FORMAL ALGEBRAIC STACKS

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Abstract. We develop the basic theory of formal algebraic stacks, closely following the development of the theory of formal algebraic spaces in the Stacks Project [Stacks].

Contents

1. Introduction

Our goal in this paper is to introduce some basic definitions and results related to the notion of a formal algebraic stack over a scheme S.

1.1. **Overview.** In the theory of formal schemes, one finds a tension between two points of view. Firstly, a formal scheme admits an open cover by affine formal schemes, and an affine formal scheme may be regarded as a geometric object (the formal spectrum) associated to a certain kind of complete ring. In particular, we can ultimately refer back to these complete rings for defining various local properties of a formal scheme, or for making geometric constructions involving formal schemes. As simple examples: the notion of a locally Noetherian formal scheme rests on having such ring-based models for the local structure of formal schemes, as does the notion of a p-adic formal scheme being flat over $\mathbb{Z}_p$, and the construction of the flat part of a p-adic formal scheme which is not itself flat over $\mathbb{Z}_p$.

On the other hand, formal schemes are also often profitably regarded as certain kinds of Ind-schemes, namely Ind-schemes with respect to transition morphisms that are thickenings (in the sense that they induce isomorphisms on the underlying

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reduced subschemes). This point of view fits well into the functor-of-points perspective, and in practice, formal schemes often arise as Ind-schemes in this way. However, certain properties of formal schemes are difficult to access from this viewpoint: consider the difficulty of deciding whether or not a certain p-adic formal scheme is flat over $\mathbb{Z}_p$, when it is given as a direct limit of schemes, on each of which p is locally nilpotent! (Even less obvious is how to construct the $\mathbb{Z}_p$-flat part of such a formal scheme, when it is described as an Ind-scheme in this way.)

In the Stacks Project [Stacks] (whose overall framework we follow) the theory of formal schemes is incorporated into the more general theory of formal algebraic spaces [Stacks, Tag 0AHW], and the latter theory is developed in a manner that takes into account both of the viewpoints described above. Formal algebraic spaces are always regarded as sheaves on the big *fppf* site of some base scheme S (so the functor-of-points viewpoint is built in to the foundations), and affine formal algebraic spaces are defined as Ind-affine schemes [Stacks, Tag 0AI7] with thickenings as transition morphisms (the inductive limit being formed in the category of *fppf* sheaves), while general formal algebraic spaces are defined as sheaves that admit an étale cover by a disjoint union of affine formal algebraic spaces [Stacks, Tag 0AIL].

However, under mild hypotheses (quasi-compactness and quasi-separatedness), one can show that a formal algebraic space admits a description as an Ind-algebraic space with thickenings as transition morphisms [Stacks, Tag 0AJE] (see also [Yas09, Prop. 3.32], where this result is originally proved, although in a somewhat different foundational set-up). Conversely, such an Ind-algebraic space *is* in fact a formal algebraic space [Stacks, Tag 0AIU]. This latter result, while not difficult to prove (the key input is the invariance of the étale site with respect to thickenings), is crucial. It shows that the Ind-object under consideration may be described (étale locally) in terms of certain complete rings, which, as already noted above, provides a key input to many of the basic definitions and constructions in the theory of formal schemes and formal algebraic spaces.

In this paper we develop the theory of formal algebraic stacks along similar lines to the Stacks Project's development of the theory of formal algebraic spaces. In particular, while our definition of a formal algebraic stack in Section 5 will be in terms of having local models that are affine formal algebraic spaces (local now in the smooth topology), in Section 6 we show that (under the hypotheses of being quasi-compact and quasi-separated) they admit a description as Ind-algebraic stacks, and we also show that certain Ind-algebraic stacks *are* formal algebraic stacks. (The main obstacle to be overcome in the context of stacks is that the smooth site is not invariant under thickenings, so that the construction of smooth local affine formal models for the Ind-algebraic stacks under consideration is more involved in general than in the case of formal algebraic spaces.)

In Section 7 we define the notion of adic morphisms, which provides the relative version of the notion of an adic formal algebraic stack. (For example, if A is a Noetherian ring complete with respect to an ideal I, then an I-adic formal algebraic stack consists of a formal algebraic stack equipped with an adic morphism to Spf A.) In Section 8 we develop a general framework for defining various properties of formal algebraic stacks in terms of properties of affine formal algebraic spaces, and then apply it to define a number of such properties.

The remaining sections are somewhat more specialized to the applications that we have in mind. In Section 9, we develop a framework for defining closed substacks of formal algebraic stacks in a functorial manner. For example, this allows us to define the flat part of a locally Noetherian formal algebraic stack over $\operatorname{Spf} \mathbb{Z}_p$. In Section 10, we prove a technical result which, in applications, will allow us to prove that, in certain situations, the "scheme-theoretic image" of a p-adic formal algebraic stack is again a p-adic formal algebraic stack. Finally, in Section 11, we provide a criterion (phrased in terms of the existence of a certain obstruction theory) for a formal algebraic stack over a locally Noetherian base scheme to itself be locally Noetherian.

The earlier sections develop necessary preliminary material. Section 2 gives a rapid development of the theory of colimits in 2-categories, which forms the technical basis for the theory of Ind-algebraic stacks. We follow the very concrete view-point on 2-categories adopted in [Stacks], and also in [GR04]. We recommend that the reader skip or at best skim this section, and rely instead on their intuition regarding the formation of Ind-stacks. (We also note that in this draft, some arguments in Section 2 are incomplete or missing. We hope to fill them in eventually.)

Section 3 discusses several other stack-theoretic notions that will be required in our later development. The emphasis is on extending various geometric properties and constructions (properties of morphisms, underlying reduced substacks, topological invariance of the small étale site) from the context of algebraic stacks to more general contexts, so that we may apply them when studying formal algebraic stacks or Ind-algebraic stacks. Ind-algebraic stacks themselves are formally defined in Section 4.

We anticipate that much of the material in this paper will ultimately be incorporated into the Stacks Project, and indeed, some of it already has been, sometimes undergoing improvements in the process (e.g. Lemma 6.4 below).

1.2. **Conventions.** We will work in the overall framework of the Stacks Project [Stacks], and so will work with stacks on the *fppf* site of S, although we could equally well have worked with stacks on the étale site.

Throughout the paper, all stacks under consideration will be stacks in groupoids, and so we usually write simply *stack* rather than the more cumbersome *stack in groupoids*. (We make an exception in the foundational Section 2, where we will write *stacks in groupoids*, just to make the context of our discussion clear.)

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2. 2-categorical background

In this section, we recall and develop various 2-categorical notions that we will require in the sequel. The most important of these is the notion of 2-colimits in a 2-category, which we will use in Section 4 to define the notion of Ind-algebraic stacks. We also recall the notion of a strict pseudo-functor between 2-categories, which will be useful for describing the functorial properties of certain constructions we make later on (such as underlying reduced substacks), as well as some related notions.

The discussion that follows would perhaps be better expressed in a more flexible higher categorical language. However, we follow the more concrete approach of the stacks project [Stacks] (which is also the approach adopted in [GR04]) by working explicitly with 2-categories in the sense defined there; this explicit approach also accords well with our own mathematical limitations.

We first remind the reader that the notion of 2-*category* is defined in [Stacks, Tag 003H] (the notion defined there is also sometimes referred to as a *strict* 2-category), while the notion of *pseudo-functor* from a category to a 2-category is defined in [Stacks, Tag 003N] (and see also Definition 2.1 below).

The main examples of 2-categories that we will consider are the 2-category of all categories lying over some base category $\mathcal{D}$, the 2-category of categories fibred over $\mathcal{D}$, the 2-category of categories fibred in groupoids over $\mathcal{D}$, and (assuming that $\mathcal{D}$ is equipped with the structure of a site) the 2-category of stacks in groupoids over $\mathcal{D}$.

As already remarked, our first goal is to define the notion of a 2-diagram in a 2-category, and the related notion of 2-colimit. In fact, just as a diagram in a category $\mathcal{C}$ indexed by a category $\mathcal{I}$ is simply a functor from $\mathcal{I}$ to $\mathcal{C}$, a 2-diagram is in a 2-category $\mathcal{C}$ indexed by a category $\mathcal{I}$ is simply a pseudo-functor from $\mathcal{I}$ to $\mathcal{C}$. We formalize this in the following definition (which may then be regarded simply as a recollection of the notion of pseudo-functor).

Definition 2.1. If $\mathcal{I}$ is a category and $\mathcal{C}$ is a 2-category, then a 2-*diagram in* $\mathcal{C}$ *indexed by* $\mathcal{I}$ is a pseudo-functor (φ, α) from $\mathcal{I}$ to $\mathcal{C}$. Explicitly, then, such a 2-diagram consists of the following data:

(1) For each $x \in \mathcal{I}$, an object $\varphi(x)$ of $\mathcal{C}$.
(2) For each morphism $f : x \to y$ in $\mathcal{I}$, a 1-morphism $\varphi(f) : \varphi(x) \to \varphi(y)$ in $\mathcal{C}$.
(3) For every object x of $\mathcal{I}$, a 2-morphism $\alpha_x : \mathrm{id}_{\varphi(x)} \to \varphi(\mathrm{id}_x)$ in $\mathcal{C}$.
(4) For every composable pair of morphisms $f : x \to y$, $g : y \to z$ in $\mathcal{I}$, a 2-morphism $\alpha_{g,f} : \varphi(g \circ f) \to \varphi(g) \circ \varphi(f)$ in $\mathcal{C}$.

This data is required to satisfy the following conditions:

(4) The various 2-morphisms α_x and $\alpha_{g,f}$ are isomorphisms.
(5) For any morphism $f : x \to y$ in $\mathcal{I}$, we have
$$\alpha_{\mathrm{id}_y, f} = \alpha_y \star \mathrm{id}_{\varphi(f)}$$
and
$$\alpha_{f, \mathrm{id}_x} = \mathrm{id}_{\varphi(f)} \star \alpha_x.$$
(6) For any composable triple $f : w \to x$, $g : x \to y$, $h : y \to z$ of morphisms in $\mathcal{I}$, we have
$$(\mathrm{id}_{\varphi(h)} \star \alpha_{g,f}) \circ \alpha_{h, g\circ f} = (\alpha_{h,g} \star \mathrm{id}_{\varphi(f)}) \circ \alpha_{h\circ g, f}.$$

Remark 2.2. In this section we typically denote a 2-diagram in $\mathcal{C}$ indexed by $\mathcal{I}$ by the pair (φ, α); this is understood to be short-hand for the data described in the preceding definition. In later sections (when we apply the general theory here in the case of Ind-algebraic stacks), the indexing category $\mathcal{I}$ will typically be a partially ordered set, the 2-category $\mathcal{C}$ will be the category of stacks in groupoids over some site, and we will often denote a 2-diagram in this category simply by $\{\mathcal{X}_i\}$, where i runs over the elements of $\mathcal{I}$. In the more precise notation introduced above, we have $\mathcal{X}_i := \varphi(i)$, while the 2-morphisms α are left implicit (and are presumed to be evident from a consideration of the situation at hand).

Remark 2.3. For the brave reader who attempts to compare our discussion with that in [GR04], we note the the direction of the coherence constraint $\alpha_{g,f}$ in Definition 2.1 (which follows the conventions of [Stacks]) is opposite to that of [GR04, Def. 2.1.14]. Since this coherence constraint is required to be an isomorphism, this doesn't have any affect on the subsequent development of the theory, but should be taken into account when comparing formulas here or in [Stacks] to those of [GR04].

A key fact is that the 2-diagrams indexed by $\mathcal{I}$ with values in $\mathcal{C}$ may naturally be regarded as the objects of a 2-category (see e.g. [GR04, Def. 2.2.12 (ii)]). The next several definitions and remarks explicate this additional structure.

We begin by defining the notion of a 1-morphism between two 2-diagrams.

Definition 2.4. Let $\mathcal{I}$ be a category, let $\mathcal{C}$ be a 2-category, and let (φ_1, α_1) and (φ_2, α_2) be 2-diagrams indexed by $\mathcal{I}$ with values in $\mathcal{C}$.

A 1-*morphism* (F, ψ) *from* (φ_1, α_1) *to* (φ_2, α_2) consists of the following data:

(1) For each object x of $\mathcal{I}$, a 1-morphism $F_x : \varphi_1(x) \to \varphi_2(x)$ in $\mathcal{C}$.
(2) For each morphism $f : x \to y$ in I, a 2-morphism $\psi_f : \varphi_2(f) \circ F_x \to F_y \circ \varphi_1(f)$ in $\mathcal{C}$.

This data is required to satisfy the following conditions:

(3) The various 2-morphisms ψ_f are isomorphisms.
(4) For any object x of $\mathcal{I}$, the two 2-morphisms

$$\mathrm{id}_{F_x} \star \alpha_{1,x} : F_x = F_x \circ \mathrm{id}_{\varphi_1(x)} \to F_x \circ \varphi_1(\mathrm{id}_x)$$

and

$$\psi_{\mathrm{id}_x} \circ (\alpha_{2,x} \star \mathrm{id}_{F_x}) : F_x = \mathrm{id}_{\varphi_2(x)} \circ F_x \to \varphi_2(\mathrm{id}_x) \circ F_x \to F_x \circ \varphi_1(x)$$

coincide.

(5) For any pair $f : x \to y$, $g : y \to z$ of composable morphisms in $\mathcal{I}$, the 2-morphisms

$$(\mathrm{id}_{F_z} \star \alpha_{1,g,f}) \circ \psi_{g \circ f} : \varphi_2(g \circ f) \circ F_x \to F_z \circ \varphi_1(g \circ f) \to F_z \circ \varphi_1(g) \circ \varphi_1(f)$$

and

$$\begin{aligned}(\psi_g \star \mathrm{id}_{\varphi_1(f)}) \circ (\mathrm{id}_{\varphi_2(g)} \star \psi_f) \circ (\alpha_{2,g,f} \star \mathrm{id}_{F_x}) : \varphi_2(g \circ f) \circ F_x &\to \varphi_2(g) \circ \varphi_2(f) \circ F_x \\ &\to \varphi_2(g) \circ F_y \circ \varphi_1(f) \to F_z \circ \varphi_1(g) \circ \varphi_1(f)\end{aligned}$$

coincide.

Remark 2.5. A 1-morphism from (φ_1, α_1) to (φ_2, α_2), as in Definition 2.4, is the same as a *pseudo-natural transformation* (in the sense of [GR04, Def. 2.2.1]) between the pseudo-functors (φ_1, α) and (φ_2, α_2).

Definition 2.6. Suppose that $\mathcal{I}$ is a category and $\mathcal{C}$ is a 2-category, that (φ_1, α_1), (φ_2, α_2), and (φ_3, α_3) are 2-diagrams indexed by $\mathcal{I}$ with values in $\mathcal{C}$, that $(F_1, \psi_1) : (\varphi_1, \alpha_1)$ and $(F_2, \psi_2) : (\varphi_2, \alpha_2) \to (\varphi_3, \alpha)_3$ are 1-morphisms of 2-diagrams, in the sense of Definition 2.4. Then we define the composite 1-morphism

$$(F_3, \psi_3) := (F_2, \psi_2) \circ (F_1, \psi_1) : (\varphi_1, \alpha_1) \to (\varphi_3, \alpha_3)$$

as follows:

For an object x of $\mathcal{I}$, the 1-morphism $F_{3,x}$ is defined via

$$F_{3,x} := F_{2,x} \circ F_{1,x} : \varphi_1(x) \to \varphi_3(x),$$

while for a morphism $f : x \to y$ of $\mathcal{I}$, the 2-morphism $\psi_{3,f}$ is defined via

$$\begin{aligned}\psi_{3,f} := (\psi_{2,f} \star \mathrm{id}_{F_{1,x}}) \circ (\mathrm{id}_{F_{2,y}} \star \psi_{1,2}) : \varphi_3(x) \circ F_{3,x} = \varphi_3(x) \circ F_{2,x} \circ F_{1,x} \\ \to F_{2,y} \circ \varphi_2(f) \circ F_{1,x} \to F_{2,y} \circ F_{1,y} \circ \varphi_1(f) = F_{3,y} \circ \varphi_1(f).\end{aligned}$$

We leave it to the reader to check that, defined in this manner, the composite is indeed a 1-morphism of 2-diagrams. (See also [GR04, Rem. 2.2.5 (iii)].)

We next define the notion of a 2-morphism between two 1-morphisms of 2-diagrams.

Definition 2.7. If (φ_1, α_1) and (φ_2, α_2) are two 2-diagrams in the 2-category $\mathcal{C}$ indexed by the category $\mathcal{I}$, and if (F_1, ψ_1) and (F_2, ψ_2) are two 1-morphisms from (φ_1, α_1) to (φ_2, α_2), then a 2-morphism ξ from (F_1, ψ_1) to (F_2, ψ_2) consists of a 2-morphism $\xi_x : F_{1,x} \to F_{2,x}$ for each object x of $\mathcal{I}$, such that for each morphism $f : x \to y$ in $\mathcal{I}$, we have

$$(\xi_y \star \mathrm{id}_{\varphi_1(f)}) \circ \psi_{1,f} = \psi_{2,f} \circ (\mathrm{id}_{\varphi_2(f)} \star \xi_x).$$

Remark 2.8. If we interpret 1-morphisms from (φ_1, α_1) to (φ_2, α_2) as pseudo-natural transformations, as in Remark 2.5, then we may interpret 2-morphisms between 1-morphisms as *modifications* between those pseudo-natural transformations, in the sense of [GR04, Def. 2.2.9].

We now define the various forms of composition for 2-morphisms between 1-morphisms of diagrams.

