Completed cohomology and the p-adic Langlands program

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Abstract. We discuss some known and conjectural properties of the completed cohomology of congruence quotients associated to reductive groups over $\mathbb{Q}$. We also discuss the conjectural relationships to local and global Galois representations, and a possible p-adic local Langlands correspondence.

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1. Introduction

The idea that there could be a p-adic version of the local Langlands correspondence was originally proposed by C. Breuil, in the case of the group $\mathrm{GL}_2(\mathbb{Q}_p)$, in his papers [11, 12, 13]. In [13], he also proposed a local-global compatibility between this (at the time conjectural) correspondence and p-adically completed cohomology of modular curves. Since then, the theory of p-adic Langlands has been extensively developed in the case of the group GL_2 over $\mathbb{Q}$; see e.g. [6, 29, 50, 30, 32, 46, 52]. Since excellent expositions of much of this material already exist [5, 14, 15], we have decided here to describe some aspects of the p-adic Langlands program that make sense for arbitrary groups. This has led us to focus on completed cohomology, since this is a construction that makes sense for arbitrary groups, and about which it is possible to establish some general results, and make some general conjectures.

Many of these conjectures are very much motivated by the conjectural relationship with Galois representations, and we have also tried to outline our expectations regarding this relationship, while trying not get bogged down in the myriad technical details that would necessarily attend a more careful discussion of this topic. Finally, we have tried to indicate how completed cohomology may be related, by the principle of local-global compatibility, to a still largely conjectural p-adic local Langlands correspondence for groups other than $\mathrm{GL}_2(\mathbb{Q}_p)$. Our focus is on drawing inferences about the possible structure of the local correspondence by interpolating from our expectations regarding completed cohomology. In this regard we mention also the paper [27], which somewhat literally interpolates p-adically

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completed cohomology (via the Taylor–Wiles–Kisin method) in order to construct a candidate for the p-adic local Langlands correspondence for GL_n of an arbitrary p-adic field.

We close this introduction by simply listing a number of additional important papers [17, 18, 19, 20] which also have the goal of extending the p-adic local Langlands correspondence, and local-global compatibility, beyond the case of $\mathrm{GL}_2(\mathbb{Q}_p)$. Unfortunately, lack of space prevents us from saying more about them here.

1.1. Notation. Throughout, we fix a prime p. We also fix a finite extension L of $\mathbb{Q}_p$, contained inside some given algebraic closure $\overline{\mathbb{Q}}_p$. We let $\mathcal{O}$ denote the ring of integers of L, and let ϖ denote a uniformizer of $\mathcal{O}$. The ring $\mathcal{O}$, and field L, will serve as our coefficients.

As usual, $\mathbb{A}$ denotes the ring of adèles over $\mathbb{Q}$, $\mathbb{A}_f$ denotes the ring of finite adèles, and $\mathbb{A}_f^p$ denotes the ring of prime-to-p finite adéles. For a given algebraic group $\mathbb{G}$ over $\mathbb{Q}$, we will consistently write $G_\infty := \mathbb{G}(\mathbb{R})$, and $G := \mathbb{G}(\mathbb{Q}_p)$. Additional notation will be introduced as necessary.

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2. Completed cohomology

Ideally, in the p-adic Langlands program, we would like to define spaces of p-adic automorphic forms, to serve as p-adic analogues of spaces of classical automorphic forms in the usual Langlands program. Unfortunately, for general reductive groups, no such definition is currently available. For groups that are compact at infinity, we can make such a definition (see (2.2.4) below), while for groups giving rise to Shimura varieties, we can use algebro-geometric methods to define spaces of p-adic automorphic forms, which however seem less representation-theoretic in nature than classical automorphic forms. Since one of our main goals is to employ representation-theoretic methods, we are thus led to find an alternative approach.

The approach we take here is to work with p-adically completed cohomology. This has the advantages of being definable for arbitrary groups, and of being of a representation-theoretic nature. For our purposes, it will thus serve as a suitable surrogate for a space of p-adic automorphic forms. For groups that are compact at infinity, it does recover the usual notion of p-adic automorphic forms that was already mentioned. (Its relationship to algebro-geometric notions of p-adic automorphic forms in the Shimura variety context is less clear; P. Scholze's paper [53] makes fundamental progress in this — and many other! — directions, but we won't attempt to discuss this here.) We therefore begin our discussion of global p-adic Langlands by recalling the basic definitions and facts related to completed cohomology, referring to [24] and [31] for more details.

2.1. Definitions and basic properties.

Let us suppose that $\mathbb{G}$ is a reductive linear algebraic group over $\mathbb{Q}$. We write $G_\infty := \mathbb{G}(\mathbb{R})$ for the group of real-valued points of $\mathbb{R}$; this is a reductive Lie group. We let A_∞ denote the $\mathbb{R}$-points of the maximal $\mathbb{Q}$-split torus in the centre of $\mathbb{G}$, and let K_∞ denote a choice of maximal compact subgroup of G_∞. For any Lie group H, we let H° denote the subgroup consisting of the connected component of the identity.

The quotient $G_\infty/A_\infty^\circ K_\infty^\circ$ is a symmetric space on which G_∞ acts. We denote its dimension by d. We also write $l_0 :=$ rank of $G_\infty -$ rank of $A_\infty K_\infty$, and $q_0 := (d - l_0)/2$. If G_∞ is semisimple (so that in particular A_∞ is trivial) then these quantities coincide with the quantities denoted by the same symbols in [10] (which is why we notate them as we do). We note that q_0 is in fact an integer. (For the role played by these two quantities in our discussion, see (2.1.6), as well as Conjectures 3.1 and 3.2, below.)

If $K_f \subset \mathbb{G}(\mathbb{A}_f)$ is a compact open subgroup, then we write

$$Y(K_f) := \mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A})/A_\infty^\circ K_\infty^\circ K_f.$$

The double quotient $Y(K_f)$ is a finite union of quotients of the symmetric space $G_\infty^\circ/A_\infty^\circ K_\infty^\circ$ by cofinite volume discrete subgroups of G_∞.

2.1.1. Definitions.

We fix a tame level, i.e. a compact open subgroup $K_f^p \subset \mathbb{G}(\mathbb{A}_f^p)$. If $K_p \subset G := \mathbb{G}(\mathbb{Q}_p)$ is a compact open subgroup, then of course $K_pK_f^p$ is a compact open subgroup of $\mathbb{G}(\mathbb{A}_f)$, and so we may form the space $Y(K_pK_f^p)$. We define the completed (co)homology at the tame level K_f^p as follows [24]:

$$\widetilde{H}^i := \varprojlim_s \varinjlim_{K_p} H^i\big(Y(K_pK_f^p), \mathcal{O}/\varpi^s\big) \quad \text{and} \quad \widetilde{H}_i := \varprojlim_{K_p} H_i\big(Y(K_pK_f^p), \mathcal{O}\big), \qquad (1)$$

where in both limits K_p ranges over all compact open subgroups of $\mathbb{G}(\mathbb{Q}_p)$.

We equip each of $\widetilde{H}^i$ and $\widetilde{H}_i$ with its evident inverse limit topology. In the case of $\widetilde{H}^i$, each of the $\mathcal{O}/\varpi^s$-modules $\varinjlim_{K_p} H^i\big(Y(K_pK_f^p), \mathcal{O}/\varpi^s\big)$ appearing in the projective limit is equipped with its discrete topology, and the projective limit topology on $\widetilde{H}^i$ coincides with its ϖ-adic topology; $\widetilde{H}^i$ is complete with respect to this topology. In the case of $\widetilde{H}_i$, each of the terms $H_i\big(Y(K_pK_f^p), \mathcal{O}\big)$ (which is a finitely generated $\mathcal{O}$-module) appearing in the projective limit is equipped with its ϖ-adic topology. The topology on $\widetilde{H}_i$ is then a pro-finite topology, and so in particular $\widetilde{H}_i$ is compact.

The completed cohomology and homology are related to one another in the usual way by duality over $\mathcal{O}$ [24]. In particular, if we ignore $\mathcal{O}$-torsion, then $\widetilde{H}_i$ is the $\mathcal{O}$-dual of $\widetilde{H}^i$ (equipped with its weak topology), and $\widetilde{H}^i$ is the continuous $\mathcal{O}$-dual of $\widetilde{H}_i$.

We can also form the limit

$$H^i := \varinjlim_{K_p} \varprojlim_{s} H^i\big(Y(K_pK_f^p), \mathcal{O}/\varpi^s\big) \cong \varinjlim_{K_p} H^i\big(Y(K_pK_f^p), \mathcal{O}\big). \tag{2}$$

There is a natural morphism

$$H^i \to \widetilde{H}^i, \tag{3}$$

which induces an embedding

$$\widehat{H}^i \hookrightarrow \widetilde{H}^i, \tag{4}$$

whose source is the ϖ-adic completion of H^i. Note that the morphism (3) need not be injective: although each of the terms $H^i\big(Y(K_pK_f^p), \mathcal{O}\big)$ in the direct limit defining H^i is a finitely generated $\mathcal{O}$-module, the limit H^i is merely countably generated as an $\mathcal{O}$-module, and hence may contain divisible elements. These then become zero after we pass to the ϖ-adic completion to obtain the embedding (4). (We give an example of this in (2.2.3) below.)

We furthermore remark that the transition maps in the direct limits (1) and (2) need not be injective. (This is also illustrated by the example of (2.2.3).) However, if $K_p' \subset K_p$, then a trace argument shows that the restriction map

$$H^i\big(Y(K_pK_f^p), \mathcal{O}) \to H^i\big(Y(K_p'K_f^p), \mathcal{O})$$

becomes injective after tensoring with L. Thus the kernel of the natural map $H^i\big(Y(K_pK_f^p), \mathcal{O}\big) \to H^i$ consists of torsion classes.

On the other hand, the kernel of the natural map

$$H^i\big(Y(K_pK_f^p), \mathcal{O}) \to \widetilde{H}^i \tag{5}$$

need not consist only of torsion classes; it is possible for non-torsion classes to have infinitely divisible image in H^i, and hence have vanishing image in $\widehat{H}^i$ (and so also have vanishing image in $\widetilde{H}^i$). (Again, we refer to (2.2.3) for an example of this.).

The morphism (4), although injective, need not be surjective. Its cokernel is naturally identified with $T_pH^{i+1} := \varprojlim_s H^{i+1}[\varpi^s]$; in other words we have a short exact sequence

$$0 \to \widehat{H}^i \to \widetilde{H}^i \to T_pH^{i+1} \to 0. \tag{6}$$

(Here we have written $H^{i+1}[\varpi^s]$ to denote the submodule of ϖ^s-torsion elements in H^{i+1}, and the projective limit is taken with respect to the multiplication-by-ϖ map from $H^{i+1}[\varpi^{s+1}]$ to $H^{i+1}[\varpi^s]$. The notation "T_p" is for *Tate module*.) In particular, if all the cohomology modules $H^{i+1}\big(Y(K_pK_f^p), \mathcal{O}\big)$ are torsion free, so that H^{i+1} is torsion free, then the morphism (4) *is* an isomorphism.

Although the restriction maps (5) can have non-trivial kernels, one can recover the cohomology at the various finite levels $K_pK_f^p$ via the Hochschild–Serre spectral sequence discussed in (2.1.3) below. The precise manner in which the cohomology at finite levels gets encoded in the completed cohomology is somewhat complicated in general. For instance, infinitely divisible torsion elements in H^i give rise to elements of T_pH^i, which by the discussion of the previous paragraph is naturally a quotient of $\widetilde{H}^{i-1}$. However, infinitely divisible non-torsion elements of H^i have no obvious incarnation as elements of completed cohomology, and the manner in which they are recovered by the Hochschild–Serre spectral sequence can be subtle.

2.1.2. Group actions. There is a natural continuous action of $G := \mathbb{G}(\mathbb{Q}_p)$ on each of $\widetilde{H}^i$ and $\widetilde{H}_i$, whose key property is encapsulated in the following result.

