\mathfrak{n}_0 \xrightarrow{\sim} N_0 \xrightarrow{\sim} \mathbb{Z}_p$, and hence a dual isomorphism $\mathbb{Z}_p \xrightarrow{\sim} \mathfrak{n}_0^\vee$, while the complex (3.5.5) gives rise to an isomorphism

$$H^1(N_0, V) \xrightarrow{\sim} V/(\gamma - 1)V \xrightarrow{\sim} V_{N_0}.$$

Combining this isomorphism with the preceding one, we obtain an isomorphism

$$H^1(N_0, V) \xrightarrow{\sim} V_{N_0} \otimes_{\mathbb{Z}_p} \mathfrak{n}_0^\vee,$$

which the reader may verify is independent of the choice of γ.

We now turn to the general case. Applying the Hochschild–Serre spectral sequence to (3.5.3), and taking into account Lemma 3.5.4, we obtain an isomorphism

$$H^d(N_0, V) \xrightarrow{\sim} H^1\big(N_0'', H^{d-1}(N_0', V)\big).$$

If we let $\mathfrak{n}_0'$ (resp. $\mathfrak{n}_0''$) denote the preimage of N_0' (resp. N_0'') under the exponential map, then by induction we have a natural isomorphism

$$H^{d-1}(N_0', V) \xrightarrow{\sim} V_{N_0'} \otimes_{\mathbb{Z}_p} \det(\mathfrak{n}_0')^\vee, \tag{3.5.7}$$

and hence an isomorphism

$$H^d(N_0, V) \xrightarrow{\sim} \big(V_{N_0'} \otimes_{\mathbb{Z}_p} \det(\mathfrak{n}_0')^\vee\big)_{N_0''} \otimes_{\mathbb{Z}_p} (\mathfrak{n}_0'')^\vee.$$

Since $\det(\mathfrak{n}_0')^\vee$ is a rank one free $\mathbb{Z}_p$-module, the N_0''-action on it is necessarily trivial, and hence

$$\big(V_{N_0'} \otimes_{\mathbb{Z}_p} \det(\mathfrak{n}_0')^\vee\big)_{N_0''} \xrightarrow{\sim} (V_{N_0'})_{N_0''} \otimes_{\mathbb{Z}_p} \det(\mathfrak{n}_0')^\vee \xrightarrow{\sim} V_{N_0} \otimes_{\mathbb{Z}_p} \det(\mathfrak{n}_0')^\vee.$$

Thus we may rewrite (3.5.7) as an isomorphism

$$H^d(N_0, V) \xrightarrow{\sim} V_{N_0} \otimes_{\mathbb{Z}_p} \det(\mathfrak{n}_0')^\vee \otimes_{\mathbb{Z}_p} (\mathfrak{n}_0'')^\vee. \tag{3.5.8}$$

The short exact sequence (3.5.3) gives rise to a short exact sequence

$$0 \to \mathfrak{n}_0' \to \mathfrak{n}_0 \to \mathfrak{n}_0'' \to 0,$$

and thus there is an isomorphism

$$\det(\mathfrak{n}_0')^\vee \otimes_{\mathbb{Z}_p} (\mathfrak{n}_0'')^\vee \xrightarrow{\sim} \det \mathfrak{n}_0^\vee.$$

Combining this isomorphism with (3.5.8) gives the isomorphism of the lemma. We leave to the reader the task of verifying that this isomorphism is natural (and, in particular, is independent of the choice of dévissage (3.5.3)). $\square$

Let $N_1 \subset N_0$ be an inclusion of standard compact open subgroups of N, corresponding under exponential map to the inclusion $\mathfrak{n}_1 \subset \mathfrak{n}_0$ of $\mathbb{Z}_p$-Lie subalgebras of $\mathfrak{n}$. This inclusion induces an embedding of rank one free $\mathbb{Z}_p$-modules

$$\det \mathfrak{n}_0^\vee \hookrightarrow \det \mathfrak{n}_1^\vee,$$

whose cokernel has order equal to the index $[\mathfrak{n}_0 : \mathfrak{n}_1]$. Thus this embedding induces an isomorphism

$$\det \mathfrak{n}_0^\vee \xrightarrow{\sim} [\mathfrak{n}_0 : \mathfrak{n}_1] \det \mathfrak{n}_1^\vee.$$

Multiplication by $[\mathfrak{n}_0 : \mathfrak{n}_1]$, composed with the inverse of this isomorphism, thus induces an isomorphism

$$\det \mathfrak{n}_1^\vee \xrightarrow{\sim} \det \mathfrak{n}_0^\vee. \tag{3.5.9}$$

3.5.10. **Lemma.** *If V is a smooth N_0-module, then we have the following commutative diagram*

$$\begin{array}{ccc} H^d(N_1, V) & \xrightarrow{\sim} & V_{N_1} \otimes_{\mathbb{Z}_p} \det \mathfrak{n}_1^\vee \\ \big\downarrow{\scriptstyle \text{cores}} & & \big\downarrow \\ H^d(N_0, V) & \xrightarrow{\sim} & V_{N_0} \otimes_{\mathbb{Z}_p} \det \mathfrak{n}_0^\vee, \end{array}$$

in which the horizontal arrows are the isomorphisms of Proposition 3.5.6*, the left-hand vertical arrow is given by corestriction, and the right-hand vertical arrow is the tensor product of the natural surjection $V_{N_1} \to V_{N_0}$ and the isomorphism* (3.5.9)*.*

Proof. To begin with, suppose that N_0, and hence N_1, has dimension one. Write $I := [N_0 : N_1]$. Let γ denote a topological generator of N_0, so that γ^I is a topological generator of N_1. The explicit description of the isomorphism of Proposition 3.5.6 given in the proof of that proposition shows that we may rewrite the diagram in the statement of the lemma in the form

$$\begin{array}{ccc} H^1(N_1, V) & \longrightarrow & V/(\gamma - 1)V \\ \Big\downarrow{\scriptstyle \text{cores}} & & \Big\downarrow \\ H^1(N_0, V) & \longrightarrow & V/(\gamma^I - 1)V, \end{array} \tag{3.5.11}$$

in which the left-hand vertical arrow is given by corestriction, the right-hand vertical arrow is given by the natural surjection, and the horizontal isomorphisms are given by using the complexes

$$V \xrightarrow{\gamma - 1} V$$

and

$$V \xrightarrow{\gamma^I - 1} V$$

to compute N_0 cohomology and N_1 cohomology respectively. Now $1, \gamma, \ldots, \gamma^{I-1}$ serve as coset representatives for N_0 in N_1, and so cores : $V^{N_1} \to V^{N_0}$ is given by multiplication by $\sum_{i=0}^{I-1} \gamma^i$. The diagram

$$\begin{array}{ccc} V & \xrightarrow{\gamma - 1} & V \\ {\scriptstyle 1+\ldots+\gamma^{I-1}}\Big\downarrow & & \Big\| \\ V & \xrightarrow{\gamma^I - 1} & V \end{array}$$

evidently commutes, and so we see that cores : $H^1(N_1, V) \to H^1(N_0, V)$ corresponds to the natural surjection $V/(\gamma - 1)V \to V/(\gamma^I - 1)V$. Thus (3.5.11) does indeed commute.

We now consider the general case. Intersecting the members of the short exact sequence (3.5.3) induces a short exact sequence

$$0 \to N_1' \to N_1 \to N_1'' \to 0. \tag{3.5.12}$$

If (3.5.3) is the image under the exponential map of

$$0 \to \mathfrak{n}_0' \to \mathfrak{n}_0 \to \mathfrak{n}_0' \to 0, \tag{3.5.13}$$

then (3.5.12) is the image of

$$0 \to \mathfrak{n}_1' \to \mathfrak{n}_1 \to \mathfrak{n}_1' \to 0,$$

obtained by intersecting (3.5.13) with $\mathfrak{n}_1$.

The Hochschild–Serre spectral sequence induces isomorphisms

$$H^d(N_0, V) \xrightarrow{\sim} H^1\big(N_0'', H^{d-1}(N_0', V)\big)$$

and

$$H^d(N_1, V) \xrightarrow{\sim} H^1\big(N_1'', H^{d-1}(N_1', V)\big),$$

in terms of which corestriction may be written as the composite

$$(3.5.14)\quad H^1\big(N_1'', H^{d-1}(N_1', V)\big) \to H^1\big(N_1'', H^{d-1}(N_0', V)\big) \to H^1\big(N_0'', H^{d-1}(N_0', V)\big),$$

in which the first arrow is induced by applying the functor $H^1(N_1'', -)$ to the corestriction map $H^{d-1}(N_1', V) \to H^{d-1}(N_0', V)$, and the second arrow is given by corestriction from N_1'' to N_0''.

As noted in the proof of Proposition 3.5.6, there are isomorphisms

$$\big(V_{N_1'} \otimes_{\mathbb{Z}_p} (\det \mathfrak{n}_1')^\vee\big)_{N_1''} \otimes (\mathfrak{n}_1'')^\vee \xrightarrow{\sim} V_{N_1} \otimes_{\mathbb{Z}_p} \det \mathfrak{n}_1^\vee$$

and

$$\big(V_{N_0'} \otimes_{\mathbb{Z}_p} (\det \mathfrak{n}_0')^\vee\big)_{N_0''} \otimes (\mathfrak{n}_0'')^\vee \xrightarrow{\sim} V_{N_0} \otimes_{\mathbb{Z}_p} \det \mathfrak{n}_0^\vee.$$

One easily sees that with respect to these isomorphisms, the right-hand vertical column of the diagram in the statement of the lemma coincides with the composite

$$(3.5.15)\quad \big(V_{N_1'} \otimes_{\mathbb{Z}_p} (\det \mathfrak{n}_1')^\vee\big)_{N_1''} \otimes_{\mathbb{Z}_p} (\mathfrak{n}_1'')^\vee \to \big(V_{N_0'} \otimes_{\mathbb{Z}_p} (\det \mathfrak{n}_0')^\vee\big)_{N_1''} \otimes_{\mathbb{Z}_p} (\mathfrak{n}_1'')^\vee \to \big(V_{N_0'} \otimes_{\mathbb{Z}_p} (\det \mathfrak{n}_0')^\vee\big)_{N_0''} \otimes_{\mathbb{Z}_p} (\mathfrak{n}_0'')^\vee,$$

where the first arrow is induced from the map

$$V_{N_1'} \otimes_{\mathbb{Z}_p} (\det \mathfrak{n}_1')^\vee \to V_{N_0'} \otimes_{\mathbb{Z}_p} (\det \mathfrak{n}_0')^\vee,$$

obtained by tensoring the natural surjection with the analogue of the isomorphism (3.5.9) for the inclusion $\mathfrak{n}_1' \subset \mathfrak{n}_0'$, by passing to N_1''-invariants and tensoring with $(\mathfrak{n}_1'')^\vee$, while the second arrow is the tensor product of the natural surjection

$$\big(V_{N_0'} \otimes_{\mathbb{Z}_p} (\det \mathfrak{n}_0')^\vee\big)_{N_1''} \to \big(V_{N_0'} \otimes_{\mathbb{Z}_p} (\det \mathfrak{n}_0')^\vee\big)_{N_0''}$$

with the analogue of the isomorphism (3.5.9) for the inclusion $\mathfrak{n}_1'' \subset \mathfrak{n}_0''$.

Consider now the diagram

$$\begin{array}{ccc}
H^1\big(N_1'', H^{d-1}(N_1', V)\big) & \xrightarrow{\sim} & \big(V_{N_1'} \otimes_{\mathbb{Z}_p} (\det \mathfrak{n}_1')^\vee\big)_{N_1''} \otimes (\mathfrak{n}_1'')^\vee \\
\downarrow & & \downarrow \\
H^1\big(N_1'', H^{d-1}(N_0', V)\big) & \xrightarrow{\sim} & \big(V_{N_0'} \otimes_{\mathbb{Z}_p} (\det \mathfrak{n}_0')^\vee\big)_{N_1''} \otimes (\mathfrak{n}_1'')^\vee \\
\downarrow & & \downarrow \\
H^1\big(N_0'', H^{d-1}(N_0', V)\big) & \xrightarrow{\sim} & \big(V_{N_0'} \otimes_{\mathbb{Z}_p} (\det \mathfrak{n}_0')^\vee\big)_{N_0''} \otimes (\mathfrak{n}_0'')^\vee,
\end{array}$$

in which the horizontal isomorphisms are provided by Proposition 3.5.6 (applied variously to N_1'', N_1', N_0', and N_0''), the left-hand column is equal to (3.5.14), and the right-hand column is equal to (3.5.15).

By induction on d, each of the squares commutes, and thus so does the outer square. On the other hand, we have already identified the two columns of this outer square with two vertical arrows of the diagram in the statement of the lemma. The explicit description of the isomorphism of Proposition 3.5.6 given in the proof of that proposition further allows us to identify the horizontal arrows of this diagram with the horizontal arrows in the diagram in the statement of the lemma. Thus the diagram in the statement of the lemma commutes, as claimed. □

3.6. **An application to Jacquet modules.** We employ our standard notation (G, P, M, N, etc.). Let d denote the dimension of N.

3.6.1. **Proposition.** *The functors $H^\bullet\mathrm{Ord}_P$ on $\mathrm{Mod}_P^{\mathrm{sm}}(A)$ vanish in degrees greater than d.*

Proof. This follows immediately from Lemma 3.5.4 together with the definition of $H^\bullet\mathrm{Ord}$. ☐

In the remainder of this subsection we demonstrate a relation between the functor $H^d\mathrm{Ord}_P$ and the so-called Jacquet module functor associated to P. Recall that if V is a smooth G-representation, then the Jacquet module of V with respect to P is defined to be the space of coinvariants V_N. It is naturally an M-representation (since M normalizes N), and is characterized by the adjunction formula

$$\mathrm{Hom}_G(V, \mathrm{Ind}_P^G U) \xrightarrow{\sim} \mathrm{Hom}_M(V_N, U)$$

for any smooth representation U of M.

The action by conjugation of M on N induces an action (the so-called adjoint action) of M on $\mathfrak{n}$, the Lie algebra of N. Passing to top exterior powers, we obtain an action of M on $\det \mathfrak{n}$. Since $\det \mathfrak{n}$ is one-dimensional, this action is given by a character $\alpha : M \to \mathbb{Q}_p^\times$. If we let $|\ \ | : \mathbb{Q}_p^\times \to p^{\mathbb{Z}}$ denote the p-adic absolute value (normalized so that $|p| = p^{-1}$), then we write $\alpha^u := \alpha|\alpha| : M \to \mathbb{Z}_p^\times$ for the unit part of α. Of course, we can and do also regard α^u as an A-valued character of M, by composing it with the natural map $\mathbb{Z}_p^\times \to A^\times$.

Choose N_0 to be a standard compact open subgroup of N, so that $N_0 = \exp \mathfrak{n}_0$, as in the setting of Proposition 3.5.6. If we set $M_0 := \{m \in M \mid mN_0m^{-1} = N_0\}$, then M_0 is a compact open subgroup of M, and so $P_0 := M_0N_0$ is a compact open subgroup of P with respect to which we may compute the functor $H^\bullet\mathrm{Ord}_P$.

Under the exponential map $\mathfrak{n} \xrightarrow{\sim} N$, the action of M on N by conjugation corresponds to the adjoint action of M on $\mathfrak{n}$. In particular, if $m \in M^+$ (so that by definition $mN_0m^{-1} \subset N_0$) then Ad_m restricts to an injective endomorphism of $\mathfrak{n}_0$, and hence $\det \mathrm{Ad}_m$ induces an injective endomorphism

$$\det \mathrm{Ad}_m : \det \mathfrak{n}_0 \hookrightarrow \det \mathfrak{n}_0.$$

Of course, this action is given simply by multiplication by $\alpha(m)$. Since its source and target are free of rank one over $\mathbb{Z}_p$, its cokernel is cyclic of order $|\alpha(m)|^{-1}$. Thus composing $\det \mathrm{Ad}_m$ with multiplication by $|\alpha(m)|$, we obtain an isomorphism

$$\det \mathfrak{n}_0 \xrightarrow{\sim} \det \mathfrak{n}_0,$$

which is given by multiplication by $\alpha^u(m)$.

3.6.2. **Proposition.** *There is a natural isomorphism of functors*

$$V_N \xrightarrow{\sim} H^d\mathrm{Ord}_P(V) \otimes \alpha^u$$

on the category $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$.

Proof. We compute $H^d\mathrm{Ord}_P$ with respect to the subgroup P_0 constructed above. Proposition 3.5.6 gives a natural isomorphism $H^d(N_0, V) \xrightarrow{\sim} V_{N_0} \otimes_{\mathbb{Z}_p} \det \mathfrak{n}_0^\vee$. Via this isomorphism we may transfer the Hecke M^+-action on the left-hand side of this isomorphism to the right-hand side, and we will begin by describing this action explicitly.

If $m \in M^+$ then the Hecke action of m on $H^d(N_0, V)$ is the equal to the composite $H^d(N_0, V) \xrightarrow{\sim} H^d(mN_0m^{-1}, V) \to H^d(N_0, V)$, in which the first isomorphism is induced by conjugation by m, and the second by corestriction. We may make this composite into the left-hand column of the following commutative diagram:

$$\begin{array}{ccc} H^d(N_0, V) & \xrightarrow{\sim} & V_{N_0} \otimes_{\mathbb{Z}_p} \det \mathfrak{n}_0^\vee \\ \downarrow \wr & & \downarrow \wr \\ H^d(mN_0m^{-1}, V) & \xrightarrow{\sim} & V_{mN_0m^{-1}} \otimes_{\mathbb{Z}_p} (\det \mathrm{Ad}_m)(\det \mathfrak{n}_0)^\vee \\ \downarrow \text{cores} & & \downarrow \\ H^d(N_0, V) & \xrightarrow{\sim} & V_{N_0} \otimes_{\mathbb{Z}_p} \det \mathfrak{n}_0^\vee, \end{array}$$

in which the horizontal isomorphisms are those of Proposition 3.5.6, the upper right-hand vertical arrow is given by the tensor product of the isomorphism

$$V_{N_0} \xrightarrow{\sim} V_{mN_0m^{-1}} \tag{3.6.3}$$

induced by multiplication by m and the inverse of the isomorphism

$$\det \mathrm{Ad}_m^\vee : (\det \mathrm{Ad}_m)(\det \mathfrak{n}_0)^\vee \xrightarrow{\sim} \det \mathfrak{n}_0^\vee,$$

and the lower right-hand vertical arrow is given by the tensor product of the natural surjection

$$V_{mN_0m^{-1}} \to V_{N_0}, \tag{3.6.4}$$

and the isomorphism (3.5.9) (taking $\mathfrak{n}_1$ to be $\mathrm{Ad}_m(\mathfrak{n}_0)$). The commutativity of the upper square is immediate, while the commutativity of the lower square follows from Lemma 3.5.10.