Definition 2.9. (1) If (φ_1, α_1) and (φ_2, α_2) are two 2-diagrams in the 2-category $\mathcal{C}$ indexed by the category $\mathcal{I}$, if (F_1, ψ_1), (F_2, ψ_2), and (F_3, ψ_3) are three 1-morphisms from (φ_1, α_1) to (φ_2, α_2), and if ξ_1 and ξ_2 are 2-morphisms from from (F_1, ψ_1) to (F_2, ψ_2) and from (F_2, ψ_2) to (F_3, ψ_3) respectively, then we define the composite $\xi_3 := \xi_2 \circ \xi_1$ from (F_1, ψ_1) to (F_3, ψ_3) via

$$\xi_{3,x} := \xi_{2,x} \circ \xi_{1,x} : F_{1,x} \to F_{2,x} \to F_{3,x},$$

for each object x of $\mathcal{I}$. (The reader may easily check that ξ_3 satisfies the requirements to be a 2-morphism.)

(2) If (φ_1, α_1), (φ_2, α_2), and (φ_3, α_3) are three 2-diagrams in the 2-category $\mathcal{C}$ indexed by the category $\mathcal{I}$, if (F_1, ψ_1) and (G_1, η_1) are two 1-morphisms from (φ_1, α_1) to (φ_2, α_2), if (F_2, ψ_2) and (G_2, η_2) are two 1-morphisms from (φ_2, α_2) to (φ_3, α_3), if ξ_1 is a 2-morphism from (F_1, ψ_1) to (G_1, η_1), and if ξ_2 is a 2-morphism from (F_2, ψ_2) to (G_2, η_2), then we define $\xi_3 := \xi_2 \star \xi_1$ to be the 2-morphism from $(F_2, \psi_2) \circ (F_1, \psi_1)$ to $(G_2, \eta_2) \circ (G_1, \eta_1)$ defined via

$$\xi_{3,x} := \xi_{2,x} \star \xi_{1,x}$$

for each object $x \in \mathcal{I}$.

We can now explain how the 2-diagrams indexed by the category $\mathcal{I}$, with values in the 2-category $\mathcal{C}$, themselves form a 2-category.

Definition 2.10. Fix a category $\mathcal{I}$ and a 2-category $\mathcal{C}$.

(1) If (φ_1, α_1) and (φ_2, α_2) are two 2-diagrams indexed by $\mathcal{I}$ with values in $\mathcal{C}$, then we let $\mathrm{Mor}_{\mathcal{C}}\big((\varphi_1, \alpha_1), (\varphi_2, \alpha_2)\big)$ denote the category whose objects are the 1-morphisms from (φ_1, α_1) to (φ_2, α_2), and whose morphisms are the 2-morphisms between 1-morphisms. The composition of morphisms is given by Definition 2.9 (1). (The reader may easily verify that with composition defined in this manner, we do indeed obtain a category.)

(2) If (φ_1, α_1), (φ_2, α_2), and (φ_3, α_3) are three 2-diagrams in the 2-category $\mathcal{C}$ indexed by the category $\mathcal{I}$, then we define the composition bifunctor

$$\mathrm{Mor}_{\mathcal{C}}\big((\varphi_1, \alpha_1), (\varphi_2, \alpha_2)\big) \times \mathrm{Mor}_{\mathcal{C}}\big((\varphi_2, \alpha_3), (\varphi_3, \alpha_3)\big) \to \mathrm{Mor}_{\mathcal{C}}\big((\varphi_1, \alpha_1), (\varphi_3, \alpha_3)\big)$$

via Definition 2.6 (for objects) and Definition 2.9 (2) (for morphisms). (The reader may verify that with these definitions we do obtain a bifunctor.)

(3) We let $2\text{-}\mathrm{Diag}(\mathcal{I}, \mathcal{C})$ denote the 2-category whose objects are the 2-diagrams indexed by $\mathcal{I}$ with values in $\mathcal{C}$, whose morphisms categories are defined as in part (1), and whose composition bifunctors are defined as in part (2). (The reader may verify that with these definitions, we do indeed obtain a 2-category.)

Remark 2.11. The reader may immediately verify that in the context of Definition 2.10, if $\mathcal{C}$ is a $(2,1)$-category, then the same is true of $2\text{-}\mathrm{Diag}(\mathcal{I}, \mathcal{C})$.

Remark 2.12. The 2-category structure on $2\text{-}\mathrm{Diag}(\mathcal{I}, \mathcal{C})$ is one aspect of the 3-category structure on the collection of all 2-categories, which we don't discuss here. (See also [GR04, Rem. 2.2.5 (vi)].)

We wish to explain how the 2-categories of 2-diagrams behave with respect to a change of indexing category. Since these are genuine 2-categories, we will require an appropriate notion of pseudo-functor between 2-categories, neither of which a 1-category. The stacks project doesn't discuss the general notion of a pseudo-functor between 2-categories (the reader can consult [GR04] for a discussion of this notion in the same spirit as that of the Stacks Project), and we won't require this notion either. However, we will require the following very special case of the general definition. (See e.g. [GR04, Def. 2.1.14 (ii)].)

Definition 2.13. If $\mathcal{C}$ and $\mathcal{C}'$ are both 2-categories, then a *strict pseudo-functor* $F : \mathcal{C} \to \mathcal{C}'$ consists of a function mapping the set of objects of $\mathcal{C}$ to the set of objects of $\mathcal{C}'$ (denoted $\mathcal{X} \mapsto F(\mathcal{X})$), and for each pair of objects $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{C}$, a functor $F_{\mathcal{X},\mathcal{Y}} : \mathrm{Mor}_{\mathcal{C}}\big(\mathcal{X}, \mathcal{Y}\big) \to \mathrm{Mor}_{\mathcal{C}}\big(F(\mathcal{X}), F(\mathcal{Y})\big)$, which takes the unit object to the unit object in the case $\mathcal{X} = \mathcal{Y}$, and such that these functors are compatible with composition in $\mathcal{C}$ and $\mathcal{C}'$.

Remark 2.14. The *strictness* in the preceding definition is the requirement that the functor F respects unit objects and composition *precisely*, rather than up to various 2-isomorphisms in $\mathcal{C}'$. Again, we refer to [GR04] for the general definition of a pseudo-functor (when the source as well as the target is allowed to be a general 2-category, rather than just a 1-category, so that [Stacks, Tag 003N] no longer applies).

Example 2.15. The 2-fibre product provides an example of a pseudo-functor between two 2-categories that is not strict in general. More precisely, if $\mathcal{C}$ is a 2-category that admits 2-fibre products, if $\mathcal{Y}$ is an object of $\mathcal{C}$, and if $\mathcal{Z} \to \mathcal{Y}$ is a 1-morphism in $\mathcal{C}$, then the map which on objects is defined via $\mathcal{X} \mapsto \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ extends to a pseudo-functor $\mathcal{C}/\mathcal{Y} \to \mathcal{C}/\mathcal{Z}$ which is not strict in general (and hence is outside the scope of our discussion). In Remark 3.25, however, we consider a particular case in which the 2-fibre product can be constructed in such a way that this pseudo-functor *does* become strict.

For later use, we also recall what it means for a strict pseudo-functor to be fully faithful, or to be a 2-equivalence. (The definitions actually make sense for a general pseudo-functor; see e.g. [GR04, Def. 2.4.9 (ii), (iv)].)

Definition 2.16. Let $F : \mathcal{C} \to \mathcal{C}'$ be a strict pseudo-functor between a pair of two categories.

(1) We say that F is *fully faithful* (resp. *strongly faithful*) if for each pair of objects $\mathcal{X}, \mathcal{Y} \in \mathcal{C}$, the functor $F_{\mathcal{X},\mathcal{Y}}$ is an equivalence of categories (resp. an isomorphism of categories).

(2) We say that F is a 2-*equivalence* (resp. a *strong* 2-*equivalence*) is F is fully faithful (resp. strongly faithful) and if each object of $\mathcal{C}'$ is equivalent (resp. 1-isomorphic) to an object in the image of F.

Remark 2.17. One can define the notion of strict pseudo-natural transformation between strict pseudo-functors in a fairly evident sense (see e.g. [GR04, Def. 2.2.1 (ii)]), and one may form a 2-category whose objects are all 2-categories,[1] whose 1-morphisms are the strict pseudo-functor, and whose 2-morphisms are the strict pseudo-natural transformations. One can then show that a strict pseudo-functor $F : \mathcal{C} \to \mathcal{C}'$ is a strong 2-equivalence if and only if it is an equivalence in this 2-category; in particular, we may find a strict pseudo-functor $G : \mathcal{C}' \to \mathcal{C}$ which is quasi-inverse to F in a fairly evident sense. (The reader can see [GR04, Rem. 2.4.29] for more details.)

If a strict pseudo-functor $F : \mathcal{C} \to \mathcal{C}'$ is a 2-equivalence, then it also admits a certain kind of quasi-inverse pseudo-functor G; see [GR04, Cor. 2.4.30]. However, the pseudo-functor G will not be strict in general (and the expression of the quasi-inverse property involves the notion of modification between pseudo-natural transformations), and so we don't discuss this point any further here.

The next definition makes precise the sense in which we can precompose a 2-diagram with a functor between index categories.

Definition 2.18. Let $\mathcal{C}$ be a 2-category, and let $G : \widetilde{\mathcal{I}} \to \mathcal{I}$ be a functor between categories. We define a strict pseudo-functor (in the sense of Definition 2.13)

$$G^* : 2\text{-}\mathrm{Diag}(\mathcal{I}, \mathcal{C}) \to 2\text{-}\mathrm{Diag}(\widetilde{\mathcal{I}}, \mathcal{C})$$

as follows:

(1) If (φ, α) is a 2-diagram with values in the 2-category $\mathcal{C}$, indexed by the category $\mathcal{I}$, then we let $G^*(\varphi, \alpha) \circ G$ denote the 2-diagram $(\tilde{\varphi}, \tilde{\alpha})$ with values in $\mathcal{C}$, indexed by $\widetilde{\mathcal{I}}$, whose values on objects $\tilde{x}$ of $\widetilde{\mathcal{I}}$ are defined by $\tilde{\varphi}(\tilde{x}) := \varphi\big(G(\tilde{x})\big)$, whose values on morphisms $\tilde{f} : \tilde{x} \to \tilde{y}$ in $\widetilde{\mathcal{I}}$ are defined by $\tilde{\varphi}(\tilde{f}) := \varphi\big(G(\tilde{f})\big)$, for which the natural transformations $\tilde{\alpha}_{\tilde{x}}$, for an object $\tilde{x}$ of $\widetilde{\mathcal{I}}$, are defined by $\tilde{\alpha}_{\tilde{x}} := \alpha_{G(\tilde{x})}$,

[1] We ignore set-theoretic issues here.

and for which the natural transformations $\tilde{\alpha}_{\tilde{f},\tilde{g}}$, for composable pairs of morphisms $(\tilde{f},\tilde{g})$ in $\widetilde{\mathcal{I}}$, are defined by $\tilde{\alpha}_{\tilde{g},\tilde{f}} := \alpha_{G(g),G(g)}$. One immediately verifies (using the functorial properties of G) that $(\tilde{\varphi},\tilde{\alpha})$ is a 2-diagram.

(2) If (F,ψ) is a 1-morphism between the objects (φ_1,α_1) and (φ_2,α_2) of $2\text{-}\mathrm{Diag}(\mathcal{I},\mathcal{C})$, then we let $G^*(F,\psi)$ denote the 1-morphism $(\widetilde{F},\tilde{\psi})$ from $G^*(\varphi_1,\alpha_1)$ to $G^*(\varphi_2,\alpha_2)$, whose values on objects $\tilde{x}$ of $\widetilde{\mathcal{I}}$ are defined by $\widetilde{F}_{\tilde{x}} = F_{G(\tilde{x})}$, and whose values on morphisms $\tilde{f}:\tilde{x}\to\tilde{y}$ in $\widetilde{\mathcal{I}}$ are defined by $\tilde{\psi}_{\tilde{f}} := \psi_{G(\tilde{f})}$. One immediately verifies (using the functorial properties of G) that $(\widetilde{F},\tilde{\psi})$ is a 1-morphism.

(3) If (φ_1,α_1) and (φ_2,α_2) are objects of $2\text{-}\mathrm{Diag}(\mathcal{I},\mathcal{C})$, if (F_1,ψ_1) and (F_2,ψ_2) are two 1-morphisms between these objects, and if ξ is a 2-morphism between these 1-morphisms, then we define the 2-morphism $G^*\xi$ between $G^*(F_1,\psi_1)$ and $G^*(F_2,\psi_2)$ to be the 2-morphism $\tilde{\xi}$ whose values on objects $\tilde{x}$ of $\widetilde{\mathcal{I}}$ are defined by $\tilde{\xi}_{\tilde{x}} := \xi_{G(\tilde{x})}$. One immediately verifies (using the functorial properties of G) that $\tilde{\xi}$ is a 2-morphism.

The reader may similarly verify that the above definitions, taken together, define a strict pseudo-functor $G^* : 2\text{-}\mathrm{Diag}(\mathcal{I},\mathcal{C}) \to 2\text{-}\mathrm{Diag}(\widetilde{\mathcal{I}},\mathcal{C})$.

Our aim, now, is to define the notion of a 2-colimit of a 2-diagram. In order to do this, we have to define the notation of a *constant* 2-diagram. Before doing this, we recall the notion of a sub-2-category of a 2-category, as well as some related notions. (See e.g. [Stacks, Tag 02X7] and [GR04, Def. 2.4.9 (iii)].)

Definition 2.19. If $\mathcal{C}$ is a 2-category, then a sub-2-category $\mathcal{C}'$ of $\mathcal{C}$ consists of a subset of the set of objects of $\mathcal{C}$, and for each pair of objects $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{C}$, a subcategory $\mathrm{Mor}_{\mathcal{C}'}(\mathcal{X},\mathcal{Y})$ of $\mathrm{Mor}_{\mathcal{C}}(\mathcal{X},\mathcal{Y})$, which contains $\mathrm{id}_{\mathcal{X}}$ when $\mathcal{X}$ and $\mathcal{Y}$ coincide, and such that composition of morphisms in $\mathcal{C}$ restricts to induce a composition of morphisms in $\mathcal{C}'$.

We say that $\mathcal{C}'$ is a *full* sub-2-category if the inclusion $\mathrm{Mor}_{\mathcal{C}'}(\mathcal{X},\mathcal{Y}) \hookrightarrow \mathrm{Mor}_{\mathcal{C}}(\mathcal{X},\mathcal{Y})$ is an equivalence of categories, for each pair of objects $\mathcal{X},\mathcal{Y}$ of $\mathcal{C}'$. We say that $\mathcal{C}'$ is a *strong* sub-2-category of $\mathcal{C}$ if $\mathrm{Mor}_{\mathcal{C}'}(\mathcal{X},\mathcal{Y}) = \mathrm{Mor}_{\mathcal{C}}(\mathcal{X},\mathcal{Y})$ for each pair of objects $\mathcal{X},\mathcal{Y}$ of $\mathcal{C}'$. (Thus, choosing a strong sub-2-category of $\mathcal{C}$ amounts to choosing a subset of the set of objects of $\mathcal{C}$.)

Remark 2.20. Evidently, if $\mathcal{C}$ is a $(2,1)$-category, so is any full sub-2-category of $\mathcal{C}$.

Example 2.21. The image of a fully faithful strict pseudo-functor is a full sub-2-category of its target, while the image of a strongly faithful strict pseudo-functor is a strong sub-2-category of its target.

We now explain how the 2-category $\mathcal{C}$ may be embedded as a sub-2-category of $2\text{-}\mathrm{Diag}(\mathcal{I},\mathcal{C})$ (for any index category $\mathcal{I}$).

Definition 2.22. If $\mathcal{C}$ is a 2-category and $\mathcal{I}$ is an index category, then for any object $\mathcal{X}$ of $\mathcal{C}$, we define the *constant 2-diagram* $\Phi_{\mathcal{X}}$ indexed by $\mathcal{I}$ to be the object (φ,α) of $2\text{-}\mathrm{Diag}(\mathcal{I},\mathcal{C})$ defined by setting $\varphi(x) := \mathcal{X}$ for every object x of $\mathcal{I}$, $\varphi(f) = \mathrm{id}_{\mathcal{X}}$ for every morphism f of $\mathcal{I}$, and by defining all the 2-morphisms α to be $\mathrm{id}_{\mathrm{id}_{\mathcal{X}}}$.

Remark 2.23. In the context of Definition 2.22, let $\mathcal{I}^{\mathrm{gp}}$ denote the *groupoid completion* of $\mathcal{I}$, i.e. the localization of $\mathcal{I}$ at all its morphisms. The reader may easily verify that if $\mathcal{X}$ and $\mathcal{Y}$ are two objects of $\mathcal{C}$, then $\mathrm{Mor}_{\mathcal{C}}(\Phi_{\mathcal{X}},\Phi_{\mathcal{Y}})$ is naturally isomorphic to the functor category $\mathrm{Fun}\big(\mathcal{I}^{\mathrm{gp}},\mathrm{Mor}_{\mathcal{C}}(\mathcal{X},\mathcal{Y})\big)$ (whose objects are functors from $\mathcal{I}^{\mathrm{gp}}$

to $\mathrm{Mor}_{\mathcal{C}}(\mathcal{X},\mathcal{Y})$, and whose morphisms are natural transformations between these functors).

In particular, by mapping each object of $\mathrm{Mor}_{\mathcal{C}}(\mathcal{X},\mathcal{Y})$ to the corresponding constant functor (and mapping morphisms to the corresponding natural transformation), we obtain an inclusion of categories $\mathrm{Mor}_{\mathcal{C}}(\mathcal{X},\mathcal{Y}) \subseteq \mathrm{Mor}_{\mathcal{C}}(\Phi_{\mathcal{X}},\Phi_{\mathcal{Y}})$. In this way we regard $\mathcal{C}$ as a sub-2-category of $2\text{-}\mathrm{Diag}(\mathcal{I},\mathcal{C})$.