Theorem 2.1. [24, 31] *The G-action on $\widetilde{H}^i$ makes it a ϖ-adically admissible representation of G; i.e. it is ϖ-adically complete as an $\mathcal{O}$-module, and each quotient $\widetilde{H}^i/\varpi^s$ (which is a smooth representation of G over $\mathcal{O}/\varpi^s$) is admissible in the usual sense (for each compact open subgroup $K_p \subset G$, the submodule of K_p-invariants is finitely generated over $\mathcal{O}$).*

If $K_p \subset G$ is compact open, then we write $\mathcal{O}[[K_p]] := \varprojlim_{K'_p} \mathcal{O}[K_p/K'_p]$, where the projective limit is taken over all open normal subgroups $K'_p \subset K_p$. Since G acts continuously on $\widetilde{H}^i$ and $\widetilde{H}_i$, it follows that for any such K_p, the K_p-action on each of these modules may be promoted to an action of $\mathcal{O}[[K_p]]$. One then has the following reformulation of the admissibility of the G-action on $\widetilde{H}^i$.

Theorem 2.2. *Each of the $\mathcal{O}[[K_p]]$-modules $\widetilde{H}_i$ is finitely generated.*

(We remark that if this finite generation statement holds for one choice of K_p then it holds for any such choice.)

2.1.3. Cohomology of local systems and the Hochschild–Serre spectral sequence. If W is a finitely generated $\mathcal{O}$-module equipped with a continuous representation of an open subgroup $K_p \subset G$, and K_p is sufficiently small, so that $\mathbb{G}(\mathbb{Q})$ acts with trivial stabilizers on $\mathbb{G}(\mathbb{A})/A_\infty^\circ K_\infty^\circ K_p K_f^p$, then, for each open subgroup $K'_p \subset K_p$, the representation W determines a local system $\mathcal{W}$ of $\mathcal{O}$-modules on $Y(K_pK_f^p)$, defined via $\mathcal{W} := \mathbb{G}(\mathbb{Q})\backslash\Big(\big(\mathbb{G}(\mathbb{A}_f)/A_\infty^\circ K_\infty^\circ K_f^p\big) \times W\Big)/K'_p$. Suppose that W is furthermore torsion-free as an $\mathcal{O}$-module, and let $W^\vee$ denote the $\mathcal{O}$-dual of W, endowed with the contragredient K_p-action. There is then [24, 31] a Hochschild–Serre spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathcal{O}[[K_p]]}(W^\vee, \widetilde{H}^j) \implies H^{i+j}\big(Y(K_pK_f^p), \mathcal{W}\big).$$

This gives a precise sense to the idea that p-adically completed cohomology captures *all* the cohomology (with arbitrary coefficients) at finite levels. It is the realization, in the context of completed cohomology, of the general philosophy (brought out especially in Hida's work, e.g. [44]) that when working with p-adic automorphic forms, by passing to infinite p-power level one automatically encompasses automorphic forms of all possible weights. (For the precise relationship with classical automorphic forms, see (2.1.6) below.)

In particular, if we take $W = \mathcal{O}$ (with the trivial K_p-action), we obtain a spectral sequence $E_2^{i,j} = H^i\big(K_p, \widetilde{H}^j\big) \implies H^{i+j}\big(Y(K_pK_f^p), \mathcal{O}\big)$ (where H^i denotes continuous cohomology), which recovers the cohomology at the finite level $K_pK_f^p$ from the completed cohomology. If we take a direct limit over all K_p, we obtain a spectral sequence $E_2^{i,j} = \varinjlim_{K_p} H^i\big(K_p, \widetilde{H}^j\big) \implies H^{i+j}$, which recovers the limits H^{i+j} of the cohomology at finite levels. The edge map $H^i \to \varinjlim_{K_p} (\widetilde{H}^i)^{K_p}$ is induced by the morphism (3). As already noted, the relationship between this spectral sequence, the morphism (3), and the exact sequence (6) is subtle. (See the example of (2.2.3) below.)

More generally, if we write $H^i(\mathcal{W}) := \varinjlim_{K'_p} H^i\big(Y(K'_pK_f^p), \mathcal{W}\big)$ (where K'_p runs over all open subgroups of the fixed K_p, of which W is a representation), then we have an edge map $H^i(\mathcal{W}) \to \varinjlim_{K'_p} \mathrm{Hom}_{K'_p}(W^\vee, \widetilde{H}^i\big)$ which relates the cohomology at finite levels with coefficients in $\mathcal{W}$ to the $W^\vee$-isotypic parts (for open subgroups K'_p of K_p) of $\widetilde{H}^i$.

2.1.4. Hecke actions. There is a finite set of primes Σ_0 (containing p) such that for $\ell \notin \Sigma_0$, we may factor $K_f^p = K_f^{p,\ell} K_\ell$ where $K_f^{p,\ell}$ is compact open in $\mathbb{G}(\mathbb{A}_f^{p,\ell})$, and K_ℓ is a hyperspecial maximal compact subgroup of $\mathbb{G}(\mathbb{Q}_\ell)$. (In particular, $\mathbb{G}$ is unramified at such a prime ℓ, so that it admits a hyperspecial maximal compact subgroup.) We may then consider the spherical Hecke algebra (i.e. the double coset algebra) $\mathcal{H}_\ell := \mathcal{H}\big(\mathbb{G}(\mathbb{Q}_\ell)//K_\ell, \mathcal{O}\big)$ with coefficients in $\mathcal{O}$. This algebra is commutative [41, 42], and acts naturally (by continuous operators) on any of the cohomology groups $H^i\big(Y(K_pK_f^p), \mathcal{W}\big)$ considered in (2.1.3).

If we let i range over all cohomological degrees, let K_p range over all compact open subgroups of G, and let W range over all representations of K_p on finitely generated *torsion* $\mathcal{O}$-modules, then $\prod_i \prod_{K_p} \prod_W \mathrm{End}\, H^i\big(Y(K_pK_f^p), \mathcal{W}\big)$ is a profinite ring. We fix a set of primes Σ containing Σ_0, and we define the global Hecke algebra $\mathbb{T}_\Sigma$ to be the closure in this profinite ring of the $\mathcal{O}$-subalgebra generated by the image of $\mathcal{H}_\ell$ for all $\ell \notin \Sigma$. (The reasons for allowing the possibility of Σ being larger than Σ_0 are essentially technical; it is not misleading to simply imagine that Σ is equal to Σ_0 and is fixed once and for all, e.g. by being taken to be as small as possible, given our fixed choice of tame level K_f^p.) By construction, $\mathbb{T}_\Sigma$ acts on each of the cohomology groups $H^i\big(Y(K_pK_f^p), \mathcal{W}\big)$ considered in (2.1.3) (this is obvious if W is torsion, and follows in general by writing W as a projective limit $W \cong \varprojlim_s W/\varpi^s$), and also on $\widetilde{H}^i$ and $\widetilde{H}_i$. (In the case of $\widetilde{H}^i$ this follows from its definition in terms of limits of cohomology groups $H^i\big(Y(K_pK_f^p), \mathcal{O}/\varpi^s\big)$. It then follows for $\widetilde{H}_i$ by duality; more precisely, we can write $\widetilde{H}_i$ as the projective limit over s and K_p of $H_i\big(Y(K_pK_f^p), \mathcal{O}/\varpi^s\big)$, and this latter module is the $\mathcal{O}/\varpi^s$-dual of $H^i\big(Y(K_pK_f^p), \mathcal{O}/\varpi^s\big)$.)

These Hecke actions on $\widetilde{H}^i$ and $\widetilde{H}_i$ commute with the action of G. Thus there is an action of Hecke on the E_2-terms, as well the E_∞-terms, of the various Hochschild–Serre spectral sequences considered in (2.1.3), and these are in fact spectral sequences of $\mathbb{T}_\Sigma$-modules.

The $\mathcal{O}$-algebra $\mathbb{T}_\Sigma$ is a complete semi-local ring, i.e. it is a product of finitely many factors $\mathbb{T}_\mathfrak{m}$, each of which is a complete local ring. (This finiteness statement is proved via the methods of [3].) The topology on $\mathbb{T}_\Sigma$ is the product of the $\mathfrak{m}$-adic topologies on each of the complete local rings $\mathbb{T}_\mathfrak{m}$.

2.1.5. Variants in the non-compact case, and Poincaré duality. There are some variants of completed (co)homology which are useful in the case when the quotients $Y(K_f)$ are non-compact. (We recall that the quotients $Y(K_f)$ are compact precisely when the semisimple part of the group $\mathbb{G}$ is *anisotropic*, i.e. contains no torus that is split over $\mathbb{Q}$.)

Namely, replacing cohomology by cohomology with compact supports, we can define G-representations $\widetilde{H}^i_c$, H^i_c, $\widehat{H}^i_c$, and $T_pH^i_c$, and we have Hochschild–Serre spectral sequences for compactly supported cohomology. Similarly, we can define completed Borel–Moore homology $\widetilde{H}^{\mathrm{BM}}_i$, which is related to $\widetilde{H}^i_c$ by duality over $\mathcal{O}$.

There is a more subtle duality over $\mathcal{O}[[K_p]]$ (for a compact open subgroup $K_p \subset G$) which relates the usual and compactly supported variants of completed cohomology. It is most easily expressed on the homology side, where it takes the form of the Poincaré duality spectral sequence

$$E_2^{i,j} := \mathrm{Ext}^i_{\mathcal{O}[[K_p]]}(\widetilde{H}_j, \mathcal{O}[[K_p]]) \implies \widetilde{H}^{\mathrm{BM}}_{d-(i+j)}. \tag{7}$$

Of course, when the quotients $Y(K_f)$ are compact, compactly supported and usual cohomology coincide, as do usual and Borel–Moore homology. In general, we can relate them by considering the Borel–Serre compactifications $\overline{Y}(K_f)$. If we let $\partial(K_f)$ denote the boundary of $\overline{Y}(K_f)$, then we may compute the compactly supported cohomology of $Y(K_f)$ as the relative cohomology

$$H^i_c\big(Y(K_f), \mathcal{O}/\varpi^s\big) = H^i\big(\overline{Y}(K_f), \partial(K_f); \mathcal{O}/\varpi^s\big)$$

(and similarly we may compute Borel–Moore homology as relative homology), and so we obtain the long exact sequence of the pair relating the cohomology of $Y(K_f)$, the compactly supported cohomology of $Y(K_f)$, and the cohomology of $\partial(K_f)$. Passing to the various limits, we obtain a long exact sequence

$$\cdots \to \widetilde{H}^i_c \to \widetilde{H}^i \to \widetilde{H}^i(\partial) := \varprojlim_s \varinjlim_{K_p} H^i\big(\partial(K_pK^p_f), \mathcal{O}/\varpi^s\big) \to \widetilde{H}^{i+1}_c \to \cdots .$$

The completed cohomology of the boundary $\widetilde{H}^i(\partial)$ can be computed in terms of completed cohomology of the various Levi subgroups of $\mathbb{G}$; see [24] for more details.

We also mention that there is a $\mathbb{T}_\Sigma$-action on each of the objects introduced here, and that the various spectral sequences and long exact sequences considered here are all compatible with these actions.

2.1.6. The relationship to automorphic forms. We fix an isomorphism $\imath : \overline{\mathbb{Q}}_p \cong \mathbb{C}$. Since $L \subset \overline{\mathbb{Q}}_p$, we obtain induced embeddings $\mathcal{O} \subset L \hookrightarrow \mathbb{C}$.

Suppose that V is an algebraic representation of $\mathbb{G}$ defined over L, and suppose further that W is a K_p-invariant $\mathcal{O}$-lattice in V, for some compact open subgroup K_p of $G := \mathbb{G}(\mathbb{Q}_p) \subset \mathbb{G}(L)$. Write $V_{\mathbb{C}} := \mathbb{C} \otimes_L V = \mathbb{C} \otimes_{\mathcal{O}} W$ (the tensor products being taken with respect to the embeddings induced by $\imath$). If, for any compact open subgroup K_p' of K_p, we let $\mathcal{V}_{\mathbb{C}}$ denote the local system of $\mathbb{C}$-vector spaces on $Y(K_p' K_f^p)$ associated to $V_{\mathbb{C}}$, then we obtain a natural isomorphism

$$\mathbb{C} \otimes_{\mathcal{O}} H^i(\mathcal{W}) \cong \varinjlim_{K_p'} H^i\big(Y(K_p' K_f^p), \mathcal{V}_{\mathbb{C}}\big) =: H^i(\mathcal{V}_{\mathbb{C}}).$$

We may compute this cohomology using the space of automorphic forms on $\mathbb{G}$ [39].