We may equip V_{N_0} with an M^+-representation by defining the action of m on V_{N_0} to be the composite of (3.6.4) and (3.6.3). We may equip $\det \mathfrak{n}_0^\vee$ with an M^+-representation by defining the action of m on $\det \mathfrak{n}_0^\vee$ to be the composite of the isomorphism (3.5.9), with $\mathfrak{n}_1$ taken to be $\mathrm{Ad}_m(\mathfrak{n}_0)$, and the isomorphism

$$(\det \mathrm{Ad}_m^\vee)^{-1} : \det \mathfrak{n}_0^\vee \xrightarrow{\sim} \big((\det \mathrm{Ad}_m)(\det \mathfrak{n}_0)\big)^\vee.$$

The discussion preceding the statement of the proposition shows that this action of M^+ on $\det \mathfrak{n}_0^\vee$ is given by the character $(\alpha^u)^{-1}$.

By construction, the composite of the two vertical arrows of the above diagram coincides with the tensor product action of m on $V_{N_0} \otimes_{\mathbb{Z}_p} \det \mathfrak{n}_0^\vee$, and thus this commutative diagram exhibits a natural M^+-equivariant isomorphism

$$H^d(N_0, V) \xrightarrow{\sim} V_{N_0} \otimes_{\mathbb{Z}_p} \det \mathfrak{n}_0^\vee \xrightarrow{\sim} V_{N_0} \otimes_{\mathbb{Z}_p} (\alpha^u)^{-1},$$

which we may rewrite as an isomorphism

$$V_{N_0} \xrightarrow{\sim} \alpha^u \otimes H^d(N_0, V). \tag{3.6.5}$$

Now fix an element $z \in Z_M^+$ such that z^{-1} and Z_M^+ generate Z_M as a monoid, as we may be Lemma 3.1.5. We may then form the increasing chain of compact open subgroups

$$N_0 \subset z^{-1}N_0z \subset \cdots \subset z^{-i}N_0z^i \subset \cdots ;$$

Lemma 3.1.3 (2) shows that $N = \bigcup_{i=0}^\infty z^{-i}N_0z^i$. Thus there is a natural isomorphism $\varinjlim_i V_{z^{-i}N_0z^i} \xrightarrow{\sim} V_N$, with the transition map $V_{z^{-i}N_0z^i} \to V_{z^{-(i+1)}N_0z^{i+1}}$

being the natural projection. If we let C^+ (resp. C) denote the submonoid (resp. subgroup) of Z_M generated by z, then we may rewrite the preceding isomorphism more succinctly in the form

$$A[C] \otimes_{A[C^+]} V_{N_0} \xrightarrow{\sim} V_N. \tag{3.6.6}$$

Part (1) of Theorem 3.4.7, implies that $H^d(N_0, V)$ is a union of finitely generated, Z_M^+-invariant A-modules. Thus Lemma 3.2.1 shows that

$$\begin{aligned} H^d\mathrm{Ord}_P(V) &:= \mathrm{Hom}_{A[Z_M^+]}\big(A[Z_M], H^d(N_0, V)\big) \\ &\xrightarrow{\sim} \mathrm{Hom}_{A[C_M^+]}\big(A[C_M], H^d(N_0, V)\big) \xrightarrow{\sim} A[C_M] \otimes_{A[C_M^+]} H^d(N_0, V). \end{aligned}$$

Combining this with the isomorphism (3.6.5), we find that

$$A[C] \otimes_{A[C^+]} V_{N_0} \xrightarrow{\sim} \alpha^u \otimes H^d\mathrm{Ord}_P(V).$$

Combining this in turn with the isomorphism (3.6.6) proves the proposition. □

3.6.7. **Corollary.** *If V is a (locally) admissible (finitely generated) smooth G-representation over A, then V_N is a (locally) admissible (finitely generated) smooth M-representation over A.*

Proof. The (local) admissibility of V_N follows from part (2) of Theorem 3.4.7, which shows that $H^d\mathrm{Ord}_P(V)$ is (locally) admissible if V is. If V is finitely generated over $A[G]$, then writing $G = \Omega P$ for some compact open subset Ω of G, we see that V is in fact finitely generated over $A[P]$, and thus that V_N is finitely generated over $A[M]$. □

3.6.8. **Remark.** In the representation theory of p-adic reductive groups on complex vector spaces, locally admissible smooth representations were introduced by Harish-Chandra under the name of quasi-admissible smooth representations (see [7]), and it is shown in [7] that in this context, the Jacquet functor takes quasi-admissible smooth G-representations to quasi-admissible smooth M-representations. In [5] the corresponding result is proved for admissible smooth representations.

3.7. **The relation to derived functors.** We write $R^\bullet\mathrm{Ord}_P$ to denote the derived functors of the functor Ord_P on the category $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$. Since the derived functors may be characterized as the universal δ-functor whose 0th term is equivalent to Ord_P, there is a natural transformation (of δ-functors)

$$R^\bullet\mathrm{Ord}_P \to H^\bullet\mathrm{Ord}_P \tag{3.7.1}$$

extending the equivalence $R^0\mathrm{Ord}_P \xrightarrow{\sim} H^0\mathrm{Ord}_P \xrightarrow{\sim} \mathrm{Ord}_P$. We make the following conjecture regarding this transformation.

3.7.2. **Conjecture.** *The natural transformation* (3.7.1) *is a natural equivalence.*

The derived functors are also characterized by their effaceability. Thus Conjecture 3.7.2 is equivalent to conjecturing that the functors $H^{\geq 1}\mathrm{Ord}_P$ are effaceable.

In an appendix, joint with Paškūnas [11], we prove Conjecture 3.7.2 in the case when $G = \mathrm{GL}_2(F)$, for some finite extension F of $\mathbb{Q}_p$, and P is a Borel subgroup of G.

3.7.3. **Remark.** The natural transformation $R^1\mathrm{Ord}_P \to H^1\mathrm{Ord}_P$ is an injection. (This is a general property of the natural transformation from a universal δ-functor to any other δ-functor which is an equivalence on 0th terms.)

In [10, Thm. 4.4.6], we proved that $\mathrm{Ord}_P : \mathrm{Mod}_G^{\mathrm{adm}}(A) \to \mathrm{Mod}_M^{\mathrm{adm}}(A)$ is right adjoint to the functor $\mathrm{Ind}_{\overline{P}}^G : \mathrm{Mod}_M^{\mathrm{adm}}(A) \to \mathrm{Mod}_G^{\mathrm{adm}}(A)$. (Note that $\mathrm{Ind}_{\overline{P}}^G$ does indeed induce a functor between these categories, by [10, Lem 4.1.7].) Since both functors commute with the formation of inductive limits [10, Lems. 3.2.2, 4.1.4], the same statement is true with $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ and $\mathrm{Mod}_M^{\mathrm{adm}}(A)$ replaced by $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$ and $\mathrm{Mod}_M^{\mathrm{l.adm}}(A)$. Since $\mathrm{Ind}_{\overline{P}}^G$ is furthermore exact [10, Prop. 4.1.5], we find that Ord_P takes injective objects in $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$ to injective objects in $\mathrm{Mod}_M^{\mathrm{l.adm}}(A)$. Thus the adjointness between Ord_P and $\mathrm{Ind}_{\overline{P}}^G$ extends to a derived adjointness, i.e., for any two objects U of $\mathrm{Mod}_M^{\mathrm{l.adm}}(A)$ and V of $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$, there is a natural quasi-isomorphism

$$R\,\mathrm{Hom}_M\big(U, R\mathrm{Ord}_P(V)\big) \xrightarrow{\sim} R\,\mathrm{Hom}_G(\mathrm{Ind}_{\overline{P}}^G U, V).$$

More concretely, there is a spectral sequence

$$E_2^{i,j} := \mathrm{Ext}_M^i\big(U, R^j\mathrm{Ord}_P(V)\big) \Rightarrow \mathrm{Ext}_G^{i+j}(\mathrm{Ind}_{\overline{P}}^G U, V). \tag{3.7.4}$$

(Here the $R\,\mathrm{Hom}$ and Ext functors are computed in the categories of locally admissible representations.)

The spectral sequence (3.7.4) gives rise in particular to an exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}_M^1\big(U, \mathrm{Ord}_P(V)\big) \to \mathrm{Ext}_G^1(\mathrm{Ind}_{\overline{P}}^G U, V) \to \mathrm{Hom}_M\big(U, R^1\mathrm{Ord}_P(V)\big) \\ \to \mathrm{Ext}_M^2\big(U, \mathrm{Ord}_P(V)\big) \to \mathrm{Ext}_G^2(\mathrm{Ind}_{\overline{P}}^G U, V). \end{aligned} \tag{3.7.5}$$

As we noted in Remark 3.7.3, there is a natural injection

$$R^1\mathrm{Ord}_P(V) \hookrightarrow H^1\mathrm{Ord}_P(V),$$

and so from (3.7.5) we obtain an exact sequence

$$0 \to \mathrm{Ext}_M^1\big(U, \mathrm{Ord}_P(V)\big) \to \mathrm{Ext}_G^1(\mathrm{Ind}_{\overline{P}}^G U, V) \to \mathrm{Hom}_M\big(U, H^1\mathrm{Ord}_P(V)\big), \tag{3.7.6}$$

which one can combine with a computation of $H^1\mathrm{Ord}_P(V)$ in various cases to obtain an upper bound on the size of $\mathrm{Ext}_G^1(\mathrm{Ind}_{\overline{P}}^G U, V)$.

3.7.7. **Remark.** The maps in (3.7.6) have a concrete interpretation. Given an extension of locally admissible smooth M-representations

$$0 \to \mathrm{Ord}_P(V) \to E \to U \to 0,$$

applying the exact functor $\mathrm{Ind}_{\overline{P}}^G$ yields an extension of smooth G-representations

$$0 \to \mathrm{Ind}_{\overline{P}}^G \mathrm{Ord}_P(V) \to \mathrm{Ind}_{\overline{P}}^G E \to \mathrm{Ind}_{\overline{P}}^G U \to 0,$$

which we may push forward along the canonical map $\mathrm{Ind}_{\overline{P}}^G \mathrm{Ord}_P(V) \to V$ to obtain an extension

$$0 \to V \to E' \to \mathrm{Ind}_{\overline{P}}^G U \to 0.$$

Thus we have constructed a map

$$\mathrm{Ext}_M^1\big(U, \mathrm{Ord}_P(V)\big) \to \mathrm{Ext}_G^1(\mathrm{Ind}_{\overline{P}}^G U, V),$$

which, up to a sign, coincides with the first non-trivial arrow in (3.7.6).

On the other hand, given an extension

$$0 \to V \to E' \to \mathrm{Ind}_{\overline{P}}^G U \to 0$$

of locally admissible smooth G-representations, we may apply the δ-functor $H^\bullet\mathrm{Ord}_P$ to obtain an exact sequence

$$0 \longrightarrow \mathrm{Ord}_P(V) \longrightarrow \mathrm{Ord}_P(E') \longrightarrow \mathrm{Ord}_P(\mathrm{Ind}_{\overline{P}}^G U) \xrightarrow{\delta} H^1\mathrm{Ord}_P(V) \to \cdots .$$

In particular, the boundary map δ yields a map

$$\mathrm{Ord}_P(\mathrm{Ind}_{\overline{P}}^G U) \to H^1\mathrm{Ord}_P(V).$$

Composing this with the canonical isomorphism $U \xrightarrow{\sim} \mathrm{Ord}_P(\mathrm{Ind}_{\overline{P}}^G U)$ of [10, Cor. 4.3.5] we obtain an M-equivariant map $U \to H^1\mathrm{Ord}_P(V)$. Thus we have constructed a map

$$\mathrm{Ext}^1_G(\mathrm{Ind}_{\overline{P}}^G U, V) \to \mathrm{Hom}_M\big(U, H^1\mathrm{Ord}_P(V)\big),$$

which, up to a sign, coincides with the second non-trivial arrow in (3.7.6).

3.7.8. **Remark.** An unfortunate complication in studying the homological algebra of admissible smooth G-representations is that there are (at least) three relevant categories in which one might compute the Ext spaces of a pair of admissible smooth representations: namely $\mathrm{Mod}_G^{\mathrm{adm}}(A)$, $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$, and $\mathrm{Mod}_G^{\mathrm{sm}}(A)$ (in the first, via Yoneda Exts, and in the second and third either via Yoneda Exts, or equivalently, via derived functors). While working in the first of these categories might seem to be the most natural choice, working in the second has the advantage of putting at ones disposal the formalism of derived functors (since, by Proposition 2.1.1, it has enough injectives), and certain useful results, such as Frobenius reciprocity, are most naturally available when one works in the category of all smooth representations. It follows from [10, Prop. 2.2.13] that one obtains the same value of Ext^1, no matter which of the three categories one computes in. One might hope that this is true for all Ext^i, but one does not know this at present. Thus we will be careful below to specify in which categories any values of Ext^i $(i > 1)$ are computed.

4. The case of $\mathrm{GL}_2(\mathbb{Q}_p)$

In this section we set $G := \mathrm{GL}_2(\mathbb{Q}_p)$, and we work in the category $\mathrm{Mod}_G^{\mathrm{adm}}(k)$, where k is a finite field of characteristic p. We take P (resp. $\overline{P}$) to be the Borel subgroup of G consisting of upper triangular (resp. lower triangular) matrices, so that M is the maximal torus consisting of diagonal matrices. We will compute $H^\bullet\mathrm{Ord}_P(V)$ for all absolutely irreducible objects V of this category, and then, as an application, we will compute the spaces $\mathrm{Ext}^1_G(V_1, V_2)$ for various pairs of absolutely irreducible objects V_1, V_2. As we remarked in the introduction, similar computations have been performed by Breuil and Paškūnas [4] (who in fact consider the case of $\mathrm{GL}_2(F)$ for any finite extension F of $\mathbb{Q}_p$) and Colmez [6], using different methods. Furthermore, in [15] Paškūnas completes the computation of the spaces $\mathrm{Ext}^1_G(V_1, V_2)$ for absolutely irreducible V_1 and V_2 (at least for $p > 2$), by treating the cases when V_1 is supersingular and V_2 is supersingular or a twist of the Steinberg (cases which are not treated here, nor in [4] or [6].)

4.1. **Absolutely irreducible representations.** We begin with some generalities regarding irreducible and absolutely irreducible admissible smooth representations of a p-adic analytic group H over the finite field k.

4.1.1. **Lemma.** *If H is a p-adic analytic group and V is a non-zero irreducible object of $\mathrm{Mod}_G^{\mathrm{adm}}(k)$, then $\mathrm{End}_{k[H]}(V)$ is a finite field extension of k.*

Proof. If V is irreducible, then in particular it is finitely generated, and hence (being smooth) is generated by V^{H_0} for some compact open subgroup H_0 of V. Thus restriction induces an embedding $\mathrm{End}_{k[H]}(V) \hookrightarrow \mathrm{End}_k(V^{H_0})$. Since V is admissible, the space V^{H_0} is finite dimensional over k, and hence the target of the above embedding is finite dimensional. Thus $\mathrm{End}_{k[H]}(V)$ is finite dimensional k. It is also a division ring (since V is irreducible), and thus is a finite dimensional division algebra over k. Since k is finite, so is $\mathrm{End}_{k[H]}(V)$, and so it is in fact commutative, and hence is a field. □

4.1.2. **Lemma.** *If H is a p-adic analytic group, and V is an irreducible admissible smooth representation of H over k, then for any field extension l of k, the H-representation $l \otimes_k V$ is a direct sum of finitely many irreducible H-representations over l, and the natural map $l \otimes_k \mathrm{End}_{k[H]}(V) \to \mathrm{End}_{l[H]}(l \otimes_k V)$ is an isomorphism.*

Proof. Since V is irreducible, it is in particular finitely generated over $k[H]$, and so the functor $\mathrm{Hom}_{k[H]}(V, –)$ commutes with the formation of inductive limits with respect to an inductive system having injective transition maps. Thus we see that

$$\mathrm{Hom}_{l[H]}(l \otimes_k V, l \otimes_k V) \xrightarrow{\sim} \mathrm{Hom}_{k[H]}(V, l \otimes_k V) \xrightarrow{\sim} l \otimes_k \mathrm{Hom}_{k[H]}(V, V),$$

proving the second claim of the lemma.

Now suppose that W is an $l[H]$-submodule of $l \otimes_k V$. The inclusion of W into $l \otimes_k V$ induces an $l \otimes_k \mathrm{End}_{k[H]}(V)$-linear embedding

$$\mathrm{Hom}_{k[H]}(V, W) \hookrightarrow \mathrm{Hom}_{k[H]}(V, l \otimes_k V) \xrightarrow{\sim} l \otimes_k \mathrm{End}_{k[H]}(V). \tag{4.1.3}$$

Tensoring on the right with V over $\mathrm{End}_{k[H]}(V)$, we obtain an $l[H]$-linear map

$$\mathrm{Hom}_{k[H]}(V, W) \otimes_{\mathrm{End}_{k[H]}(V)} V \to l \otimes_k \mathrm{End}_{k[H]}(V) \otimes_{\mathrm{End}_{k[H]}(V)} V \xrightarrow{\sim} l \otimes_k V, \tag{4.1.4}$$

which is again an embedding, since by the previous lemma $\mathrm{End}_{k[H]}(V)$ is a field. The image of this map is precisely the image of the natural evaluation map

$$\mathrm{Hom}_{k[H]}(V, W) \otimes_{\mathrm{End}_{k[H]}(V)} V \to W. \tag{4.1.5}$$

Since $l \otimes_k V$ is semi-simple as a $k[H]$-representation, and is in fact just a direct sum of copies of the irreducible representation V, we see that as a $k[H]$-representation, W is also isomorphic to a direct sum of copies of V. Hence (4.1.5) is surjective, and thus the image of (4.1.4) is equal to W. Thus we may factor (4.1.4) as

$$\mathrm{Hom}_{k[H]}(V, W) \otimes_{\mathrm{End}_{k[H]}(V)} V \xrightarrow{\sim} W \hookrightarrow l \otimes_k V. \tag{4.1.6}$$

Since k is finite, and so separable, the l-algebra $l \otimes_k \mathrm{End}_{k[H]}(V)$ is separable, and so (4.1.3) must split as a map of $l \otimes_k \mathrm{End}_{k[H]}(V)$-modules. Thus the embedding (4.1.4) must be split as a map of $l[H]$-modules. A consideration of (4.1.6) then shows that W must be a direct summand of $l \otimes_k V$ as an $l[H]$-module.

Clearly, if $W_1 \subsetneq W_2$ is a strict inclusion of $l[H]$-submodules of $l \otimes_k V$, then we have a corresponding inclusion $\mathrm{Hom}_{k[H]}(V, W_1) \subset \mathrm{Hom}_{k[H]}(V, W_2)$ of ideals in $l \otimes_k \mathrm{End}_{k[H]}(V)$, which, by a consideration of (4.1.6), is seen to also be strict. Since $l \otimes_k \mathrm{End}_{k[H]}(V)$ is a finite dimensional l-algebra, any strictly increasing chain of ideals in this algebra is of some uniformly bounded length. Another consideration of (4.1.6) then shows that any strictly increasing chain of $l[H]$-submodules of $l \otimes_k V$ must be of finite length. Combining this with the conclusion of the preceding paragraph, we infer that $l \otimes_k V$ is indeed the direct sum of finitely many irreducible $l[H]$-modules, completing the proof of the lemma. □

4.1.7. **Lemma.** *Let H be a p-adic analytic group. If V is a non-zero irreducible object of* $\mathrm{Mod}_G^{\mathrm{adm}}(k)$*, then the following are equivalent:*

(1) $\mathrm{End}_{k[H]}(V) = k$.
(2) $l \otimes_k V$ *is irreducible (as a representation of H over l) for any finite field extension l of k.*
(3) $l \otimes_k V$ *is irreducible (as a representation of H over l) for any field extension l of k.*

Proof. Suppose that (1) holds, and that l is an extension of k. Lemma 4.1.2 implies that $l \otimes_k V$ is a semi-simple representation of H over l, and that $\mathrm{End}_{l[H]}(l \otimes_k V) = l$. It follows that in fact $l \otimes_k V$ is irreducible, and thus that (3) holds.