If $F:\mathcal{X}\to\mathcal{Y}$ is a 1-morphism in $\mathcal{C}$, we will let $\Phi_F:\Phi_{\mathcal{X}}\to\Phi_{\mathcal{Y}}$ denote the corresponding constant 1-morphism in $2\text{-}\mathrm{Diag}(\mathcal{I},\mathcal{C})$, and we will use analogous notation for constant 2-morphisms, if necessary.

Because of its importance in the definition of 2-colimits, we spell out the notion of 1-morphism of 2-diagrams in the particular case when the target is a constant 2-diagram.

Definition 2.24. If (φ,α) is a 2-diagram in the 2-category $\mathcal{C}$, indexed by the category $\mathcal{I}$, and if $\mathcal{X}$ is an object of $\mathcal{C}$, then a *1-morphism (F,ψ) from (φ,α) to the constant 2-diagram $\Phi_{\mathcal{X}}$* consists of the the following data:

(1) For each object x of $\mathcal{I}$, a 1-morphism $F_x:\varphi(x)\to\mathcal{X}$ in $\mathcal{C}$.
(2) For each morphism $f:x\to y$ in I, a 2-morphism $\psi_f:F_x\to F_y\circ\varphi(f)$ in $\mathcal{C}$.

This data is required to satisfy the following conditions:

(3) The various 2-morphisms ψ_f are isomorphisms.
(4) For any object x of $\mathcal{I}$, we have $\psi_{\mathrm{id}_x}=\mathrm{id}_{F_x}\star\alpha_x$.
(5) For any pair $f:x\to y$, $g:y\to z$ of composable morphisms in $\mathcal{I}$, we have

$$(\mathrm{id}_{F_z}\star\alpha_{g,f})\circ\psi_{g\circ f}=(\psi_g\star\mathrm{id}_{\varphi(f)})\circ\psi_f.$$

We similarly spell out the definition of 2-morphisms between 1-morphisms in this special case.

Definition 2.25. If (φ,α) is a 2-diagram in the 2-category $\mathcal{C}$ indexed by the category $\mathcal{I}$, if $\mathcal{X}$ is an object of $\mathcal{C}$, and if (F_1,ψ_1) and (F_2,ψ_2) are two 1-morphisms from (φ,α) to $\Phi_{\mathcal{X}}$, then a 2-morphism ξ from (F_1,ψ_1) to (F_2,ψ_2) consists of a 2-morphism $\xi_x:F_{1,x}\to F_{2,x}$ for each object x of $\mathcal{I}$, and such that for each morphism $f:x\to y$ in $\mathcal{I}$, we have

$$(\xi_y\star\mathrm{id}_{\varphi(f)})\circ\psi_{1,f}=\psi_{2,f}\circ\xi_x.$$

We may now define the notion of 2-colimit (see e.g. [GR04, Def. 2.5.1], but note that they work in the more general context in which $\mathcal{I}$ is also allowed to be a 2-category). To do this, we note that if (φ,α) is an object of $2\text{-}\mathrm{Diag}(\mathcal{I},\mathcal{C})$, if $\mathcal{X}$ and $\mathcal{Y}$ are two objects of $\mathcal{C}$, and if (F,ψ) is an object of $\mathrm{Mor}_{\mathcal{C}}\big((\varphi,\alpha),\Phi_{\mathcal{X}}\big)$, then by evaluating the composition bifunctor

$$\mathrm{Mor}_{\mathcal{C}}\big((\varphi,\alpha),\Phi_{\mathcal{X}}\big)\times\mathrm{Mor}_{\mathcal{C}}(\Phi_{\mathcal{X}},\Phi_{\mathcal{Y}})\to\mathrm{Mor}_{\mathcal{C}}\big((\varphi,\alpha),\mathcal{Y}\big)$$

of $2\text{-}\mathrm{Diag}(\mathcal{I},\mathcal{C})$ on (F,ψ), we obtain a functor $\mathrm{Mor}_{\mathcal{C}}(\Phi_{\mathcal{X}},\Phi_{\mathcal{Y}})\to\mathrm{Mor}_{\mathcal{C}}\big((\varphi,\alpha),\Phi_{\mathcal{Y}}\big)$, which we may restrict to the constant subcategory $\mathrm{Mor}_{\mathcal{C}}(\mathcal{X},\mathcal{Y})$ of $\mathrm{Mor}_{\mathcal{C}}(\Phi_{\mathcal{X}},\Phi_{\mathcal{Y}})$ (see Remark 2.23) so as to obtain a functor

$$\mathrm{Mor}_{\mathcal{C}}(\mathcal{X},\mathcal{Y})\to\mathrm{Mor}_{\mathcal{C}}\big((\varphi,\alpha),\mathcal{Y}\big). \tag{2.26}$$

Definition 2.27. If (φ,α) is a 2-diagram in the 2-category $\mathcal{C}$, then we say that a 1-morphism $(F,\psi):(\varphi,\alpha)\to\Phi_{\mathcal{X}}$, for some object $\mathcal{X}$ of $\mathcal{C}$, realizes $\mathcal{X}$ as a *2-colimit* of the 2-diagram (φ, α), if for every object $\mathcal{Y}$ of $\mathcal{C}$, the induced functor (2.26) is an equivalence of categories.

If $\mathcal{X}$ is a 2-colimit of (φ, α), then we will use the notation

$$2\text{-}\operatorname{colim}_{\mathcal{I}}(\varphi, \alpha) \xrightarrow{\sim} \mathcal{X}.$$

(The notation on the arrow should be understood as indicating an equivalence, and is justified by the following remark.)

Remark 2.28. Standard arguments show that if $\mathcal{X}_0$ and $\mathcal{X}_1$ are two objects of $\mathcal{C}$, equipped with 1-morphisms $(F_0, \psi_0) : (\varphi, \alpha) \to \Phi_{\mathcal{X}_0}$ and $(F_1, \psi_1) : (\varphi, \alpha) \to \Phi_{\mathcal{X}_1}$ realizing each of them as the 2-colimit of (φ, α) in $\mathcal{C}$, then there is a 1-morphism $F : \mathcal{X}_0 \to \mathcal{X}_1$ which is an equivalence in $\mathcal{C}$, and for which $\Phi_F \circ (F_0, \psi_0)$ is isomorphic to (F_1, ψ_1) (in the category $\operatorname{Mor}_{\mathcal{C}}\big((\varphi, \alpha), \Phi_{\mathcal{X}_1}\big)$).

Before making our next remark, we recall the notion of an initial object in a 2-category.

Definition 2.29. If $\mathcal{C}$ is a 2-category, an object of x of $\mathcal{C}$ is said to be an *initial object* of $\mathcal{C}$ if for each object y of $\mathcal{C}$, the category $\operatorname{Mor}_{\mathcal{C}}(x, y)$ is a contractible groupoid, or equivalently a connected setoid[2], i.e. a category equivalent to the one-point category.

Remark 2.30. As noted in [Stacks, Tag 003O], where the analogous definition of a final object is given, it is perhaps best to restrict this definition to the situation in which $\mathcal{C}$ is actually a $(2, 1)$-category, and indeed, this is the only case in which we will apply it.

Remark 2.31. In the case when $\mathcal{C}$ is a $(2, 1)$-category, there is an alternate characterization of 2-colimits, as initial objects in a certain 2-category. Namely, if (φ, α) is a 2-diagram in the 2-category $\mathcal{C}$, we define a new 2-category $\operatorname{Mor}_{\mathcal{C}}\big((\varphi, \alpha)\big)$ as follows: its set of objects is the disjoint union, indexed by the objects $\mathcal{X}$ of $\mathcal{C}$, of the set of objects of $\operatorname{Mor}_{\mathcal{C}}\big((\varphi, \alpha), \mathcal{X}\big)$; a 1-morphism in $\operatorname{Mor}_{\mathcal{C}}\big((\varphi, \alpha)\big)$ between two 1-morphisms $(F, \psi) : (\varphi, \alpha) \to \mathcal{X}$ and $(G, \eta) : (\varphi, \alpha) \to \mathcal{Y}$ consists of a 1-morphism $H : \mathcal{X} \to \mathcal{Y}$ in $\mathcal{C}$, together with a 2-morphism ξ in $\operatorname{Mor}_{\mathcal{C}}\big((\varphi, \alpha), \mathcal{Y}\big)$ between $\Phi_H \circ (F, \psi)$ and (G, η); a 2-morphism in $\operatorname{Mor}_{\mathcal{C}}\big((\varphi, \alpha)\big)$ between 1-morphisms (H, ξ) and (H', ξ') consists of a 2-morphism $\theta : H \to H'$ in $\mathcal{C}$ such that $\xi'_x \circ (\theta \star \mathrm{id}_{F_x}) = \xi_x$ for each object x of the index category $\mathcal{I}$. We leave it to the reader to deduce the definitions of the various compositions in $\operatorname{Mor}_{\mathcal{C}}\big((\varphi, \alpha)\big)$. If $\mathcal{C}$ is a $(2, 1)$-category, then so is $\operatorname{Mor}_{\mathcal{C}}\big((\varphi, \alpha)\big)$.

We furthermore leave it to the reader to then verify, in the case when $\mathcal{C}$ is a $(2, 1)$-category, that a morphism $(F, \psi) : (\varphi, \alpha) \to \mathcal{X}$ realizes the object $\mathcal{X}$ as the 2-colimit of (φ, α) if and only if this morphism, thought of as an object of $\operatorname{Mor}_{\mathcal{C}}\big((\varphi, \alpha)\big)$, is an initial object in this 2-category, in the sense of Definition 2.29. (The following lemma, whose proof we leave to the reader, aids in this.)

Lemma 2.32. *If $F : \mathcal{A} \to \mathcal{B}$ is a morphism of groupoids, then F is an equivalence if and only if, for every object b of $\mathcal{B}$, there exists an object $a \in \mathcal{A}$ and an isomorphism $\xi : F(a) \xrightarrow{\sim} b$, and if, furthermore, for any other object $a' \in \mathcal{A}$ endowed with an*

[2] We follow the terminology of [Stacks], and refer to a groupoid, each of whose connected components is contractible, as a *setoid*. Such a groupoid is equivalent to a discrete set, hence the name.

isomorphism $\xi' : F(a') \xrightarrow{\sim} b$*, there is a unique isomorphism* $\theta : a \xrightarrow{\sim} a'$ *such that* $\xi = \xi' \circ F(\theta)$*.*

Remark 2.33. There is a completely analogous theory of 2-limits in a 2-category $\mathcal{C}$, which admit a characterization analogous to the characterization of 2-colimits given in Definition 2.27, and also, when $\mathcal{C}$ is actually a $(2,1)$-category, a characterization in terms of final objects in a 2-category, as in Remark 2.31.

As one example, if the index category $\mathcal{I}$ is the partially ordered set

$$\begin{tikzcd} \bullet \arrow[dr] & & \bullet \arrow[dl] \\ & \bullet & \end{tikzcd}$$

then forming the 2-limit in $\mathcal{C}$ of a diagram indexed by $\mathcal{I}$ amounts to forming the 2-fibre product of objects in $\mathcal{C}$. (The treatment of 2-fibre products in [Stacks, Tag 003O] follows the "final object in a 2-category" approach.)

We next establish some basic functorial property of 2-colimits.

Lemma 2.34. *Let* (φ, α) *be a* 2*-diagram indexed by the category* $\mathcal{I}$*, with values in the* 2*-category* $\mathcal{C}$*, and let* $G : \widetilde{\mathcal{I}} \to \mathcal{I}$ *be a functor between categories. Suppose that* $(F, \psi) : (\varphi, \alpha) \to \Phi_{\mathcal{X}}$ *is a* 1*-morphism realizing the object* $\mathcal{X}$ *of* $\mathcal{C}$ *as a* 2*-colimit of* (φ, α)*, and that* $(\widetilde{F}, \tilde{\psi}) : G^*(\varphi, \alpha) \to \widetilde{\Phi}_{\mathcal{Y}}$ *is a* 1*-morphism realizing the object* $\mathcal{Y}$ *as a* 2*-colimit of* $G^*(\varphi, \alpha)$*. (Here we use* Φ*, with an appropriate subscript, to denote a constant object indexed by* $\mathcal{I}$*, and use* $\widetilde{\Phi}$*, with an appropriate subscript, to denote a constant object indexed by* $\widetilde{\mathcal{I}}$*.) Then there is a* 1*-morphism* $H : \mathcal{Y} \to \mathcal{X}$ *in* $\mathcal{C}$ *for which there exists a* 2*-isomorphism of* 1*-morphisms* $\widetilde{\Phi}_H \circ (\widetilde{F}, \tilde{\psi}) \xrightarrow{\sim} G^*(F, \psi)$*.*

Proof. The pull-back $G^*(F, \psi)$ is a 1-morphism from $G^*(\varphi, \alpha)$ to $\widetilde{\Phi}_{\mathcal{X}}$, i.e. is an object of the category $\mathrm{Mor}_{\mathcal{C}}\big(G^*(\varphi, \alpha), \widetilde{\Phi}_{\mathcal{X}}\big)$. Since $(\widetilde{F}, \tilde{\psi})$ realizes $\mathcal{Y}$ as a 2-colimit of (φ, α), composition with $(\widetilde{F}, \tilde{\psi})$ induces an equivalence of categories

$$\mathrm{Mor}_{\mathcal{C}}(\mathcal{Y}, \mathcal{X}) \xrightarrow{\sim} \mathrm{Mor}_{\mathcal{X}}\big(G^*(\varphi, \alpha), \widetilde{\Phi}_{\mathcal{X}}\big).$$

In particular, there is an object H of the domain of this equivalence, i.e. a 1-morphism $H : \mathcal{Y} \to \mathcal{X}$, such that $\widetilde{\Phi}_H \circ (\widetilde{F}, \tilde{\psi})$ (which is its image under the equivalence) is isomorphic in $\mathrm{Mor}_{\mathcal{X}}\big(G^*(\varphi, \alpha), \widetilde{\Phi}_{\mathcal{X}}\big)$ to the object $G^*(F, \psi)$ of the target. This proves the lemma. $\square$

In the particular case of the preceding lemma when $G : \widetilde{\mathcal{I}} \to \mathcal{I}$ is *cofinal* (in the sense of [Stacks, Tag 04E6]), we can say more.

Since we will need to apply it, we recall the definition of cofinality.

Definition 2.35. We say that a functor $G : \widetilde{\mathcal{I}} \to \mathcal{I}$ is *cofinal* if the following conditions hold:

(1) For any object x of $\mathcal{I}$, there is an object $\tilde{x}$ of $\widetilde{\mathcal{I}}$ and a morphism $x \to G(\tilde{x})$ in $\mathcal{I}$.
(2) For any object x of $\mathcal{I}$, any pair of objects $\tilde{x}, \tilde{x}'$ of $\widetilde{\mathcal{I}}$, and any pair of morphisms $x \to G(\tilde{x}), x \to G(\tilde{x}')$, there exists a sequence of morphisms

$$\tilde{x} = \tilde{x}_0 \leftarrow \tilde{x}_1 \to \tilde{x}_2 \leftarrow \tilde{x}_3 \to \cdots \leftarrow \tilde{x}_{2n-3} \to \tilde{x}_{2n-2} \leftarrow \tilde{x}_{2n-1} \to \tilde{x}_{2n} = \tilde{x}'$$

in $\widetilde{\mathcal{I}}$, and morphisms $x \to G(\tilde{x}_i)$ for each $i = 0, \ldots, 2n$, such that the morphisms for $i = 0$ and $2n$ coincide with the given morphisms, and such that each diagram

$$\begin{tikzcd} & x \arrow[dl] \arrow[d] \arrow[dr] & \\ G(\tilde{x}_{2i-2}) & G(\tilde{x}_{2i-1}) \arrow[l] \arrow[r] & G(\tilde{x}_{2i}) \end{tikzcd}$$

commutes.

Lemma 2.36. *If, in the context of Lemma* 2.34*, the functor G is cofinal, for any object $\mathcal{Z}$ of $\mathcal{C}$, the pull-back functor*

$$G^* : \mathrm{Mor}_{\mathcal{C}}\big((\varphi, \alpha), \Phi_{\mathcal{Z}}\big) \to \mathrm{Mor}_{\mathcal{C}}\big(G^*(\varphi, \alpha), \widetilde{\Phi}_{\mathcal{Z}}\big) \tag{2.37}$$

is an equivalence of categories. In particular, the 2*-diagram (φ, α) admits a* 2*-colimit if and only if $G^*(\varphi, \alpha)$ does, and furthermore, the* 1*-morphism H* (*given by Lemma* 2.34) *between these* 2*-colimits is an equivalence.*

Proof. The claim regarding 2-colimits follows directly from the definition, together with the claimed equivalence of categories. Thus, for any object $\mathcal{Z}$ of $\mathcal{C}$, we must show that (2.37) is an equivalence. To do this, we will show in turn that G^* is faithful, full, and essentially surjective.

To begin with, suppose that (F_1, ψ_1) and (F_2, ψ_2) are two 1-morphisms from (φ, α) to $\Phi_{\mathcal{X}}$, that ξ_1 and ξ_2 are two 2-morphisms between them, and that $G^*\xi_1 = G^*\xi_2$; by definition, this means that

$$\xi_{1,G(\tilde{x})} = \xi_{2,G(tx)} \tag{2.38}$$

for any object $\tilde{x}$ of $\widetilde{\mathcal{I}}$.

Now let x be an object of $\widetilde{\mathcal{I}}$, and choose (as we may) an object $\tilde{x}$ of $\widetilde{\mathcal{I}}$, and a morphism $f : x \to G(\tilde{x})$. Then for each $i = 1, 2$, we have an equality

$$(\xi_{i,G(\tilde{x})} \star \mathrm{id}_{\varphi(f)}) \circ \psi_{1,f} = \psi_{2,f} \circ \xi_{i,x}. \tag{2.39}$$

Since the 2-morphisms $\psi_{1,f}$ and $\psi_{2,f}$ are isomorphisms, we deduce from (2.38) and (2.39) that $\xi_{1,x} = \xi_{2,x}$, as required.