Precisely, write $\mathcal{A}(K_f)$ for the space of automorphic forms on $\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A})/K_f$, for any compact open subgroup $K_f \subset \mathbb{G}(\mathbb{A}_f)$, and write $\mathcal{A}(K_f^p) = \varinjlim_{K_p} \mathcal{A}(K_p K_f^p)$, where the direct limit is taken (as usual) over the compact open subgroups $K_p \subset G := \mathbb{G}(\mathbb{Q}_p)$. For simplicity, suppose that A_∞° acts on $V_{\mathbb{C}}$ through some character χ (this will certainly hold if V is absolutely irreducible), and let $\mathcal{A}(K_f^p)_{\chi^{-1}}$ denote the subspace of $\mathcal{A}(K_f^p)$ on which A_∞° acts through the character χ^{-1}. Let $\widetilde{G}_\infty$ denote the group of real points of the intersection of the kernels of all the rational characters of $\mathbb{G}$, and let $\widetilde{\mathfrak{g}}$ and $\mathfrak{k}$ denote the Lie algebras of $\widetilde{G}_\infty$ and K_∞ respectively. Then it is proved by J. Franke in [39] that there is a natural isomorphism

$$H^i(\mathcal{V}_{\mathbb{C}}) \cong H^i\big(\widetilde{\mathfrak{g}}, \mathfrak{k}; \mathcal{A}(K_f^p)_{\chi^{-1}} \otimes V_{\mathbb{C}}\big). \tag{8}$$

(In the case when the the quotients $Y(K_f)$ are compact, this result is known as *Matsushima's formula* [49]. In the case when $\mathbb{G} = \mathrm{GL}_2$ and $i = 1$, it is known as the *Eichler–Shimura isomorphism* [54].) The action of $\mathbb{T}_\Sigma$ on $H^i(\mathcal{W})$ induces an action of $\mathbb{T}_\Sigma$ on $H^i(\mathcal{V}_{\mathbb{C}})$, and the resulting systems of Hecke eigenvalues that appear in $H^i(\mathcal{V}_{\mathbb{C}})$ (which we may think of as being simultaneously $\mathbb{C}$-valued and $\overline{\mathbb{Q}}_p$-valued, by employing the isomorphism $\imath$) are automorphic, i.e. arise as systems of Hecke eigenvalues on the space of automorphic forms $\mathcal{A}(K_f^p)$.

Thus, to first approximation, we may regard the Hecke algebra $\mathbb{T}_\Sigma$ (or, more precisely, the $\overline{\mathbb{Q}}_p$-valued points of its Spec) as being obtained by ϖ-adically interpolating the automorphic systems of Hecke eigenvalues that occur in H^i. However, as we already noted, the map from H^i to $\widehat{H}^i$ can have a non-trivial kernel. More significantly (given that the Hochschild–Serre spectral sequence shows that, despite this possibility, the systems of Hecke eigenvalues appearing in H^i can be recovered from the completed cohomology), the inclusion $\widehat{H}^i \hookrightarrow \widetilde{H}^i$ can have a non-trivial cokernel $T_p H^{i+1}$, which arises from infinitely divisible torsion in H^{i+1}. Thus $\operatorname{Spec} \mathbb{T}_\Sigma$ sees not only the systems of eigenvalues arising from classical automorphic forms (and their ϖ-adic interpolations), but also systems of Hecke eigenvalues arising from torsion cohomology classes (and *their* ϖ-adic interpolations). We will discuss this point further in (3.1.3) below.

It will be helpful to say a little more about how one can use (8) to analyze cohomology. To this end, we decompose the space of automorphic forms $\mathcal{A}(K_f^p)_{\chi^{-1}}$ as the direct sum

$$\mathcal{A}(K_f^p)_{\chi^{-1}} = \mathcal{A}_{\mathrm{cusp}}(K_f^p)_{\chi^{-1}} \oplus \mathcal{A}_{\mathrm{Eis}}(K_f^p)_{\chi^{-1}} \tag{9}$$

of the cuspforms and the forms which are orthogonal to the cuspforms (in $\mathcal{A}(K_f^p)_\chi$) under the Petersson inner product (i.e. the L^2 inner product of functions on $\mathbb{G}(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A})/A_\infty^\circ K_f^p)$). We label this complement to the space of cuspforms with the subscript Eis for *Eisenstein*, although its precise relationship with the space of Eisenstein series can be complicated; see e.g. [47], [39], and [40]. Note that if the quotients $Y(K_f)$ are compact, then $\mathcal{A}(K_f^p)_{\chi^{-1}} = \mathcal{A}_{\text{cusp}}(K_f^p)_{\chi^{-1}}$.

The decomposition (9) is a direct sum of $G_\infty \times G$-representations, and so the cohomology $H^i\big(\widetilde{\mathfrak{g}}, \mathfrak{k}; \mathcal{A}_{\text{cusp}}(K_f^p)_{\chi^{-1}} \otimes V_{\mathbb{C}}\big)$ is a G-invariant direct summand of the cohomology $H^i\big(\widetilde{\mathfrak{g}}, \mathfrak{k}; \mathcal{A}(K_f^p)_{\chi^{-1}} \otimes V_{\mathbb{C}}\big)$, which via the isomorphism (8) we identify with a G-invariant direct summand $H^i_{\text{cusp}}(\mathcal{V}_{\mathbb{C}})$ of $H^i(\mathcal{V}_{\mathbb{C}})$. It is also $\mathbb{T}_\Sigma$-invariant.

We may choose an everywhere positive element of $\mathcal{A}(K_f^p)$ on which $\mathbb{G}(\mathbb{A})$ acts via a character extending χ^2, multiplication by which induces an isomorphism between $\mathcal{A}_{\text{cusp}}(K_f^p)_{\chi^{-1}}$ and $\mathcal{A}_{\text{cusp}}(K_f^p)_\chi$. The Petersson (i.e. the L^2) inner product between $\mathcal{A}_{\text{cusp}}(K_f^p)_{\chi^{-1}}$ and $\mathcal{A}_{\text{cusp}}(K_f^p)_\chi$, taken together with this isomorphism, thus induces a pre-Hilbert space structure on $\mathcal{A}_{\text{cusp}}(K_f^p)_{\chi^{-1}}$, with respect to which the $G_\infty \times G$-action is unitary up to a character. In particular, we find that $\mathcal{A}_{\text{cusp}}(K_f^p)_{\chi^{-1}}$ is semisimple as a $G_\infty \times G$-representation, and so decomposes as a direct sum $\mathcal{A}_{\text{cusp}}(K_f^p)_{\chi^{-1}} = \bigoplus_{\pi_\infty\otimes\pi_p} \pi_\infty \otimes \pi_p \otimes M(\pi_\infty \otimes \pi_p)$, where the direct sum is taken over (a set of isomorphism class representatives of) all the irreducible admissible representations of $G_\infty \otimes G$ over $\mathbb{C}$ on which A_∞° acts via χ^{-1} (which factor as the tensor product $\pi_\infty \otimes \pi_p$ of an irreducible admissible representation π_∞ of G_∞ on which A_∞° acts via χ^{-1}, and an irreducible admissible smooth representation π_p of G), and $M(\pi_\infty \otimes \pi_p) := \operatorname{Hom}_{G_\infty\times G}\big(\pi_\infty \otimes \pi_p, \mathcal{A}_{\text{cusp}}(K_f^p)_{\chi^{-1}}\big)$ is the (finite-dimensional) multiplicity space of $\pi_\infty \otimes \pi_p$ in $\mathcal{A}_{\text{cusp}}(K_f^p)_{\chi^{-1}}$. Consequently, we obtain a direct sum decomposition

$$\begin{aligned} H^i_{\text{cusp}}(\mathcal{V}_{\mathbb{C}}) &\cong H^i\big(\widetilde{\mathfrak{g}}, \mathfrak{k}; \mathcal{A}_{\text{cusp}}(K_f^p)_{\chi^{-1}} \otimes V_{\mathbb{C}}\big) \\ &\cong \bigoplus_{\pi_\infty\otimes\pi_p} H^i(\widetilde{\mathfrak{g}}, \mathfrak{k}; \pi_\infty \otimes V_{\mathbb{C}}) \otimes \pi_p \otimes M(\pi_\infty \otimes \pi_p) \qquad (10) \end{aligned}$$

(the point here being that the $(\widetilde{\mathfrak{g}}, \mathfrak{k})$-cohomology depends only on the structure of $\pi_\infty \otimes \pi_p$ as a G_∞-representation, which is to say, only on π_∞).

In this way, we can speak of the contribution of each of the irreducible summands $\pi_\infty \otimes \pi_p$ of $\mathcal{A}_{\text{cusp}}(K_f^p)_{\chi^{-1}}$ to the cohomology $H^i_{\text{cusp}}(\mathcal{V}_{\mathbb{C}})$. In the case when $H^i(\widetilde{\mathfrak{g}}, \mathfrak{k}; \pi_\infty \otimes V_{\mathbb{C}})$ is non-zero, so that $\pi_\infty \otimes \pi_p$ actually does contribute to cohomology, the multiplicity space $M(\pi_\infty \otimes \pi_p)$ is naturally a $\mathbb{T}_\Sigma$-module, and the direct sum decomposition (10) is compatible with the actions of G and $\mathbb{T}_\Sigma$ (where G-acts on π_p, and $\mathbb{T}_\Sigma$ acts on $M(\pi_\infty \otimes \pi_p)$).

We can now explain the significance of the quantities l_0 and q_0 introduced above. Namely, q_0 is the lowest degree in which *tempered* π_∞ can admit non-trivial $(\widetilde{\mathfrak{g}}, \mathfrak{k})$-cohomology. Furthermore, among the tempered representations of G_∞, it is precisely the *fundamental tempered representations* (i.e. those which are induced from discrete series representations of the Levi subgroup of a fundamental parabolic subgroup of G_∞) which can admit non-zero cohomology at all, and they do so precisely in degrees between q_0 and $q_0 + l_0$ [10, Ch. III, Thm. 5.1]. (This is a range of degrees of length $l_0 + 1$ symmetric about $d/2$ — which is one-half of the dimension of the quotients $Y(K_f)$.) A fundamental theorem of Harish-Chandra states that, when $\mathbb{G}$ is semisimple, the group G_∞ admits discrete series representations precisely if $l_0 = 0$; in this case q_0 is equal to $d/2$, and the fundamental tempered representations are precisely the discrete series representations. Key examples for which $l_0 = 0$ are given by groups $\mathbb{G}$ for which G_∞ is compact, and (the semisimple parts of) groups giving rise to Shimura varieties.

One representation of $G_\infty \times G$ that always appears in $\mathcal{A}(K_f^p)$ is the trivial representation $\mathbb{1}$. Thus there are induced morphisms

$$H^i(\widetilde{\mathfrak{g}}, \mathfrak{k}; \mathbb{1}) \to H^i\big(\widetilde{\mathfrak{g}}, \mathfrak{k}; \mathcal{A}(K_f^p)_{\mathbb{1}}\big) \cong \mathbb{C} \otimes_{\mathcal{O}} H^i, \tag{11}$$

whose sources are naturally identified with the cohomology spaces of the compact dual to the symmetric space $G_\infty^\circ / A_\infty^\circ K_\infty^\circ$. If the $Y(K_f)$ are not compact, then $\mathbb{1}$ lies in $\mathcal{A}_{\mathrm{Eis}}(K_f^p)_{\mathbb{1}}$, but it is typically not a direct summand, and so the morphisms (11) are typically not injective. Nevertheless, when i is small they will be injective, and indeed (if e.g. $\mathbb{G}$ is split, semisimple, and simply connected) an isomorphism, and these isomorphisms play a key role in the theory of homological stability [9]. We discuss the interaction of these isomorphisms with the theory of completed cohomology (in the case when $\mathbb{G} = \mathrm{SL}_N$) in (2.2.3) below.