On the other hand, if (1) does not hold, then Lemma 4.1.1 shows that $\mathrm{End}_{k[H]}(V)$ is a proper finite extension l of k. Lemma 4.1.2 then shows that $\mathrm{End}_{l[H]}(l \otimes_k V) = l \otimes_k l$, which is not a field. Thus (2) does not hold either. Since clearly (3) implies (2), the lemma follows. □

4.1.8. **Definition.** If H is a p-adic analytic group, then we say that an object V of $\mathrm{Mod}_H^{\mathrm{adm}}(k)$ is absolutely irreducible if it is irreducible, and furthermore satisfies the equivalent conditions of Lemma 4.1.7.

The following lemma shows that the study of irreducible admissible smooth representations may always be reduced to the study of absolutely irreducible ones by making a finite extension of scalars.

4.1.9. **Lemma.** *If V is an irreducible object of* $\mathrm{Mod}_H^{\mathrm{adm}}(k)$*, then there is a finite extension l of k such $l \otimes_k V$ is a finite direct sum of absolutely irreducible objects of* $\mathrm{Mod}_H^{\mathrm{adm}}(l)$.

Proof. If we write $l = \mathrm{End}_{k[H]}(V)$, then Lemma 4.1.1 shows that l is a finite extension of k, and Lemma 4.1.2 shows that $l \otimes_k V$ is a semi-simple H-representation and that $\mathrm{End}_{l[H]}(l \otimes_k V) \xrightarrow{\sim} l \otimes_k l$. Now $l \otimes_k l \xrightarrow{\sim} \prod_i l_i$, where each l_i is an isomorphic copy of l. We conclude that $l \otimes_k V = \prod_i V_i$, where each V_i is irreducible and satisfies $\mathrm{End}_{l[H]}(V_i) = l$. Thus each V_i is in fact absolutely irreducible, and the lemma is proved. □

We now recall a result from [1], which provides a (partial) classification of the absolutely irreducible objects of $\mathrm{Mod}_G^{\mathrm{adm}}(k)$. (Recall that $G := \mathrm{GL}_2(\mathbb{Q}_p)$.)

4.1.10. **Proposition.** *If V is a non-zero absolutely irreducible object of* $\mathrm{Mod}_G^{\mathrm{adm}}(k)$*, then V has precisely one of the following forms:*

(1) $V \xrightarrow{\sim} \mathrm{Ind}_{\overline{P}}^G \chi_1 \otimes \chi_2$, *where $\chi_1 \neq \chi_2$ are distinct characters $\mathbb{Q}_p^\times \to k^\times$ that are uniquely determined by V.*
(2) $V \xrightarrow{\sim} (\chi \circ \det) \otimes \mathrm{St}$, *where* St *denotes the Steinberg representation and $\chi : \mathbb{Q}_p^\times \to k^\times$ is a uniquely determined character.*
(3) $V \xrightarrow{\sim} \chi \circ \det$ *for some uniquely determined character $\chi : \mathbb{Q}_p^\times \to k^\times$.*
(4) V *is supersingular, which is to say, V is neither a quotient nor a subobject of any representation of the form $\mathrm{Ind}_{\overline{P}}^G U$, with U an object of* $\mathrm{Mod}_M^{\mathrm{l.adm}}(k)$.

Proof. The paper [1] is set in a slightly different context to the one that we are working in; they classify smooth irreducible representations over the algebraically closed field $\overline{k}$ which furthermore admit a central character.

If V is an absolutely irreducible object of $\mathrm{Mod}_G^{\mathrm{adm}}(k)$, then Lemma 4.1.7 (1) implies that V admits a central character, while part (3) of the same lemma implies that $\overline{k}\otimes_k V$ remains irreducible. The results of [1] then imply that precisely one of the following holds:

(1′) $\overline{k}\otimes_k V \xrightarrow{\sim} \mathrm{Ind}_{\overline{P}}^G \chi_1\otimes\chi_2$, where $\chi_1\neq\chi_2$ are distinct characters $\mathbb{Q}_p^\times\to\overline{k}^\times$ that are uniquely determined by V.

(2′) $\overline{k}\otimes_k V \xrightarrow{\sim} (\chi\circ\det)\otimes\mathrm{St}$, where St denotes the Steinberg representation and $\chi:\mathbb{Q}_p^\times\to\overline{k}^\times$ is a uniquely determined character.

(3′) $\overline{k}\otimes_k V \xrightarrow{\sim} \chi\circ\det$ for some uniquely determined character $\chi:\mathbb{Q}_p^\times\to\overline{k}^\times$.

(4′) $\overline{k}\otimes_k V$ is supersingular, which is to say, V is neither a quotient nor a subobject of any representation of the form $\mathrm{Ind}_{\overline{P}}^G\chi_1\otimes\chi_2$ for any pair of characters $\chi_1,\chi_2:\mathbb{Q}_p^\times\to\overline{k}^\times$.

Since in each of cases (1′), (2′), and (3′) the characters are uniquely determined, a simple descent argument then shows that in fact these characters take values in $k^\times$, and that V satisfies the corresponding condition (1), (2), or (3) of the Proposition.

Suppose then that $\overline{k}\otimes_k V$ satisfies condition (4′), and consider a map

$$\mathrm{Ind}_{\overline{P}}^G U\to V,$$

where U lies in $\mathrm{Mod}_M^{\mathrm{l.adm}}(k)$; we will show that this map necessarily vanishes. Since U is the inductive limit of finitely generated admissible smooth M-representations, and since parabolic induction commutes with the formation of inductive limits, it is no loss of generality to assume that U is fact finitely generated and admissible, and hence finite dimensional (since M is a torus). Extending scalars to $\overline{k}$, we obtain a map

$$\mathrm{Ind}_{\overline{P}}^G(\overline{k}\otimes_k U)\to\overline{k}\otimes_k V, \tag{4.1.11}$$

and it suffices to show that (4.1.11) vanishes. Since $\overline{k}\otimes_k U$ is finite dimensional representation of M over $\overline{k}$, it is a successive extension of characters, and thus $\mathrm{Ind}_{\overline{P}}^G(\overline{k}\otimes_k U)$ is a successive extension of inductions of characters. The vanishing of (4.1.11) thus follows from condition (4′).

A similar argument shows that any map $V\to\mathrm{Ind}_{\overline{P}}^G U$, with U an object of $\mathrm{Mod}_M^{\mathrm{l.adm}}(k)$, must vanish. Thus if $\overline{k}\otimes_k V$ satisfies condition (4′), then V satisfies condition (4) of the lemma. □

4.1.12. **Remark.** The absolutely irreducible supersingular representations are completely classified in [3], but we will not need this classification for our computations.

4.2. Computations of $H^\bullet\mathrm{Ord}_P(V)$.

Fix a ring of coefficients A in $\mathrm{Art}(\mathbb{Z}_p)$, with residue field k. Suppose that U is a locally admissible smooth representation of M over A. We may also regard U as a P-representation via the natural surjection $P\to P/N\xrightarrow{\sim}M$. We equip $\mathcal{C}_c^{\mathrm{sm}}(N,A)$ with the P-action described in [10, §4.1], and equip $\mathcal{C}_c^{\mathrm{sm}}(N,A)\otimes_A U$ with the tensor product P-action (making it a smooth P-representation).

4.2.1. **Remark.** There is a natural isomorphism of P-representations

$$\mathcal{C}_c^{\mathrm{sm}}(N)\otimes_A U\xrightarrow{\sim} c\text{–}\mathrm{Ind}_M^P U. \tag{4.2.2}$$

To construct this isomorphism, note first that restricting functions from P to N induces an isomorphism

$$c\text{–}\mathrm{Ind}_M^P U \xrightarrow{\sim} \mathcal{C}_c^{\mathrm{sm}}(N,U). \tag{4.2.3}$$

Secondly, there is a natural isomorphism

$$\mathcal{C}_c^{\mathrm{sm}}(N,A)\otimes_A U \xrightarrow{\sim} \mathcal{C}_c^{\mathrm{sm}}(N,U). \tag{4.2.4}$$

Composing the inverse of (4.2.3) with (4.2.4) yields the desired isomorphism (4.2.2).

We let w denote the involution of M induced by the non-trivial element of the Weyl group of M in G; concretely, $w\left(\begin{smallmatrix} a & 0\\ 0 & d\end{smallmatrix}\right) := \left(\begin{smallmatrix} d & 0\\ 0 & a\end{smallmatrix}\right)$. If U is an M-representation, we let U^w denote the twist of the representation U by w, i.e. the U^w has the same underlying A-module as U, and writing $\overset{U}{\cdot}$ to denote the action on U, and $\overset{U^w}{\cdot}$ to denote the action on U^w, we have $m \overset{U^w}{\cdot} u := w(m) \overset{U}{\cdot} u$ for all $m\in M$ and $u\in U$.

If $\chi_1,\chi_2 : \mathbb{Q}_p^\times \to A^\times$ are two smooth characters, we write $\chi_1\otimes\chi_2$ to denote the character of M defined by $\big(\chi_1\otimes\chi_2\big)\left(\begin{smallmatrix} a & 0\\ 0 & d\end{smallmatrix}\right) := \chi_1(a)\chi_2(d)$. If $\alpha^u : M\to A^\times$ denotes the character defined in Subsection 3.6, then $\alpha^u = \varepsilon\otimes\varepsilon^{-1}$, where ε denotes the cyclotomic character (thought of as taking values in A): concretely,

$$\varepsilon : \mathbb{Q}_p^\times = p^{\mathbb{Z}}\times\mathbb{Z}_p^\times \to \mathbb{Z}_p^\times \to A^\times,$$

where the first arrow is given by projection onto the second factor, and the second arrow arises from the fact that A is a $\mathbb{Z}_p$-algebra.

4.2.5. **Proposition.** *If U is a locally admissible smooth M-representation over A, then there are isomorphisms:*

$$\mathrm{Ord}_P\big(U\otimes\mathcal{C}_c^{\mathrm{sm}}(N,A)\big) \xrightarrow{\sim} U, \tag{4.2.6}$$

$$\mathrm{Ord}_P(U) = 0, \tag{4.2.7}$$

$$H^1\big(N_0, U\otimes\mathcal{C}_c^{\mathrm{sm}}(N,A)\big) = 0, \tag{4.2.8}$$

and

$$H^1(N_0,U) \xrightarrow{\sim} (\alpha^u)^{-1}\otimes U, \tag{4.2.9}$$

Proof. The isomorphism (4.2.6) is constructed in [10, Prop. 4.2.7]. The isomorphism (4.2.7) follows from the fact that if $m\in M^+$ is chosen so that mN_0m^{-1} is strictly contained in N_0, then the operator $h_{N_0,m}$ is locally nilpotent on U. The isomorphism (4.2.8) follows from the isomorphism of N_0-representations

$$\mathcal{C}_c^{\mathrm{sm}}(N,A) \xrightarrow{\sim} \bigoplus_{n\in N/N_0} \mathcal{C}^{\mathrm{sm}}(N_0,A),$$

while the isomorphism (4.2.9) follows from (3.6.5). □

4.2.10. **Corollary.** *For any locally admissible smooth representation U of M over A, there are isomorphisms* $\mathrm{Ord}_P(\mathrm{Ind}_P^G U) \xrightarrow{\sim} U$ *and* $H^1\mathrm{Ord}_P(\mathrm{Ind}_P^G U) \xrightarrow{\sim} (\alpha^u)^{-1}\otimes U^w$.

Proof. Let $G = \overline{P}N \coprod \overline{P}w$ denote the Bruhat decomposition of G. (Here w denotes some representative of the non-trivial element of the Weyl group of M in G, e.g. one can take $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.) Evaluation of elements of $\mathrm{Ind}_{\overline{P}}^{G} U$ at w yields a P-equivariant surjection

$$\mathrm{Ind}_{\overline{P}}^{G} U \to U^w,$$

where P acts on U^w via the natural quotient map $P \to M$. The kernel of this surjection is precisely $(\mathrm{Ind}_{\overline{P}}^{G} U)(N)$, the P-invariant submodule of $\mathrm{Ind}_{\overline{P}}^{G} U$ consisting of functions whose support lies in $\overline{P}N$. As noted in [10, §4.1], there is a P-equivariant isomorphism

$$\mathcal{C}_c^{\mathrm{sm}}(N, A) \otimes_A U \xrightarrow{\sim} (\mathrm{Ind}_{\overline{P}}^{G} U)(N).$$

Altogether, we find that as a P-representation, $\mathrm{Ind}_{\overline{P}}^{G} U$ sits in a short exact sequence

$$0 \to \mathcal{C}_c^{\mathrm{sm}}(N, A) \otimes_A U \to \mathrm{Ind}_{\overline{P}}^{G} U \to U^w \to 0. \qquad (4.2.11)$$

Applying Ord_P, and taking into account its left exactness, together with the isomorphisms (4.2.6) and (4.2.7), we find that

$$\mathrm{Ord}_P(\mathrm{Ind}_{\overline{P}}^{G} U) \xrightarrow{\sim} U,$$

while forming the long exact sequence of continuous N_0-cohomology, and taking into account the isomorphisms (4.2.8) and (4.2.9), we find that $H^1(N_0, \mathrm{Ind}_{\overline{P}}^{G} U) \xrightarrow{\sim} (\alpha^u)^{-1} \otimes U^w$, and thus also that

$$H^1\mathrm{Ord}_P(\mathrm{Ind}_{\overline{P}}^{G} U) \xrightarrow{\sim} (\alpha^u)^{-1} \otimes U^w.$$

This proves the corollary. □

4.2.12. **Theorem.** *Let V be an absolutely irreducible admissible representation of G over k. (Recall that these are classified by Proposition 4.1.10.)*

1. *If $V \xrightarrow{\sim} \mathrm{Ind}_{\overline{P}}^{G} \chi_1 \otimes \chi_2$, for a pair of smooth characters $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to k^\times$, then $\mathrm{Ord}_P(V) \xrightarrow{\sim} \chi_1 \otimes \chi_2$, while $H^1\mathrm{Ord}_P(V) \xrightarrow{\sim} \chi_2 \overline{\varepsilon}^{-1} \otimes \chi_1 \overline{\varepsilon}$.*
2. *If $V \xrightarrow{\sim} (\chi \circ \det) \otimes \mathrm{St}$, for some smooth character $\chi : \mathbb{Q}_p^\times \to k^\times$, then $\mathrm{Ord}_P(V) \xrightarrow{\sim} \chi \otimes \chi$, while $H^1\mathrm{Ord}_P(V) = 0$.*
3. *If $V \xrightarrow{\sim} \chi \circ \det$ for some smooth character $\chi : \mathbb{Q}_p^\times \to k^\times$, then $\mathrm{Ord}_P(V) = 0$, while $H^1\mathrm{Ord}_P(V) = \chi \otimes \chi$.*
4. *If V is a supersingular absolutely irreducible admissible smooth representation of G, then $\mathrm{Ord}_P(V) = H^1\mathrm{Ord}_P(V) = 0$.*

Proof. Claim (1) is a special case of Corollary 4.2.10. By definition, the representation $(\chi \circ \det) \otimes \mathrm{St}$ sits in the short exact sequence

$$0 \to \chi \circ \det \to \mathrm{Ind}_{\overline{P}}^{G} \chi \otimes \chi \to (\chi \circ \det) \otimes \mathrm{St} \to 0.$$

Comparing this with the short exact sequence (4.2.11) (in the case when $U = \chi \otimes \chi$), we find that $\mathrm{St} \xrightarrow{\sim} (\chi \otimes \chi) \otimes_A \mathcal{C}_c^{\mathrm{sm}}(N, k)$ as P-representations. Thus claim (2) follows from Proposition 4.2.5. As a P-representation, the representation $\chi \circ \det$ is isomorphic to the M-representation $\chi \otimes \chi$. Thus claim (3) also follows from Proposition 4.2.5. Claim (4) follows from Proposition 3.6.2, and the adjointness properties of $V \mapsto \mathrm{Ord}_P(V)$ and $V \mapsto V_N$. (Recall from Proposition 4.1.10 (4) that if V is supersingular, then it is neither a subrepresentation of nor a quotient of a representation induced from P or $\overline{P}$.) □

4.2.13. **Remark.** If V is supersingular, then, using the description of V in terms of (φ, Γ)-modules provided by [2, Thm. 2.2.1], one can show that in fact $H^1(N_0, V)$ (which up to a twist is isomorphic to the space of N_0-coinvariants V_{N_0}) vanishes for any compact open subgroup N_0 of N.

Thus we see that for any absolutely irreducible object V of $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ the natural surjection $V_{N_0} \to V_N$ is in fact an isomorphism. It then follows that this surjection is an isomorphism for any object V of $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$. Indeed, beginning with the absolutely irreducible case, one uses Lemma 4.1.9 to extend to the irreducible case, and then proceeds by induction on length to prove it for any finite length object. Since any object $\mathrm{Mod}_G^{\mathrm{l.adm}}(A)$ is the inductive limit of finitely generated admissible – and hence finite length [10, Thm. 2.3.8] – objects, and since the formation of coinvariants commutes with the formation of inductive limits, we conclude that it holds for arbitrary such V. We don't know whether or not the analogous result holds for other groups G.

We conclude this subsection by presenting an application of the preceding computations.