Next, we suppose given a 2-morphism $\tilde{\xi} : G^*(F_1, \psi_1) \to G^*(F_2, \psi_2)$; we must show that $\tilde{\xi} = G^*\xi$ for some 2-morphism $\xi : (F_1, \psi_1) \to (F_2, \psi_2)$. By definition, the data of $\tilde{\xi}$ amounts to a 2-morphism $\tilde{\xi}_{\tilde{x}} : F_{1,G(\tilde{x})} \to F_{2,G(\tilde{x})}$, for each object $\tilde{x}$ of $\widetilde{\mathcal{I}}$. These 2-morphisms satisfy the identity

$$(\tilde{\xi}_{\tilde{y}} \star \mathrm{id}_{\varphi(G(\tilde{f}))}) \circ \psi_{1,G(\tilde{f})} = \psi_{2,G(\tilde{f})} \circ \tilde{\xi}_{\tilde{x}} \tag{2.40}$$

for any morphism $\tilde{f} : \tilde{x} \to \tilde{y}$ in $\widetilde{\mathcal{I}}$.

By the assumption of cofinality, given any object x of $\mathcal{I}$, we may find an object $\tilde{x}$ of $\widetilde{\mathcal{I}}$, and a morphism $f : x \to G(\tilde{x})$. We now use (2.39) to *define* ξ_x, namely we set

$$\xi_x := \psi_{2,f}^{-1} \circ (\tilde{\xi}_{\tilde{x}} \star \mathrm{id}_{\varphi(f)}) \circ \psi_{1,f}.$$

Of course, we need to check that this is well-defined. A consideration of property (2) in Definition 2.35 shows that it suffices to show that if $\tilde{f} : \tilde{x} \to \tilde{y}$ is a morphism in $\widetilde{\mathcal{I}}$, then the 2-morphism ξ_x obtained by using the preceding formula coincides with the

2-morphism obtained by replacing $\tilde{x}$ with $\tilde{y}$, and replacing f with the composite $x \to G(\tilde{x}) \to G(\tilde{y})$. In other words, we have to show that

$$\psi_{2,f}^{-1} \circ (\tilde{\xi}_{\tilde{x}} \star \mathrm{id}_{\varphi(f)}) \circ \psi_{1,f} = \psi_{2,G(\tilde{f})\circ f}^{-1} \circ (\tilde{\xi}_{\tilde{y}} \star \mathrm{id}_{\varphi(\mathcal{G}(\tilde{f})\circ f)}) \circ \psi_{1,G(\tilde{f})\circ f}.$$

This follows from a consideration of the commutative diagram

$$\begin{tikzcd}
 & F_{1,x} \arrow[rr, "\psi_{1,G(\tilde{f})\circ f}"] \arrow[dl, "\psi_{1,f}"'] & & F_{1,G(\tilde{y})} \circ \varphi\big(G(\tilde{f}) \circ f\big) \arrow[dl, "\mathrm{id}_{F_{1,G(\tilde{y})}} \star \alpha_{G(\tilde{f}),f}"] \arrow[dd, "\tilde{\xi}_{\tilde{y}} \star \mathrm{id}_{\varphi(G(\tilde{f})\circ f)}"] \\
F_{1,G(\tilde{x})} \circ \varphi(f) \arrow[rr, "\psi_{1,G(\tilde{f})} \star \mathrm{id}_{\varphi(f)}"] \arrow[dd, "\tilde{\xi}_{\tilde{x}} \star \mathrm{id}_{\varphi(f)}"'] & & F_{1,G(\tilde{y})} \circ \varphi\big(G(\tilde{f})\big) \circ \varphi(f) \arrow[dd, "\tilde{\xi}_{\tilde{y}} \star \mathrm{id}_{\varphi(\tilde{f})\circ\varphi(f)}"' near start] & \\
 & F_{2,x} \arrow[rr, "\psi_{2,f\circ G(\tilde{f})}"' near end] \arrow[dl, "\psi_{2,f}"'] & & F_{2,G(\tilde{y})} \circ \varphi\big(G(\tilde{f}) \circ f\big) \arrow[dl, "\mathrm{id}_{F_{2,G(\tilde{y})}} \star \alpha_{G(\tilde{f}),f}"] \\
F_{2,G(\tilde{x})} \circ \varphi(f) \arrow[rr, "\psi_{2,G(\tilde{f})} \star \mathrm{id}_{\varphi(f)}"] & & F_{2,G(\tilde{y})} \circ \varphi\big(G(\tilde{f})\big) \circ \varphi(f) &
\end{tikzcd}$$

(The commutativity of the top and bottom faces follows from the fact that (F_1, ψ_1) and (F_2, ψ_2) are 1-morphisms, the commutativity of the front face follows from (2.40), and the commutativity of the remaining face is clear.) In carrying out this verification, it is helpful to remember that all the 2-morphisms ψ and α are in fact 2-isomorphisms.

Now that the 2-morphisms ξ_x are defined, we have to verify that they satisfy the necessary conditions to define a 2-morphism between (F_1, ψ_1) and (F_2, ψ_2). This is achieved by a consideration of various commutative diagrams similar to the one appearing above. (Details will appear in a future draft.)

□

Lemma 2.41. *Let $\mathcal{I}$ be a category, let $\mathcal{C}$ be a 2-category, and let (φ_1, α_1) and (φ_2, α_2) be 2-diagrams with values in $\mathcal{C}$, indexed by $\mathcal{I}$. Suppose that $(F_1, \psi_1) : (\varphi_1, \alpha_1) \to \Phi_{\mathcal{X}_1}$ and $(F_2, \psi_2) : (\varphi_2, \alpha_2) \to \Phi_{\mathcal{X}_2}$ are 1-morphisms realizing $\mathcal{X}_1$ (resp. $\mathcal{X}_2$) as the 2-colimit of (φ_1, α_1) (resp. (φ_2, α_2)). Then any 1-morphism of 2-diagrams $(F, \psi) : (\varphi_1, \psi_1) \to (\varphi_2, \psi_2)$ induces a 1-morphism $H : \mathcal{X}_1 \to \mathcal{X}_2$ in $\mathcal{C}$, such that the 1-morphisms of 2-diagrams $\Phi_H \circ (F_1, \psi_1)$ and $(F_2, \psi_2) \circ (F, \psi)$ are 2-isomorphic.*

Proof. Since $(F_2, \psi_2) \circ (F, \psi)$ is an object of $\mathrm{Mor}_{\mathcal{C}}\big((\varphi_1, \alpha_1), \Phi_{\mathcal{X}_2}\big)$, this follows from the defining property of $\mathcal{X}_1$ as a 2-colimit of (φ_1, α_1). □

Example 2.42. If the index category $\mathcal{I}$ is filtered and admits a countable skeleton, then we may find an equivalence $\mathbb{N} \xrightarrow{\sim} \mathcal{I}$. In this case any 2-colimit computed along $\mathcal{I}$ can be replaced by an equivalent 2-colimit computed along $\mathbb{N}$. Furthermore, any 2-diagram (φ, α) indexed by $\mathbb{N}$ may be replaced by a strictly commutative diagram $(\widetilde{\varphi}, \mathrm{id})$: for each $n \in \mathbb{N}$, we define $\widetilde{\varphi}(n) := \varphi(n)$, define $\widetilde{\varphi}_{n,n} := \mathrm{id}$, and define $\widetilde{\varphi}_{n,n+1} := \varphi_{n,n+1}$. But in general, given any pair $m < n$, we define

$$\widetilde{\varphi}_{m,n} := \varphi_{n-1,n} \circ \varphi_{n-2,n-1} \circ \cdots \circ \varphi_{m+1,m+2} \circ \varphi_{m,m+1}.$$

The data α may be used to construct an equivalence between (φ, α) and $(\widetilde{\varphi}, \mathrm{id})$.

We turn to proving that 2-colimits exist in various situations, beginning with the case of the $(2,1)$-category of categories fibred in groupoids over some base category. In order to do this, we will use the notion of *relative groupoid completion.*

Lemma 2.43. *If $\mathcal{X}$ is a category fibred over the base category $\mathcal{D}$, then there is a category $\mathcal{X}^{\mathrm{gp}}$ fibred in groupoids over $\mathcal{D}$, which we call the* relative groupoid completion of $\mathcal{X}$ over $\mathcal{D}$, *and a* 1*-morphism $F : \mathcal{X} \to \mathcal{X}^{\mathrm{gp}}$ of categories fibred over $\mathcal{D}$, with the property that for any category $\mathcal{Y}$ fibred in groupoids over $\mathcal{D}$, the functor $F^* : \mathrm{Fun}_{\mathcal{D}}(\mathcal{X}^{\mathrm{gp}}, \mathcal{Y}) \to \mathrm{Fun}_{\mathcal{D}}(\mathcal{X}, \mathcal{Y})$ is an equivalence of categories.*

Proof. Our construction of $\mathcal{X}^{\mathrm{gp}}$, which is somewhat clumsy, proceeds by *strictification.* Namely, we know from [Stacks, Tag 004A] that $\mathcal{X}$ is equivalent to a presheaf of categories on $\mathcal{D}$, and so we may assume that $\mathcal{X}$ is in fact a presheaf of categories. Now given any category, we may localize it at all its morphisms, to obtain a groupoid which receives a functor from the given category, and is universal for functors from the category to groupoids. (This is the usual groupoid completion.) Applying this construction to the values of the given presheaf of categories, we obtain a presheaf of groupoids, which then serves as a relative groupoid completion of $\mathcal{X}$ over $\mathcal{D}$. $\square$

Lemma 2.44. *If (φ, α) is a 2-diagram in the $(2,1)$-category $\mathcal{C}$ of categories fibered in groupoids over a base category $\mathcal{D}$, then (φ, α) admits a 2-colimit in $\mathcal{C}$.*

Proof. We begin by constructing a category $\mathcal{X}$ fibred over $\mathcal{D}$, which *almost* receives a morphism from (φ, α), More precisely, we will construct the data (F, ψ) attached to such a morphism, but the various 2-morphisms ψ_f will not necessarily be isomorphisms. Modulo this difficulty, and the fact that $\mathcal{X}$ will not be fibred in groupoids, we will see that $\mathcal{X}$ satisfies the universal property to be a 2-colimit of (φ, α). We will then pass to a groupoid completion $\mathcal{X}^{\mathrm{gp}}$ of $\mathcal{C}$. Doing this will force the 2-morphisms ψ_f to become isomorphisms, so that obtain a genuine morphism from (φ, α) to $\mathcal{X}^{\mathrm{gp}}$. The already-proved universal property of $\mathcal{X}$, together with the universal property of $\mathcal{X}^{\mathrm{gp}}$, when taken together, will then show that $\mathcal{X}^{\mathrm{gp}}$ is a 2-colimit of (φ, α).

We give an explicit description of $\mathcal{X}$, and of the functors $F_x : \varphi(x) \to \mathcal{X}$ and natural transformations ψ_f. We define the set of objects of $\mathcal{X}$ to be the disjoint union of the set of objects of each of the categories $\varphi(x)$, as x ranges over the objects of the index category $\mathcal{I}$. For any two objects x and y of $\mathcal{I}$, if a is an object of $\varphi(x)$ and if b is an object of $\varphi(y)$, then we also define $\mathrm{Mor}_{\mathcal{X}}(a,b)$ as a disjoint union:

$$\mathrm{Mor}_{\mathcal{X}}(a,b) := \coprod_{f \in \mathrm{Mor}_{\mathcal{I}}(x,y)} \mathrm{Mor}_{\varphi(y)}\big(\varphi(f)(a), b\big).$$

For notational purpose, we also write, for any $f \in \mathrm{Mor}_{\mathcal{I}}(x,y)$,

$$\mathrm{Mor}_{\mathcal{X}}(a,b)^{[f]} := \mathrm{Mod}_{\varphi(y)}\big(\varphi(f)(a), b\big),$$

so that we can rewrite the preceding disjoint union as

$$\mathrm{Mor}_{\mathcal{X}}(a,b) := \coprod_{f \in \mathrm{Mor}_{\mathcal{I}}(x,y)} \mathrm{Mor}_{\mathcal{X}}(a,b)^{[f]};$$

and if $l \in \mathrm{Mor}_{\varphi(y)}\big(\varphi(f)(a), b\big)$, then we write $l^{[f]}$ to denote l when thought of an an element of $\mathrm{Mor}_{\mathcal{X}}(a,b)^{[f]} \subseteq \mathrm{Mor}_{\mathcal{X}}(a,b)$.

The composition of morphisms in $\mathcal{X}$ is defined as follows: for a pair of composable morphisms $f : x \to y$ and $g : y \to z$, and objects a of $\varphi(x)$, b of $\varphi(y)$, and c of $\varphi(z)$, we define the pairing

$$p_{g,f} : \mathrm{Mor}_{\mathcal{X}}(b,c)^{[g]} \times \mathrm{Mor}_{\mathcal{X}}(a,b)^{[f]} \to \mathrm{Mor}_{\mathcal{X}}(a,c)^{[gf]}$$

via

$$(m^{[g]}, l^{[f]}) \mapsto \big(m \circ \varphi(g)(l) \circ (\alpha_{g,f})_a\big)^{[gf]};$$

the composition

$$\mathrm{Mor}_{\mathcal{X}}(b,c) \times \mathrm{Mor}_{\mathcal{X}}(a,b) \to \mathrm{Mor}_{\mathcal{X}}(a,c)$$

is then defined as the disjoint union of the various pairings $p_{g,f}$. We leave the verification that $\mathcal{X}$ forms a category to the reader.

If $x \in \mathcal{I}$, then we let π_x denote the functor $\varphi(x) \to \mathcal{D}$ realizing $\varphi(x)$ as a category over $\mathcal{D}$. We define a functor $\pi : \mathcal{X} \to \mathcal{D}$ as follows: on objects it is defined to be the disjoint union of the given maps $\mathrm{Ob}\big(\varphi(x)\big) \to \mathrm{Ob}(\mathcal{D})$; on morphisms, it is defined as follows: if $f : x \to y$ is a morphism in $\mathcal{I}$, if a and b are objects of $\varphi(x)$ and $\varphi(y)$ respectively, and if $l : \varphi(f)(a) \to b$ is a morphism in $\varphi(y)$, thought of as a morphism in $\mathcal{X}$, then (noting that $\pi_x(a) = \pi_y\big(\varphi(f)(a)\big)$ because $\varphi(f)$ is a 1-morphism between categories lying over $\mathcal{D}$) we define $\pi(l) : \pi_x(a) = \pi_y\big(\varphi(f)(a)\big) \to \pi_y(b)$ to be the morphism $\pi_y(l)$. Again, we leave to the reader the verification that π is in fact a functor from $\mathcal{X}$ to $\mathcal{D}$. (When making this verification, it helps to remember that, by the definition of the 2-category of categories fibred over $\mathcal{D}$ [Stacks, Tag 02XP], all the 2-morphisms in this category are required to project to identity morphisms in $\mathcal{D}$, so that if x is any object of $\mathcal{I}$, then $\pi_x\big((\alpha_x)_a\big) = \mathrm{id}_{\pi_x(a)}$ for all objects a of $\varphi(x)$; and similarly, if $x \xrightarrow{f} y \xrightarrow{g} z$ are morphisms in $\mathcal{I}$, then

$$\pi_z\big((\alpha_{g,f})_a\big) = \mathrm{id}_{\pi_x(a)} = \mathrm{id}_{\pi_z\left((g\circ f)(a)\right)}$$

for all objects a of $\varphi(x)$.)

The functor $\pi : \mathcal{X} \to \mathcal{D}$ realizes $\mathcal{X}$ as a category fibred over $\mathcal{D}$. To see this, we have to show that for any object $b \in \mathcal{X}$, and any morphism $l : u \to \pi(b)$ in $\mathcal{D}$, there is a morphism $\tilde{l} : a \to b$ in $\mathcal{X}$ lying over l, and which is strongly Cartesian. Suppose that b is an object of $\varphi(x)$. Then $\pi(b) = \pi_x(b)$ by definition, and by the assumption that $\varphi(x)$ is fibred over $\mathcal{D}$, there exists a morphism $m : a \to b$ in $\varphi(x)$ such that $\pi_x(m) = l$, and which is strongly Cartesian. Now define

$$\tilde{l} := (m \circ \alpha_{x,a}^{-1})^{[\mathrm{id}_x]} \in \mathrm{Mor}_{\mathcal{X}}(a,b)^{[\mathrm{id}_x]} \subseteq \mathrm{Mor}_{\mathcal{X}}(a,b).$$

Using the fact that m is strongly Cartesian, the reader can verify that $\tilde{l}$ is strongly Cartesian, as required.

We define what we might call a *proto-morphism* $(F,\psi) : (\varphi,\alpha) \to \mathcal{X}$. Namely, for any $x \in \mathcal{I}$, we define a morphism $F_x : \varphi(x) \to \mathcal{X}$, which we take to be the obvious inclusion on objects, while on morphisms, we define F_x as follows: if $l : a \to b$ is a morphism in $\varphi(x)$, then we define $F_x(l)$ to be the morphism $l \circ \alpha_{x,a}^{-1} : \varphi(\mathrm{id}_x)(a) \to b \in \mathrm{Mor}_{\varphi(x)}\big(\varphi(\mathrm{id}_x)(a), b\big)$ (interpreted as the subset $\mathrm{Mor}_{\mathcal{X}}(a,b)^{[\mathrm{id}_x]}$ of $\mathrm{Mor}_{\mathcal{X}}(a,b)$). We leave the verification that F_x is a functor to the reader. Note, though, that by construction (and recalling the point already remarked upon above, that 2-morphisms in $\mathcal{C}$ project to identity morphisms in $\mathcal{D}$), we have that $\pi \circ F_x = \pi_x$. Also, the very construction of strongly Cartesian morphisms in $\mathcal{X}$ given above shows that F_x takes strongly Cartesian morphisms in $\varphi(x)$ to strongly Cartesian morphisms in $\mathcal{X}$. Thus F_x is in fact a 1-morphism in $\mathcal{C}$.