2.1.7. Dimension theory. For any compact open subgroup $K_p \subset G$, the completed group ring $\mathcal{O}[[K_p]]$ is (left and right) Noetherian, and of finite injective dimension as a module over itself; more precisely, its injective dimension is equal to $\dim \mathbb{G} + 1$. This allows us to consider a (derived) duality theory for finitely generated $\mathcal{O}[[K_p]]$-modules: to any such module M we associate the Ext-modules $E^i(M) := \mathrm{Ext}^i_{\mathcal{O}[[K_p]]}(M, \mathcal{O}[[K_p]])$, for $i \geq 0$ (which necessarily vanish if $i > \dim \mathbb{G} + 1$); these are again naturally $\mathcal{O}[[K_p]]$-modules. (We use the left $\mathcal{O}[[K_p]]$-module structure on $\mathcal{O}[[K_p]]$ to compute the E^i, and then use the right $\mathcal{O}[[K_p]]$-module structure on $\mathcal{O}[[K_p]]$, converted to a left module structure by the usual device of applying the anti-involution $k \mapsto k^{-1}$ on K_p, to give the E^i an $\mathcal{O}[[K_p]]$-module structure.)

An important point is that $E^i(M)$ is canonically independent of the choice of K_p (in the sense that we may regard M as an $\mathcal{O}[[K_p']]$-module, for any open subgroup K_p' of K_p, but $E^i(M)$ is canonically independent of which choice of K_p' we make to define it). This has the consequence that if the K_p-representation on M extends to a G-representation, then the $E^i(M)$ are also naturally G-representations.

The Poincaré duality spectral sequence (7) may thus be rewritten as

$$E_2^{i,j} = E^i(\widetilde{H}_j) \implies \widetilde{H}^{\mathrm{BM}}_{d-(i+j)}.$$

All the objects appearing in it are G-representations, and it is G-equivariant.

If K_p is p-torsion free (which will be true provided K_p is sufficiently small) then the ring $\mathcal{O}[[K_p]]$ is in fact Auslander regular [55]. If K_p is furthermore pro-p, then the ring $\mathcal{O}[[K_p]]$ is an integral domain (in the sense that it contains no non-trivial left or right zero divisors). This has various implications for the theory of finitely generated $\mathcal{O}[[K_p]]$-modules. For example, it is reasonable to define such a module to be *torsion* if every element has a non-zero annihilator in $\mathcal{O}[[K_p]]$ (where K_p is chosen small enough to be pro-p and p-torsion free). More generally, we may make the following definition.

Definition 2.3. If M is a finitely generated $\mathcal{O}[[K_p]]$-module, then the *codimension* $\mathrm{cd}(M)$ of M is defined to be the least value of i for which $E^i(M) \neq 0$; the dimension $\dim M$ of M is defined to be $\dim \mathbb{G} + 1 - \mathrm{cd}(M)$.

If $\mathbb{G}$ is a torus, so that K_p is commutative, then $\dim M$ is precisely the dimension of the support of the localization of M over $\mathrm{Spec}\, \mathcal{O}[[K_p]]$. In general $\mathcal{O}[[K_p]]$ is non-commutative, and hence doesn't have a Spec over which we can localize a finitely generated module M. Nevertheless, the quantity $\dim M$ behaves as if it were the dimension of the support of M in the (non-existent) Spec of $\mathcal{O}[[K_p]]$. (See e.g. [55, Prop. 3.5].) As one example, we note that M is torsion (in the sense defined above) if and only if $\mathrm{cd}(M) \geq 1$. As another, we note that $\mathrm{cd}\, E^i(M) \geq i$, with equality when $i = \mathrm{cd}\, M$. Also, we note that $\mathrm{cd}\, M = \infty$ (so, morally, the support of M is empty) precisely when $M = 0$.

By Theorem 2.2, completed homology is finitely generated over $\mathcal{O}[[K_p]]$, and so these dimension-theoretic notions apply to it. Information about the (co)dimension of $\widetilde{H}_i$ can be obtained by analyzing the Poincaré duality spectral sequence, since the functors E^i appear explicitly in it, and there is a tension in this spectral sequence between the top degree for the duality of $\mathcal{O}[[K_p]]$-modules (which is $\dim \mathbb{G} + 1$) and the top degree for usual Poincaré duality (which is d), which can sometimes be exploited. (See e.g. (2.2.2) below.) In making any such analysis, it helps to have some *a priori* information about the (co)dimensions of the $\widetilde{H}^i$. This is provided by the following result.

Theorem 2.4. [23] *Suppose that $\mathbb{G}$ is semisimple. Then $\widetilde{H}^i$ is torsion* (*i.e. its codimension is positive*) *unless $l_0 = 0$ and $i = q_0$, in which case* $\mathrm{cd}\, \widetilde{H}_i = 0$.

We make a much more precise conjecture about the codimensions of the $\widetilde{H}^i$ in Conjecture 3.2 below.

2.2. Examples.

We illustrate the preceding discussion with some examples.

2.2.1. $\mathbf{GL_2}$ of $\mathbb{Q}$. If $\mathbb{G} = \mathrm{GL}_2$, then the quotients $Y(K_f)$ are classical modular curves. In particular, they have non-vanishing cohomology only in degrees 0 and 1, and so we consider completed cohomology in degrees 0 and 1. Since the cohomology of a curve is torsion free, we have $\widetilde{H}^0 = \widehat{H}^0 \cong \mathcal{C}\big(\Delta \times \mathbb{Z}_p^\times, \mathcal{O}\big)$, the space of continuous $\mathcal{O}$-valued functions on the product $\Delta \times \mathbb{Z}_p^\times$, where Δ is a finite group that depends on the choice of tame level, and so $\widetilde{H}_0$ (which is simply the $\mathcal{O}$-dual of $\widehat{H}^0$) is isomorphic to $\mathcal{O}[[\Delta \times \mathbb{Z}_p^\times]]$, which is of dimension two as a module over $\mathcal{O}[[K_p]]$ (for any compact open $K_p \subset \mathrm{GL}_2(\mathbb{Q}_p)$). We also have $\widetilde{H}^1 = \widehat{H}^1$. Again, $\widetilde{H}_1$ is the $\mathcal{O}$-dual of $\widetilde{H}^1$, and Theorem 2.4 shows that $\mathrm{cd}\, \widetilde{H}_1 = 0$. (Strictly speaking, we have to apply the theorem to SL_2 rather than GL_2, but it is then easy to deduce the corresponding result for GL_2, by considering the cup-product action of H^0 on H^1.)

2.2.2. $\mathbf{SL_2}$ of an imaginary quadratic field. Suppose that $\mathbb{G} = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{SL}_2$, where F is an imaginary quadratic extension of $\mathbb{Q}$. The quotients $Y(K_f)$ are then connected, non-compact three-manifolds. The relevant (co)homological degrees are thus $i = 0, 1$, and 2. Since the $Y(K_f)$ are connected we see that $\widetilde{H}^0 = \widetilde{H}_0 = \mathcal{O}$. Theorem 2.4 implies that $\widetilde{H}_1$ has positive codimension. A consideration of the Poincaré duality spectral sequence then shows that $\widetilde{H}_2 = 0$, and that $\widetilde{H}_1$ is of codimension 1. This computation exploits both the fact that $\widetilde{H}_0 \neq 0$, and the gap between 3 (the dimension of the $Y(K_f)$) and 6 (the dimension of $\mathbb{G}$).

Since H^1 with coefficients in $\mathcal{O}$ of any space is ϖ-torsion free, we see that $\widetilde{H}^1$ is ϖ torsion-free, and coincides with the $\mathcal{O}$-dual of $\widetilde{H}_1$, while $\widetilde{H}^2$ is ϖ-torsion, and is naturally identified with the Pontrjagin dual of the $\mathcal{O}$-torsion submodule of $\widetilde{H}_1$ [24, Thm. 1.1]. We conjecture that in fact $\widetilde{H}_1$ is ϖ-torsion free (see Conjecture 3.2), and thus that $\widetilde{H}^2 = 0$.

2.2.3. $\mathbf{SL}_{\boldsymbol{N}}$ of $\mathbb{Q}$ in low degrees. We first discuss $\widetilde{H}^0$ and $\widetilde{H}^1$, before turning to a discussion of higher degree cohomology from the point of view of homological stability.

The quotients $Y(K_f)$ are connected, and so $\widetilde{H}_0 = \mathcal{O}$. If $N \geq 3$, then SL_N satisfies the congruence subgroup property. Furthermore, the groups $\mathrm{SL}_N(\mathbb{Z}_\ell)$ are perfect for all values of ℓ. Combining these two facts, we see that if $\Gamma(p^r)$ denotes the usual congruence subgroup of level p^r (i.e. the kernel of the surjection $\mathrm{SL}_N(\mathbb{Z}) \to \mathrm{SL}_N(\mathbb{Z}/p^r)$), then $H_1\big(\Gamma(p^r), \mathcal{O}\big) = \mathcal{O} \otimes_{\mathbb{Z}_p} \Gamma(p^r)^{\mathrm{ab}} \cong \mathcal{O} \otimes_{\mathbb{Z}_p} \Gamma(p^r)/\Gamma(p^{2r})$. Thus, if we take the tame level to be trivial (i.e. "level one"), then we see that the transition maps in the projective system defining $\widetilde{H}_1$ are eventually zero, implying that $\widetilde{H}_1 = 0$. Similarly, we see that $H^1 = \widetilde{H}^1 = 0$. (If we allowed a more general tame level, then $\widetilde{H}_1$ could be non-zero, but would be finite. Since H^1 and $\widetilde{H}^1$ are always torsion free, they would continue to vanish.)

We now consider cohomology in higher degree, but in the stable range, in the sense that we now explain. (The tame level can now be taken to be arbitrary.) Borel's result [9] on homological stability for SL_N shows that when i is sufficiently small (compared to N), the cohomology $L \otimes_{\mathcal{O}} H^i$ is independent of N; indeed, it consists precisely of the contributions to cohomology arising from the trivial automorphic representation, in the sense discussed in (2.1.6). If we define H^i_{stab} to be the stable value of H^i, then Borel shows that $\bigoplus H^i_{\mathrm{stab}}$ (as an algebra under cup product) is isomorphic to an exterior algebra generated by a single element in each degree $i = 0, 5, 9, 13, \ldots$.

In fact, stability also holds for completed cohomology.

Theorem 2.5. [25] *If i is sufficiently small compared to N, then $\widetilde{H}^i$ is independent of N, is finitely generated as an $\mathcal{O}$-module, and affords the trivial representation of G.*

We write $\widetilde{H}^i_{\mathrm{stab}}$ to denote the stable value of $\widetilde{H}^i$. F. Calegari has succeeded in computing $\widetilde{H}^i_{\text{stab}}$ modulo torsion, contingent on a natural non-vanishing conjecture for certain special values of the p-adic ζ-function (a conjecture which holds automatically if p is a regular prime).

Theorem 2.6. [22, Thm. 2.3] *Suppose either that p is a regular prime, or that appropriate special values of the p-adic zeta function are non-zero. Then there is an isomorphism of graded vector spaces $\bigoplus_{i\geq 0} L\otimes_{\mathcal{O}} \widetilde{H}^i_{\text{stab}} \cong L[x_2, x_6, x_{10}, \ldots]$ (where $L[x_2, x_6, x_{10}, \ldots]$ denotes the graded ring generated by the indicated sequence of polynomial variables x_i, $i \equiv 2 \bmod 4$, with x_i placed in degree i).*

Note in particular that $\bigoplus_i L\otimes_{\mathcal{O}} \widetilde{H}^i_{\text{stab}}$ vanishes in all odd degrees, while $\bigoplus_i L\otimes_{\mathcal{O}} H^i_{\text{stab}}$ is generated by classes in odd degrees. Thus, when $\mathbb{G} = \mathrm{SL}_N$, the map (3) is identically zero (modulo torsion) when i lies in the stable range!