4.2.14. **Proposition.** *Let V be an object of $\mathrm{Mod}_G^{\mathrm{adm}}(A)$ which is faithful as an A-module, and for which $k \otimes_A V \xrightarrow{\sim} \mathrm{Ind}_{\overline{P}}^G \overline{\chi}_1 \otimes \overline{\chi}_2$, where $\overline{\chi}_1, \overline{\chi}_2 : \mathbb{Q}_p^\times \to k^\times$ are smooth characters for which $\overline{\chi}_1 \overline{\chi}_2^{-1} \neq \underline{1}, \overline{\varepsilon}$. Then $V \xrightarrow{\sim} \mathrm{Ind}_{\overline{P}}^G \chi_1 \otimes \chi_2$, where χ_1 and χ_2 are lifts of $\overline{\chi}_1$ and $\overline{\chi}_2$ to smooth characters $\mathbb{Q}_p^\times \to A^\times$.*

Proof. We prove this by induction on the length of A. Let $\mathfrak{m}$ denote the maximal ideal of A, let $A[\mathfrak{m}]$ denote the ideal in A consisting of elements annihilated by $\mathfrak{m}$ (note that is it non-zero, since A is Artinian), and let $I \subset A$ be the ideal generated by some non-zero element $a \in A[\mathfrak{m}]$. By our inductive hypothesis, there is an isomorphism $V/IV \xrightarrow{\sim} \mathrm{Ind}_{\overline{P}}^G \chi_1' \otimes \chi_2'$, for some lifts χ_1', χ_2' of $\overline{\chi}_1$ and $\overline{\chi}_2$ to $(A/I)^\times$-valued characters. Since V is a faithful A-module, $IV \neq 0$, and thus $IV \xrightarrow{\sim} \mathrm{Ind}_{\overline{P}}^G \overline{\chi}_1 \otimes \overline{\chi}_2$. (The latter representation is irreducible, since $\overline{\chi}_1 \neq \overline{\chi}_2$ by assumption.) Thus V sits in the short exact sequence

$$0 \to \mathrm{Ind}_{\overline{P}}^G \overline{\chi}_1 \otimes \overline{\chi}_2 \to V \to \mathrm{Ind}_{\overline{P}}^G \chi_1' \otimes \chi_2' \to 0. \tag{4.2.15}$$

Applying $H^\bullet \mathrm{Ord}_P$ to (4.2.15), and taking into account Corollary 4.2.10, we obtain the exact sequence

$$0 \to \overline{\chi}_1 \otimes \overline{\chi}_2 \to \mathrm{Ord}_P(V) \to \chi_1' \otimes \chi_2' \to \chi_2 \overline{\varepsilon}^{-1} \otimes \chi_1 \overline{\varepsilon}^{-1}. \tag{4.2.16}$$

Since $\chi_1 \neq \chi_2 \overline{\varepsilon}^{-1}$ by assumption, we find that the final arrow in (4.2.16) vanishes, and thus that $\mathrm{Ord}_P(V)$ is an extension of $\chi_1' \otimes \chi_2'$ by $\overline{\chi}_1 \otimes \overline{\chi}_2$. Now consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ind}_{\overline{P}}^G \overline{\chi}_1 \otimes \overline{\chi}_2 & \longrightarrow & \mathrm{Ind}_{\overline{P}}^G \mathrm{Ord}_P(V) & \longrightarrow & \mathrm{Ind}_{\overline{P}}^G \overline{\chi}_1 \otimes \overline{\chi}_2 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{Ind}_{\overline{P}}^G \overline{\chi}_1 \otimes \overline{\chi}_2 & \longrightarrow & V & \longrightarrow & \mathrm{Ind}_{\overline{P}}^G \chi_1' \otimes \chi_2' & \longrightarrow & 0,
\end{array}$$

induced by the adjointness between $\mathrm{Ord}_P(V)$ and $\mathrm{Ind}_{\overline{P}}$. Since the outer two vertical arrows are isomorphisms, so is the middle isomorphism. We conclude that $\mathrm{Ord}_P(V)$ is given by an $A^\times$-valued character $\chi_1 \otimes \chi_2$, and that $\mathrm{Ind}_{\overline{P}}^G \chi_1 \otimes \chi_2 \xrightarrow{\sim} V$. $\square$

4.2.17. **Remark.** The preceding result shows that if $\overline{\chi}_1, \overline{\chi}_2 : \mathbb{Q}_p^\times \to k^\times$ are smooth characters for which $\overline{\chi}_1\overline{\chi}_2^{-1} \neq \underline{1}, \overline{\varepsilon}^{-1}$, then the map from the deformation functor of $\overline{\chi}_1 \otimes \overline{\chi}_2$ to the deformation functor of $\mathrm{Ind}_P^G \overline{\chi}_1 \otimes \overline{\chi}_2$ induced by Ind_P^G is an equivalence.

4.3. **Extension computations.** In this subsection we let k denote a finite field of characteristic p. Our goal is to compute the spaces $\mathrm{Ext}^1_G(V_1, V_2)$, where V_1 and V_2 are absolutely irreducible objects of $\mathrm{Mod}_G^{\mathrm{adm}}(k)$. We do this for every possible choice of V_1 and V_2 other than for the cases when V_1 and V_2 are both supersingular, when V_1 is a character and V_2 is a twist of Steinberg or supersingular (although we do treat the extensions of a character by Steinberg in one particular case, originally considered by Colmez – see the second isomorphism of claim (2) of Proposition 4.3.22), and when V_1 is a supersingular and V_2 is a twist of Steinberg. We also omit certain cases when V_1 is a twist of Steinberg and V_2 is a character when $p = 2$ or 3. Breuil and Paškūnas, and Colmez, have each found a different approach to many of the Ext calculations that we make; see Remark 4.3.35 below.

Our basic tool will be the exact sequence (3.7.5), together with the computations of the preceding subsection, and of course Corollary 3.12 of [11], verifying that Conjecture 3.7.2 holds in our context, which we will apply without further mention. We also require some additional technical computations which we have relegated to an appendix.[1] We will also exploit the short exact sequence

$$0 \to \chi \circ \det \to \mathrm{Ind}_P^G \chi \otimes \chi \to (\chi \circ \det) \otimes \mathrm{St} \to 0, \tag{4.3.1}$$

which gives rise to an exact sequence

$$\begin{aligned}\mathrm{Hom}_G(\chi \circ \det, V) \to \mathrm{Ext}^1_G\big((\chi \circ \det) \otimes \mathrm{St}, V\big) \\ \to \mathrm{Ext}^1_G(\mathrm{Ind}_P^G \chi \otimes \chi, V) \to \mathrm{Ext}^1_G(\chi \circ \det, V),\end{aligned} \tag{4.3.2}$$

for any object V of $\mathrm{Mod}_G^{\mathrm{adm}}(k)$. Finally, we will utilize the following lemmas:

4.3.3. **Lemma.** *If V is an object of $\mathrm{Mod}_G^{\mathrm{sm}}(k)$, and W is an object of $\mathrm{Mod}_P^{\mathrm{sm}}(k)$, then there are natural isomorphisms*

$$\mathrm{Ext}^\bullet_P(V, W) \xrightarrow{\sim} \mathrm{Ext}^\bullet_G(V, \mathrm{Ind}_P^G W).$$

(*Here the* Ext *spaces are computed in* $\mathrm{Mod}_P^{\mathrm{sm}}(k)$ *and* $\mathrm{Mod}_G^{\mathrm{sm}}(k)$, *respectively.*)

Proof. If $W \hookrightarrow I^\bullet$ is an injective resolution in $\mathrm{Mod}_P^{\mathrm{sm}}(k)$, then $\mathrm{Ind}_P^G W \hookrightarrow \mathrm{Ind}_P^G I^\bullet$ is an injective resolution in $\mathrm{Mod}_G^{\mathrm{sm}}(k)$. (Indeed, the functor Ind_P^G is right adjoint to the exact forgetful functor $\mathrm{Mod}_G^{\mathrm{sm}}(k) \to \mathrm{Mod}_P^{\mathrm{sm}}(k)$, and hence preserves injectives, and is furthermore exact [10, Prop. 4.1.5].) The adjunction property of Ind_P^G gives an isomorphism $\mathrm{Hom}_P(V, I^\bullet) \xrightarrow{\sim} \mathrm{Hom}_G(V, \mathrm{Ind}_P^G I^\bullet)$. Passing to cohomology, and taking into account the preceding discussion, we obtain the isomorphism in the statement of the lemma. □

4.3.4. **Lemma.** *If V_1 and V_2 are smooth P-representation over k, with V_1 being finite-dimensional over k, then restricting from P to M induces an isomorphism $\mathrm{Hom}_P(V_1, V_2) \xrightarrow{\sim} \mathrm{Hom}_M(V_1, V_2)$, and consequently an embedding $\mathrm{Ext}^1_P(V_1, V_2) \hookrightarrow$*

[1] In an earlier version of this paper, these computations were quite extensive, and were largely necessitated by the fact that we didn't know Conjecture 3.7.2, and so were forced to rely on the weaker exact sequence (3.7.6), rather than (3.7.5). With the results of [11] available, this complication has been obviated, and the computations of the appendix have become correspondingly shorter and comparatively less technical.

$\mathrm{Ext}^1_M(V_1, V_2)$. (*Here the* Ext^1 *spaces are computed in the categories of smooth representations over* k.)

Proof. Let $\phi \in \mathrm{Hom}_M(V_1, V_2)$. Since V_1 and $\phi(V_1)$ are both finite-dimensional smooth representations of P, we may find a compact open subgroup N_1 of N which acts trivially on each of V_1 and $\phi(V_1)$. The map ϕ is then both M-equivariant and N_1-equivariant. Since M and N_1 together generate P, the map ϕ is in fact P-equivariant. This proves the first statement of the lemma. The claim regarding Ext spaces follows, either by a general argument with δ-functors, or more directly by noting that any M-equivariant section to an extension of V_1 by V_2 is then necessarily P-equivariant. □

4.3.5. **Lemma.** *If* V_1 *and* V_2 *are two objects of* $\mathrm{Mod}^{\mathrm{sm}}_G(k)$, *and if* V_1 *is finite-dimensional over* k, *then the restriction map* $\mathrm{Hom}_G(V_1, V_2) \to \mathrm{Hom}_P(V_1, V_2)$ *is an isomorphism, and consequently, the restriction map* $\mathrm{Ext}^1_G(V_1, V_2) \to \mathrm{Ext}^1_P(V_1, V_2)$ (*in which the* Hom *and* Ext *spaces are computed in the categories* $\mathrm{Mod}^{\mathrm{sm}}_G(k)$ *and* $\mathrm{Mod}^{\mathrm{sm}}_P(k)$ *respectively*) *is injective.*

Proof. This can be proved in a similar manner to the preceding lemma. (In particular, the claim regarding Ext^1 spaces follows formally from the claim regarding Hom spaces.) Namely, let $\phi \in \mathrm{Hom}_P(V_1, V_2)$. Since V_1 is finite-dimensional over k, so is $\phi(V_1)$, and so there is some open subgroup G_0 of G which fixes both V_1 and $\phi(V_1)$ elementwise. The map ϕ is then both G_0 and P-equivariant. Since G_0 and P generate G as a group, we see that ϕ is in fact G-equivariant.

Alternatively, one can argue directly from (an obvious variant, with P replaced by $\overline{P}$, of) the preceding lemma: if ϕ is P-equivariant, then it is in particular M-equivariant, and hence also $\overline{P}$-equivariant. Since P and $\overline{P}$ generated G, we see that ϕ is G-equivariant, as claimed. □

4.3.6. **Remark.** The main theorem of [14] shows that the analogue of the preceding result holds for any absolutely irreducible object V_1 of $\mathrm{Mod}^{\mathrm{adm}}_G(k)$ which is not a twist of St.

We also have the following variant of Lemma 4.3.4.

4.3.7. **Lemma.** *If* W_1 *and* W_2 *are two objects of* $\mathrm{Mod}^{\mathrm{sm}}_M(k)$ *that are finite dimensional over* k, *then restriction from* P *to* M *induces isomorphisms*

$$\mathrm{Ext}^i_P(W_1, W_2) \xrightarrow{\sim} \mathrm{Ext}^i_M(W_1, W_2).$$

for $0 \leq i \leq 2$. (*In the source, the* Ext *space is computed in the category* $\mathrm{Mod}^{\mathrm{sm}}_P(k)$, *and* W_1 *and* W_2 *are regarded as* P*-representations via the surjection* $P \to M$, *while in the target, the* Ext *space is computed in* $\mathrm{Mod}^{\mathrm{sm}}_M(k)$.)

Proof. When $i = 0$ this is a special case of Lemma 4.3.4. To treat the case when $i = 1$, note that Lemma 4.3.8 below shows that N acts trivially on W_1 and W_2, and on any extension of W_1 by W_2 in the category $\mathrm{Mod}^{\mathrm{sm}}_P(k)$. Thus computing extensions of W_1 by W_2 as P-representations is the same as computing extensions of W_1 by W_2 as $M = P/N$-representations.

The argument in the case when $i = 2$ is slightly more involved. (We also note that this case is required only in Section A.1 of the appendix.) We begin by showing that if V is a finite-dimensional representation of M over k, regarded as a P-representation by having N act trivially, then the inflation map $H^i(M, V) \to$

$H^i(P, V)$ is an isomorphism. Indeed, by the Hochschild–Serre spectral sequence for continuous cohomology, it suffices to verify that $H^i(N, V) = 0$ for $i > 0$ when V is regarded as a trivial N-representation.

Of course there is an isomorphism $\mathbb{Q}_p \xrightarrow{\sim} N$; for each $j \geq 0$, write N^j to denote the image of $p^{-j}\mathbb{Z}_p$ under this isomorphism. We then have that $N = \bigcup N^j$, and for each $i \geq 1$ there is a short exact sequence

$$0 \to R^1 \varprojlim_j H^{i-1}(N_j, V) \to H^i(N, V) \to \varprojlim_j H^i(N_j, V) \to 0.$$

Since each $H^{i-1}(N_j, V)$ is finite-dimensional (as V is), the projective system of $H^{i-1}(N_j, V)$ satisfies the Mittag–Leffler condition, and so $R^1 \varprojlim_j H^{i-1}(N_j, V) = 0$. Thus in fact $H^i(N, V) \xrightarrow{\sim} \varprojlim_j H^i(N_j, V)$. Now $H^i(N_j, V) = 0$ if $i > 1$, and thus $H^i(N, V) = 0$ for such i. On the other hand, $H^1(N_j, V) = \mathrm{Hom}_{\mathrm{cont}}(N_j, V) \xrightarrow{\sim} V$ (since N, and so N_j, acts trivially on V). As $[N^{j+1} : N^j] = p$, we see that the transition map $H^1(N_{j+1}, V) \to H^1(N_j, V)$ vanishes (since V is a k-vector space, and k has characteristic p); thus in fact $H^1(N, V) = 0$ as well.

We now return to proving that the map $\mathrm{Ext}^2_P(W_1, W_2) \to \mathrm{Ext}^2_M(W_1, W_2)$ is an isomorphism. Since W_1 is finite-dimensional, we see that if $W_1^\vee$ denotes the contragredient representation to W_1, then there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1_P(W_1, W_2) & \longrightarrow & \mathrm{Ext}^1_M(W_1, W_2) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^1_P(\underline{1} \otimes \underline{1}, W_1^\vee \otimes W_2) & \longrightarrow & \mathrm{Ext}^1_M(\underline{1} \otimes \underline{1}, W_1^\vee \otimes W_2). \end{array}$$

Thus it suffices to consider the case when W_1 is the trivial representation.

Now consider the diagram

$$\begin{array}{ccccc} \mathrm{Ext}^2_M(\underline{1}, W_2) & \longrightarrow & \mathrm{Ext}^2_P(\underline{1}, W_2) & \longrightarrow & \mathrm{Ext}^2_M(\underline{1}, W_2) \\ \downarrow & & \downarrow & & \downarrow \\ H^2(M, W_2) & \longrightarrow & H^2(P, W_2) & \longrightarrow & H^2(M, W_2), \end{array}$$

where the first horizontal arrows are given by inflation from M to P, the second horizontal arrows are given by restriction from P to M, and the vertical arrows are given by (2.2.1). Composing either the upper or lower horizontal arrows yields the identity, and we showed above that the first of the lower horizontal arrows is an isomorphism. Thus both lower horizontal arrows are isomorphisms. On the other hand, it follows from Corollary 2.2.4 that the vertical arrows are injective. From these facts we conclude that the upper horizontal arrows are injective; since they compose to give the identity, they must be isomorphisms. This completes the proof of the lemma for $i = 2$. □

The following lemma is proved in [6, Lem. III.1.3]. For the reader's convenience, we give a proof here.

4.3.8. **Lemma.** *Any smooth N-action on a finite-dimensional k-vector space is trivial.*

Proof. Suppose that W is finite-dimensional over k and equipped with a smooth N-action. Write $N = \bigcup N^i$, where each N^i is a compact open subgroup. Since N^i is in particular pro-p, we see that $W^{N^i} \neq 0$. Since W is finite-dimensional, we find that $W^N = \bigcap W^{N^i} \neq 0$. (A nested intersection of non-zero subspaces of a finite-dimensional space is non-zero.) Thus, proceeding by induction, we find that the Jordan–Hölder factors of W, as N-representation, are copies of the trivial representation. The lemma then follows from the fact that

$$\mathrm{Ext}^1_N(\underline{1}, \underline{1}) \xrightarrow{\sim} H^1(N, \underline{1}) = \mathrm{Hom}_{\mathrm{cont}}(N, k) = 0.$$

(Here the isomorphism is provided by Proposition 2.2.2, the first equality is the standard description of H^1 with coefficients in the trivial representations as a space of homomorphisms, and the second equality holds since N is a divisible group, while k is of characteristic p.) □

We also have the following closely related lemma.

4.3.9. **Lemma.** *Any smooth* $\mathrm{SL}_2(\mathbb{Q}_p)$*-action on a finite-dimensional* k*-vector space is trivial.*

Proof. If V is a finite-dimensional k-vector space equipped with a smooth $\mathrm{SL}_2(\mathbb{Q}_p)$-action, then there exists an open subgroup H of $\mathrm{SL}_2(\mathbb{Q}_p)$ which acts trivially on V. Since by the preceding lemma N also acts trivially on V, and N and H together generate $\mathrm{SL}_2(\mathbb{Q}_p)$, the present lemma follows. (Alternatively, one can combine the preceding lemma with its obvious analogoue for $\overline{N}$ to conclude that both N and $\overline{N}$ act trivially on V, and then note that N and $\overline{N}$ together generate $\mathrm{SL}_2(\mathbb{Q}_p)$.) □

4.3.10. **Lemma.** *If* $\alpha, \alpha', \beta, \beta' : \mathbb{Q}_p^\times \to k^\times$ *are smooth characters, then* $\mathrm{Ext}^i_M(\alpha \otimes \beta, \alpha' \otimes \beta') = 0$ *for all* $i \geq 0$*, unless* $\alpha = \alpha'$ *and* $\beta = \beta'$*. (Here the* Ext^i *are computed in the category* $\mathrm{Mod}^{\mathrm{sm}}_M(k)$*.)*

Proof. Since M is commutative, for any two smooth M-representations U and V, there is a natural M-action on the space $\mathrm{Hom}_M(U, V)$ (induced by the M-action on either U or V), and hence (by passing to derived functors) a natural M-action on $\mathrm{Ext}^i_M(U, V)$, for each $i \geq 0$. If now we take $U = \alpha \otimes \beta$ and $V = \alpha' \otimes \beta'$, we see that this M-action is simultaneously given by the character $\alpha \otimes \beta$ and by the character $\alpha' \otimes \beta'$. Thus, unless these characters coincide, the space $\mathrm{Ext}^i_M(\alpha \otimes \beta, \alpha' \otimes \beta')$ must vanish. □

4.3.11. **Remark.** We also remark that if $\alpha, \beta : \mathbb{Q}_p^\times \to k^\times$ are smooth characters, so that $\alpha \otimes \beta$ is a smooth character of M, then

$$\mathrm{Ext}^1_M(\alpha \otimes \beta, \alpha \otimes \beta) \xrightarrow{\sim} \mathrm{Ext}^1_M(\underline{1}, \underline{1}) \xrightarrow{\sim} H^1(M, \underline{1}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{cont}}(M, \underline{1}),$$

by Proposition 2.2.2.

We are now ready to begin our calculation of Ext spaces for G-representations. We present the calculations in a series of propositions. We begin with the case of extensions by a principal series.