For a morphism $f : x \to y$ in $\mathcal{I}$, a natural transformation $\psi_f : F_x \xrightarrow{\sim} F_y \circ \varphi(f)$ is defined as follows: for an object a of $\varphi(x)$, the morphism $(\psi_f)_a : a \to \varphi(f)(a)$ is defined to be the element

$$(\mathrm{id}_{\varphi(f)(a)})^{[f]} \in \mathrm{Mor}_{\mathcal{X}}\big(a, \varphi(f)(a)\big)^{[f]} \subseteq \mathrm{Mod}_{\mathcal{X}}\big(a, \varphi(f)(a)\big).$$

The natural transformations ψ_f project under π to identity natural transformations in $\mathcal{D}$, and thus are indeed 2-morphisms in $\mathcal{C}$.

As the reader can verify, the couple (F, ψ) satisfies all the conditions of Definition 2.24 other than condition (3): in general, the natural transformations ψ_f will not be isomorphisms.

Now suppose that $\mathcal{Y}$ is a category fibred in groupoids over $\mathcal{D}$, and that $\Phi : \mathcal{X} \to \mathcal{Y}$ is a morphism of categories fibred over $\mathcal{D}$. Composing (F, ψ) with Φ in the evident way, we obtain a genuine morphism $\Phi \circ (F, \psi) : (\varphi, \alpha) \to \mathcal{Y}$, since all the 2-morphisms in $\mathcal{Y}$ are necessarily isomorphisms (so that condition (3) of Definition 2.24 is automatically satisfied, while the remaining conditions are evidently satisfied, since they are satisfied by (F, ψ)). In this way we obtain a functor

$$\mathrm{Fun}_{\mathcal{D}}(\mathcal{X}, \mathcal{Y}) \to \mathrm{Mor}_{\mathcal{C}}\big((\varphi, \alpha), \mathcal{Y}\big) \tag{2.45}$$

whose source is the category of morphisms between $\mathcal{X}$ and $\mathcal{Y}$ (thought of as categories fibred over $\mathcal{D}$) and whose target is the category of morphisms from (φ, α) to $\mathcal{Y}$. We will show that this functor is an isomorphism of categories.

To this end, we first re-express some of the preceding constructions in a simple formula. Namely, if $f : x \to y$ is a morphism in $\mathcal{I}$, if a is an object of $\varphi(x)$, if b is an object of $\varphi(y)$, then we then have the formula

$$l^{[f]} = F_y(l) \circ (\psi_f)_a. \tag{2.46}$$

Thus, if $\Phi : \mathcal{X} \to \mathcal{Y}$ is a 1-morphism, and if we define $(G, \eta) := \Phi \circ (F, \psi)$, then we see from this formula that

$$\Phi(l^{[f]}) = G_y(l) \circ (\eta_f)_a, \tag{2.47}$$

for each morphism $l^{[f]} \in \mathrm{Mor}_{\mathcal{X}}(a, b)^{[f]}$. It is now straightforward to verify that, conversely, if one is given a morphism $(G, \eta) : (\varphi, \alpha) \to \mathcal{Y}$ to a category $\mathcal{Y}$ fibred in groupoids over $\mathcal{D}$, and if one defines the functor $\Phi : \mathcal{X} \to \mathcal{Y}$ on objects via mapping an object $a \in \varphi(x)$ (thought of as an object of $\mathcal{X}$) to $G_x(a)$, and on morphisms via the formula (2.47), then Φ is indeed a 1-morphism from $\mathcal{X}$ to $\mathcal{Y}$, and that the functor (2.45) induces a bijection on objects. It is similarly verified to induce a bijection on morphisms, and thus is indeed an isomorphism of categories.

Now let $\mathcal{X} \to \mathcal{X}^{\mathrm{gp}}$ be a groupoid completion of $\mathcal{X}$. By the preceding discussion, the proto-morphism (F, ψ) induces a genuine morphism $(F^{\mathrm{gp}}, \psi^{\mathrm{gp}}) : (\varphi, \alpha) \to \mathcal{X}^{\mathrm{gp}}$. Furthermore, for any category fibred in groupoids $\mathcal{Y}$ over $\mathcal{D}$, the induced functor $\mathrm{Fun}_{\mathcal{D}}(\mathcal{X}^{\mathrm{gp}}, \mathcal{Y}) \to \mathrm{Fun}_{\mathcal{D}}(\mathcal{X}, \mathcal{Y})$ is an equivalence of categories. We thus see that $(F^{\mathrm{gp}}, \psi^{\mathrm{gp}})$ realizes $\mathcal{X}^{\mathrm{gp}}$ as a 2-colimit of (φ, α). ∎

Lemma 2.48. *If (φ, α) is a 2-diagram in the $(2,1)$-category $\mathcal{C}$ of stacks in groupoids over a site $\mathcal{D}$, then (φ, α) admits a 2-colimit in $\mathcal{C}$.*

Proof. By Lemma 2.44, we may form a 2-colimit $\mathcal{X}$ of (φ, α) in the $(2,1)$-category of categories fibred in groupoids over $\mathcal{D}$. If we let $\widetilde{\mathcal{X}}$ denote the stackification of $\mathcal{X}$

over $\mathcal{D}$ (in the sense, e.g., of [Stacks, Tag 02ZO]), then the 2-categorical universal property of $\widetilde{\mathcal{X}}$ (see [Stacks, Tag 0436] and also [Stacks, Tag 04W9]) shows that $\widetilde{\mathcal{X}}$ forms a 2-colimit of (φ, α) in the $(2,1)$-category of stacks in groupoids over $\mathcal{D}$. □

Lemma 2.49. *Let $\mathcal{C}$ be a $(2,1)$-category of categories fibred in groupoids over the base category $\mathcal{D}$, let (φ, α) be a 2-diagram in $\mathcal{C}$, indexed by the filtered category $\mathcal{I}$, and let $(F, \psi) : (\varphi, \alpha) \to \mathcal{X}$ be a 2-colimit of (φ, α). If each of the functors $\varphi(f)$ (as f runs over morphisms in $\mathcal{I}$) is faithful (resp. fully faithful), then each of the functors $F_x : \varphi(x) \to \mathcal{X}$ (as x runs over the objects of $\mathcal{I}$) is faithful (resp. fully faithful).*

Proof. The proof will appear in a subsequent draft. □

The following result concerning the values of a 2-colimit of stacks is often useful.

Lemma 2.50. *If $\{\mathcal{X}_i\}_{i \in \mathcal{I}}$ is a 2-diagram in the $(2,1)$-category of stacks in groupoids over the site $\mathcal{D}$, and if T is an object $\mathcal{D}$, then there is a natural morphism of groupoids $2 - \operatorname{colim}_i \mathcal{X}_i(T) \to (2 - \operatorname{colim}_i \mathcal{X}_i)(T)$. If $\mathcal{I}$ is furthermore filtered, and if T is quasi-compact then this morphism is fully faithful. Continuing to assume that $\mathcal{I}$ is filtered, if in addition either T is quasi-separated or the transition morphisms in the 2-diagram $\{\mathcal{X}_i\}$ are monomorphisms, then this morphism is an equivalence.*

Proof. This is standard; a proof should appear in a future draft. □

We introduce one more 2-categorical notion, the 2-categorical analogue of the usual categorical construction $\mathcal{C}/y$ (which for any object y of a category $\mathcal{C}$, is the category whose objects are $\mathcal{C}$-morphisms $f : x \to y$, with a morphism between $f : x \to y$ and $f' : x' \to y$ consisting of a $\mathcal{C}$-morphism $g : x \to x'$ such that $f = f' \circ g$).

Definition 2.51. If $\mathcal{C}$ is a 2-category, and $\mathcal{Y}$ is an object of $\mathcal{C}$, then we let $\mathcal{C}/\mathcal{Y}$ denote the 2-category whose objects consist of 1-morphisms $F : \mathcal{X} \to \mathcal{Y}$ in $\mathcal{C}$. If $F : \mathcal{X} \to \mathcal{Y}$ and $F' : \mathcal{X}' \to \mathcal{Y}$ are two objects of $\mathcal{C}/\mathcal{Y}$, then the objects of $\operatorname{Mor}_{\mathcal{C}/\mathcal{Y}}(F, F')$ (which are 1-morphisms from F to F' in $\mathcal{C}/\mathcal{Y}$) consist of pairs (G, α), where $G : \mathcal{X} \to \mathcal{X}'$ is a 1-morphism and $\alpha : F \to F' \circ G$ is a 2-morphism, while the morphisms $(G, \alpha) \to (H, \beta)$ (which are the 2-morphisms from (G, α) to (H, β) in $\mathcal{C}/\mathcal{X}$) consist of of 2-morphisms $\gamma : G \to H$ such that $(\mathrm{id}_{F'} \star \gamma) \circ \alpha = \beta$. (See [GR04, Ex. 2.1.11 (ii)].)

3. Stack-theoretic background

In this section we introduce and study various stack-theoretic notions that will be needed in our developing the theory of formal algebraic stacks. One key point is that in this theory, one studies many stacks (in groupoids) that are *not* algebraic stacks (e.g. formal algebraic stacks themselves, as well as Ind-algebraic stacks), and so it is important to have a strong and flexible theory of the geometric properties of morphisms between such stacks. We develop some of that theory here, with an emphasis on so-called *morphisms that are representable by algebraic stacks*.

Another important construction in the context of formal algebraic stacks is the formation of the underlying reduced algebraic stack. We develop a general version of that theory in this section (after introducing our conventions related to substacks), which is flexible enough to recover the existing theory for algebraic stacks, but which also applies to formal algebraic stacks and Ind-algebraic stacks.

Related to this, we also prove a form of topological invariance of the small étale site which applies to not-necessarily-algebraic stacks.

Throughout this section, we fix a scheme S, and by a stack over S, we mean a category fibred in groupoids over the category Sch/S,[3] which is a stack with respect to the *fppf* topology on Sch/S.

We begin by recalling the following definition, which generalizes the notion of a morphism between stacks being representable by algebraic spaces.

Definition 3.1. (1) We say that a morphism $\mathcal{X} \to \mathcal{Y}$ of stacks over S is *representable by Deligne–Mumford stacks*, or *DM* for short, if for any morphism of stacks $\mathcal{Z} \to \mathcal{Y}$ whose source is a Deligne–Mumford stack, the fibre product $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is also a Deligne–Mumford stack.

(2) We say that a morphism $\mathcal{X} \to \mathcal{Y}$ of stacks over S is *representable by algebraic stacks*, or *algebraic*,[4] if for any morphism of stacks $\mathcal{Z} \to \mathcal{Y}$ whose source is an algebraic stack, the fibre product $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is again an algebraic stack.

We will be primarily interested in the second of these two definitions, but the first is also useful. (In particular, it appears naturally in the context of étale morphisms of algebraic stacks.)

Remark 3.2. It is easy to check, in the context of the preceding definition, that in verifying either condition (1) or (2), it suffices to restrict to the case when $\mathcal{Z}$ is actually a scheme. Indeed, suppose that condition (1) is satisfied in that case, and then consider the case of a morphism $\mathcal{Z} \to \mathcal{Y}$ whose source is a Deligne–Mumford stack. We may choose an étale surjection $U \to \mathcal{Z}$ whose source is a scheme (and which is representable by algebraic spaces, since $\mathcal{Z}$ is an algebraic stack), and consider the base-changed morphism $\mathcal{X} \times_{\mathcal{Y}} U \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$; this is again representable by algebraic spaces, étale, and surjective. By assumption, the source is a Deligne–Mumford stack, and thus we may find an étale surjection $V \to \mathcal{X} \times_{\mathcal{Y}} U$ whose source is a scheme (and which again is representable by algebraic spaces). The composite $V \to \mathcal{X} \times_{\mathcal{Y}} U \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is then a morphism that is representable by algebraic spaces, étale, and surjective, from a scheme to the stack $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$. Thus this fibre product is itself a Deligne–Mumford stack (since it is an algebraic stack, by [Stacks, Tag 05UL], which admits a surjective étale morphism from a scheme).

Similarly, if condition (2) is satisfied when $\mathcal{Z}$ is a scheme, then consider the case of a morphism $\mathcal{Z} \to \mathcal{Y}$ whose source is an algebraic stack. We choose a smooth surjection $U \to \mathcal{Z}$ whose source is a scheme (and which is representable by algebraic spaces, since $\mathcal{Z}$ is an algebraic stack), and consider the base-changed morphism $\mathcal{X} \times_{\mathcal{Y}} U \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$; this is again representable by algebraic spaces, smooth, and surjective. By assumption, the source is an algebraic stack, and thus we may find a smooth surjection $V \to \mathcal{X} \times_{\mathcal{Y}} U$ whose source is a scheme (and which again is representable by algebraic spaces). The composite $V \to \mathcal{X} \times_{\mathcal{Y}} U \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is then a morphism that is representable by algebraic spaces, smooth, and surjective, from a scheme to the stack $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$. Thus this fibre product is itself an algebraic stack [Stacks, Tag 05UL].

In particular (since schemes are DM algebraic stacks), we see that a DM morphism is representable by algebraic stacks.

[3] We ignore all set-theoretic issues throughout this write-up. The concerned reader should add their preferred flavour of foundations.

[4] This latter term is the one used in the Stacks Project [Stacks, Tag 06CF].

Finally, we remark that since any scheme is locally affine, and since the property of a stack over a base scheme T being algebraic, or Deligne–Mumford, may be checked Zariski locally on T, we see that in order to verify either condition (1) or (2), it even suffices to consider the case when $\mathcal{Z}$ is an *affine* scheme.

The following lemma is sometimes useful for verifying the representability properties of a morphism.

Lemma 3.3. *If $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ are morphisms of stacks over S, then the first arrow is representable (i.e. representable by schemes, resp. representable by algebraic spaces, resp. representable by Deligne–Mumford stacks, resp. representable by algebraic stacks) if and only if, for each morphism $T \to \mathcal{Z}$ whose source is a scheme, the induced morphism $T \times_{\mathcal{Z}} \mathcal{X} \to T \times_{\mathcal{Z}} \mathcal{Y}$ is representable (resp. representable by algebraic spaces, resp. representable by Deligne–Mumford stacks, resp. representable by algebraic stacks).*

Proof. If we are given a morphism $f : T \to \mathcal{Y}$, whose source is a scheme, then we can also consider the composite $g : T \to \mathcal{Y} \to \mathcal{Z}$, and the morphism f induces a section $s : T \to T \times_{g,\mathcal{Z}} \mathcal{Y}$. The projection $T \times_{f,\mathcal{Y}} \mathcal{X} \to T$ can then be obtained as the pull-back under s of the morphism $T \times_{g,\mathcal{Z}} \mathcal{X} \to T \times_{g,\mathcal{Z}} \mathcal{Y}$. If this latter morphism has a given representability property, so does its pull-back under s. (Here we are also taking into account Remark 3.2, of course.) $\square$

Lemma 3.4. *If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of stacks over S that is representable by algebraic stacks, then the diagonal morphism $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable by algebraic spaces. In addition, the morphism f is DM (resp. representable by algebraic spaces) if and only if Δ_f is furthermore unramified (resp. a monomorphism).*

Proof. We apply the criterion of Lemma 3.3 to the sequence of morphisms $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Y}$. Let $Z \to \mathcal{Y}$ be a morphism whose source is a scheme, and write $\mathcal{W} := \mathcal{X} \times_{\mathcal{Y}} Z$. Then $\mathcal{W}$ is an algebraic stack over Z, and the morphism Δ_f pulls-back to the diagonal morphism $\Delta_{\mathcal{W}/Z}$. This morphism is representable by algebraic spaces, and so (since $Z \to \mathcal{Y}$ was arbitrary) so is the morphism Δ_f.

The morphism Δ_f is furthermore unramified (resp. a monomorphism) if and only if each of the base-changed morphisms $\Delta_{\mathcal{W}/Z}$ is unramified (resp. a monomorphism). Each of these morphism is unramified (resp. a monomorphism), in turn, if and only if each of the algebraic stacks $\mathcal{W}$ is in fact a Deligne–Mumford stack [Stacks, Tag 06N3] (resp. an algebraic space [Stacks, Tag 04SZ]). This completes the proof of the lemma. $\square$

Lemma 3.5. *A morphism $X \to \mathcal{Y}$ from a sheaf over S to a stack over S is representable by algebraic stacks if and only if it is representable by algebraic spaces.*

Proof. The if direction is clear. For the only if direction, suppose that $X \to \mathcal{Y}$ is representable by algebraic stacks, and consider a morphism $T \to \mathcal{Y}$ whose source is a scheme. Then the fibre product $X \times_{\mathcal{Y}} T$ is an algebraic stack, by assumption, but it is also a sheaf. Thus it is in fact an algebraic space [Stacks, Tag 04SZ]. $\square$

The following lemmas give additional criteria for deducing that certain morphisms of stacks are representable (in various senses).

Lemma 3.6. *Suppose that $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ are morphisms of stacks over S, such that the second morphism, and also the composite morphism, are both representable* (*i.e. representable by schemes, resp. representable by algebraic spaces, resp. representable by Deligne–Mumford stacks, resp. representable by algebraic stacks*). *Then the morphism* $\mathcal{X} \to \mathcal{Y}$ *is representable* (*resp. representable by algebraic spaces, resp. representable by Deligne–Mumford stacks, resp. representable by algebraic stacks*).