Assuming the hypothesis of the theorem, we conclude that the Borel classes (i.e. the non-zero elements of $L\otimes_{\mathcal{O}} H^i_{\text{stab}}$) become infinitely divisible when we pull them back up the p-power level tower. It is interesting to consider how they reappear in the Hochschild–Serre spectral sequence. If we again ignore torsion, then $H^i\big(\mathrm{SL}_N(\mathbb{Z}_p), L\big)$ coincides with the Lie algebra cohomology of $\mathfrak{sl}_N$ [48], which is (stably in N) an exterior algebra on generators in degrees $3, 5, 7, \ldots$. Since all of the non-zero $\widetilde{H}^i_{\text{stab}}$ have trivial G-action, we see that we obtain non-trivial Ext terms in odd degrees in the Hochschild–Serre spectral sequence, and the Borel classes are recovered from these.

If we consider the short exact sequence (6) for non-zero even degrees, we see (continuing to assume the hypothesis of the theorem) that $\widehat{H}^i$ vanishes (modulo torsion), while $\widetilde{H}^i$ is non-zero. Thus the term T_pH^{i+1} must be non-zero; this provides a rather compelling illustration of the manner in which torsion classes can accumulate as we pass to the limit of the p-power level tower.

2.2.4. Groups that are compact at infinity. If G_∞ (or, more generally, the quotient G_∞/A_∞) is compact, then $A^\circ_\infty K^\circ_\infty$ equals G°_∞, and so the quotients $Y(K_f)$ are simply finite sets. Thus the only interesting degree of cohomology is $i = 0$. In this case the inverse limit $Y(K_f^p) := \varprojlim_{K_p} Y(K_pK_f^p)$ is a pro-finite set, which is in fact a compact p-adic analytic manifold, equipped with an analytic action of G. If K_p is taken sufficiently small (small enough that the $\mathbb{G}(\mathbb{Q})$-action on $\mathbb{G}(\mathbb{A})/G^\circ_\infty K_pK_f^p$ is fixed-point free), then K_p acts with trivial stabilizers on $Y(K_pK_f^p)$, and $Y(K_pK_f^p)$ is the disjoint union of finitely many (say n) principal homogeneous spaces over K_p.

One immediately verifies that $\widetilde{H}^0 \cong \mathcal{C}\big(Y(K_f^p), \mathcal{O}\big)$, the space of continuous $\mathcal{O}$-valued functions on $Y(K_f^p)$, while $\widetilde{H}_0 \cong \mathcal{O}[[Y(K_f^p)]]$, the space of $\mathcal{O}$-valued measures on $Y(K_f^p)$ (which is the $\mathcal{O}$-dual of $\mathcal{C}\big(Y(K_f^p), \mathcal{O}\big)$). In particular, if K_p is sufficiently small, then we see that $\widetilde{H}_0$ is free of rank n over $\mathcal{O}[[K_p]]$.

It is natural (e.g. in light of Gross's definition of algebraic modular forms in this context [42]) to define $\mathcal{C}\big(Y(K_f^p), \mathcal{O}\big)$ to be the space of p-adic automorphic forms on $\mathbb{G}(\mathbb{A})$ of tame level K_f^p, and so in this case we see that completed cohomology does indeed coincide with the natural notion of p-adic automorphic forms.

3. *p*-adic Langlands

We describe the manner in which we expect completed cohomology, and the Hecke algebra acting on it, to be related to deformation rings of global Galois representations. This conjectural relationship suggests various further conjectures, as we explain, as well as a relationship to a hypothetical p-adic local Langlands correspondence.

3.1. The connection between completed cohomology and Galois representations. If π is an automorphic representation of $\mathbb{G}(\mathbb{A})$ for which $H^i(\widetilde{\mathfrak{g}}, \mathfrak{k}; \pi_\infty \otimes V_{\mathbb{C}}) \neq 0$ for some algebraic representation V of $\mathbb{G}$ and some degree i, then π is C-algebraic [21, Lemma 7.2.2]. Thus, we expect that associated to π there should be a continuous representation of $G_{\mathbb{Q}}$ into the $\overline{\mathbb{Q}}_p$-valued points of the C-group of $\mathbb{G}$ [21, Conj. 5.3.4]. In light of the isomorphism (8), we thus expect that the systems of Hecke eigenvalues that occur in $H^i(\mathcal{V})$ should have associated representations of $G_{\mathbb{Q}}$ into the C-group of $\mathbb{G}$. For systems of Hecke eigenvalues occurring on torsion classes in cohomology, conjectures of Ash [2] again suggest that there should be associated Galois representations.

These expectations have been proved correct in many cases; for example if $\mathbb{G}$ is the restriction of scalars to $\mathbb{Q}$ of GL_n over a totally real or a CM number field. (See [45] in the case of characteristic zero systems of eigenvalues, and [53] in the case of torsion systems of eigenvalues. See also [43] for an overview of these results.)

Returning for a moment to the general case, one further expects that the Galois representations obtained should be *odd*, in the sense that complex conjugation in $G_{\mathbb{Q}}$ maps to a certain prescribed conjugacy class in the C-group of $\mathbb{G}$. (See [7, Prop. 6.1] for a description of the analogous conjugacy class in the L-group of $\mathbb{G}$.)

Let $\mathfrak{m}$ denote a maximal ideal in $\mathbb{T}_\Sigma$, write $\mathbb{F} := \mathbb{T}_\Sigma/\mathfrak{m}$, and let $\overline{\mathbb{F}}$ denote a chosen algebraic closure of $\mathbb{F}$. Then, by the above discussion, we believe that associated to $\mathfrak{m}$ there should be a continuous representation of $G_{\mathbb{Q}}$ into the $\overline{\mathbb{F}}$-valued points of the C-group of $\mathbb{G}$. Assuming that this representation exists, we will denote it by $\overline{\rho}_{\mathfrak{m}}$.

To be a little more precise: the representation $\overline{\rho}_{\mathfrak{m}}$ should have the property that it is unramified at the primes outside Σ, and that for any $\ell \notin \Sigma$, the semisimple part of $\overline{\rho}_{\mathfrak{m}}(\mathrm{Frob}_\ell)$ should be associated to the local-at-ℓ part of the system of Hecke eigenvalues by a suitable form of the Satake isomorphism (see [21] and [42]). In general, this condition may not serve to determine $\overline{\rho}_{\mathfrak{m}}$ uniquely; even in the case $\mathbb{G} = \mathrm{GL}_n$, it determines $\overline{\rho}_{\mathfrak{m}}$ only up to semisimplification; for other choices of $\mathbb{G}$, in addition to this issue, one can have the additional phenomenon that even a representation which doesn't factor through any parabolic subgroup of the C-group is not determined by its pointwise conjugacy classes ("local conjugacy does not determine global conjugacy"). We do not attempt to deal with this issue in general here; rather we simply ignore it, and continue our discussion as if $\overline{\rho}_{\mathfrak{m}}$ were unambiguously defined.

As we observed above, we may regard $\mathbb{T}_{\mathfrak{m}}$ (or its Spec) as interpolating systems of Hecke eigenvalues associated to classical C-algebraic automorphic forms, and/or to torsion classes in cohomology. Since the deformations of $\overline{\rho}_{\mathfrak{m}}$ can be interpolated into a formal deformation space, it is then reasonable to imagine that we may deform $\overline{\rho}_{\mathfrak{m}}$ to a representation of $G_{\mathbb{Q}}$ into the $\mathbb{T}_{\mathfrak{m}}$-valued points of the C-group of $\mathbb{G}$. Here again there are many caveats: firstly, if $\overline{\rho}_{\mathfrak{m}}$ is "reducible" (i.e. factors through a proper parabolic of the C-group), then we would have to work with some form of *pseudo-character* or *determinant*, as in [28] and [53]; also, there is a question of rationality, or "Schur index" — it may be that if we want our deformation of $\overline{\rho}_{\mathfrak{m}}$ to be defined over $\mathbb{F}$, then we may have to extend our scalars, or else replace the C-group by an inner twist. Again, we don't attempt to address these issues here.

Rather, we begin with some general conjectures about dimension and vanishing that are motivated by the preceding discussion, and then continue by discussing the relationships between completed cohomology and p-adic Hodge theory and a possible p-adic local Langlands correspondence. Finally, we turn to a discussion of some specific examples, where we can make our generalities more precise.

3.1.1. Conjectures on dimension and vanishing. We begin by making a somewhat vague conjecture, which can be thought of as a rough expression of our hopes for reciprocity in the context of global p-adic Langlands: namely (continuing with the notation introduced in the preceding discussion), we conjecture that $\mathbb{T}_{\mathfrak{m}}$ is universal (in some suitable sense) for parameterizing *odd* formal deformations of $\overline{\rho}_{\mathfrak{m}}$ whose ramification away from p is compatible with the tame level structure K_f^p (via some appropriate form of ϖ-adic local Langlands for the group $\mathbb{G}$ at primes $\ell \nmid p$ which, again, we won't attempt to formulate here; but see [36] in the case when $\mathbb{G} = \mathrm{GL}_n$). The global Euler characteristic formula for Galois cohomology lets us compute the expected dimension of such a universal deformation ring, and this motivates in large part the following concrete conjecture [24, §8].

Conjecture 3.1. *Each local factor $\mathbb{T}_{\mathfrak{m}}$ of $\mathbb{T}_{\Sigma}$ is Noetherian, reduced, and ϖ-torsion free, of Krull dimension equal to* $\dim \mathbb{B} + 1 - l_0$, *where* $\dim \mathbb{B}$ *denotes the dimension of a Borel subgroup $\mathbb{B}$ of $\mathbb{G}$.*

This is known in some cases (which we will recall below). We know of no way to prove the Noetherianness of $\mathbb{T}_{\mathfrak{m}}$, or the statement about its Krull dimension, other than to relate the Hecke algebra to a deformation ring of Galois representations, and then use techniques from the theory of Galois representations to compute the dimension.

In some situations we *can* prove that $\mathbb{T}_{\mathfrak{m}}$ is torsion free and reduced (see e.g. the discussion of (3.1.2) below), and it seems reasonable to conjecture these properties in general. Indeed, these properties are closely related to the following conjecture [24, Conj. 1.5].

Conjecture 3.2. *$\widetilde{H}_i = 0$ if $i > q_0$, while* $\operatorname{cd} \widetilde{H}_{q_0} = l_0$, *and* $\operatorname{cd} \widetilde{H}_i > l_0 + q_0 - i$ *if $i < q_0$. Furthermore, $\widetilde{H}_{q_0}$ is ϖ-torsion free, and $\mathbb{T}_{\Sigma}$ acts faithfully on $\widetilde{H}_{q_0}$.*

As noted in [24, Thm. 1.6], the truth of this conjecture for $\mathbb{G}$ and all its Levi subgroups implies the analogous statement for $\widetilde{H}_i^{\mathrm{BM}}$. Also, the vanishing conjecture for $\mathbb{G} \times \mathbb{G}$ implies (via the Künneth formula) the torsion-freeness of $\widetilde{H}_{q_0}$.

One of the ideas behind this conjecture is that "all the interesting Hecke eigenvalues should appear in degree q_0". This is inspired by the fact (recalled in (2.1.6)) that tempered automorphic representations don't contribute to cohomology in degrees below q_0, so that the Galois representations associated to the systems of Hecke eigenvalues appearing in degrees below q_0 should be "reducible" (i.e. factor through a proper parabolic subgroup of the C-group). In a Galois deformation space with unrestricted ramification at p, the reducible representations should form a proper closed subset, and so should be approximable by "irreducible" Galois representations (i.e. representations which don't factor through a proper parabolic). Thus we don't expect to see any systems of Hecke eigenvalues in degrees below q_0 which can't already be observed in degree q_0, and hence we expect that $\mathbb{T}_\Sigma$ will act faithfully on $\widetilde{H}_{q_0}$.