4.3.12. **Proposition.** *Let* $\chi : \mathbb{Q}_p^\times \to k^\times$ *be a smooth character.*

(1) *If* $\chi_1 \otimes \chi_2 \neq \chi \otimes \chi$*, then* $\mathrm{Ext}^1_G\big(\mathrm{Ind}_{\overline{P}}^G \chi_1 \otimes \chi_2, (\chi \circ \det) \otimes \mathrm{St}\big) = 0$.
(2) *There is a natural isomorphism*

$$\mathrm{Ext}^1_M(\chi \otimes \chi, \chi \otimes \chi) \xrightarrow{\sim} \mathrm{Ext}^1_G\big(\mathrm{Ind}_{\overline{P}}^G \chi \otimes \chi, (\chi \circ \det) \otimes \mathrm{St}\big).$$

Proof. This is a direct application of (3.7.5), setting $U = \chi_1 \otimes \chi_2$ and $V = (\chi \circ \det) \otimes \mathrm{St}$, and taking into account part (2) of Theorem 4.2.12. □

4.3.13. **Proposition.** *Let $\chi : \mathbb{Q}_p^\times \to k^\times$ be a smooth character.*

(1) *If $\chi_1 \otimes \chi_2 \neq \chi\overline{\varepsilon}^{-1} \otimes \chi\overline{\varepsilon}$, then $\mathrm{Ext}^1_G\big(\mathrm{Ind}_P^G \chi_1 \otimes \chi_2, (\chi \circ \det)\big) = 0$.*
(2) *The space $\mathrm{Ext}^1_G\big(\mathrm{Ind}_P^G \chi\overline{\varepsilon}^{-1} \otimes \chi\overline{\varepsilon}, (\chi \circ \det)\big)$ is one-dimensional.*

Proof. These claims follow directly from (3.7.5), together with part (3) of Theorem 4.2.12. □

4.3.14. **Proposition.** *The space $\mathrm{Ext}^1_G(\mathrm{Ind}_P^G \chi_1 \otimes \chi_2, V)$ vanishes for any pair of smooth characters $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to k^\times$, and any supersingular absolutely irreducible admissible smooth representation V of G over k.*

Proof. This follows directly from (3.7.5) (taking $U = \chi_1 \otimes \chi_2$), together with part (4) of Theorem 4.2.12. □

4.3.15. **Proposition.** *Let $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to k^\times$ be a pair of smooth characters.*

(1) *If $\chi_1', \chi_2' : \mathbb{Q}_p^\times \to k^\times$ is a second pair of smooth characters, then*
$$\mathrm{Ext}^1_G(\mathrm{Ind}_P^G \chi_1 \otimes \chi_2, \mathrm{Ind}_P^G \chi_1' \otimes \chi_2')$$
vanishes unless either $\chi_1 = \chi_1'$ and $\chi_2 = \chi_2'$, or else $\chi_1 = \chi_2'\overline{\varepsilon}^{-1}$ and $\chi_2 = \chi_1'\overline{\varepsilon}$.
(2) *If $\chi_1 \neq \chi_2\overline{\varepsilon}^{-1}$, then*
$$\mathrm{Ext}^1_G(\mathrm{Ind}_P^G \chi_1 \otimes \chi_2, \mathrm{Ind}_P^G \chi_2\overline{\varepsilon}^{-1} \otimes \chi_1\overline{\varepsilon})$$
is one-dimensional over k.
(3) *If either $\chi_1 \neq \chi_2\overline{\varepsilon}^{-1}$ or $p \neq 2$, then the natural map*
$$\mathrm{Ext}^1_M(\chi_1 \otimes \chi_2, \chi_1 \otimes \chi_2) \to \mathrm{Ext}^1_G(\mathrm{Ind}_P^G \chi_1 \otimes \chi_2, \mathrm{Ind}_P^G \chi_1 \otimes \chi_2)$$
given by applying Ind_P^G is an isomorphism.
(4) *If $p = 2$ and $\chi_1 = \chi_2$ $(= \chi_2\overline{\varepsilon}^{-1})$, then the natural map*
$$\mathrm{Ext}^1_M(\chi_1 \otimes \chi_2, \chi_1 \otimes \chi_2) \to \mathrm{Ext}^1_G(\mathrm{Ind}_P^G \chi_1 \otimes \chi_2, \mathrm{Ind}_P^G \chi_1 \otimes \chi_2)$$
given by applying Ind_P^G is injective, with a one-dimensional cokernel.

Proof. We apply the exact sequence (3.7.5), taking U to be $\chi_1 \otimes \chi_2$, and V to be $\mathrm{Ind}_P^G \chi_1' \otimes \chi_2'$. Taking into account Theorem 4.2.12 (1), we obtain an exact sequence

$$0 \to \mathrm{Ext}^1_M(\chi_1 \otimes \chi_2, \chi_1' \otimes \chi_2') \to \mathrm{Ext}^1_G(\mathrm{Ind}_P^G \chi_1 \otimes \chi_2, \mathrm{Ind}_P^G \chi_1' \otimes \chi_2') \to \mathrm{Hom}_M(\chi_1 \otimes \chi_2, \chi_2'\overline{\varepsilon}^{-1} \otimes \chi_1'\overline{\varepsilon}) \to \mathrm{Ext}^2_M(\chi_1 \otimes \chi_2, \chi_1' \otimes \chi_2') \to \mathrm{Ext}^2_G(\mathrm{Ind}_P^G \chi_1 \otimes \chi_2, \mathrm{Ind}_P^G \chi_1' \otimes \chi_2'). \tag{4.3.16}$$

Claim (1) is now seen to follow immediately from the exactness of (4.3.16) together with Lemma 4.3.10.

If we set $\chi_1' = \chi_2\overline{\varepsilon}^{-1}$ and $\chi_2' = \chi_1\overline{\varepsilon}$, and assume that $\chi_1' \neq \chi_1$ (or equivalently, that $\chi_2' \neq \chi_2$), then the exact sequence (4.3.16), together with Lemma 4.3.10, gives rise to an isomorphism
$$\mathrm{Ext}^1_G(\mathrm{Ind}_P^G \chi_1 \otimes \chi_2, \mathrm{Ind}_P^G \chi_2\overline{\varepsilon}^{-1} \otimes \chi_1\overline{\varepsilon}) \xrightarrow{\sim} \mathrm{Hom}_M(\chi_1 \otimes \chi_2, \chi_1 \otimes \chi_2),$$
whose target is one-dimensional. This establishes claim (2).

If we set $\chi_1' = \chi_1$ and $\chi_2' = \chi_2$, then the exact sequence (4.3.16) includes the subsequence

$$0 \to \mathrm{Ext}^1_M(\chi_1 \otimes \chi_2, \chi_1 \otimes \chi_2) \to \mathrm{Ext}^1_G(\mathrm{Ind}_{\overline{P}}^G \chi_1 \otimes \chi_2, \mathrm{Ind}_{\overline{P}}^G \chi_1 \otimes \chi_2) \\ \to \mathrm{Hom}_M(\chi_1 \otimes \chi_2, \chi_2\overline{\varepsilon}^{-1} \otimes \chi_1\overline{\varepsilon}).$$

If in addition $\chi_1 \neq \chi_2\overline{\varepsilon}^{-1}$ (or equivalently, $\chi_2 \neq \chi_1\overline{\varepsilon}$), we find that the last term in this exact sequence vanishes, and so claim (3) follows in this case. To complete the proof of (3), suppose that $p \neq 2$ and that $\chi_1 = \chi_2\overline{\varepsilon}^{-1}$. In this case the exact sequence (4.3.16) takes the form

$$(4.3.17) \quad 0 \to \mathrm{Ext}^1_M(\chi \otimes \chi\overline{\varepsilon}, \chi \otimes \chi\overline{\varepsilon}) \to \mathrm{Ext}^1_G(\mathrm{Ind}_{\overline{P}}^G \chi \otimes \chi\overline{\varepsilon}, \mathrm{Ind}_{\overline{P}}^G \chi \otimes \chi\overline{\varepsilon}) \\ \to \mathrm{Hom}_M\big(\chi \otimes \chi\overline{\varepsilon}, R^1\mathrm{Ord}_P(\mathrm{Ind}_{\overline{P}}^G \chi \otimes \chi\overline{\varepsilon})\big) \to \mathrm{Ext}^2_M\big(\chi \otimes \chi\overline{\varepsilon}, \chi \otimes \chi\overline{\varepsilon}\big) \\ \to \mathrm{Ext}^2_G(\mathrm{Ind}_{\overline{P}}^G \chi \otimes \chi\overline{\varepsilon}, \mathrm{Ind}_{\overline{P}}^G \chi \otimes \chi\overline{\varepsilon}).$$

We show in Subsection A.1 of the appendix that the final arrow in this exact sequence is *not* injective. On the other hand, we have shown in Theorem 4.2.12 that $R^1\mathrm{Ord}_P(\mathrm{Ind}_{\overline{P}}^G \chi \otimes \chi\overline{\varepsilon})$ is one-dimensional, and so we conclude that the first non-trivial arrow in (4.3.17) is an isomorphism, as required.

We now turn to claim (4). Thus we assume that $p = 2$, and that $\chi_1 = \chi_2 = \chi$. To simplify the notation, we apply, as we may, a twist by $\chi^{-1} \circ \det$, and thus in fact assume that $\chi = \underline{1}$. The short exact sequence (4.3.1) gives rise to a long exact sequence

$$0 \to \mathrm{Hom}_G(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \underline{1}, \underline{1}) \to \mathrm{Hom}_G(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \underline{1}, \mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \underline{1}) \to \mathrm{Hom}_G(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \underline{1}, \mathrm{St}) \\ \to \mathrm{Ext}^1_G(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \underline{1}, \underline{1}) \to \mathrm{Ext}^1_G(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \underline{1}, \mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \underline{1}) \to \mathrm{Ext}^1_G(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \underline{1}, \mathrm{St}).$$

Since (4.3.1) is non-split, this long exact sequence induces an exact sequence

$$(4.3.18) \quad 0 \to \mathrm{Ext}^1_G(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \underline{1}, \underline{1}) \to \mathrm{Ext}^1_G(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \underline{1}, \mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \underline{1}) \\ \to \mathrm{Ext}^1_G(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \underline{1}, \mathrm{St}).$$

We claim that the last arrow is surjective; indeed, it is easily checked that the composite

$$\mathrm{Ext}^1_M(\underline{1} \otimes \underline{1}, \underline{1} \otimes \underline{1}) \to \mathrm{Ext}^1_G(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \underline{1}, \mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \underline{1}) \to \mathrm{Ext}^1_G(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \underline{1}, \mathrm{St}),$$

with the first arrow being induced by the functor $\mathrm{Ind}_{\overline{P}}^G$, coincides with the isomorphism of Proposition 4.3.12 (2). Thus not only have we shown that (4.3.18) extends to a short exact sequence, but in fact we have provided a splitting, and thus have established an isomorphism

$$\mathrm{Ext}^1_G(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \underline{1}, \underline{1}) \oplus \mathrm{Ext}^1_M(\underline{1} \otimes \underline{1}, \underline{1} \otimes \underline{1}) \xrightarrow{\sim} \mathrm{Ext}^1_G(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \underline{1}, \mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \underline{1}).$$

Since Proposition 4.3.13 (2) shows that the first summand is one-dimensional (recall that $p = 2$, so that $\overline{\varepsilon} = \underline{1}$), claim (4) follows. □

We now consider the case of extensions by Steinberg and by characters.

4.3.19. **Proposition.** *Let $\chi, \chi_1, \chi_2 : \mathbb{Q}_p^\times \to k^\times$ be smooth characters.*

1. *If $\chi_1 \otimes \chi_2 \neq \chi \otimes \chi$ or $\chi\overline{\varepsilon}^{-1} \otimes \chi\overline{\varepsilon}$, then $\mathrm{Ext}^1_G\big((\chi \circ \det) \otimes \mathrm{St}, \mathrm{Ind}_{\overline{P}}^G \chi_1 \otimes \chi_2\big)$ vanishes.*
2. *If $p \neq 2$, then $\mathrm{Ext}^1_G\big((\chi \circ \det) \otimes \mathrm{St}, \mathrm{Ind}_{\overline{P}}^G \chi\overline{\varepsilon}^{-1} \otimes \chi\overline{\varepsilon}\big)$ is one-dimensional.*

Proof. Claim (1) follows directly from a consideration of (4.3.2), together with Proposition 4.3.15. Claim (2) follows in a similar manner, since

$$\mathrm{Ext}^1_G(\chi\circ\det, \mathrm{Ind}_{\overline{P}}^G \chi\overline{\varepsilon}^{-1}\otimes\chi\overline{\varepsilon}) \xrightarrow{\sim} \mathrm{Ext}^1_M(\chi\otimes\chi, \chi\overline{\varepsilon}\otimes\chi\overline{\varepsilon}^{-1})$$

by Lemma 4.3.3 and (the analogue for $\overline{P}$ of) Lemma 4.3.7, and the latter Ext space vanishes, by Lemma 4.3.10, since $\chi\otimes\chi$ and $\chi\overline{\varepsilon}\otimes\chi\overline{\varepsilon}^{-1}$ are distinct characters of M (as $p\neq 2$). ☐

Recall that in Subsection 4.2 we defined the action of the Weyl group element w on the category of smooth M-representations. Clearly, if $\chi_1,\chi_2:\mathbb{Q}_p^\times\to k^\times$ are smooth characters, then $(\chi_1\otimes\chi_2)^w = \chi_2\otimes\chi_1$. In particular, if $\chi:\mathbb{Q}_p^\times\to k^\times$ is a smooth character, then $(\chi\otimes\chi)^w=\chi\otimes\chi$, and hence the action of w on M-representations induces an involution of $\mathrm{Ext}^1_M(\chi\otimes\chi,\chi\otimes\chi)$. We let $\mathrm{Ext}^1_M(\chi\otimes\chi,\chi\otimes\chi)^w$ (resp. $\mathrm{Ext}^1_M(\chi\otimes\chi,\chi\otimes\chi)^{w=-1}$) denote the fixed space under this involution (resp. the anti-fixed space of the involution w).

4.3.20. **Lemma.** (1) *The subspace of* $\mathrm{Ext}^1_M(\chi\otimes\chi,\chi\otimes\chi)^w$ *of* $\mathrm{Ext}^1_M(\chi\otimes\chi,\chi\otimes\chi)$ *consists of extensions which are trivial when restricted to* $M\bigcap \mathrm{SL}_2(\mathbb{Q}_p)$.
(2) *The subspace* $\mathrm{Ext}^1_M(\chi\otimes\chi,\chi\otimes\chi)^{w=-1}$ *of* $\mathrm{Ext}^1_M(\chi\otimes\chi,\chi\otimes\chi)$ *consists of extensions which are trivial when restricted to the centre* Z_G *of* G.

Proof. Under the composition of the isomorphisms

$$\mathrm{Ext}^1_M(\chi\otimes\chi,\chi\otimes\chi)\xrightarrow{\sim}\mathrm{Ext}^1_M(\underline{1},\underline{1})\xrightarrow{\sim}\mathrm{Hom}_{\mathrm{cont}}(M,\underline{1})$$

discussed in Remark 4.3.11, the action of w on the source corresponds to the w-action $\phi\mapsto\Big(\phi^w : m\mapsto \phi\big(w(m)\big)\Big)$ on the target. Also, an element of the source is trivial when restricted to $M\bigcap\mathrm{SL}_2(\mathbb{Q}_p)$ (resp. Z_G) precisely when the corresponding homomorphism ϕ is trivial on $M\bigcap\mathrm{SL}_2(\mathbb{Q}_p)$ (resp. Z_G). An elementary computation shows that this latter condition holds if and only if $\phi^w=\phi$ (resp. $\phi^w=-\phi$), and so the lemma follows. ☐

We note that $\mathrm{Ext}^1_M(\chi\otimes\chi,\chi\otimes\chi)^{w=-1}$ is linearly independent of (resp. coincides with) $\mathrm{Ext}^1_M(\chi\otimes\chi,\chi\otimes\chi)^w$ when $p\neq 2$ (resp. $p=2$).

4.3.21. **Proposition.** *Let* $\chi_1,\chi_2:\mathbb{Q}_p^\times\to k^\times$ *be smooth characters.*
(1) *If* $\chi_1\neq\chi_2$, *then* $\mathrm{Ext}^1_G(\chi_1\circ\det,\chi_2\circ\det)$ *vanishes.*
(2) *If* $\chi_1=\chi_2=\chi$ *(say), then restricting to* M *induces an isomorphism*

$$\mathrm{Ext}^1_G(\chi\circ\det,\chi\circ\det)\xrightarrow{\sim}\mathrm{Ext}^1_M(\chi\otimes\chi,\chi\otimes\chi)^w.$$

Proof. Note that Lemma 4.3.9 shows that any extension of $\chi_1\circ\det$ by $\chi_2\circ\det$ is trivial on $\mathrm{SL}_2(\mathbb{Q}_p)$, and so factors through det. Thus the natural map

$$\mathrm{Ext}^1_{\mathbb{Q}_p^\times}(\chi_1,\chi_2)\to\mathrm{Ext}^1_G(\chi_1\circ\det,\chi_2\circ\det),$$

given by pulling back $\mathbb{Q}_p^\times$ representations along $\det: G\to\mathbb{Q}_p^\times$, is an isomorphism. Now det also induces an isomorphism $M/\big(M\bigcap\mathrm{SL}_2(\mathbb{Q}_p)\big)\xrightarrow{\sim}\mathbb{Q}_p^\times$, and so we see that restriction to M induces an isomorphism between $\mathrm{Ext}^1_G(\chi_1\circ\det,\chi_2\circ\det)$ and the subspace of extensions of $\mathrm{Ext}^1_M(\chi_1\otimes\chi_1,\chi_2\otimes\chi_2)$ which are trivial when restricted to $M\bigcap\mathrm{SL}_2(\mathbb{Q}_p)$. Claim (1) now follows from Lemma 4.3.10 (which shows that $\mathrm{Ext}^1_M(\chi_1\otimes\chi_1,\chi_2\otimes\chi_2)=0$ when $\chi_1\neq\chi_2$), while claim (2) follows from Lemma 4.3.20 (1). ☐

4.3.22. **Proposition.** *Let* $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to k^\times$ *be smooth characters.*

(1) *If* $\chi_1 \neq \chi_2$, *then* $\mathrm{Ext}^1_G\big((\chi_1 \circ \det) \otimes \mathrm{St}, (\chi_2 \circ \det) \otimes \mathrm{St}\big)$ *vanishes.*
(2) *If* $\chi_1 = \chi_2 = \chi$ *(say), then there are natural isomorphisms*

$$\mathrm{Ext}^1_M(\chi \otimes \chi, \chi \otimes \chi)^w \xrightarrow{\sim} \mathrm{Ext}^1_G\big((\chi \circ \det) \otimes \mathrm{St}, (\chi \circ \det) \otimes \mathrm{St}\big) \tag{4.3.23}$$

and

$$\mathrm{Ext}^1_M(\chi \otimes \chi, \chi \otimes \chi)^{w=-1} \xrightarrow{\sim} \mathrm{Ext}^1_G\big(\chi \circ \det, (\chi \circ \det) \otimes \mathrm{St}\big). \tag{4.3.24}$$

Proof. Claim (1) follows from (4.3.2) and the vanishing result of Proposition 4.3.12. To prove claim (2), we again consider the exact sequence (4.3.2) (taking $V = (\chi \circ \det) \otimes \mathrm{St}$), and place it into the following commutative diagram, as the left-hand column:

$$\begin{array}{ccc}
0 & & \\
\downarrow & & \\
\mathrm{Ext}^1_G\big((\chi \circ \det) \otimes \mathrm{St}, (\chi \circ \det) \otimes \mathrm{St}\big) & & \\
\downarrow & & \\
\mathrm{Ext}^1_G\big(\mathrm{Ind}_P^G \chi \otimes \chi, (\chi \circ \det) \otimes \mathrm{St}\big) & \xleftarrow{\sim} & \mathrm{Ext}^1_M(\chi \otimes \chi, \chi \otimes \chi) \\
\downarrow & & \downarrow \\
\mathrm{Ext}^1_G\big(\chi \circ \det, (\chi \circ \det) \otimes \mathrm{St}\big) & \longrightarrow & \mathrm{Ext}^1_M(\chi \otimes \chi, \chi \otimes \chi)^{w=-1},
\end{array}$$

in which the upper horizontal arrow is given by part (2) of Proposition 4.3.12, the lower horizontal arrow will be specified below, and the right-hand vertical arrow is defined to make the diagram commute.