Proof. We apply the criterion of Lemma 3.3, and so find that it suffices to show that if $T \to \mathcal{Z}$ is any morphism whose source is a scheme, the base-changed morphism $T \times_{\mathcal{Z}} \mathcal{X} \to T \times_{\mathcal{Z}} \mathcal{Y}$ is representable in the appropriate way. This follows from the fact that a morphism of schemes (resp. algebraic spaces, resp. Deligne–Mumford stacks, resp. algebraic stacks) is representable by schemes (resp. algebraic spaces, resp. Deligne–Mumford stacks, resp. algebraic stacks). $\square$

Lemma 3.7. *If* $\mathcal{X} \to \mathcal{Y}$ *and* $\mathcal{Y} \to \mathcal{Z}$ *are morphisms of stacks over* S *for which the composite* $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ *is representable by algebraic spaces* (*resp. Deligne–Mumford stacks, resp. algebraic stacks*) *and the diagonal morphism* $\mathcal{Y} \to \mathcal{Y} \times_{\mathcal{Z}} \mathcal{Y}$ *is representable by algebraic spaces,*[5] *then* $\mathcal{X} \to \mathcal{Y}$ *is representable by algebraic spaces* (*resp. Deligne–Mumford stacks, resp. algebraic stacks*).

Proof. This follows from the usual graph argument: we factor $\mathcal{X} \to \mathcal{Y}$ as the composite of the graph $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ and the projection from $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ to $\mathcal{Y}$. The first morphism is representable by algebraic spaces (being a base-change of the diagonal morphism $\mathcal{Y} \to \mathcal{Y} \times_{\mathcal{Z}} \mathcal{Y}$), and the second morphism is representable by algebraic spaces (resp. Deligne–Mumford stacks, resp. algebraic stacks), being the base-change of the morphism $\mathcal{X} \to \mathcal{Z}$. Thus their composite is representable by algebraic spaces (resp. Deligne–Mumford stacks, resp. algebraic stacks), as claimed. $\square$

Recall the following definition [Stacks, Tag 07Y8].

Definition 3.8. We say that a category fibred in groupoids $\mathcal{X}$ over a locally Noetherian scheme S satisfies the strong Rim–Schlessinger criterion, denoted (RS*), if, given an affine open subset $\operatorname{Spec}\Lambda \subseteq S$, and a fibre product diagram of Λ-algebras

$$\begin{tikzcd} B' \arrow[r] & B \\ A' = A \times_B B' \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

in which $B' \to B$ surjective with square zero kernel, the induced functor on fibre categories

$$\mathcal{X}_{\operatorname{Spec}(A')} \longrightarrow \mathcal{X}_{\operatorname{Spec}(A)} \times_{\mathcal{X}_{\operatorname{Spec}(B)}} \mathcal{X}_{\operatorname{Spec}(B')}$$

is an equivalence of categories.

We now show that the strong Rim–Schlessinger condition can be pulled back along morphisms representable by algebraic stacks.

Lemma 3.9. *If* $\mathcal{X} \to \mathcal{Y}$ *is a morphism of stacks over a locally Noetherian scheme* S *which is representable by algebraic stacks, and if* $\mathcal{Y}$ *satisfies* (RS*), *then so does* $\mathcal{X}$.

[5] Such a diagonal morphism is always representable by sheaves, and hence if it representable by algebraic stacks, it is in fact representable by algebraic spaces; this explains our apparently restrictive hypothesis.

Proof. By [Stacks, Tag 06N7] (and [Stacks, Tag 042X]), we may (and do) replace $\mathcal{X}$ by an isomorphic stack such that the morphism $\mathcal{X} \to \mathcal{Y}$ realizes $\mathcal{X}$ as a category fibred in groupoids over $\mathcal{Y}$. Let

$$\begin{tikzcd} X = \operatorname{Spec} B \arrow[r] \arrow[d] & X' = \operatorname{Spec} B' \arrow[d] \\ Y = \operatorname{Spec} A \arrow[r] & Y' = \operatorname{Spec} A' \end{tikzcd}$$

be a pushout diagram in the category of schemes obtained by taking Spec of a Cartesian square of rings as in Definition 3.8. We must show that the natural functor between groupoids

$$\mathcal{X}(Y') \to \mathcal{X}(X') \times_{\mathcal{X}(X)} \mathcal{X}(Y) \tag{3.10}$$

is an equivalence. Since $\mathcal{Y}$ satisfies (RS*), the functor

$$\mathcal{Y}(Y') \to \mathcal{Y}(X') \times_{\mathcal{Y}(X)} \mathcal{Y}(Y)$$

is an equivalence, and so we may place the functor (3.10) into a 2-commutative diagram of groupoids

$$\begin{tikzcd} \mathcal{X}(Y') \arrow[r] \arrow[d] & \mathcal{X}(X') \times_{\mathcal{X}(X)} \mathcal{X}(Y) \arrow[dl] \\ \mathcal{Y}(Y') & \end{tikzcd}$$

Since $\mathcal{X} \to \mathcal{Y}$ realizes the source as being fibred in groupoids over the target, the same is true of the vertical arrow in this diagram. Thus, in order to check that the horizontal functor is an equivalence, it suffices to check that it induces an equivalence on the fibre categories over each object in the base. (It is straightforward to verify directly that this suffices. Alternatively, note that the diagonal arrow is easily verified to also realize its source as a category fibred in groupoids over its target, and then apply part (1) of [Stacks, Tag 003Z].)

This reduces our problem to verifying that the functor analogous to (3.10), obtained by replacing $\mathcal{X}$ by $\mathcal{X} \times_{\mathcal{Y}} Y'$ for some given morphism $Y' \to \mathcal{Y}$, is an equivalence. Since $\mathcal{X} \to \mathcal{Y}$ is representable by algebraic stacks, this fibre product is an algebraic stack. We are therefore done, since algebraic stacks satisfy (RS*), by [Stacks, Tag 0CXP]. ☐

Following [Stacks, Tag 03YK, Tag 04XB], we can define properties of morphisms representable by algebraic stacks in the following way.

Definition 3.11. If P is a property of morphisms of algebraic stacks which is *fppf* local on the target, and preserved by arbitrary base-change, then we say that a morphism $f : \mathcal{X} \to \mathcal{Y}$ of stacks which is representable by algebraic stacks *has property* P if and only if for every algebraic stack $\mathcal{Z}$ and morphism $\mathcal{Z} \to \mathcal{Y}$, the base-changed morphism of algebraic stacks $\mathcal{Z} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{Z}$ has property P.

Remark 3.12. When applying Definition 3.11, it suffices to consider the case when $\mathcal{Z}$ is actually a scheme, or even an affine scheme (since any algebraic stack admits a smooth, and so in particular *fppf*, cover by a scheme, and hence even by a disjoint union of affine schemes, and by assumption the property P may be tested for *fppf* locally).

Remark 3.13. Some of the properties to which we will apply Definition 3.11 are *flat*, *locally of finite presentation*, *locally of finite type*, *proper*, *smooth*, and *surjective*.

Remark 3.14. We may also apply Definition 3.11 to the property of being *étale*. Recall that a morphism $\mathcal{X} \to \mathcal{Y}$ of algebraic stacks is said to be étale if it is unramified, flat, and locally of finite presentation, or equivalently, if it is a DM morphism, with the property that for any morphism $Z \to \mathcal{Y}$ whose source is a scheme, the base-changed morphism $\mathcal{X} \times_{\mathcal{Y}} Z \to Z$ (whose source is a Deligne–Mumford stack, by assumption) is étale, in the sense that for some (or, equivalently, any) surjective étale morphism $U \to \mathcal{X} \times_{\mathcal{Y}} Z$ whose source is a scheme, the composite morphism $U \to \mathcal{X} \times_{\mathcal{Y}} Z \to Z$ is étale.

We then find that a morphism $\mathcal{X} \to \mathcal{Y}$ of stacks over S which is representable by algebraic stacks is étale (in the sense given by Definition 3.11 and the discussion of the preceding paragraph) if and only if it is a DM morphism, with the additional property that for any morphism $Z \to \mathcal{Y}$ whose source is a scheme, the base-changed morphism $\mathcal{X} \times_{\mathcal{Y}} Z \to Z$ (whose source is a Deligne–Mumford stack, by assumption) is an étale morphism in the sense described in the preceding paragraph.

Remark 3.15. If we apply Definition 3.11 to the property of being a *monomorphism*, then we find that a morphism $f : \mathcal{X} \to \mathcal{Y}$, which is representable by algebraic stacks, is a monomorphism if and only if $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is an isomorphism (since this may be checked after pulling back over morphisms $T \to \mathcal{Y}$ whose source is a scheme), and thus if and only if f is fully faithful as a functor.

Furthermore, Lemma 3.4 then shows that if a morphism representable by algebraic stacks is a monomorphism, then it is actually representable by algebraic spaces (since if the associated diagonal morphism is an isomorphism, it is certainly a monomorphism). It then follows from [Stacks, Tag 040X] that such a morphism is also *universally injective* (where this latter property is again interpreted via Definition 3.11).

Example 3.16. If $\mathcal{Y}$ is a stack whose diagonal is representable by algebraic spaces, then any morphism $\mathcal{X} \to \mathcal{Y}$ whose source is an algebraic stack is representable by algebraic stacks [EG19b, Lem. 2.3.1], and so Definition 3.11 applies in such cases. Many such properties are discussed in detail in [EG19b, § 2.3]. (Note that Definition 3.11 is compatible with [EG19b, Def. 2.3.4], although the latter definition is more expansive than the former, in that it does not require that the property P be *fppf* local on the target. In practice, all the example properties considered in detail in [EG19b, § 2.3] are in fact *fppf* local on the target.)

It is sometimes useful to describe a given stack as a quotient stack, and we state and prove a lemma that gives a general criterion for this, together with some related results. We begin by establishing a general context in which to consider such a result. To this end, suppose that $\mathcal{X}$ is a stack over S, and that $U \to \mathcal{X}$ is a morphism whose source is an *fppf* sheaf on Sch/S, and which is representable by algebraic spaces and locally of finite presentation. We may then form the fibre product $R := U \times_{\mathcal{X}} U$ which is again an *fppf* sheaf on Sch/S, having the structure of a groupoid in sheaves over U. Suppose now that R' is another groupoid in sheaves over U, for which each of the projections $R' \rightrightarrows U$ are representable by algebraic spaces, flat, and locally of finite presentation, and that we are given a morphism $R' \to R$ of groupoids in sheaves over U.

Following the discussion of [Stacks, Tag 0440], we may form the quotient stack $\mathcal{X}' := [U/R']$. (In the cited discussion, the quotient stack is defined when U and R are in fact algebraic spaces, but the discussion there — and in particular, the definition of the quotient stack given in [Stacks, Tag 044Q] — makes sense whenever U is a sheaf and R' is a groupoid in sheaves over U whose projections to U are representable by algebraic spaces, flat, and locally of finite presentation.) Note that, by construction, there is a canonical isomorphism $R' \xrightarrow{\sim} U \times_{\mathcal{X}'} U$.

In the preceding situation, we have the following lemma relating the various objects involved.

Lemma 3.17. *We put ourselves in the context of the preceding discussion.*

(1) The morphism $U \to \mathcal{X}$ factors (up to 2-isomorphism) through a morphism $\mathcal{X}' \to \mathcal{X}$, which is furthermore representable by algebraic stacks.

(2) There is a 2-Cartesian square

$$\begin{tikzcd} R' \arrow[r] \arrow[d] & R \arrow[d] \\ \mathcal{X}' \arrow[r, "\Delta_{\mathcal{X}'/\mathcal{X}}"] & \mathcal{X}' \times_{\mathcal{X}} \mathcal{X}' \end{tikzcd}$$

in which the horizontal arrows are representable by algebraic spaces, and in which the vertical arrows are representable by algebraic spaces, flat, locally of finite presentation, and surjective.

Proof. The stack $[U/R']$ is defined as the stackification of the presheaf in groupoids over S whose objects are morphisms $T \to U$, and whose morphisms are morphisms $T \to R'$ projecting to a given pair of morphisms $T \rightrightarrows U$. The given morphism $R' \to R = U \times_{\mathcal{X}} U$ allows us construct a morphism from this presheaf to $\mathcal{X}$, and hence a morphism from its stackification $\mathcal{X}'$ to $\mathcal{X}$.

Consider the composite $U \to \mathcal{X}' \to \mathcal{X}$ (which by assumption is representable by algebraic spaces). If we base-change this along a morphism $T \to \mathcal{X}$ whose source is a scheme, we obtain morphisms

$$T \times_{\mathcal{X}} U \to T \times_{\mathcal{X}} \mathcal{X}' \to T.$$

The source of the first arrow is an algebraic space, and first arrow itself is representable by algebraic spaces, flat, and surjective (since it a base-change of the morphism $U \to \mathcal{X}'$, which has these properties). Thus $T \times_{\mathcal{X}} \mathcal{X}'$ is indeed an algebraic stack [Stacks, Tag 05UL]; this proves (1).

It follows from (1) together with Lemma 3.4 that $\Delta_{\mathcal{X}'/\mathcal{X}}$ is representable by algebraic spaces, proving the first claim of (2). Consider now the morphism $R = U \times_{\mathcal{X}} U \to \mathcal{X}' \times_{\mathcal{X}} \mathcal{X}'$. Since $U \to \mathcal{X}'$ is representable by algebraic spaces, flat, locally of finite presentation, and surjective, the same is true of this morphism. It provides the right-hand arrow of the diagram in (2). If we pull this map back along $\Delta_{\mathcal{X}'/\mathcal{X}}$, we obtain the 2-Cartesian square

$$\begin{tikzcd} U \times_{\mathcal{X}'} U \arrow[r] \arrow[d] & R \arrow[d] \\ \mathcal{X}' \arrow[r, "\Delta_{\mathcal{X}'/\mathcal{X}}"] & \mathcal{X}' \times_{\mathcal{X}} \mathcal{X}' \end{tikzcd}$$

Taking into account the isomorphism $R' \xrightarrow{\sim} U \times_{\mathcal{X}'} U$, we obtain the 2-Cartesian square of (2). $\square$

The preceding lemma has various applications, such as to the following criterion for presenting a stack as a quotient.

Lemma 3.18. *Suppose that $\mathcal{X}$ is a stack over S, and that $U \to \mathcal{X}$ is a morphism from a sheaf to $\mathcal{X}$ which is representable by algebraic spaces, flat, surjective, and locally of finite presentation. Then, if we write $R := U \times_{\mathcal{X}} U$, the morphism $U \to \mathcal{X}$ induces an isomorphism $[U/R] \xrightarrow{\sim} \mathcal{X}$.*

Proof. We apply Lemma 3.17, taking $R' = R$ (as the hypotheses of the present lemma allow us to do). Since the upper right arrow in the diagram of part (2) of that lemma is then an isomorphism, we conclude that $\Delta_{\mathcal{X}'/\mathcal{X}}$ is also an isomorphism (as the property of being an isomorphism may be checked *fppf* locally on the target), and thus that $\mathcal{X}' \to \mathcal{X}$ is a monomorphism. Since an isomorphism is in particular a monomorphism, and since $\mathcal{X}' \to \mathcal{X}$ is representable by algebraic stacks, by Lemma 3.17 (1), it is in fact representable by algebraic spaces, by Lemma 3.4.

Consider the sequence of morphisms $U \to \mathcal{X}' \to \mathcal{X}$, each of which we know to be representable by algebraic spaces. Both the first arrow and the composite are furthermore flat, locally of finite presentation, and surjective. Since each of these properties may be checked *fppf* locally on the source, we find that the second arrow also has these properties. In conclusion, the morphism $\mathcal{X}' \to \mathcal{X}$ is representable by algebraic spaces, a monomorphism, flat, locally of finite presentation, and surjective. It is thus an isomorphism [Stacks, Tag 05W5]. □

We also have the following variant of the preceding result.

Lemma 3.19. *Suppose that $\mathcal{X}$ is a stack over S, that $U \to \mathcal{X}$ is a morphism from a sheaf to $\mathcal{X}$ which is representable by algebraic spaces, and that the projections $R := U \times_{\mathcal{X}} U \rightrightarrows U$ (which are again representable by algebraic spaces, being the base-change of morphisms which are so representable) are flat and locally of finite presentation. Then the morphism $U \to \mathcal{X}$ induces a monomorphism $[U/R] \to \mathcal{X}$.*

Proof. As in the proof of Lemma 3.18, we may apply Lemma 3.17 (2), taking $R' = R$. We find that $\Delta_{\mathcal{X}'/\mathcal{X}}$ is an isomorphism, as required. □

Remark 3.20. In the particular case when $\mathcal{X}$ itself is an algebraic stack, results along the lines of the preceding lemmas are presented in [Stacks, Tag 04T4] and [Stacks, Tag 04T5].

If $\mathcal{X}$ is an algebraic stack and $U = \operatorname{Spec} k$, and the morphism $\operatorname{Spec} k \to \mathcal{X}$ is locally of finite presentation, then the projections $R := U \times_{\mathcal{X}} U \rightrightarrows U$ are automatically flat, as well as being locally of finite presentation, and the preceding lemma then gives a construction of the residual gerbe [Stacks, Tag 06ML] at the image $x \in |\mathcal{X}|$ of $\operatorname{Spec} k$.

In what follows, we will be given a stack $\mathcal{X}$, and will want to talk about various substacks of $\mathcal{X}$. We briefly digress to explain our conventions regarding substacks.