We also expect that $\widetilde{H}_i$ should be "small" if $i < q_0$, since the possible systems of Hecke eigenvalues which it can carry are (or should be) constrained. However, if the $\widetilde{H}_i$ are small enough for $i < q_0$, then the Poincaré duality spectral sequence (combined with an analysis of the completed cohomology of the boundary, which can be treated inductively, by reducing to the case of lower rank groups) implies the vanishing of $\widetilde{H}_i$ in degrees $> q_0$.

If we believe that $\mathbb{T}_\Sigma$ acts faithfully on $\widetilde{H}_{q_0}$, and that $\mathbb{T}_\Sigma$ has dimension $\dim \mathbb{B} + 1 - l_0$ (as predicted by Conjecture 3.1), then conjecturing that $\widetilde{H}_{q_0}$ has codimension l_0 is morally equivalent to conjecturing that the fibres of $\widetilde{H}_{q_0}$ over the points of $\operatorname{Spec} \mathbb{T}_\Sigma$ are of dimension $\dim \mathbb{G}/\mathbb{B}$. This latter statement fits nicely with the fact that generic irreducible representations of $G := \mathbb{G}(\mathbb{Q}_p)$ have Gelfand–Kirillov dimension equal to $\dim \mathbb{G}/\mathbb{B}$, and the analogy between the dimension of $\mathcal{O}[[K_p]]$-modules and Gelfand–Kirillov dimension [24, Remark 1.19]. Unfortunately, we don't know how to make this idea precise, since we don't know how to prove (in any generality) this relationship between the Krull dimension of $\mathbb{T}_\Sigma$, the dimension of $\widetilde{H}_{q_0}$, and the dimension of the fibres of the latter over points of $\operatorname{Spec} \mathbb{T}_\Sigma$, even assuming that $\mathbb{T}_\Sigma$ acts faithfully on $\widetilde{H}_{q_0}$. Nevertheless, the idea that these dimensions should be related is an important motivation for the conjecture.

We note that if $\mathbb{T}_\Sigma$ acts faithfully on $\widetilde{H}_{q_0}$, and $\widetilde{H}_{q_0}$ is ϖ-torsion free, then $\mathbb{T}_\Sigma$ is ϖ-torsion free. This motivates the conjecture of ϖ-torsion freeness in Conjecture 3.1.

Conjecture 3.2 has essentially been proved by P. Scholze in many cases for which the group $\mathbb{G}$ gives rise to Shimura varieties [53, Cor. IV.2.3]. (He has non-strict inequalities rather than strict inequalities for the codimension of the cohomology in degrees $< q_0$.)

We mention one more example here, namely the case when $\mathbb{G}$ is the restriction of scalars of SL_2 from an imaginary quadratic field. In this case we have $l_0 = 1$, and the codimension statement of Conjecture 3.2 follows from the computation sketched in (2.2.2).

3.1.2. Localization at a non-Eisenstein system of Hecke eigenvalues. Suppose that $\mathfrak{m}$ is a maximal ideal in $\mathbb{T}_\Sigma$, and that the associated representation $\overline{\rho}_{\mathfrak{m}}$ (which we assume exists) is "irreducible", i.e. does not factor through any proper parabolic subgroup of the C-group of $\mathbb{G}$. We refer to such a $\mathfrak{m}$ as *non-Eisenstein.* For any $\mathbb{T}_\Sigma$-module M, we write $M_{\mathfrak{m}} := \mathbb{T}_{\mathfrak{m}} \otimes_{\mathbb{T}_\Sigma} M$ to denote the localization of M at $\mathfrak{m}$.

As already noted, we expect that all the systems of Hecke eigenvalues appearing in cohomological degrees $< i$ are "reducible", i.e. do factor through a proper parabolic subgroup, and so we conjecture that, when $\mathfrak{m}$ is a non-Eisenstein maximal ideal, $H^i\big(Y(K_pK_f^p), \mathcal{W}\big)_{\mathfrak{m}} = 0$ for all $i < q_0$ and all local systems $\mathcal{W}$ associated to K_p-representations W on finitely generated $\mathcal{O}$-modules (where K_p is any sufficiently small compact open subgroup of G).

We remark that if we ignore torsion, then this follows (at least morally) from Arthur's conjectures [1]. (Namely, the result is true for the boundary cohomology, and hence it suffices to check it for the interior cohomology, i.e. the image of compactly supported cohomology in the usual cohomology. However, the interior cohomology is contained in the L^2-cohomology, and now Arthur's conjectures imply that any automorphic representation π contributing to L^2-cohomology that is non-tempered at ∞ — as must be the case for an automorphic representation that contributes to cohomology in degree $< q_0$ — is non-tempered at every prime, and hence should give rise to a reducible Galois representation.) We believe that this statement should be true for torsion cohomology as well.

Let us suppose, then, that $H^i\big(Y(K_pK_f^p), \mathcal{W}\big)_{\mathfrak{m}} = 0$ for all $i < q_0$. Just considering the case when $W = \mathcal{O}$ (the trivial representation), we find that $\widetilde{H}^i_{\mathfrak{m}} = 0$ for $i < q_0$. Certainly it should be the case that $H^i\big(\partial(K_pK_f^p), \mathcal{W}\big)_{\mathfrak{m}} = 0$ when $\mathfrak{m}$ is non-Eisenstein. We then find that $H^i_c\big(Y(K_pK_f^p), \mathcal{W}\big)_{\mathfrak{m}} = 0$ for all $i < q_0$ as well. *Suppose now that* $l_0 = 0$, so that $q_0 = d/2$. Classical Poincaré duality then gives that $H^i\big(Y(K_pK_f^p), \mathcal{W}\big)_{\mathfrak{m}} = 0$ for all $i > q_0$. Presuming that Conjecture 3.2 is true, so that $\widetilde{H}^i = 0$ for $i > q_0$ and $\widetilde{H}^{q_0}$ is ϖ-torsion free, a consideration of the Hochschild–Serre spectral sequence shows that $\mathrm{Ext}^i_{\mathcal{O}[[K_p]]}(W^\vee, \widetilde{H}^{q_0}_{\mathfrak{m}}) = 0$ for all $i > 0$ and all representations of K_p on finitely generated torsion free $\mathcal{O}$-modules. From this one easily deduces that $(\widetilde{H}_{q_0})_{\mathfrak{m}}$ is projective as an $\mathcal{O}[[K_p]]$-module.

In short, we have given a plausibility argument for the following conjecture, which refines Conjecture 3.2 in the context of localizing at a non-Eisenstein maximal ideal.

Conjecture 3.3. *If* $\mathfrak{m}$ *is a* non-Eisenstein *maximal ideal in* $\mathbb{T}_\Sigma$*, and if* $l_0 = 0$*, then* $(\widetilde{H}_i)_{\mathfrak{m}} = 0$ *for* $i \neq q_0$*, and* $(\widetilde{H}_{q_0})_{\mathfrak{m}}$ *is a projective* $\mathcal{O}[[K_p]]$*-module, for any sufficiently small subgroup* K_p *of* G*. (Here* sufficiently small *means that* $\mathbb{G}(\mathbb{Q})$ *acts with trivial stabilizers on* $\mathbb{G}(\mathbb{A})/A_\infty^\circ K_\infty^\circ K_p K_f^p$*.)*

If $l_0 = 0$, and if Conjecture 3.3 holds for some non-Eisenstein maximal ideal $\mathfrak{m}$, then we see (from the Hochschild–Serre spectral sequence) that for any sufficiently small K_p (as in the statement of the conjecture) and any representation of K_p on a finitely generated torsion free $\mathcal{O}$-module W, we have $H^i\big(Y(K_pK_f^p), \mathcal{W}\big)_{\mathfrak{m}} = 0$ if $i \neq q_0$, while $H^{q_0}\big(Y(K_pK_f^p), \mathcal{W}\big)_{\mathfrak{m}} \cong \mathrm{Hom}_{\mathcal{O}[[K_p]]}(W^\vee, \widetilde{H}^{q_0}_{\mathfrak{m}})$. In particular, we see that $H^{q_0}\big(Y(K_pK_f^p), \mathcal{W}\big)_{\mathfrak{m}}$ is ϖ-torsion free. We also see that $\widehat{H}^{q_0}_{\mathfrak{m}} = \widetilde{H}^{q_0}_{\mathfrak{m}}$ (since

$H^{q_0+1}_{\mathfrak{m}}$ vanishes, and hence so does $T_p H^{q_0+1}_{\mathfrak{m}}$), and that $\mathbb{T}_{\mathfrak{m}}$ acts faithfully on $\widetilde{H}^{q_0}_{\mathfrak{m}}$.

Presuming that $H^i\big(\partial(K_pK^p_f),\mathcal{W}\big)_{\mathfrak{m}} = 0$ (which should certainly be true when $\mathfrak{m}$ is non-Eisenstein), we conclude that the natural map

$$H^{q_0}_c\big(Y(K_pK^p_f),\mathcal{W})_{\mathfrak{m}} \to H^{q_0}\big(Y(K_pK^p_f),\mathcal{W})_{\mathfrak{m}}$$

is an isomorphism, and thus that $H^{q_0}\big(Y(K_pK^p_f),\mathcal{W})_{\mathfrak{m}}$ consists entirely of interior cohomology. In particular (being torsion free) it embeds into the L^2-cohomology. From this we deduce that the image of $\mathbb{T}_\Sigma$ acting on $H^{q_0}\big(Y(K_pK^p_f),\mathcal{W})_{\mathfrak{m}}$ is reduced, and hence (considering all possible K_p and W) that $\mathbb{T}_{\mathfrak{m}}$ itself is reduced, as well as being ϖ-torsion free. This provides some evidence for the reducedness and torsion-freeness statements in Conjecture 3.1.

We now recall some known results in the direction of Conjecture 3.3. In the case when $\mathbb{G} = \mathrm{GL}_2$, the concept of a non-Eisenstein maximal ideal is well-defined, and the above conjecture holds [32, Cor. 5.3.19]. In the case when $\mathbb{G} = \mathrm{GL}_N$, this notion is again well-defined, and the vanishing of $(\widetilde{H}_i)_{\mathfrak{m}}$ is proved for i in the stable range in [25]. Again, if $\mathbb{G}$ is a form of $U(n-1,1)$ over $\mathbb{Q}$, then the notion of a non-Eisenstein maximal ideal is well-defined, and in [34] we prove vanishing of $(\widetilde{H}_i)_{\mathfrak{m}}$ for i in a range of low degrees, for certain maximal ideals $\mathfrak{m}$. In particular, in the case of $U(2,1)$, we are able to deduce Conjecture 3.3, provided not only that $\mathfrak{m}$ is non-Eisenstein, but that the associated Galois representation $\overline{\rho}_{\mathfrak{m}}$ has sufficiently large image, and is irreducible locally at p, satisfying a certain regularity condition.

As one more example, note that if G_∞/A_∞ is compact (in which case certainly $l_0 = 0$), then Conjecture 3.3 holds. Indeed, in this case we saw in (2.2.4), for sufficiently small K_p, that $\widetilde{H}_0$ is free over $\mathcal{O}[[K_p]]$, even without localizing at a non-Eisenstein maximal ideal.

If $l_0 \neq 0$, then we don't expect that $(\widetilde{H}_{q_0})_{\mathfrak{m}}$ should be projective over $\mathcal{O}[[K_p]]$; indeed, this would be incompatible with Conjecture 3.2. Rather, we expect that it should be pure of codimension l_0, in the sense of [55, Def. 3.1].