Let us now specify the lower horizontal arrow. There is a P-equivariant isomorphism

$$\mathrm{St} \xrightarrow{\sim} \mathcal{C}_c^{\mathrm{sm}}(N, k) \otimes_k (\chi \otimes \chi).$$

Evaluation at the identity of N induces an M-equivariant surjection

$$\mathcal{C}_c^{\mathrm{sm}}(N, k) \otimes_k (\chi \otimes \chi) \to \chi \otimes \chi.$$

Composing these two, we obtain an M-equivariant surjection

$$(\chi \otimes \chi) \otimes \mathrm{St} \to \chi \otimes \chi,$$

which induces a map

$$\mathrm{Ext}^1_P\big(\chi \circ \det, (\chi \circ \det) \otimes \mathrm{St}\big) \to \mathrm{Ext}^1_M(\chi \otimes \chi, \chi \otimes \chi)^{w=-1}. \tag{4.3.25}$$

(Note that it does indeed land in $\mathrm{Ext}^1_M(\chi \otimes \chi, \chi \otimes \chi)^{w=-1}$, by Lemma 4.3.20 (2), because any P-equivariant extension of $\chi \circ \det$ by $(\chi \circ \det) \otimes \mathrm{St}$ must have a central character, there being no P-equivariant maps from the former representation to the latter.) We take the lower horizontal arrow to be the map

$$\mathrm{Ext}^1_P\big(\chi \circ \det, (\chi \circ \det) \otimes \mathrm{St}\big) \to \mathrm{Ext}^1_M(\chi \otimes \chi, \chi \otimes \chi)^{w=-1}. \tag{4.3.26}$$

obtained by composing of (4.3.25) with the restriction map

$$\mathrm{Ext}^1_G\big(\chi \circ \det, (\chi \circ \det) \otimes \mathrm{St}\big) \to \mathrm{Ext}^1_P\big(\chi \circ \det, (\chi \circ \det) \otimes \mathrm{St}\big). \tag{4.3.27}$$

The map (4.3.26) was first constructed by Colmez, who proved that it is an isomorphism [6, Thm. VII.4.19]. We will give another proof of this fact in the course of the present argument.

Corollary A.2.4 of the appendix shows that (4.3.25) is injective. Lemma 4.3.5 shows that (4.3.27) in also injective. Thus their composite (4.3.26) is again injective. A computation shows that (up to a sign) the right-hand vertical arrow in our diagram is given by the formula $\phi \mapsto \phi - \phi^w$. Thus this arrow is surjective. Since the lower horizontal arrow is injective, we discover that it is in fact an isomorphism. Thus we obtain the isomorphism (4.3.24). Also, restricting the upper horizontal arrow to $\mathrm{Ext}^1_M(\chi \otimes \chi, \chi \otimes \chi)^w$, we obtain the isomorphism (4.3.23). $\square$

4.3.28. **Remark.** The isomorphism (4.3.23) has a concrete interpretation. Namely, if U is an extension of $\chi \otimes \chi$ by itself, corresponding to an element of $\mathrm{Ext}^1_M(\chi \otimes \chi, \chi \otimes \chi)^w$, then Proposition 4.3.21 shows that U extends uniquely to a G-representation. The induction $\mathrm{Ind}_{\overline{P}}^G U$ then contains a copy of U as a subrepresentation, and the corresponding quotient is an extension V of $(\chi \circ \det) \otimes \mathrm{St}$ by itself. This map $U \mapsto V$ gives rise to the isomorphism (4.3.23) (up to a possible sign).

4.3.29. **Remark.** As was observed above, $\mathrm{Ext}^1_M(\chi \otimes \chi, \chi \otimes \chi)^{w=-1}$ is linearly independent of (resp. coincides with) $\mathrm{Ext}^1_M(\chi \otimes \chi, \chi \otimes \chi)^w$ when $p \neq 2$ (resp. $p = 2$). Thus if $p \neq 2$, then we find that *none* of the non-trivial extensions of $(\chi \circ \det) \otimes \mathrm{St}$ by $(\chi \circ \det) \otimes \mathrm{St}$ constructed in the preceding proposition admit a central character, while if $p = 2$, then they all do.

4.3.30. **Proposition.** *Let $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to k^\times$ be smooth characters.*

1. *If $\chi_1 \neq \chi_2$, and furthermore either $\chi_1 \neq \chi_2 \overline{\varepsilon}$ or else $p \neq 3$, then $\mathrm{Ext}^1_G\big((\chi_1 \circ \det) \otimes \mathrm{St}, \chi_2 \circ \det\big)$ vanishes.*
2. *If $\chi_1 = \chi_2 = \chi$ (say), and if $p \neq 2$, then $\mathrm{Ext}^1_G\big((\chi \circ \det) \otimes \mathrm{St}, \chi \circ \det\big)$ is one-dimensional.*

Proof. Both claims follow directly from a consideration of (4.3.2), together with Proposition 4.3.13.

$\square$

4.3.31. **Proposition.** *If $\chi : \mathbb{Q}_p^\times \to k^\times$ is smooth and V is a supersingular absolutely irreducible object of $\mathrm{Mod}_G^{\mathrm{adm}}(k)$, then $\mathrm{Ext}^1_G\big((\chi \circ \det) \otimes \mathrm{St}, V\big)$ vanishes.*

Proof. This follows directly from a consideration of (4.3.2), together with the vanishing result of Proposition 4.3.14 (in the case $\chi_1 = \chi_2 = \chi$). $\square$

4.3.32. **Proposition.** *Let $\chi, \chi_1, \chi_2 : \mathbb{Q}_p^\times \to k^\times$ be smooth characters.*

1. *If $\chi_1 \otimes \chi_2 \neq \chi \otimes \chi$, then $\mathrm{Ext}^1_G(\chi \circ \det, \mathrm{Ind}_{\overline{P}}^G \chi_1 \otimes \chi_2)$ vanishes.*
2. *There is a natural isomorphism*

$$\mathrm{Ext}^1_G(\chi \circ \det, \mathrm{Ind}_{\overline{P}}^G \chi \otimes \chi) \xrightarrow{\sim} \mathrm{Ext}^1_M(\chi \otimes \chi, \chi \otimes \chi).$$

Proof. This follows directly from Lemmas 4.3.3 and 4.3.7 (or rather, the analogue of the latter for $\overline{P}$). $\square$

4.3.33. **Proposition.** *Let $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to k^\times$ be smooth characters, with $\chi_1 \neq \chi_2$. If V is a supersingular absolutely irreducible object of $\mathrm{Mod}_G^{\mathrm{adm}}(k)$, then the space $\mathrm{Ext}^1_G(V, \mathrm{Ind}_{\overline{P}}^G \chi_1 \otimes \chi_2)$ vanishes.*

Proof. Let $0 \to \mathrm{Ind}_P^G \chi_1 \otimes \chi_2 \to E \to V \to 0$ be an extension. Applying $H^\bullet\mathrm{Ord}_P$, and taking into account Theorem 4.2.12, we find that

$$H^1\mathrm{Ord}_P(E) \xrightarrow{\sim} H^1\mathrm{Ord}_P(\mathrm{Ind}_P^G \chi_1 \otimes \chi_2) \xrightarrow{\sim} \chi_2\overline{\varepsilon}^{-1} \otimes \chi_1\overline{\varepsilon},$$

and hence, by Proposition 3.6.2, we obtain an isomorphism of Jacquet modules

$$E_N \xrightarrow{\sim} \chi_2 \otimes \chi_1.$$

From the adjointness between Jacquet modules and induction, we find that there exists a map $E \to \mathrm{Ind}_P^G \chi_1 \otimes \chi_2$ extending the identity map from $\mathrm{Ind}_P^G \chi_1 \otimes \chi_2$ to itself. Thus E is a split extension, as claimed. □

4.3.34. **Proposition.** *Let $\chi : \mathbb{Q}_p^\times \to k^\times$ be a smooth character. If V is a supersingular absolutely irreducible object of $\mathrm{Mod}_G^{\mathrm{adm}}(k)$, then $\mathrm{Ext}_G^1(V, \chi \circ \det)$ vanishes.*

Proof. This is proved in an analogous manner to the preceding proposition. Namely, if $0 \to \chi \circ \det \to E \to V \to 0$ is an extension, then applying $H^\bullet\mathrm{Ord}_P$, and taking into account Theorem 4.2.12 and Proposition 3.6.2, we obtain an isomorphism of Jacquet modules $E_N \xrightarrow{\sim} \chi \otimes \chi$. Thus, from the adjointness between Jacquet modules and induction, we obtain a map $E \to \mathrm{Ind}_P^G \chi \otimes \chi$ that extends the embedding $\chi \circ \det \hookrightarrow \mathrm{Ind}_P^G \chi \otimes \chi$ Thus E is a split extension, as claimed. □

4.3.35. **Remark.** In his paper [6], Colmez constructs a functor from the full subcategory of $\mathrm{Mod}_G^{\mathrm{adm}}(k)$ consisting of finitely generated objects to the category of continuous $G_{\mathbb{Q}_p}$-representations on k-vector spaces. Using this functor, Colmez has found another approach to many of the preceding computations, and some of the above computations were first made by him. In particular, Claim (3) of the Proposition 4.3.15 (under the additional assumption that $\chi_1 \neq \chi_2$) is originally due to Colmez [6, Prop. VII.4.15], as is the isomorphism (4.3.24) [6, §VII.4.3]. In [6, §VII.4.2] Colmez gives another proof of (most of) Proposition 4.3.15, and in [6, Prop. 7.5.4], he gives alternative proofs of Propositions 4.3.14, 4.3.31, 4.3.33, and 4.3.34.

In the paper [4], Breuil and Paškūnas study the functor of passage to pro-p-Iwahori invariants (in the context of representations of $\mathrm{GL}_2(F)$, for a finite extension F of $\mathbb{Q}_p$), as well as its first derived functor, and in this way obtain results regarding extensions by principal series representations. In particular they recover (most of) Proposition 4.3.15 [4, Cor. 8.2, Cor. 8.3], as well as Proposition 4.3.33 [4, Cor. 8.4].

Appendix A

In this appendix we present some technical calculations needed to complete the computations of Subsection 4.3. We have placed these calculations in an appendix, since they are of a more *ad hoc* nature than the computations using the functors $H^\bullet\mathrm{Ord}_P$ that we have presented in the main body of the text. Throughout the appendix, G, P, etc. have the same meanings as in Section 4 (i.e. $G = \mathrm{GL}_2(\mathbb{Q}_p)$, P is the Borel subgroup of upper triangular matrices, etc.).

A.1. We maintain the notation of the preceding subsection. Our goal now is to consider those aspects of the situation when $\chi_1 = \chi_2\overline{\varepsilon}^{-1}$ needed to complete the proof of part (3) of Proposition 4.3.15. To simplify the notation, we make a twist by $\chi_1^{-1} \circ \det$, and assume that in fact $\chi_1 = \underline{1}$ from now on (so that $\chi_2 = \overline{\varepsilon}$). We also assume that $p \neq 2$. We must prove that the map

$$\text{(A.1.1)} \qquad \mathrm{Ext}^2_M(\underline{1} \otimes \overline{\varepsilon}, \underline{1} \otimes \overline{\varepsilon}) \to \mathrm{Ext}^2_G(\mathrm{Ind}_P^G \underline{1} \otimes \overline{\varepsilon}, \mathrm{Ind}_P^G \underline{1} \otimes \overline{\varepsilon})$$

induced by applying Ind_P^G is *not* an injection. Here, the Ext^2 spaces are understood to be computed in the space of locally admissible representations.

In fact, we will prove that a specific line in the source lies in the kernel. To describe this line, we will have to describe some of the Ext spaces under discussion more explicitly. To begin with, note that we have isomorphisms

$$\text{(A.1.2)} \qquad \mathrm{Ext}^1_M(\underline{1} \otimes \overline{\varepsilon}, \underline{1} \otimes \overline{\varepsilon}) \xrightarrow{\sim} \mathrm{Ext}^1_M(\underline{1} \otimes \underline{1}, \underline{1} \otimes \underline{1}) \xrightarrow{\sim} H^1(M, \underline{1} \otimes \underline{1}),$$

the first being given by twisting, and the second by Proposition 2.2.2, together with the fact that Ext^1_M between admissible smooth representations is the same whether computed in the category $\mathrm{Mod}_M^{\mathrm{l.adm}}(k)$ or $\mathrm{Mod}_M^{\mathrm{sm}}(k)$.

Now write $M' = M/Z_G \xrightarrow{\sim} \mathbb{Q}_p^\times$. Since $p \neq 2$, the group admits the following description:

$$M' \cong \mathbb{Q}_p^\times \cong \Delta \times \Gamma \times A,$$

where Δ has order $p-1$, $\Gamma = 1 + \mathbb{Z}_p \cong \mathbb{Z}_p$, and $A = p^{\mathbb{Z}}$ is the cyclic group generated by p. Since Δ has prime-to-p order, it has no cohomology above degree 0; of course $H^0(\Delta, \underline{1})$ is one-dimensional. Simple computations in continuous cohomology show that $H^0(\Gamma, \underline{1})$, $H^1(\Gamma, \underline{1})$, $H^0(A, \underline{1})$, and $H^1(A, \underline{1})$ are all one-dimensional, and that the continuous cohomology of Γ and A in degrees > 1 vanishes. Each of the tensor products $H^0(\Delta, \underline{1}) \otimes_k H^0(\Gamma, \underline{1}) \otimes_k H^0(A, \underline{1})$ and $H^0(\Delta, \underline{1}) \otimes_k H^1(\Gamma, \underline{1}) \otimes_k H^0(A, \underline{1})$ is thus one-dimensional; let α and β respectively denote basis vectors. The Künneth formula then shows that $H^1(M', \underline{1})$ is two-dimensional, spanned by α and β, and that $H^2(M', \underline{1})$ is one-dimensional, spanned by the Yoneda product (which in this context coincides with the cup product) $\alpha \cup \beta$. Another application of the Künneth formula (applied to the evident isomorphism $M \xrightarrow{\sim} Z_G \times M'$) shows that the pull-back maps $H^i(M', \underline{1}) \to H^i(M, \underline{1} \otimes \underline{1})$ are injective, for all i. We may thus regard α and β as elements of $H^1(M, \underline{1} \otimes \underline{1})$, and hence they give rise to corresponding elements α' and β' of $\mathrm{Ext}^1_M(\underline{1} \otimes \underline{1}, \underline{1} \otimes \underline{1})$ and α'' and β'' of $\mathrm{Ext}^1_M(\underline{1} \otimes \overline{\varepsilon}, \underline{1} \otimes \overline{\varepsilon})$, via the isomorphisms (A.1.2).

A.1.3. Lemma. *The Yoneda product $\alpha'' \cup \beta'' \in \mathrm{Ext}^2_M(\underline{1} \otimes \overline{\varepsilon}, \underline{1} \otimes \overline{\varepsilon})$ is non-zero.*

Proof. By twisting, it is equivalent to show that Yoneda product $\alpha' \cup \beta' \in \mathrm{Ext}^2_M(\underline{1} \otimes \underline{1}, \underline{1} \otimes \underline{1})$ is non-zero. Passing from the category $\mathrm{Mod}_M^{\mathrm{l.adm}}(k)$ to the category $\mathrm{Mod}_M^{\mathrm{sm}}(k)$ induces a corresponding map on Ext^2 spaces, which we may then compose with the map to $H^2(M, \underline{1} \otimes \underline{1})$ given by (2.2.1). Under this composite, $\alpha' \cup \beta'$ maps to $\alpha \cup \beta$, which is non-zero by the Künneth formula for continuous cohomology. □

We will show that the Yoneda product $\alpha'' \cup \beta'' \in \mathrm{Ext}^2_M(\underline{1} \otimes \overline{\varepsilon}, \underline{1} \otimes \overline{\varepsilon})$, which by the preceding lemma is non-zero, has zero image under (A.1.1). The first step in the argument is to show that it suffices to check the corresponding fact with Ext^2 computed in the category of locally admissible representations replaced by Ext^2 computed in the category of smooth representations. To see this, we first note that applying Ind_P^G to the extension classes α' and β' gives rise to extension classes X and Y lying in $\mathrm{Ext}^1_G(\mathrm{Ind}_P^G \underline{1} \otimes \overline{\varepsilon}, \mathrm{Ind}_P^G \underline{1} \otimes \overline{\varepsilon})$, and the image of $\alpha' \cup \beta'$ under (A.1.1) coincides with $X \cup Y$. The following Lemma then shows that the cup product of X and Y vanishes when computed in $\mathrm{Ext}^2_G(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon}, \mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon})$ for the category $\mathrm{Mod}_G^{\mathrm{l.adm}}(k)$ if and only if it vanishes when computed in $\mathrm{Ext}^2_G(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon}, \mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon})$ for the category $\mathrm{Mod}_G^{\mathrm{sm}}(k)$.