At the least, a substack of $\mathcal{X}$ should be a full subcategory of $\mathcal{X}$ that is fibred in groupoids over S, and satisfies an appropriate descent condition on objects (see e.g. [Stacks, Tag 04TU]). However, in order to avoid complications with the possibility of two substacks $\mathcal{Y}$ and $\mathcal{Z}$ of $\mathcal{X}$ being isomorphic as substacks of $\mathcal{X}$ (i.e. admitting an isomorphism of stacks, i.e. an equivalence of categories in groupoids fibred over S, in a way that the diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r] \arrow[d, "\sim"'] & \mathcal{X} \\ \mathcal{Z} \arrow[ur] & \end{tikzcd}$$

2-commutes; here the the horizontal and diagonal arrows are the inclusions of $\mathcal{Y}$ and $\mathcal{Z}$ as subcategories of $\mathcal{X}$) without being actually equal, we adopt the following convention. (Compare [Stacks, Tag 0AIN].)

Convention 3.21. If $\mathcal{X}$ is a stack (in groupoids, as always) over S, then a substack of $\mathcal{X}$ consists of a strictly full subcategory of $\mathcal{Y}$ of $\mathcal{X}$ which also forms a stack in groupoids over S.

Remark 3.22. This rigidification of the notion of substack is slightly unnatural from a 2-categorical view-point, since it places an emphasis on equality and isomorphism, rather than on equivalence. However, it will simplify some later constructions, by enforcing a certain rigidity (which, for example, will ensure the strictness of various pseudo-functors that we construct).

Remark 3.23. Any monomorphism $\mathcal{Z} \to \mathcal{X}$ of stacks over S factors uniquely as $\mathcal{Z} \to \mathcal{Y} \hookrightarrow \mathcal{X}$, where the first arrow is an isomorphism, and the second arrow is the inclusion of a substack $\mathcal{Y}$ into $\mathcal{X}$. (A monomorphism of stacks is just a fully faithful functor of the underlying categories, compatible with their structure of categories fibred over S. This factorization is then just the usual factorization of a fully faithful functor into an equivalence onto its essential image, followed by the inclusion of the essential image into the target category of the functor.)

Remark 3.24. If $\mathcal{X}$ is a stack over S, then we can speak of one substack of $\mathcal{X}$ containing another; this holds provided the set of objects of one is contained in the set of objects of the other. If $\{\mathcal{Y}\}_{i\in I}$ is a collection of substacks of S, then $\bigcap_{i\in I} \mathcal{Y}_i$ (the full subcategory of $\mathcal{X}$ whose set of objects is the intersection of the set of objects of each of the $\mathcal{Y}_i$) is again a substack of $\mathcal{X}$; this substack is maximal with respect to being contained in each of the $\mathcal{Y}_i$.

If $\mathcal{C}$ is a subcategory of $\mathcal{X}$, then we it makes sense to talk of the substack of $\mathcal{X}$ generated by $\mathcal{C}$: it is the intersection $\bigcap_{i\in I} \mathcal{Y}_i$, where $\mathcal{Y}_i$ runs over all substacks of $\mathcal{X}$ containing $\mathcal{C}$.

In particular, if $\mathcal{Y}$ and $\mathcal{Z}$ are two substacks of $\mathcal{X}$, then we may form the substack $\mathcal{Y} \cap \mathcal{Z}$, which realizes the 2-fibre product $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Z}$ as a substack of $\mathcal{X}$.

Remark 3.25. Related to the final paragraph of the preceding remark, we make the following observation: In general, if $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ are morphisms of stacks, then the fibre product (which of course means the 2-fibre product) $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is only defined up to isomorphism of stacks, i.e. up to equivalence of categories fibred in groupoids over S. However, if $\mathcal{Z} \to \mathcal{Y}$ is the inclusion of a substack (which is in particular a monomorphism of stacks, i.e. a fully faithful functor), then (following the prescription of Remark 3.23), we may consider the essential image of the induced monomorphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{X}$, which is a substack of $\mathcal{X}$. This substack then gives a canonical choice of model of $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$; concretely, it is the full subcategory of $\mathcal{X}$ that contains precisely those objects whose images in $\mathcal{Y}$ are objects of $\mathcal{Z}$.

If we interpret $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ in this way, then $\mathcal{X} \mapsto \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ becomes a strict pseudo-functor (in the sense of Definition 2.13) from $\mathrm{St}/\mathcal{Y} \to \mathrm{St}/\mathcal{Z}$. (Here we are denoting by St the 2-category of stacks fibred in groupoids over Sch/S with the *fppf* topology, and then following Definition 2.51 to define the 2-categories $\mathrm{St}/\mathcal{X}$ and $\mathrm{St}/\mathcal{Y}$ of stacks lying over $\mathcal{Y}$ and $\mathcal{Z}$ respectively.) We apply this observation in various contexts below.

Definition 3.26. If $\mathcal{X}$ is a stack over S, then we say that a substack $\mathcal{Y}$ of $\mathcal{X}$ is open (resp. closed, resp. locally closed) if the inclusion $\mathcal{Y} \hookrightarrow \mathcal{X}$ is representable by algebraic spaces, and induces an open (resp. closed, resp. locally closed) immersion when pulled-back along any morphism $T \to \mathcal{X}$ whose source is a scheme. (Note that this inclusion is then in fact representable (by schemes), since a locally closed sub-algebraic space of a scheme is in fact a scheme.)

We now turn to defining, and proving the basic properties of, the underlying reduced substack of a stack over S. To this end, we let $(\mathrm{Sch}/S)_{\mathrm{red}}$ denote the full subcategory of Sch/S consisting of reduced schemes.

Definition 3.27. Let $\mathcal{X}$ be a stack (fibered in groupoids, as always) over S, and let $F : \mathcal{X} \to \mathrm{Sch}/S$ be the functor which realizes $\mathcal{X}$ as a stack over S. We then let $(\mathcal{X}_{\mathrm{red}})'$ denote the full subcategory of $\mathcal{X}$ whose set of objects consists of those objects x of $\mathcal{X}$ for which $F(x)$ is an object of $(\mathrm{Sch}/S)_{\mathrm{red}}$, and let $\mathcal{X}_{\mathrm{red}}$ denote the substack of $\mathcal{X}$ generated by $(\mathcal{X}_{\mathrm{red}})'$ (in the sense discussed in Remark 3.24). We refer to $\mathcal{X}_{\mathrm{red}}$ as the *underlying reduced substack* of $\mathcal{X}$.

Remark 3.28. If $\mathcal{Y} \to \mathcal{X}$ is a morphism of stacks over S (i.e. a functor of categories fibred in groupoids over S), then this morphism evidently restricts to a functor $(\mathcal{Y}_{\mathrm{red}})' \to (\mathcal{X}_{\mathrm{red}})'$ of categories over Sch/S, and hence also restricts to a functor $\mathcal{Y}_{\mathrm{red}} \to \mathcal{X}_{\mathrm{red}}$ of the corresponding reduced substacks. Thus the formation of $\mathcal{X}_{\mathrm{red}}$ is a strict pseudo-functor (in the sense of Definition (2.13)) from the 2-category of stacks over S to itself.

Remark 3.29. If $\mathcal{X}$ is an algebraic stack, then $\mathcal{X}_{\mathrm{red}}$ is precisely the underlying reduced algebraic substack of $\mathcal{X}$. (To see this, let $\mathcal{Z}$ denote the underlying reduced algebraic substack of $\mathcal{X}$, and first note that if $T \to \mathcal{X}$ is a morphism whose source is reduced, then this morphism factors through $\mathcal{Z}$; thus $(\mathcal{X}_{\mathrm{red}})'$ is contained in the substack $\mathcal{Z}$ of $\mathcal{X}$, and hence so is $\mathcal{X}_{\mathrm{red}}$. Conversely, choose a smooth surjective morphism $U \to \mathcal{X}$ whose source is a scheme, which then induces a smooth surjective morphism $U_{\mathrm{red}} \to \mathcal{Z}$. The composite morphism $U_{\mathrm{red}} \to \mathcal{Z} \hookrightarrow \mathcal{X}$ then gives an object of $(\mathcal{X}_{\mathrm{red}})'$, and hence of $\mathcal{X}_{\mathrm{red}}$. The stack property of $\mathcal{X}_{\mathrm{red}}$ then shows that the morphism $\mathcal{Z} \hookrightarrow \mathcal{X}$ factors through $\mathcal{X}_{\mathrm{red}}$.)

Remark 3.30. If X is a formal algebraic space over S, regarded as a stack in setoids over S, then an argument essentially identical to that of Remark 3.29 shows that the substack X_{red}, which corresponds to a subfunctor of X, is equal to X_{red}, the underlying reduced algebraic space of X, as constructed in [Stacks, Tag 0AIN].

Remark 3.31. If $\mathcal{X}$ and $\mathcal{Z}$ are stacks over S, equipped with a morphism to the stack $\mathcal{Y}$ over S, then the inclusions $\mathcal{X}_{\mathrm{red}} \hookrightarrow \mathcal{X}$ and $\mathcal{Z}_{\mathrm{red}} \hookrightarrow \mathcal{Z}$ induce an isomorphism

$$(\mathcal{X}_{\mathrm{red}} \times_{\mathcal{Y}_{\mathrm{red}}} \mathcal{Z}_{\mathrm{red}})_{\mathrm{red}} \xrightarrow{\sim} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})_{\mathrm{red}}$$

(as one easily checks using the characterization of the various reduced substacks involved; note also that we are using Remark 3.28 to see that the given morphisms $\mathcal{X}, \mathcal{Z} \to \mathcal{Y}$ restrict to morphisms $\mathcal{X}_{\mathrm{red}}, \mathcal{Z}_{\mathrm{red}} \to \mathcal{Y}_{\mathrm{red}}$).

We have the following more explicit description of the substack $\mathcal{X}_{\mathrm{red}}$ of $\mathcal{X}$.

Lemma 3.32. *If $\mathcal{X}$ is a stack over S, then a morphism $T \to \mathcal{X}$ whose source is a scheme factors through the underlying reduced substack $\mathcal{X}_{\mathrm{red}}$ if and only if there exists a morphism of schemes $T' \to T$ which is surjective, flat, and locally of finite presentation, such that the composite $T' \to T \to \mathcal{X}$ factors through a reduced scheme.*

Proof. Let $\mathcal{Z}$ denote the full subcategory of $\mathcal{X}$ whose objects consist of all morphisms $T \to \mathcal{X}$ satisfying the criterion of the lemma, i.e. for which there a morphism of schemes that is surjective, flat, and locally of finite presentation, such that the composite $T' \to \mathcal{X}$ factors as $T' \to T'' \to \mathcal{X}$, where T'' is reduced.

We first note that $\mathcal{Z}$ certainly contains $(\mathcal{X}_{\mathrm{red}})'$ (which classifies the morphisms $T \to \mathcal{X}$ for which T itself is reduced). We next show that $\mathcal{Z}$ is contained in $\mathcal{X}_{\mathrm{red}}$. Indeed, if $T \to \mathcal{X}$ is a morphism which is an object of $\mathcal{Z}$, and T' and T'' are above, then certainly the morphism $T'' \to \mathcal{X}$ lies in $\mathcal{X}_{\mathrm{red}}$ (since it lies in $(\mathcal{X}_{\mathrm{red}})'$), and thus so does its pull-back $T' \to \mathcal{X}$ (since $\mathcal{X}_{\mathrm{red}}$ is a category fibred in groupoids over S). The stack property of $\mathcal{X}_{\mathrm{red}}$ then implies that the originally given morphism $T \to \mathcal{X}$ must be an object of $\mathcal{X}_{\mathrm{red}}$.

Finally, we show that $\mathcal{Z}$ is itself a substack of $\mathcal{X}$. Since it contains $(\mathcal{X}_{\mathrm{red}})'$ and is contained in $\mathcal{X}_{\mathrm{red}}$, this will show that $\mathcal{Z} = \mathcal{X}_{\mathrm{red}}$, and so prove the lemma.

Suppose, then, that $T \to \mathcal{X}$ is an object of $\mathcal{Z}$. If $U \to T$ is a morphism of schemes, then the projection $U' := T' \times_T U$ is surjective, flat, and locally of finite presentation, and the composite morphism $U' \to U \to T \to \mathcal{X}$ admits the alternate factorization $U' \to T' \to T \to \mathcal{X}$, and thus factors through a reduced scheme (since $T' \to T \to \mathcal{X}$ does). Thus $\mathcal{Z}$ is a subcategory fibred in groupoids of $\mathcal{X}$.

Lastly, suppose given a morphism $T \to \mathcal{X}$, and a morphism $T' \to T$ which is surjective, flat, and locally of finite presentation, such that the composite $T' \to T \to \mathcal{X}$ is an object of $\mathcal{Z}$. By definition, then, there exists a morphism $T'' \to T'$ that is surjective, flat, and locally of finite presentation, such that the composite $T'' \to T' \to T \to \mathcal{X}$ factors through a reduced scheme. Since the composite $T'' \to T' \to T$ is surjective, flat, and locally of finite presentation, we see that in fact the morphism $T \to \mathcal{X}$ is an object of $\mathcal{Z}$, and so $\mathcal{Z}$ is indeed a substack of $\mathcal{X}$. $\square$

We now prove some lemmas concerning underlying reduced substacks which hold in the level of generality of Definition 3.27.

Lemma 3.33. *If $\mathcal{X}$ is a stack over S, then $(\mathcal{X}_{\mathrm{red}})_{\mathrm{red}} = \mathcal{X}_{\mathrm{red}}$.*

Proof. This follows immediately from the explicit description of $\mathcal{X}_{\mathrm{red}}$ given by Lemma 3.32. $\square$

Lemma 3.34. *If $\mathcal{Y} \to \mathcal{X}$ is a morphism of stacks over S that is representable by algebraic stacks, and for which the induced morphism $\mathcal{Y}_{\mathrm{red}} \to \mathcal{X}_{\mathrm{red}}$ is also representable by algebraic stacks, then $\mathcal{Y} \to \mathcal{X}$ is surjective if and only if $\mathcal{Y}_{\mathrm{red}} \to \mathcal{X}_{\mathrm{red}}$ is surjective.*

Proof. Write $\mathcal{Y}' := \mathcal{X}_{\mathrm{red}} \times_{\mathcal{X}} \mathcal{Y}$, regarded as a substack as $\mathcal{Y}$ (following the prescription of Remark 3.25); we then have the commutative diagram of morphisms of stacks

$$\begin{tikzcd} \mathcal{Y}_{\mathrm{red}} \arrow[r] \arrow[dr] & \mathcal{Y}' \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d] \\ & \mathcal{X}_{\mathrm{red}} \arrow[r] & \mathcal{X} \end{tikzcd}$$

in which the horizontal arrows are inclusions of substacks, while the diagonal and vertical arrows are representable by algebraic stacks. Passing to underlying reduced substacks of the various stacks appearing in the top row, and taking into account Lemma 3.33, we find that $(\mathcal{Y}')_{\mathrm{red}} = \mathcal{Y}_{\mathrm{red}}$.

Let $T \to \mathcal{X}$ be a morphism whose source is a scheme, and let $T_{\mathrm{red}} \to \mathcal{X}_{\mathrm{red}}$ be the induced morphism. Since $T_{\mathrm{red}} \hookrightarrow T$ is a thickening of schemes,

$$T_{\mathrm{red}} \times_{\mathcal{X}_{\mathrm{red}}} \mathcal{Y}' \xrightarrow{\sim} T_{\mathrm{red}} \times_{\mathcal{X}} \mathcal{Y} \hookrightarrow T \times_{\mathcal{X}} \mathcal{Y}$$

is a thickening of algebraic stacks, and so $T \times_{\mathcal{X}} \mathcal{Y} \to T$ is surjective if and only $T_{\mathrm{red}} \times_{\mathcal{X}_{\mathrm{red}}} \mathcal{Y}' \to T_{\mathrm{red}}$ is surjective. Replacing $\mathcal{X}$ by $\mathcal{X}_{\mathrm{red}}$ and $\mathcal{Y}$ by $\mathcal{Y}'$ (as noted above, these latter two stacks have the same underlying reduced substack), we may thus assume that $\mathcal{X} = \mathcal{X}_{\mathrm{red}}$ (so that also $\mathcal{Y} = \mathcal{Y}'$), and it suffices to show that if T is a scheme, then the projection

$$T \times_{\mathcal{X}} \mathcal{Y}_{\mathrm{red}} \to T \tag{3.35}$$

is surjective if and only if the projection

$$T \times_{\mathcal{X}} \mathcal{Y} \to T \tag{3.36}$$

is surjective.

Let $V \to T \times_{\mathcal{X}} \mathcal{Y}$ be a surjective morphism whose source is a scheme (such a morphism exists, since the target is an algebraic stack). Then the composite $V_{\mathrm{red}} \to V \to T \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$ factors through $\mathcal{Y}_{\mathrm{red}}$. Thus the composite $V_{\mathrm{red}} \to V \to T \times_{\mathcal{X}} \mathcal{Y}$ factors through a morphism $V_{\mathrm{red}} \to T \times_{\mathcal{X}} \mathcal{Y}_{\mathrm{red}}$.

Since $T \times_{\mathcal{X}} \mathcal{Y}_{\mathrm{red}} \hookrightarrow T \times_{\mathcal{X}} \mathcal{Y}$ is a monomorphism of algebraic stacks (and thus universally injective; this is a special case of Remark 3.15), while the composite

$$V_{\mathrm{red}} \to T \times_{\mathcal{X}} \mathcal{Y}_{\mathrm{red}} \hookrightarrow T \times_{\mathcal{X}} \mathcal{Y}$$

is surjective, we conclude that the second arrow in this composite is in fact universally *bijective*. Thus either of the projections (3.35) or (3.36) is surjective if and only if the other is. This completes the proof of the lemma. □

Lemma 3.37. *If $\mathcal{U} \to \mathcal{X}$ is a morphism of stacks over S which is representable by algebraic stacks and smooth, then the induced morphism $\mathcal{U}_{\mathrm{red}} \to \mathcal{U} \times_{\mathcal{X}} \mathcal{X}_{red}$ is an isomorphism.*

Proof. Following the prescription of Remark 3.25, we regard $\mathcal{U} \times_{\mathcal{X}} \mathcal{X}_{\mathrm{red}}$ as a substack of $\mathcal{U}$. As already observed in the statement of the lemma, the fact that the given morphism $\mathcal{U} \to \mathcal{X}$ induces a morphism $\mathcal{U}_{\mathrm{red}} \to \mathcal{X}_{\mathrm{red}}$ shows that this substack contains the substack $\mathcal{U}_{\mathrm{red}}$ of $\mathcal{U}$. To prove the lemma, we have to show the reverse inclusion.