3.1.3. The relationship with p-adic Hodge theory. Let us continue to suppose that $l_0 = 0$ and that $\mathfrak{m}$ is a non-Eisenstein maximal ideal in $\mathbb{T}_\Sigma$, and let us suppose that Conjecture 3.3 holds. As we have observed, this implies that for any (sufficiently small) compact open subgroup K_p of G, and any representation W of K_p on a finitely generated torsion free $\mathcal{O}$-module, we have

$$H^i\big(Y(K_pK^p_f),\mathcal{W})_{\mathfrak{m}} \cong \mathrm{Hom}_{\mathcal{O}[[K_p]]}(W^\vee,\widetilde{H}^{q_0}_{\mathfrak{m}}) \cong \mathrm{Hom}_{\mathcal{O}[[K_p]]}\big((\widetilde{H}_{q_0})_{\mathfrak{m}},W\big)$$

(the first isomorphism being given by the Hochschild–Serre spectral sequence, and the second by duality).

We now suppose that W is an $\mathcal{O}$-lattice in an irreducible algebraic representation V of $\mathbb{G}$ over L. Since $H^i\big(Y(K_pK^p_f),\mathcal{W})_{\mathfrak{m}}$ is ϖ-torsion free, it is a lattice in $H^i\big(Y(K_pK^p_f),\mathcal{V})_{\mathfrak{m}}$. Considering the description of this cohomology in terms of automorphic forms (as in (2.1.6)) and the conjectures regarding Galois representations associated to automorphic forms [21, Conj. 5.3.4], we infer that the Galois representations associated to the systems of Hecke eigenvalues appearing in

$H^i\big(Y(K_pK_f^p),\mathcal{V}\big)_{\mathfrak{m}}$ should be potentially semistable locally at p, with Hodge–Tate weights related to the highest weight of the algebraic representation V.

Since $\mathfrak{m}$ is non-Eisenstein, it should make sense to speak of the formal deformation ring $R_{\overline{\rho}_{\mathfrak{m}}}$ of $\overline{\rho}_{\mathfrak{m}}$, and we conjecture that there is a natural isomorphism $\mathbb{T}_{\mathfrak{m}} \cong R_{\overline{\rho}_{\mathfrak{m}}}$. We see that the $\mathbb{T}_{\mathfrak{m}}$-modules $H^i\big(Y(K_pK_f^p),\mathcal{W}\big)_{\mathfrak{m}}$ (where W ranges over the various lattices in the various algebraic representations V) should then be supported on the set of points of $\operatorname{Spec} R_{\overline{\rho}_{\mathfrak{m}}}$ corresponding to deformations of $\overline{\rho}_{\mathfrak{m}}$ that are potentially semistable at p.

The Fontaine–Mazur conjecture [38], when combined with Langlands reciprocity, should give a precise description of the support of $H^i\big(Y(K_pK_f^p),\mathcal{W}\big)_{\mathfrak{m}}$, in purely Galois-theoretic terms. Namely, any potentially semistable deformation of $\overline{\rho}_{\mathfrak{m}}$ of the appropriate Hodge–Tate weights should be motivic, and hence should be associated to an automorphic form on the quasi-split inner form of $\mathbb{G}$. Whether this automorphic form can then be transferred to $\mathbb{G}$, and can contribute to cohomology at level $K_pK_f^p$, should be answered by an analysis of the local Langlands correspondence for $\mathbb{G}$.

Conversely, if we can prove directly that the support of $H^i\big(Y(K_pK_f^p),\mathcal{W}\big)_{\mathfrak{m}}$ is as predicted by the Fontaine–Mazur–Langlands conjecture, then we can deduce this conjecture for deformations of $\overline{\rho}_{\mathfrak{m}}$ that are classified by $\operatorname{Spec}\mathbb{T}_{\mathfrak{m}}$. We recall in (3.2.2) below how this strategy is compatible with an optimistic view-point on how p-adic local Langlands might behave. We briefly recall in (3.3.1) how this strategy was employed in [32] and [46] in the case when $\mathbb{G} = \mathrm{GL}_2$.

Let us fix a sufficiently small open subgroup K_p. Since $(\widetilde{H}_{q_0})_{\mathfrak{m}}$ is projective over $\mathcal{O}[[K_p]]$ by assumption, it is a direct summand of a finitely generated free $\mathcal{O}[[K_p]]$-module, and hence $\widetilde{H}^{q_0}_{\mathfrak{m}}$ is a direct summand of $\mathcal{C}(K_p,\mathcal{O})^{\bigoplus n}$, for some $n > 0$. Thus $L\otimes_{\mathcal{O}}\widetilde{H}^{q_0}_{\mathfrak{m}}$ is a direct summand of $\mathcal{C}(K_p,L)^{\oplus n}$, the space of continuous L-valued functions on K_p. Now the theory of Mahler expansions shows that the affine ring $L[\mathbb{G}]$ embeds with dense image in $\mathcal{C}(K_p,L)$ [51, Lemma A.1]. Recall that, as a $\mathbb{G}$-representation, we have $L[\mathbb{G}] \cong \bigoplus_V V\otimes_L \operatorname{Hom}_{\mathbb{G}}(V, L[\mathbb{G}])$, where V runs over (a set of isomorphism class representatives of) all irreducible representations of $\mathbb{G}$. (At this point, we assume for simplicity that L is chosen so that all the irreducible representations of $\mathbb{G}$ over $\overline{\mathbb{Q}}_p$ are in fact defined over L; thus these irreducible V are in fact absolutely irreducible.) Now the inclusion of $L[\mathbb{G}]$ into $\mathcal{C}(K_p,L)$ induces an isomorphism $\operatorname{Hom}_{\mathbb{G}}(V,L[\mathbb{G}]) \cong \operatorname{Hom}_{K_p}\big(V,\mathcal{C}(K_p,L)\big)$ (both are naturally identified with $V^\vee$, the contragredient of V). We conclude that the natural morphism $\bigoplus_V V\otimes_L \operatorname{Hom}_{K_p}\big(V,\mathcal{C}(K_p,L)\big) \to \mathcal{C}(K_p,L)$ is injective with dense image. This property is clearly preserved under passing to direct sums and direct summands, and hence the natural morphism $\bigoplus_V V\otimes_L \operatorname{Hom}_{K_p}(V,\widetilde{H}^{q_0}_{\mathfrak{m}}\otimes_{\mathcal{O}} L) \to \widetilde{H}^{q_0}_{\mathfrak{m}}\otimes_{\mathcal{O}} L$ is also injective with dense image. Replacing V by $V^\vee$ (which clearly changes nothing, since we are summing over all irreducible algebraic representations of $\mathbb{G}$), and recalling that (by Hochschild–Serre) $\operatorname{Hom}_{K_p}(V^\vee,\widetilde{H}^{q_0}_{\mathfrak{m}}\otimes_{\mathcal{O}} L) \cong H^{q_0}\big(Y(K_pK_f^p),\mathcal{V}\big)_{\mathfrak{m}}$, we find that $\mathbb{T}_{\mathfrak{m}}$ acts faithfully on $\bigoplus_V H^{q_0}\big(Y(K_pK_f^p),\mathcal{V}\big)_{\mathfrak{m}}$ (since, as we noted above, it follows from Conjecture 3.3 that $\mathbb{T}_{\mathfrak{m}}$ acts faithfully on $\widetilde{H}^{q_0}_{\mathfrak{m}}$).

Assuming that we can identify $\mathbb{T}_{\mathfrak{m}}$ with a deformation space of Galois representations, and that we know that $H^{q_0}\big(Y(K_pK_f^p),\mathcal{V}\big)_{\mathfrak{m}}$ is indeed supported on an appropriate locus of potentially semistable deformations, the preceding analysis shows that these loci (as V varies) are Zariski dense in $\operatorname{Spec}\mathbb{T}_{\mathfrak{m}}$.

In forthcoming work [37], the author and V. Paškūnas will apply a more sophisticated version of this argument to deduce additional Zariski density statements for various collections of potentially semistable loci in global Galois deformation spaces.

Note that these density results rely crucially on the projectivity statement of Conjecture 3.3 for their proof. As we already noted, we can't expect such a statement to be true when $l_0 > 0$, and (at least if $\mathbb{G}$ is semisimple, or, more generally, if the semisimple part of $\mathbb{G}$ satisfies $l_0 > 0$) we don't expect the potentially semistable points to be Zariski dense in global deformation spaces in this case. (See [26] for an elaboration on this point.) In particular, in such contexts, we don't expect that $\widehat{H}^{q_0}$ will equal $\widetilde{H}^{q_0}$, and so $\widetilde{H}^{q_0}$, and $\operatorname{Spec}\mathbb{T}_{\Sigma}$, should receive a non-trivial contribution from torsion classes (via the term $T_pH^{q_0+1}$ in the exact sequence (6)).

3.2. *p*-adic local Langlands.

Until now, our discussion has been entirely global in nature. We now turn to describing how these global considerations might be related to a possible p-adic local Langlands correspondence.

3.2.1. The basic idea.

Let us return for a moment to the direct sum decomposition (10). Local-global compatibility for classical Langlands reciprocity says that if $\pi_\infty\otimes\pi_p$ is a direct summand of $\mathcal{A}_{\mathrm{cusp}}(K_f^p)_{\chi^{-1}}$ which contributes to cohomology, lying in a Hecke eigenspace that corresponds to some p-adic Galois representation ρ, then the local factor π_p and the Weil–Deligne representation attached to ρ (which should be defined, since ρ should be potentially semistable at p) should correspond via the local Langlands correspondence (or perhaps more generally via the local form of Arthur's conjectures [1], if π_∞ and π_p are not tempered). In particular, the local factor at p and the local Galois representation at p should be related in a purely local manner.

The basic idea of a p-adic local Langlands correspondence is that the same should be true when we take into account the structure of completed cohomology. To explain this, we continue the discussion of the preceding paragraph, and, in addition, we place ourselves in the context introduced in (3.1.3) (in particular, we continue to assume that the hypotheses and conclusions of Conjecture 3.3 hold). Thus, we assume that ρ is a deformation of $\overline{\rho}_{\mathfrak{m}}$, and that π_∞ and π_p are tempered, so that we have an embedding $\pi_p\hookrightarrow H^{q_0}(\mathcal{V})_{\mathfrak{m}}\cong\varinjlim_{K_p}\operatorname{Hom}_{K_p}(V^\vee,L\otimes_{\mathcal{O}}\widetilde{H}^{q_0}_{\mathfrak{m}})$. We may rewrite this as an embedding $V^\vee\otimes_L\pi_p\hookrightarrow L\otimes_{\mathcal{O}}\widetilde{H}^{q_0}_{\mathfrak{m}}$ and we can then take the closure of $V^\vee\otimes\pi_p$ in $H^{q_0}(\mathcal{V})_{\mathfrak{m}}$, to obtain a unitary Banach space representation $\widehat{V^\vee\otimes_L}\,\pi_p$ of G over L. A slightly more refined procedure is to form the intersection $(V^\vee\otimes_L\pi_p)^\circ := (V^\vee\otimes_L\pi_p)\cap\widetilde{H}^{q_0}_{\mathfrak{m}}$ (the intersection being taken in $L\otimes_{\mathcal{O}}\widetilde{H}^{q_0}_{\mathfrak{m}}$). This is a G-invariant $\mathcal{O}$-lattice contained in $V^\vee\otimes_L\pi_p$, and the Banach space $\widehat{V^\vee\otimes_L}\,\pi_p$ is then obtained by completing $V^\vee\otimes_L\pi_p$ with respect to this lattice. A natural question to ask, then, in the spirit of a local Langlands correspondence and local-global compatibility, is whether $\widehat{V^\vee \otimes_L} \pi_p$ depends only on the restriction to p of the associated Galois representation ρ. Since by assumption $\mathfrak{m}$ is non-Eisenstein, the Galois representation ρ should admit an essentially unique integral model ρ°, and we could further ask whether $(V^\vee \otimes_L \pi_p)^\circ$ depends only on the restriction to p of ρ°. This question goes back to C. Breuil's first work on the p-adic Langlands correspondence in the case when $\mathbb{G} = \mathrm{GL}_2$ (see especially the introduction of [13]). It has been largely resolved in that case, but remains open in general.