A.1.4. Lemma. *Let U, V, and W be admissible smooth G-representations over k. If we are given extensions $0 \to U \to X \to V \to 0$ and $0 \to V \to Y \to W \to 0$, then the resulting 2-extension $0 \to U \to X \to Y \to W \to 0$ given by forming the Yoneda product of X with Y is trivial in the category $\mathrm{Mod}_G^{\mathrm{l.adm}}(k)$ if and only if it is trivial in the category $\mathrm{Mod}_G^{\mathrm{sm}}(k)$.*

Proof. Let $\mathcal{C}$ denote either of the categories $\mathrm{Mod}_G^{\mathrm{l.adm}}(k)$ or $\mathrm{Mod}_G^{\mathrm{sm}}(k)$. As is well-known, the Yoneda product of X with Y is trivial in the category $\mathcal{C}$ if and only there exists an extension $0 \to X \to Z \to W \to 0$, in the category $\mathcal{C}$, such that the extension $0 \to V \to Z' \to W \to 0$ obtained by pushing forward Z along the surjection $X \to V$ is isomorphic (as an extension of W by V) to Y. However, such an object Z, being a successive extension of admissible representations, is necessarily admissible [10, Prop. 2.2.13]. Thus trivializing the Yoneda product of X with Y in $\mathcal{C}$ is equivalent to trivializing it in the category $\mathrm{Mod}_G^{\mathrm{adm}}(k)$, for either choice of $\mathcal{C}$. ☐

From now on we compute all Ext^2 spaces in the appropriate category of smooth representations (as opposed to the category of locally admissible representations). As already noted, by the preceding lemma, we are reduced to showing that $\alpha'' \cup \beta'' \in \mathrm{Ext}^2_M(\underline{1} \otimes \overline{\varepsilon}, \underline{1} \otimes \overline{\varepsilon})$ has vanishing image in $\mathrm{Ext}^2_G(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon}, \mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon})$. Our strategy for proving this will be to exhibit an explicit element of $\mathrm{Ext}^2_M(\underline{1} \otimes \overline{\varepsilon}, \underline{1} \otimes \overline{\varepsilon})$ which *does* vanish when mapped to $\mathrm{Ext}^2_G(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon}, \mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon})$, and then to show that this element is equal to a non-zero scalar multiple of $\alpha'' \cup \beta''$.

We begin by writing down the diagram

(A.1.5)
$$\begin{array}{ccc}
\mathrm{Ext}^1_M(\overline{\varepsilon} \otimes \underline{1}, \overline{\varepsilon} \otimes \underline{1}) & \xrightarrow{\sim} & \mathrm{Ext}^1_P(\overline{\varepsilon} \otimes \underline{1}, \overline{\varepsilon} \otimes \underline{1}) \\
\Big\downarrow {\scriptstyle U \mapsto \mathrm{Ind}_{\overline{P}}^G U^w} & & \Big\downarrow \\
\mathrm{Ext}^1_G(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon}, \mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon}) & \xrightarrow{\sim} & \mathrm{Ext}^1_P(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon}, \overline{\varepsilon} \otimes \underline{1}) \\
 & & \Big\downarrow \\
 & & \mathrm{Ext}^1_P\big(\mathcal{C}_c^{\mathrm{sm}}(N, k) \otimes (\underline{1} \otimes \overline{\varepsilon}), \overline{\varepsilon} \otimes \underline{1}\big) \\
 & & \Big\downarrow {\scriptstyle \delta} \\
\mathrm{Ext}^2_M(\overline{\varepsilon} \otimes \underline{1}, \overline{\varepsilon} \otimes \underline{1}) & \xleftarrow{\sim} & \mathrm{Ext}^2_P(\overline{\varepsilon} \otimes \underline{1}, \overline{\varepsilon} \otimes \underline{1}) \\
\Big\downarrow {\scriptstyle U \mapsto \mathrm{Ind}_{\overline{P}}^G U^w} & & \Big\downarrow \\
\mathrm{Ext}^2_G(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon}, \mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon}) & \xrightarrow{\sim} & \mathrm{Ext}^2_P(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon}, \overline{\varepsilon} \otimes \underline{1}),
\end{array}$$

whose right-hand column is obtained by applying $\mathrm{Hom}_P(-, \chi_1 \overline{\varepsilon} \otimes \chi_2 \overline{\varepsilon}^{-1})$ to (4.2.11) (taking $U = \underline{1} \otimes \overline{\varepsilon}$), and whose horizontal arrows are provided by Lemmas 4.3.3 and 4.3.7. Both squares commute (possibly up to a sign). (Of course, all the Ext spaces in this diagram are computed in the category of smooth representations.)

Note that we have labelled by δ the boundary map

$$\delta : \mathrm{Ext}^1_P(\mathcal{C}^{\mathrm{sm}}_c(N,k) \otimes (\underline{1} \otimes \overline{\varepsilon}), \overline{\varepsilon} \otimes \underline{1}) \to \mathrm{Ext}^2_P(\overline{\varepsilon} \otimes \underline{1}, \overline{\varepsilon} \otimes \underline{1}).$$

A.1.6. Lemma. *The space* $\mathrm{Ext}^1_P(\mathcal{C}^{\mathrm{sm}}_c(N,k) \otimes (\underline{1} \otimes \overline{\varepsilon}), \overline{\varepsilon} \otimes \underline{1})$ *is non-zero.*

Proof. A simple descent argument shows that the formation of this space is compatible with extension of scalars, and thus we may enlarge k if necessary so that we can choose smooth characters $\chi_1, \chi_2 : \mathbb{Q}_p^\times \to k^\times$ such that $\chi_1 \neq \chi_2 \overline{\varepsilon}^{-1}$. Proposition 4.3.15 (2) and its proof then shows that there is an isomorphism

$$\mathrm{Ext}^1_G(\mathrm{Ind}_{\overline{P}}^G \chi_1 \otimes \chi_2, \mathrm{Ind}_{\overline{P}}^G \chi_2 \overline{\varepsilon}^{-1} \otimes \chi_1 \overline{\varepsilon}) \to \mathrm{Hom}_M(\chi_1 \otimes \chi_2, \chi_1 \otimes \chi_2),$$

which, according to the discussion of Remark 3.7.7, is given by mapping an extension

$$0 \to \mathrm{Ind}_{\overline{P}}^G \chi_2 \overline{\varepsilon}^{-1} \otimes \chi_1 \overline{\varepsilon} \to V \to \mathrm{Ind}_{\overline{P}}^G \chi_1 \otimes \chi_2 \to 0 \qquad \text{(A.1.7)}$$

to the corresponding boundary map

$$\delta_E : \mathrm{Ord}_P(\mathrm{Ind}_{\overline{P}}^G \chi_1 \otimes \chi_2) \to H^1\mathrm{Ord}_P(\mathrm{Ind}_{\overline{P}}^G \chi_2 \overline{\varepsilon}^{-1} \otimes \chi_1 \overline{\varepsilon}),$$

obtained by applying $H^\bullet \mathrm{Ord}_P$ to (A.1.7).

As is explained in the proof of Corollary 4.2.10, there is a P-equivariant embedding

$$\mathcal{C}^{\mathrm{sm}}_c(N,k) \otimes (\chi_1 \otimes \chi_2) \hookrightarrow \mathrm{Ind}_{\overline{P}}^G \chi_1 \otimes \chi_2 \qquad \text{(A.1.8)}$$

and a P-equivariant surjection

$$\mathrm{Ind}_{\overline{P}}^G \chi_2 \overline{\varepsilon}^{-1} \otimes \chi_1 \overline{\varepsilon} \to \chi_1 \overline{\varepsilon} \otimes \chi_2 \overline{\varepsilon}^{-1}. \qquad \text{(A.1.9)}$$

If we pull E back along the first of these maps, and then push the resulting extension forward along the second, we obtain an extension

$$0 \to \chi_1 \overline{\varepsilon} \otimes \chi_2 \overline{\varepsilon}^{-1} \to E' \to \mathcal{C}^{\mathrm{sm}}_c(N,k) \otimes (\chi_1 \otimes \chi_2) \to 0. \qquad \text{(A.1.10)}$$

We may apply $H^\bullet \mathrm{Ord}_P$ to this short exact sequence to obtain a complex of the form (3.3.4). In particular, we obtain a map

$$\mathrm{Ord}_P\big(\mathcal{C}^{\mathrm{sm}}_c(N,k) \otimes (\chi_1 \otimes \chi_2)\big) \to H^1\mathrm{Ord}_P(\chi_1 \overline{\varepsilon} \otimes \chi_2 \overline{\varepsilon}^{-1}). \qquad \text{(A.1.11)}$$

It is shown in the course of the proof of Corollary 4.2.10 that (A.1.8) induces an isomorphism upon applying Ord_P, and that (A.1.9) induces an isomorphism upon applying $H^1\mathrm{Ord}_P$. It follows from this that the map (A.1.11) coincides with the map δ_E (after identifying their respective domains and codomains via these isomorphisms).

Combining the results of the preceding two paragraphs, we find that if E is non-trivial, so that δ_E is non-trivial, then the map (A.1.11) is non-trivial, and hence (A.1.10) is non-split. Twisting this extension by $\chi_1^{-1} \otimes \chi_2^{-1}\overline{\varepsilon}$, we obtain a non-trivial extension of the form required to establish the lemma. □

If $c \in \mathrm{Ext}^1_P\big(\mathcal{C}^{\mathrm{sm}}_c(N,k) \otimes (\underline{1} \otimes \overline{\varepsilon}), \overline{\varepsilon} \otimes \underline{1}\big)$ is non-zero, then applying first the third horizontal isomorphism in (A.1.5), and then the automorphism w, to $\delta(c)$, we see from a consideration of (A.1.5) (in particular, using the fact that its right-hand column is exact) that $\delta(c)$ gives rise to an element $\gamma \in \mathrm{Ext}^2_M(\underline{1} \otimes \overline{\varepsilon}, \underline{1} \otimes \overline{\varepsilon})$ whose image under $\mathrm{Ind}_{\overline{P}}^G$ vanishes. We will be done once we prove the following result.

A.1.12. Proposition. *The element* γ *is a non-zero scalar multiple of* $\alpha'' \cup \beta''$ *in* $\mathrm{Ext}^2_M(\underline{1} \otimes \overline{\varepsilon}, \underline{1} \otimes \overline{\varepsilon})$.

Proof. If we let $d \in \operatorname{Ext}^1_P\big(\overline{\varepsilon} \otimes \underline{1}, \mathcal{C}_c^{\mathrm{sm}}(N,k) \otimes (\underline{1} \otimes \overline{\varepsilon})\big)$ be the extension class corresponding to the short exact sequence (4.2.11) (with U taken to be $\underline{1} \otimes \overline{\varepsilon}$), then $\delta(c) = \pm c \cup d$ (we don't concern ourselves with the precise value of the sign), and so to prove the lemma, we have to show that $c_{|M} \cup d_{|M} = (c \cup d)_{|M}$ is a non-zero scalar multiple of $\alpha'' \cup \beta''$. (Here the subscript "$|M$" denotes the restriction of a class from P to M.)

We identify N with $\mathbb{Q}_p$ in the usual fashion, via $\mathbb{Q}_p \ni x \mapsto \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in N$, and we identify $M' = M/Z_G$ both with $\mathbb{Q}_p^\times$ and with a subgroup of M via $\mathbb{Q}_p^\times \ni a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \in M' \subset M$. With these identifications we obtain isomorphisms $\mathcal{C}_c^{\mathrm{sm}}(N,k) \xrightarrow{\sim} \mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p,k)$, and $M' = \mathbb{Q}_p^\times$ acts via the formula

$$(\langle a \rangle f)(x) = f(a^{-1}x),$$

if $a \in \mathbb{Q}_p^\times$ and $x \in \mathbb{Q}_p$, where we have written $\langle a \rangle$ to denote a thought of as an element of M' (to avoid any possible confusion with the element a thought of as a scalar).

As already observed above, $M' \times Z_G \xrightarrow{\sim} M$. Since c and d are both trivial when restricted to Z_G, so is their Yoneda product. Thus it suffices to verify that $(c \cup d)_{|M'}$ is a non-zero scalar multiple of $(\alpha'' \cup \beta'')_{|M'}$, and this is what we will do.

Evaluation at $0 \in \mathbb{Q}_p$ gives an M'-equivariant short exact sequence

$$0 \to \mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p^\times, k) \to \mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p, k) \to \underline{1} \to 0. \tag{A.1.13}$$

Now $(\overline{\varepsilon} \otimes \underline{1})_{|M'} = \overline{\varepsilon}$, while $(\underline{1} \otimes \overline{\varepsilon})_{|M'} = \underline{1}$. Thus $c_{|M'} \in \operatorname{Ext}^1_{M'}\big(\overline{\varepsilon}, \mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p,k)\big)$. Since $p \neq 2$, the character $\overline{\varepsilon}$ is non-trivial, and thus the long exact sequences of Exts arising from the short exact sequence (A.1.13) gives rise to an isomorphism

$$\operatorname{Ext}^1_{M'}\big(\mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p, \underline{1}), \overline{\varepsilon}\big) \xrightarrow{\sim} \operatorname{Ext}^1_{M'}\big(\mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p^\times, \underline{1}), \overline{\varepsilon}\big).$$

We let $c' \in \operatorname{Ext}^1_{M'}\big(\mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p^\times, \underline{1}), \overline{\varepsilon}\big)$ denote the element which corresponds to $c_{|M'}$ via this isomorphism. Similarly, there is an isomorphism

$$\operatorname{Ext}^1_{M'}\big(\overline{\varepsilon}, \mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p^\times, k)\big) \xrightarrow{\sim} \operatorname{Ext}^1_{M'}\big(\overline{\varepsilon}, \mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p, k)\big),$$

and we let $d' \in \operatorname{Ext}^1_{M'}\big(\overline{\varepsilon}, \mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p^\times, k)\big)$ denote the element which corresponds to $d_{|M'}$ via this isomorphism. There are equalities $c' \cup d' = c_{|M'} \cup d_{|M'} = (c \cup d)_{|M'}$.

As above, write $A = p^{\mathbb{Z}} \subset \mathbb{Q}_p^\times$, so that $\mathbb{Q}_p^\times = \mathbb{Z}_p^\times \times A$. There is an isomorphism of M'-representations $\mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p^\times, k) \xrightarrow{\sim} c\text{–}\operatorname{Ind}_A^{M'} \mathcal{C}^{\mathrm{sm}}(\mathbb{Z}_p^\times, k)$, and hence an isomorphism

$$\operatorname{Ext}^1_{M'}\big(\mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p^\times, k), \overline{\varepsilon}\big) \xrightarrow{\sim} \operatorname{Ext}^1_{\mathbb{Z}_p^\times}\big(\mathcal{C}^{\mathrm{sm}}(\mathbb{Z}_p^\times, k), \overline{\varepsilon}\big).$$

We let $c'' \in \operatorname{Ext}^1_{\mathbb{Z}_p^\times}\big(\mathcal{C}^{\mathrm{sm}}(\mathbb{Z}_p^\times, k), \overline{\varepsilon}\big)$ denote the element which corresponds to c' via this isomorphism. If c'' is given by the extension

$$0 \to \overline{\varepsilon} \to E \to \mathcal{C}^{\mathrm{sm}}(\mathbb{Z}_p^\times, k) \to 0,$$

then we in fact regard c'' as an extension of M'-representations, by having A act trivially on all the terms of this extension. The extension class c' is then obtained by first tensoring E with $\mathcal{C}_c^{\mathrm{sm}}(A,k)$ (regarded as an M'-representation by having $\mathbb{Z}_p^\times$ act trivially) to obtain an extension

$$0 \to \mathcal{C}_c^{\mathrm{sm}}(A,k) \otimes_k \overline{\varepsilon} \to \mathcal{C}_c^{\mathrm{sm}}(A,k) \otimes_k E \to \mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p^\times, k) \to 0,$$

and then pushing forward via the map $\mathcal{C}_c^{\mathrm{sm}}(A) \otimes_k \overline{\varepsilon} \to \overline{\varepsilon}$, obtained by tensoring the M'-equivariant map $I : \mathcal{C}_c^{\mathrm{sm}}(A, k) \to \underline{1}$ (given by "integrating with respect to Haar measure" on the discrete group A) with the character $\overline{\varepsilon}$.

Now tensoring I with the identity automorphism of $\mathcal{C}^{\mathrm{sm}}(\mathbb{Z}_p^\times, k)$ (the latter being regarded as an M'-representation by having A act trivially) similarly induces an M'-equivariant map

$$J : \mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p^\times, k) \to \mathcal{C}^{\mathrm{sm}}(\mathbb{Z}_p^\times, k),$$

and we write $d'' := J_* d$ (the pushforward of d with respect to J), an element of $\mathrm{Ext}^1_{M'}\big(\mathcal{C}^{\mathrm{sm}}(\mathbb{Z}_p^\times, k), \overline{\varepsilon}\big)$. It follows from the description of the relationship between c' and c'' in the preceding paragraph that $c' \cup d' = c'' \cup d''$.

The inclusion of the scalar multiples of the character $\overline{\varepsilon}_{|\mathbb{Z}_p^\times}$ in the space of all locally constant functions gives an embedding $\imath : \overline{\varepsilon} \hookrightarrow \mathcal{C}^{\mathrm{sm}}(\Gamma, k)$, and Lemma A.1.14 below shows that $d'' = \imath_* d'''$ for some class $d''' \in \mathrm{Ext}^1_{M'}(\overline{\varepsilon}, \overline{\varepsilon})$ which is non-trivial when restricted to A. We then compute that

$$c'' \cup d'' = c'' \cup \imath_* d''' = (\imath^* c'') \cup d'''.$$

Lemma A.1.20 below shows that $\imath^* c'' \in \mathrm{Ext}^1_{M'}(\overline{\varepsilon}, \overline{\varepsilon})$ is non-trivial when restricted to $\mathbb{Z}_p^\times$ (or equivalently, when restricted to Γ). It follows that indeed $(\imath^* c'') \cup d'''$ is a non-zero multiple of $\alpha''_{|M'} \cup \beta''_{|M'}$, and the proposition is proved. □

A.1.14. Lemma. *If $\imath : \overline{\varepsilon} \hookrightarrow \mathcal{C}^{\mathrm{sm}}(\mathbb{Z}_p^\times, k)$ denotes the inclusion of the scalar multiples of the character $\overline{\varepsilon}$ into the space of locally constant functions on $\mathbb{Z}_p^\times$, then there exists a class $d''' \in \mathrm{Ext}^1_{M''}(\overline{\varepsilon}, \overline{\varepsilon})$, whose restriction to A is non-trivial, and such that $d'' = \imath_* d'''$.*

Proof. To see this, we begin by recalling the construction of d'', beginning with the class d, which corresponds to the extension

$$0 \to \mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p, k) \to \mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon} \to \overline{\varepsilon} \otimes \underline{1} \to 0. \tag{A.1.15}$$

We extend the identification of N with $\mathbb{Q}_p$ to an identification of $\overline{P}\backslash G$ with $\mathbb{P}^1(\mathbb{Q}_p) = \mathbb{Q}_p \cup \{\infty\}$. Any element f of $\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon}$ has a support in G, which is invariant under left translation by $\overline{P}$, and so projects to a compact open subset of $\overline{P}\backslash G$, which as in [10, §4] we again refer to as the support of f. If we write

$$\Omega = \mathbb{P}^1(\mathbb{Q}_p) \setminus \{0\} \subset \mathbb{P}^1(\mathbb{Q}_p) \xrightarrow{\sim} \overline{P} \setminus G,$$

and let $(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon})(\Omega)$ denote the subspace of $\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon}$ consisting of elements whose support is contained in Ω, then the class d' corresponds to (the restriction to M' of) the extension of M-representations

$$0 \to \mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p^\times, k) \to (\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon})(\Omega) \to \overline{\varepsilon} \otimes \underline{1} \to 0 \tag{A.1.16}$$

induced by (A.1.15). The class $d'' := I_* d'$ then corresponds to the extension

$$0 \to \mathcal{C}^{\mathrm{sm}}(\mathbb{Z}_p^\times, k) \to F \to \overline{\varepsilon} \otimes \underline{1} \to 0 \tag{A.1.17}$$

obtained by pushing (A.1.16) forward via the map $J : \mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p^\times, k) \to \mathcal{C}^{\mathrm{sm}}(\mathbb{Z}_p^\times, k)$ given by "integrating along the fibres" (the fibres of the projection $\mathbb{Q}_p^\times \xrightarrow{\sim} A \times \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ being cosets of A, and hence discrete).

Let

$$U := (\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon})(\Omega)^{\mathbb{Z}_p^\times = \overline{\varepsilon}} \subset (\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon})(\Omega)$$

denote the subspace consisting of elements which transform under $\mathbb{Z}_p^\times$ via the character $\overline{\varepsilon}$. This is clearly an M-invariant subspace of $(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon})(\Omega)$. Certainly, the image of $U \bigcap \mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p^\times, k)$ under J is one-dimensional. Thus there are two possibilities for the image V of U under the projection

$$(\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon})(\Omega) \to F, \tag{A.1.18}$$

namely: either V is one-dimensional (or equivalently, U is contained in $\mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p^\times, k)$), or else V is two-dimensional, and thus gives a subextension

$$0 \to \overline{\varepsilon} \otimes \underline{1} \to V \to \overline{\varepsilon} \otimes \underline{1} \to 0 \tag{A.1.19}$$

of (A.1.17).

Let f denote the characteristic function of $\mathbb{Z}_p$, thought of as an element of $\mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p, k) \otimes (\underline{1} \otimes \overline{\varepsilon})$. Then f is an element of $\mathrm{Ind}_{\overline{P}}^G \underline{1} \otimes \overline{\varepsilon}$, and one easily verifies that $wf \in U$ (where $w = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$), and that wf does not lie in $\mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p^\times, k)$, and hence has a non-trivial projection onto $\overline{\varepsilon} \otimes \underline{1}$. Thus in fact V is two-dimensional, and gives rise to the subextension (A.1.19) of (A.1.17). If we let d''' denote the class of this subextension, then indeed $d'' = \imath_* d'''$.

It remains to be shown that d''' is non-trivial when restricted to A. To see this, one explicitly computes $J\big(\langle p \rangle (wf) - wf\big)$, and checks that the result is non-zero. We leave this straightforward computation to the sufficiently interested reader. □

A.1.20. Lemma. *If $\imath : \overline{\varepsilon} \hookrightarrow \mathcal{C}^{\mathrm{sm}}(\mathbb{Z}_p^\times, k)$ is as in the statement of Lemma* A.1.14*, and if $e \in \mathrm{Ext}^1_{\mathbb{Z}_p^\times}(\mathcal{C}_c^{\mathrm{sm}}(\mathbb{Z}_p^\times, k), \overline{\varepsilon})$ is a non-trivial class, then the same is true of $\imath^* e$.*

Proof. Recall that $\mathbb{Z}_p^\times = \Delta \times \Gamma$, where Δ has order prime to p. We may thus verify the assertion of the lemma after passing to $\overline{\varepsilon}$-eigenspaces for the action of Δ in all the representations and extensions considered, and so reduce to checking the analogous statement for $\jmath^* e$, where $e \in \mathrm{Ext}^1_\Gamma(\mathcal{C}_c^{\mathrm{sm}}(\Gamma, k), \underline{1})$ is non-trivial, and $\jmath : \underline{1} \hookrightarrow \mathcal{C}^{\mathrm{sm}}(\Gamma, k)$. We do this by passing to k-duals, under which admissible smooth Γ-representations become finitely generated modules for the completed group ring $k[[\Gamma]]$.

We must then verify that if $e' \in \mathrm{Ext}^1_{k[[\Gamma]]}(\underline{1}, k[[\Gamma]])$ is non-trivial, and if $\pi : k[[\Gamma]] \to \underline{1}$ is the canonical surjection given by taking the quotient by the augmentation ideal, then the pushforward $\pi_* e'$ is again non-trivial. This is easily verified by the reader, using the well-known isomorphism $k[[\Gamma]] \xrightarrow{\sim} k[[t]]$ (whose target is a power series ring in one variable). □

A.2. Our goal in this subsection is to prove that the map (4.3.25) is injective. We begin by recalling that $P = MN$ acts on $\mathcal{C}_c^{\mathrm{sm}}(N, k)$ via the formula

$$(mnf)(n') = f(m^{-1} n' m n),$$

for $m \in M$, $n, n' \in N$, and $f \in \mathcal{C}_c^{\mathrm{sm}}(N, k)$. The same formula defines an action of P on the space $\mathcal{C}^{\mathrm{sm}}(N, k)$ of all smooth (i.e. locally constant) k-valued functions on N. The P-action on this space is not smooth, and we denote by $\mathcal{C}_u^{\mathrm{sm}}(N, k)$ the subspace of P-smooth vectors in $\mathcal{C}^{\mathrm{sm}}(N, k)$. (The subscript u is for *uniform*; a function is P-smooth if and only if it is N-smooth, and so the elements of $\mathcal{C}_u^{\mathrm{sm}}(N, k)$ are precisely the uniformly locally constant functions on N.) Evaluation at the identity of N gives a T-equivariant map

$$\mathcal{C}_u^{\mathrm{sm}}(N, k) \to \underline{1},$$

which by Frobenius reciprocity induces an isomorphism

$$\mathcal{C}_u^{\mathrm{sm}}(N,k) \xrightarrow{\sim} \mathrm{Ind}_M^P \underline{1}. \tag{A.2.1}$$

By construction the embedding $\mathcal{C}_c^{\mathrm{sm}}(N,k) \subset \mathcal{C}_u^{\mathrm{sm}}(N,k)$ is P-equivariant. Thus if $\psi : \mathbb{Q}_p^\times \to k$ is a smooth character, then we obtain a long exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_P\big(\psi \otimes \psi^{-1}, \mathcal{C}_c^{\mathrm{sm}}(N,k)\big) \to \mathrm{Hom}_P\big(\psi \otimes \psi^{-1}, \mathcal{C}_u^{\mathrm{sm}}(N,k)\big) \\ \to \mathrm{Hom}_P\big(\psi \otimes \psi^{-1}, \mathcal{C}_u^{\mathrm{sm}}(N,k)/\,\mathcal{C}_c^{\mathrm{sm}}(N,k)\big) \to \mathrm{Ext}_P^1\big(\psi \otimes \psi^{-1}, \mathcal{C}_c^{\mathrm{sm}}(N,k)\big) \\ \to \mathrm{Ext}_P^1\big(\psi \otimes \psi^{-1}, \mathcal{C}_u^{\mathrm{sm}}(N,k)\big). \end{aligned}$$

(Here and below, all Ext^1 spaces are computed in the category of smooth representations.) Clearly $\mathrm{Hom}_P\big(\psi \otimes \psi^{-1}, \mathcal{C}_c^{\mathrm{sm}}(N,k)\big)$ must vanish, while Frobenius reciprocity together with the isomorphism (A.2.1) yields isomorphisms

$$\mathrm{Hom}_P\big(\psi \otimes \psi^{-1}, \mathcal{C}_u^{\mathrm{sm}}(N,k)\big) \xrightarrow{\sim} \mathrm{Hom}_M\big(\psi \otimes \psi^{-1}, \underline{1}\big)$$

and

$$\mathrm{Ext}_P^1\big(\psi \otimes \psi^{-1}, \mathcal{C}_u^{\mathrm{sm}}(N,k)\big) \xrightarrow{\sim} \mathrm{Ext}_M^1\big(\psi \otimes \psi^{-1}, \underline{1}\big).$$

(The latter isomorphism is proved in a manner identical to Lemma 4.3.3.) Finally, Lemma 4.3.4 yields an isomorphism

$$\mathrm{Hom}_P\big(\psi\otimes\psi^{-1}, \mathcal{C}_u^{\mathrm{sm}}(N,k)/\,\mathcal{C}_c^{\mathrm{sm}}(N,k)\big) \xrightarrow{\sim} \mathrm{Hom}_M\big(\psi\otimes\psi^{-1}, \mathcal{C}_u^{\mathrm{sm}}(N,k)/\,\mathcal{C}_c^{\mathrm{sm}}(N,k)\big).$$

Thus we may rewrite the above long-exact sequence as

$$\begin{aligned} 0 \to \mathrm{Hom}_M\big(\psi \otimes \psi^{-1}, \underline{1}\big) \to \mathrm{Hom}_M\big(\psi \otimes \psi^{-1}, \mathcal{C}_u^{\mathrm{sm}}(N,k)/\,\mathcal{C}_c^{\mathrm{sm}}(N,k)\big) \\ \to \mathrm{Ext}_P^1\big(\psi \otimes \psi^{-1}, \mathcal{C}_c^{\mathrm{sm}}(N,k)\big) \to \mathrm{Ext}_M^1\big(\psi \otimes \psi^{-1}, \underline{1}\big). \end{aligned} \tag{A.2.2}$$

A.2.3. Proposition. *$\mathrm{Hom}_M\big(\psi \otimes \psi^{-1}, \mathcal{C}_u^{\mathrm{sm}}(N,k)/\,\mathcal{C}_c^{\mathrm{sm}}(N,k)\big)$ is one-dimensional.*

Proof. We identify N with $\mathbb{Q}_p$, and so identify $\mathcal{C}_u^{\mathrm{sm}}(N,k)$ with $\mathcal{C}_u^{\mathrm{sm}}(\mathbb{Q}_p,k)$. We similarly identify $\mathbb{Q}_p^\times$ with a subgroup of M, and observe that since both $\psi \otimes \psi^{-1}$ and $\mathcal{C}_u^{\mathrm{sm}}(\mathbb{Q}_p,k)$ are acted on trivially by Z, restricting from M-representations to $\mathbb{Q}_p^\times$-representations induces an isomorphism

$$\mathrm{Hom}_M\big(\psi \otimes \psi^{-1}, \mathcal{C}_u^{\mathrm{sm}}(N,k)/\,\mathcal{C}_c^{\mathrm{sm}}(N,k)\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Q}_p^\times}\big(\psi, \mathcal{C}_u^{\mathrm{sm}}(N,k)/\,\mathcal{C}_c^{\mathrm{sm}}(N,k)\big).$$

If $a \in \mathbb{Q}_p^\times$ and $f \in \mathcal{C}_u^{\mathrm{sm}}(\mathbb{Q}_p,k)$ then we will write $\langle a\rangle f$ to denote the action of a on f. (This should prevent any possible confusion with the element a thought of as a scalar.) Explicitly, the action of $\mathbb{Q}_p^\times$ on $\mathcal{C}_u^{\mathrm{sm}}(\mathbb{Q}_p,k)$ is given by the formula

$$(\langle a\rangle f)(x) := f(a^{-1}x),$$

for $a \in \mathbb{Q}_p^\times$, $x \in \mathbb{Q}_p$, and $f \in \mathcal{C}_u^{\mathrm{sm}}(\mathbb{Q}_p,k)$.

If $f \in \mathcal{C}_u^{\mathrm{sm}}(\mathbb{Q}_p,k)$, then corresponding to the decomposition

$$\mathbb{Q}_p = \mathbb{Z}_p \sqcup \coprod_{n=1}^{\infty} p^{-n}\mathbb{Z}_p^\times,$$

we may write $f = f_0 + \sum_{i=1}^{\infty} \langle p^{-i}\rangle f_i$, where f_i is a smooth function supported on $\mathbb{Z}_p^\times$, so that $\langle p^{-i}\rangle f_i$ is supported on $p^{-i}\mathbb{Z}_p^\times$. Since f is uniformly smooth, there is an $r > 0$ such that every f_i is constant on the cosets of $1 + p^r\mathbb{Z}_p$ in $\mathbb{Z}_p^\times$.

Suppose that $f \bmod \mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p, k)$ transforms via the character ψ with respect to the action of $\mathbb{Q}_p^\times$. Let $\{a_j\}$ denote a set of coset representatives for $1+p^r\mathbb{Z}_p$ in $\mathbb{Z}_p^\times$. By assumption $\langle a_j\rangle f - \psi(a_j)f$ is compactly supported. We compute that

$$\langle a_j\rangle f - \psi(a_j)f = \langle a_j\rangle f_0 - \psi(a_j)f_0 + \sum_{i=1}^{\infty}\langle p^{-i}\rangle\big(\langle a_j\rangle f_i - \psi(a_j)f_i\big).$$

Thus for each j, if i is sufficiently large, we find that $\langle a_j\rangle f_i = \psi(a_j)f_i$. Since the number of a_j is finite, we may choose i_0 so that if $i > i_0$ then $\langle a_j\rangle f_i = \psi(a_j)f_i$ for all j. Since f_i is constant on $1+p^r\mathbb{Z}_p$ cosets, we see that then in fact $\langle a\rangle f_i = \psi(a)f_i$ for all $a \in \mathbb{Z}_p^\times$, and thus that $f_i = c_i\psi^{-1}_{|\mathbb{Z}_p^\times}$, for some scalar $c_i \in k$, if $i > i_0$. Thus we may write $f = \sum_{i=0}^{i_0}\langle p^{-i}\rangle f_i + \sum_{i>i_0} c_i\langle p^{-i}\rangle\psi^{-1}_{|\mathbb{Z}_p^\times}$.

By assumption we also have that $\langle p\rangle f - \psi(p)f$ is compactly supported. Since

$$\begin{aligned}\langle p\rangle f &- \psi(p)f \\ &= \langle p\rangle f_0 + \sum_{i=0}^{i_0-1}(\langle p^{-i}\rangle f_{i+1} - \psi(p)\langle p^{-i}\rangle f_i) - \psi(p)\langle p^{-i_0}\rangle f_{i_0} + c_{i_0+1}\langle p^{-i_0}\rangle\psi_{|\mathbb{Z}_p^\times} \\ &\qquad + \sum_{i>i_0}(c_{i+1} - \psi(p)c_i)\langle p^{-i}\rangle\psi_{|\mathbb{Z}_p^\times},\end{aligned}$$

we find that $c_{i+1} - \psi(p)c_i = 0$ for sufficiently large i, and hence, enlarging i_0 if necessary, we may write

$$f = \sum_{i=0}^{i_0}\langle p^{-i}\rangle f_i + c\sum_{i>i_0}\psi(p)^i\langle p^{-i}\rangle\psi^{-1}_{|\mathbb{Z}_p^\times} = \sum_{i=0}^{i_0}\langle p^{-i}\rangle f_i + c\sum_{i>i_0}\psi^{-1}_{|p^{-i}\mathbb{Z}_p^\times},$$

for some $c \in k$. Thus we see that, away from a compact subset $\mathbb{Q}_p$, the function f coincides with a scalar multiple of ψ^{-1}. On the other hand, any function f with this property certainly lies in $\mathcal{C}_u^{\mathrm{sm}}(\mathbb{Q}_p, k)$, and $f \mod \mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p, k)$ transforms via ψ with respect to the action of $\mathbb{Q}_p^\times$. The space of $\mathcal{C}_c^{\mathrm{sm}}(\mathbb{Q}_p, k)$-cosets of such f is clearly one-dimensional, and so the proposition follows. ☐

A.2.4. Corollary. *The map* (4.3.25) *is injective.*

Proof. Take ψ to be trivial. The map (4.3.25) is then given by the final arrow in the exact sequence (A.2.2). Clearly $\mathrm{Hom}_M(\underline{1}\otimes\underline{1}, \underline{1}\otimes\underline{1})$ is one-dimensional. It thus follows from Proposition A.2.3 that the second arrow of (A.2.2) is an isomorphism. Thus the final arrow is an embedding, as claimed. ☐

A.2.5. Remark. If ψ is non-trivial, then it follows from a Proposition A.2.3 and a consideration of (A.2.2) that $\mathrm{Ext}^1_P\big(\psi\otimes\psi^{-1}, \mathcal{C}_c^{\mathrm{sm}}(N,k)\big)$ is one-dimensional. A non-trivial such extension may obtained from the restriction to P of $\mathrm{Ind}_P^G\,\underline{1}\otimes\psi$ by applying a twist by $\underline{1}\otimes\psi^{-1}$.

References

[1] Barthel L., Livné R., *Irreducible modular representations of* GL_2 *of a local field*, Duke Math. J. **75** (1994), 261–292.

[2] Berger L., *Représentations modulaires de* $\mathrm{GL}_2(\mathbb{Q}_p)$ *et représentations galoisiennes de dimension 2*, preprint (2005), available at `http://www.ihes.fr/~lberger/prepublications.html`

[3] Breuil C., *Sur quelques représentations modulaires et p-adiques de* $\mathrm{GL}_2(\mathbb{Q}_p)$ *I*, Compositio Math. **138** (2003), 165–188.

[4] Breuil C., Paškūnas V., *Towards a modulo p Langlands correspondence for* GL_2, to appear in the Memoirs of the Amer. Math. Soc.
[5] Casselman W., *Introduction to the theory of admissible representations of p-adic reductive groups*, unpublished notes distributed by P. Sally, draft May 7 1993, available electronically at `http://www.math.ubc.ca/people/faculty/cass/research.html`
[6] Colmez P., *Représentations de* $\mathrm{GL}_2(\mathbb{Q}_p)$ *et* (φ, Γ)*-modules*, preprint (2009), available at `http://math.jussieu.fr/~colmez/publications.html`
[7] van Dijk, G., *Quasi-admissible representations of p-adic groups*, in Lie groups and their representations (Budapest, 1971) (I.M. Gelfand ed.), Halsted, New York, 1975, 65–75.
[8] Emerton M., *Jacquet modules of locally analytic representations of p-adic reductive groups I. Construction and first properties*, Ann. Sci. Éc. Norm. Sup. **39** (2006), 775–839.
[9] Emerton M., *Local-global compatibility in the p-adic Langlands programme for* $\mathrm{GL}_{2/\mathbb{Q}}$, in preparation.
[10] Emerton M., *Ordinary parts of admissible representations of p-adic reductive groups I. Definition and first properties*, preprint (2007), available at `http://www.math.northwestern.edu/~emerton/preprints.html`
[11] Emerton M., Paškūnas V., *On the effaceability of certain* δ*-functors*, appendix to the present paper.
[12] Hochschild, G., Mostow, G. D., *Cohomology of Lie groups*, Illinois J. Math. **6** (1962), 367–401.
[13] Lazard M., *Groupes analytiques p-adiques*, Publ. Math. IHES **26** (1965).
[14] Paškūnas V., *On the restriction of representations of* $\mathrm{GL}_2(F)$ *to a Borel subgroup*, Compositio Math. **143** (2007), 1533–1544.
[15] Paškūnas V., *Extensions for supersingular representations of* $\mathrm{GL}_2(\mathbb{Q}_p)$, preprint (2010), available at `http://arxiv.org/abs/0710.1053`
[16] Paškūnas V., *Admissible unitary completions of locally* $\mathbb{Q}_p$*-rational representations of* $\mathrm{GL}_2(F)$, preprint (2008), available at `http://arxiv.org/abs/0805.1006`.
[17] Wigner D., *Algebraic cohomology of topological groups*, Trans. Amer. Math. Soc. **178** (1973), 83–93.

Mathematics Department, Northwestern University, 2033 Sheridan Rd., Evanston, IL 60208
E-mail address: `emerton@math.northwestern.edu`