3.2.2. An optimistic scenario. The most optimistic conjecture that one might entertain regarding a p-adic local Langlands correspondence is that for any formal deformation ring $R_{\overline{r}}$ parameterizing the deformations of a representation $\overline{r}$ of the *local* Galois group $G_{\mathbb{Q}_p}$ into the $\overline{\mathbb{F}}$-valued points of the C-group of G, there is a profinite $R_{\overline{r}}$-module M, finitely generated over $R_{\overline{r}}[[K_p]]$ for some (and hence every) compact open subgroup $K_p \subset G$, equipped with a continuous G-action extending its K_p-module structure, which realizes the p-adic local Langlands correspondence for the group $\mathbb{G}$ in the following sense: in the context of (3.2.1) (and continuing with the notation of that discussion), the fibre over the restriction to p of ρ° (which is a point in $\operatorname{Spec} R_{\overline{r}}$, if $\overline{r}$ is the restriction of $\overline{\rho}_{\mathfrak{m}}$ to $G_{\mathbb{Q}_p}$) is isomorphic to the $\mathcal{O}$-dual of $(V^\vee \otimes_L \pi_p)^\circ$. (Here we suppress the issue of whether $R_{\overline{r}}$ should be understood to be a framed deformation ring, or a pseudo-deformation ring, or ....) In short, M should be a local analogue of the completed homology space $(\widetilde{H}_{q_0})_{\mathfrak{m}}$.

Since M is finitely generated over $R_{\overline{r}}[[K_p]]$, for any representation W of K_p on a finitely generated torsion free $\mathcal{O}$-module, the $R_{\overline{r}}$-module $\operatorname{Hom}^{\mathrm{cont}}_{K_p}(M, W)^d$ (where d denotes the continuous $\mathcal{O}$-dual) is a finitely generated $R_{\overline{r}}$-module. Another property one might require of M is that when W is a K_p-invariant $\mathcal{O}$-lattice in an irreducible algebraic $\mathbb{G}$-representation V, then $\operatorname{Hom}^{\mathrm{cont}}_{K_p}(M, W)^d$ is supported on an appropriate locus of potentially semistable representations, corresponding to the fact that $\operatorname{Hom}_{\mathcal{O}[[K_p]]}\big((\widetilde{H}_{q_0})_{\mathfrak{m}}, W\big)$ (which is isomorphic to $H^{q_0}\big(Y(K_p K_f^p), \mathcal{W}\big)$) is supported on a locus of potentially semistable representations in $\operatorname{Spec} \mathbb{T}_{\mathfrak{m}}$.

Ideally, one might ask for the support of $\operatorname{Hom}^{\mathrm{cont}}_{K_p}(M, W)^d$ to be the full potentially semistable locus of appropriate Hodge–Tate weights (corresponding to the highest weight of V) and Weil–Deligne representations (corresponding to the particular choice of K_p and the nature of the local Langlands correspondence for $\mathbb{G}$). As we indicated in (3.1.3), such a result, combined with the local-global compatibility between M and $(\widetilde{H}_{q_0})_{\mathfrak{m}}$, would prove that any global Galois representation corresponding to a point of $\operatorname{Spec} \mathbb{T}_{\mathfrak{m}}$ which is potentially semistable of appropriate Hodge–Tate weights and with an appropriate Weil–Deligne representation, in fact arises from a system of Hecke eigenvalues occurring in $\widetilde{H}^{q_0}\big(Y(K_p K_f^p), \mathcal{V}\big)_{\mathfrak{m}}$, thus verifying the Fontaine–Mazur–Langlands conjecture for such points.

Note that a local Galois representation lies in the support of $\operatorname{Hom}^{\mathrm{cont}}_{K_p}(M, W)^d$ precisely if the dual to the fibre of M at this point receives a non-zero K_p-equivariant homomorphism from W. In particular, the dual to the fibre at such a point contains locally algebraic vectors. In the case of GL_2, the idea of describing p-adic local Langlands for potentially semistable representations in terms of locally algebraic vectors goes back to C. Breuil's first work on the subject [11, 12].

The optimistic scenario described here has been realized for the group $G = \mathrm{GL}_2(\mathbb{Q}_p)$; this is the theory of p-adic local Langlands for $\mathrm{GL}_2(\mathbb{Q}_p)$ [6, 29, 50, 30] (see also [5, 14]). Whether it can be realized for other groups, or is overly optimistic, remains to be seen.

3.3. Examples.

We again illustrate our discussion with some examples.

3.3.1. $\mathbf{GL_2}$ of $\mathbb{Q}$. As already mentioned, the theory of p-adic local Langlands for $\mathrm{GL}_2(\mathbb{Q}_p)$ provides a structure satisfying all the desiderata of (3.2.2). The local-global compatibility between this structure and $\widetilde{H}^1_{\mathfrak{m}}$ (the localization of completed cohomology at a non-Eisenstein maximal ideal of the Hecke algebra) has been proved in [32] (see also [14, 15]) (under a mild hypothesis on the local behaviour of $\overline{\rho}_{\mathfrak{m}}$). In particular, the strategy of (3.1.3) then applies to prove the Fontaine–Mazur–Langlands conjecture for points of $\operatorname{Spec} \mathbb{T}_{\mathfrak{m}}$.

Under the slightly stronger hypotheses that p is odd and $\overline{\rho}_{\mathfrak{m}}$ remains irreducible on restriction to $G_{\mathbb{Q}(\zeta_p)}$ (the *Taylor–Wiles condition*) one can prove that $\mathbb{T}_{\mathfrak{m}} \cong R_{\overline{\rho}_{\mathfrak{m}}}$ [8], [32, Thm. 1.2.3]. Since Conjecture 3.3 holds in this context [32, Cor. 5.3.19], the method of (3.1.3) (applied with $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$) allows us to deduce the density of the crystalline loci in $\operatorname{Spec} R_{\overline{\rho}_{\mathfrak{m}}}$; the extension of this method to be described in [37] will allow us to deduce other density results about various potentially semistable loci. Here is one such simple variant: instead of considering the restriction to $\mathrm{GL}_2(\mathbb{Z}_p)$ of the family V of algebraic representations of GL_2, we instead consider the family of representations $\sigma \otimes V$, where σ is some fixed supercuspidal type and V is algebraic. This allows us to show that potentially crystalline points of (any fixed) supercuspidal type are Zariski dense in $\mathbb{T}_{\mathfrak{m}}$ (and hence in $R_{\overline{\rho}_{\mathfrak{m}}}$, if the Taylor–Wiles condition is satisfied).

3.3.2. A definite quaternion algebra ramified at p. Let D be the quaternion algebra over $\mathbb{Q}$ that is ramified at ∞ and p, and let $\mathbb{G} = D^\times$. Then G_∞/A_∞ is compact, and so, as noted above, Conjecture 3.3 holds. The classical Jacquet–Langlands correspondence allows us to attach odd two-dimensional Galois representations to systems of Hecke eigenvalues. In particular, if $\mathfrak{m}$ is non-Eisenstein, then $\mathbb{T}_{\mathfrak{m}}$ is a quotient of $R_{\overline{\rho}_{\mathfrak{m}}}$.

Assuming that p is odd and $\overline{\rho}_{\mathfrak{m}}$ satisfies the Taylor–Wiles condition, we conclude that in fact $\mathbb{T}_{\mathfrak{m}} \cong R_{\overline{\rho}_{\mathfrak{m}}}$. Indeed, we saw in the preceding example that the potentially crystalline points of supercuspidal type are Zariski dense in $R_{\overline{\rho}_{\mathfrak{m}}}$. Since Fontaine–Mazur–Langlands holds in this context, they all arise from classical modular forms, and hence (by Jacquet–Langlands) from classical automorphic forms on $D^\times$. Thus $\operatorname{Spec} \mathbb{T}_{\mathfrak{m}}$ contains a Zariski dense set of points in $\operatorname{Spec} R_{\overline{\rho}_{\mathfrak{m}}}$, and so the two Specs coincide. As one interesting consequence of this, we note that (assuming that the Taylor–Wiles condition holds) the Hecke algebras at $\mathfrak{m}$ for GL_2 and $D^\times$ are naturally isomorphic (both being isomorphic to $R_{\overline{\rho}_{\mathfrak{m}}}$). One can think of this as a p-adic interpolation of the classical Jacquet–Langlands correspondence. It is interesting to note that even though under the classical Jacquet–Langlands correspondence automorphic forms on GL_2 that are principal series at p don't match with a corresponding Hecke eigenform on $D^\times$, they are not excluded from this p-adic Jacquet–Langlands correspondence.

As another consequence, note that if we choose K_p to be the units in the maximal order of $(D \otimes \mathbb{Q}_p)^\times$, and apply the density argument of (3.1.3), we deduce that the representations which are *genuinely semistable* at p (i.e. semistable, and not crystalline) are Zariski dense in $\operatorname{Spec} \mathbb{T}_{\mathfrak{m}}$, and hence in $\operatorname{Spec} R_{\overline{\rho}_{\mathfrak{m}}}$, provided the Taylor–Wiles condition holds.

3.3.3. Definite quaternion algebras over totally real fields. Consider the case where G is the restriction of scalars to $\mathbb{Q}$ of the units $D^\times$ in a totally definite quaternion algebra D over a totally real field F; and suppose that F is unramified at p, and D is split at every prime in F above p. Put ourselves in the situation of (3.2.1), with the additional assumption that V is the trivial representation, and π_p is a tamely ramified principal series. In this case π_p contains, as a $\mathrm{GL}_2(\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_F)$-subrepresentation, a principal series type σ, which is a representation of $\mathrm{GL}_2(\mathcal{O}_F/p)$ induced from a character of the Borel subgroup. In the paper [16], Breuil gave a conjectural description of the isomorphism class of the lattice $\sigma^\circ := \sigma \cap \widetilde{H}^0_{\mathfrak{m}}$, purely in terms of the restriction to p of ρ°. Under mild assumptions on $\overline{\rho}_{\mathfrak{m}}$, this was proved in [35]. This gives some evidence towards the possibility that $(V^\vee \otimes_L \pi_p)^\circ$ may be of a purely local nature.

3.3.4. Compact unitary groups. In [27], we apply Taylor–Wiles patching to pass from completed cohomology over a unitary group that is compact at infinity to a G-representation on a module M_∞ over the ring R_∞ obtained by adjoining a certain number of formal variables to a local deformation ring. More precisely: for any finite extension K of $\mathbb{Q}_p$, any n such that $p \nmid 2n$, and any representation $\overline{r} : G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ that admits a potentially crystalline lift which is *potentially diagonalizable* (in the sense of [4]), we can (by [33, Cor. A.7]) choose a unitary group $\mathbb{G}$ and Hecke maximal ideal $\mathfrak{m}$ so that $G := \mathbb{G}(\mathbb{Q}_p)$ is isomorphic to a product of copies of $\mathrm{GL}_n(K)$, and such that the restriction of $\overline{\rho}_{\mathfrak{m}}$ to G_K is equal to $\overline{r}$. The G-representation M_∞, which is a module over $R_{\overline{r}}[[x_1, \ldots, x_n]]$ for some some $n \geq 0$, is then obtained by patching the completed cohomology for $\mathbb{G}$.

The module M_∞ satisfies several of the desiderata of (3.2.2): it is finitely generated over $R_\infty[[K_p]]$, and the modules $\operatorname{Hom}^{\mathrm{cont}}_{K_p}(M, W)^d$ (for K_p-invariant lattices W in algebraic representations) are supported on a union of components of the appropriate potentially semistable loci. Many questions about M_∞ remain open, however: whether the support of M_∞ equals the entirety of $\operatorname{Spec} R_\infty$; whether all potentially semistable points are contained in the support of $\operatorname{Hom}^{\mathrm{cont}}_{K_p}(M, W)^d$ (for an appropriate choice of W and K_p); and whether M_∞ is in fact of a purely local nature.

One further property of M_∞ is that it is projective in the category of profinite topological $\mathcal{O}[[K_p]]$-modules. In [37], we hope to show that M_∞ is in fact of full support on $\operatorname{Spec} R_{\overline{r}}[[x_1, \ldots, x_n]]$ in many cases, and thus extend the method described in (3.1.3) to deduce density results for potentially semistable representations in local deformation spaces.

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