Thus suppose $T \to \mathcal{U}$ is an object of the fibre product, which is to say that the composite $T \to \mathcal{U} \to \mathcal{X}$ factors through $\mathcal{X}_{\mathrm{red}}$. Lemma 3.32 shows that we may find a morphism of schemes $T' \to T$ which is surjective, flat, and locally of finite presentation, for which the composite $T' \to T \to \mathcal{U} \to \mathcal{X}$ factors as $T' \to T'' \to \mathcal{X}$, with T'' is reduced. The morphisms $T' \to T''$ and $T' \to T \to \mathcal{U}$ induce a morphism

$T' \to T'' \times_{\mathcal{X}} \mathcal{U}$, whose target (by virtue of our assumptions on the morphism $\mathcal{U} \to \mathcal{X}$) is an algebraic stack for which the projection to T'' is smooth. Since T'' is reduced, so is the fibre product $T'' \times_{\mathcal{X}} \mathcal{U}$, and so we may choose a surjective smooth morphism $V \to T'' \times_{\mathcal{X}} \mathcal{U}$ whose source is a reduced scheme.

Since the diagonal of the algebraic stack $T'' \times_{\mathcal{X}} \mathcal{U}$ is representable by algebraic spaces, the fibre product $T'_{T'' \times_{\mathcal{X}} \mathcal{U}} V$ is an algebraic space, and so we may find an étale surjective morphism $V' \to T'_{T'' \times_{\mathcal{X}} \mathcal{U}} V$ whose source is a scheme. The projection $V' \to T'$ is surjective and smooth, and thus the composite $V' \to T' \to T$ is surjective, flat, and locally of finite presentation. Furthermore, the composite $V' \to T' \to T \to \mathcal{U}$ admits the alternative factorization $V' \to V \to T'' \times_{\mathcal{X}} \mathcal{U} \to \mathcal{U}$. Since V is reduced, we find (again applying Lemma 3.32) that the original morphism $T \to \mathcal{U}$ is an object of $\mathcal{U}_{\mathrm{red}}$, as required. □

We next prove a general version of the topological invariance of the small Zariski and étale sites. We begin by stating a general form of the infinitesimal lifting property for an étale morphism of stacks.

Lemma 3.38. *If $\mathcal{U} \to \mathcal{X}$ is a morphism of stacks which is DM and étale, if $\mathcal{Z}_0 \hookrightarrow \mathcal{Z}$ is a closed immersion of algebraic stacks that is a thickening, and if we are given a morphism $\mathcal{Z} \to \mathcal{X}$ (which induces the composite morphism $\mathcal{Z}_0 \hookrightarrow \mathcal{Z} \to \mathcal{X}$), then the morphism of groupoids $\mathrm{Mor}_{\mathrm{St}/\mathcal{X}}(\mathcal{Z},\mathcal{U}) \to \mathrm{Mor}_{\mathrm{St}/\mathcal{X}}(\mathcal{Z}_0,\mathcal{U})$ is an equivalence.*

Proof. We have canonical equivalences $\mathrm{Mor}_{\mathrm{St}/\mathcal{X}}(\mathcal{Z}_0,\mathcal{U}) \xrightarrow{\sim} \mathrm{Mor}_{\mathrm{St}/\mathcal{Z}}(\mathcal{Z}_0,\mathcal{U} \times_{\mathcal{X}} \mathcal{Z})$ and $\mathrm{Mor}_{\mathrm{St}/\mathcal{X}}(\mathcal{Z},\mathcal{U}) \xrightarrow{\sim} \mathrm{Mor}_{\mathrm{St}/\mathcal{Z}}(\mathcal{Z},\mathcal{U} \times_{\mathcal{X}} \mathcal{Z})$. Thus we may assume that $\mathcal{Z}$ equals $\mathcal{X}$, and so in particular reduce to the case when $\mathcal{U} \to \mathcal{X}$ is an étale morphism of algebraic stacks. Thus we suppose from now that we are in this situation (but we don't restrict to the case when $\mathcal{Z}$ equal $\mathcal{X}$).

We first suppose that $\mathcal{Z} = T$ is an affine scheme, so that T_0 is a closed subscheme, with $T_0 \hookrightarrow T$ being a thickening. In this case the statement of the lemma follows from the infinitesimal criterion as stated in [Ryd11, Cor. B.9].

Using the stack property of $\mathcal{U}$ to glue morphisms, one then extends successively to the case when $\mathcal{Z}$ is an arbitrary scheme, to the case when when $\mathcal{Z}$ is an algebraic space, and finally to the case when $\mathcal{Z}$ is an algebraic stack. (We leave the details to the reader.) □

We next define the small étale and small Zariski 2-categories attached to a stack $\mathcal{X}$ over S.

Definition 3.39. If $\mathcal{X}$ is a stack over S, and if $\mathrm{St}/\mathcal{X}$ denotes the 2-category of stacks lying over $\mathcal{X}$ (following the notation of Definition 2.51), then we let $\mathrm{Et}_{\mathcal{X}}$ denote the strong subcategory (in the sense of Definition 2.19) of $\mathrm{St}/\mathcal{X}$ whose objects are the 1-morphisms $\mathcal{U} \to \mathcal{X}$ of stacks which are DM and étale.

We let $\mathrm{Zar}_{\mathcal{X}}$ denote the strong subcategory of $\mathrm{St}/\mathcal{X}$ consisting of morphisms $\mathcal{U} \to \mathcal{X}$ which are monomorphisms.

Remark 3.40. Recall that open immersions of schemes coincide with étale monomorphisms [Stacks, Tag 025G]. This statement generalizes to morphisms $\mathcal{U} \to \mathcal{X}$ of stacks which are representable by algebraic stacks and étale — such a morphism is an open immersion if and only if its a monomorphism. (The only direction is clear. Conversely, if $\mathcal{U} \to \mathcal{X}$ is a monomorphism, then $\Delta_{\mathcal{U}/\mathcal{X}}$ is an isomorphism, and hence in particular a monomorphism, and so $\mathcal{U} \to \mathcal{X}$ is representable by algebraic spaces. If $T \to \mathcal{X}$ is a morphism from a scheme, then $T \times_{\mathcal{X}} \mathcal{U} \to T$ is an étale monomorphism from an algebraic space to a scheme, hence an étale monomorphism from a scheme to a scheme [Stacks, Tag 0418], and hence an open immersion.) This explains our choice of notation $\mathrm{Zar}_{\mathcal{X}}$.

We can now prove our result on topological invariance.

Lemma 3.41. *If $\mathcal{X}_0 \to \mathcal{X}$ is the inclusion of a closed substack into a stack over S, which is furthermore a thickening, then the strict pseudo-functor $\mathcal{U} \mapsto \mathcal{U}_0 := \mathcal{U} \times_{\mathcal{X}} \mathcal{X}_0$ (see Remark 3.25 for an explanation of the strictness) induces 2-equivalences (in the sense of Definition 2.16 (2)) of 2-categories $\mathrm{Et}_{\mathcal{X}} \to \mathrm{Et}_{\mathcal{X}_0}$ and $\mathrm{Zar}_{\mathcal{X}} \to \mathrm{Zar}_{\mathcal{X}_0}$.*

Proof. We begin by proving that the strict pseudo-functor $\mathrm{Et}_{\mathcal{X}} \to \mathrm{Et}_{\mathcal{X}_0}$ is fully faithful. If $\mathcal{U} \to \mathcal{X}$ and $\mathcal{V} \to \mathcal{X}$ are objects of $\mathrm{Et}_{\mathcal{X}}$, then we must show that the functor

$$\mathrm{Mor}_{\mathcal{X}}(\mathcal{U}, \mathcal{V}) \to \mathrm{Mor}_{\mathcal{X}}(\mathcal{U}_0, \mathcal{V}_0)$$

is an equivalence of categories. The fact that $\mathcal{X}_0 \hookrightarrow \mathcal{X}$ is a monomorphism, and the descriptions of $\mathcal{U}_0$ and $\mathcal{V}_0$ as fibre products, taken together show that

$$\mathrm{Mor}_{\mathcal{X}_0}(\mathcal{U}_0, \mathcal{V}_0) \to \mathrm{Mor}_{\mathcal{X}}(\mathcal{U}_0, \mathcal{V})$$

is an equivalence. Thus it suffices to show that the composite

$$\mathrm{Mor}_{\mathcal{X}}(\mathcal{U}, \mathcal{V}) \to \mathrm{Mor}_{\mathcal{X}}(\mathcal{U}_0, \mathcal{V}_0) \to \mathrm{Mor}_{\mathcal{X}}(\mathcal{U}_0, \mathcal{V})$$

is an equivalence. This follows from Lemma 3.38.

To complete the proof that $\mathrm{Et}_{\mathcal{X}} \to \mathrm{Et}_{\mathcal{X}_0}$ is a 2-equivalence, it remains to show that any object of $\mathrm{Et}_{\mathcal{X}_0}$ is equivalent to a morphism $\mathcal{U}_0 \to \mathcal{X}_0$, for some object $\mathcal{U} \to \mathcal{X}$ of $\mathrm{Et}_{\mathcal{X}}$.

If $T \to \mathcal{X}$ is a morphism with T a scheme, write $T_0 := T \times_{\mathcal{X}} \mathcal{X}_0$, so that T_0 is a closed subscheme of T, with $T_0 \hookrightarrow T$ being a thickening. Given $\mathcal{U}_0 \to \mathcal{X}_0$ as above, we then define, for any morphism of schemes $T \to S$, a groupoid $\mathcal{U}(T)$ via

$$\mathcal{U}(T) := \mathcal{X}(T) \times_{\mathcal{X}_0(T_0)} \mathcal{U}_0(T_0).$$

In this manner we obtain a category fibred in groupoids $\mathcal{U}$ over S, which is in fact a stack, since each of $\mathcal{X}$, $\mathcal{X}_0$, and $\mathcal{U}_0$ is a stack. By construction, we also have a morphism $\mathcal{U} \to \mathcal{X}$, and a morphism $\mathcal{U} \times_{\mathcal{X}} \mathcal{X}_0 \to \mathcal{U}_0$, which one verifies is an isomorphism. It remains to show that this morphism is DM and étale.

If we fix $T \to \mathcal{X}$, then we see that $T \times_{\mathcal{X}} \mathcal{U}$ is equivalent to the stack over T which associates, to a morphism of schemes $T' \to T$, the groupoid $(T \times_{\mathcal{X}_0} \mathcal{U}_0)(T'_0)$. To simplify notation, then, we replace $\mathcal{X}$ by T, and thus assume given a thickening $T_0 \hookrightarrow T$ and a Deligne–Mumford stack $\mathcal{U}_0$ equipped with an étale morphism $\mathcal{U}_0 \to T_0$; our goal is to show that $\mathcal{U}(T') := \mathcal{U}_0(T'_0)$ defines a Deligne–Mumford stack which is étale over T. We may write $\mathcal{U}_0 := [U_0/R_0]$ for some scheme étale over T_0, and some étale groupoid in algebraic spaces R_0 over U_0. The topological invariance of the étale site for algebraic spaces [Stacks, Tag 05ZG] then lifts U_0 to an étale morphism $U \to T$, and R_0 to an étale groupoid in algebraic spaces over U. The tautological morphism $U_0 \to [U_0/R_0] := \mathcal{U}_0$ induces a morphism $U \to \mathcal{U}$ (by definition of $\mathcal{U}$), which is easily verified to induce an isomorphism $[U/R] \xrightarrow{\sim} \mathcal{U}$. Thus $\mathcal{U}$ is a Deligne–Mumford stack which is étale over T, as required.

The statement concerning Zariski topologies follows from the fact that, since $\mathcal{X}_0 \hookrightarrow \mathcal{X}$ is a thickening, an étale morphism $\mathcal{U} \to \mathcal{X}$ is a monomorphism if and only if $\mathcal{U}_{\mathrm{red}} \to \mathcal{X}_{\mathrm{red}}$ is (by [Stacks, Tag 0CJC]). $\square$

4. Ind-algebraic stacks

In this section, we use the theory of 2-colimits presented in Section 2 to define Ind-algebraic stacks, and establish some simple results concerning them.

We first recall the notion of an Ind-algebraic space.

Definition 4.1. If S is a scheme, then a sheaf X on the *fppf* site of S is called an *Ind-algebraic space* if there exists a directed system $\{X_i\}_{i \in I}$ of algebraic spaces over S and an isomorphism $\varinjlim_i X_i \cong X$, where the inductive limit is computed in the category of sheaves on the *fppf* site of S. We say that X is an *Ind-scheme* if the X_i can be taken to be schemes.

We wish to extend this notion so as to define a notion of Ind-algebraic stack. Out of habit and old-fashioned convention, we will usually use the notation $\varinjlim$ rather than $2-\operatorname{colim}$. We also frequently write "2-directed system" to mean a 2-diagram with a filtered index category.

Definition 4.2. A stack in groupoids $\mathcal{X}$ on the *fppf* site of S is called an *Ind-algebraic stack* if there exists a 2-directed system $\{\mathcal{X}_i\}_{i \in I}$ of algebraic stacks over S and an isomorphism of stacks $\varinjlim_i \mathcal{X}_i \cong \mathcal{X}$, where the left-hand side denotes the 2-colimit computed in the 2-category of stacks in groupoids on the *fppf* site of S.

Example 4.3. We give an illustrative example of an Ind-algebraic stack. Fix a base algebraic space B (lying over the base scheme S), and let $\{X_i\}_{i \in I}$ denote a directed system of algebraic spaces that are of finite presentation and separated over B. For each X_i, we have the stack $\mathrm{Coh}_{X_i/B}$ of coherent sheaves on X_i whose support is proper over B. This is an algebraic stack [Stacks, Tag 08WB]. If $i \leq i'$, then pushforward induces a morphism $\mathrm{Coh}_{X_i/B} \to \mathrm{Coh}_{X_{i'}/B}$ [Stacks, Tag 08DS]. Since the pushforward along a composite is naturally isomorphic to the composite of the corresponding pushforwards, in a manner that is compatible with triple composites, we see that $\{\mathrm{Coh}_{X_i/B}\}_{i \in I}$ forms a 2-directed system of algebraic stacks. We may thus form the 2-colimit $\varinjlim_i \mathrm{Coh}_{X_i/B}$. This is an Ind-algebraic stack, which morally may be regarded as the stack of coherent sheaves on the Ind-algebraic space $X := \varinjlim_i X_i$ whose support is proper over B.

We now establish some simple results related to Ind-algebraic stacks and morphisms between them.

Lemma 4.4. *If $\mathcal{X}$ is an Ind-algebraic stack, and if $\varinjlim_i \mathcal{X}_i \xrightarrow{\sim} \mathcal{X}$ is an isomorphism, where $\{\mathcal{X}_i\}_{i \in \mathcal{I}}$ is a 2-directed system of algebraic stacks, then there is a natural morphism of groupoids $2-\operatorname{colim}_i \mathcal{X}_i(T) \to \mathcal{X}(T)$. If T is quasi-compact, then this morphism is fully faithful. If in addition either T is quasi-separated or the transition morphisms in the 2-directed system $\{\mathcal{X}_i\}$ are monomorphisms, then this morphism is an equivalence.*

Proof. This follows from Lemma 2.50. □

Lemma 4.5. *Let $\{\mathcal{X}_i\}_{i \in \mathcal{I}}$ be a 2-directed system of algebraic stacks, all of whose transition morphisms are monomorphisms (i.e. fully faithful). If $\varinjlim_i \mathcal{X}_i \xrightarrow{\sim} \mathcal{X}$, then each of the induced morphisms*

$$\mathcal{X}_i \to \mathcal{X} \tag{4.6}$$

is a monomorphism, and is representable by algebraic stacks.

Furthermore, if P is a property of morphisms of algebraic stacks that is preserved under base-change and fppf local on the target, then each of the transition morphisms in the 2-directed system $\{\mathcal{X}_i\}$ has property P if and only if each of the induced morphisms (4.6) *has property P (in the sense of Definition* 3.11*).*

Proof. To show that the induced morphism (4.6) is a monomorphism, it suffices to evaluate it on morphisms $T \to \mathcal{X}$, with T an affine (and hence quasi-compact and quasi-separated) scheme. By Lemma 4.4, it then suffices to show that the natural morphism $\mathcal{X}_i(T) \to 2\text{-colim}_i\, \mathcal{X}_i(T)$ is fully faithful, given that each of the transition morphisms in the 2-colimit is fully faithful. This follows from Lemma 2.49.

To prove that (4.6) is representable by algebraic spaces, we again consider morphisms $T \to \mathcal{X}$ with T being affine. As already noted, such a morphism factors through $\mathcal{X}_{i'}$ for some $i' \geq i$, and since (by the result of the preceding paragraph) the morphism $\mathcal{X}_{i'} \to \mathcal{X}$ is a monomorphism, the canonical morphism

$$T \times_{\mathcal{X}_{i'}} \mathcal{X}_i \to T \times_{\mathcal{X}} \mathcal{X}_i$$

is an isomorphism. The source of this morphism is an algebraic stack, since it is the 2-fibre product of algebraic stacks. Thus the target is also an algebraic stack, which (taking into account Remark 3.2) is what we had to show.

To prove the final claim of the lemma, suppose first that each transition morphism has property P. From the argument of the preceding paragraph, we obtain an isomorphism $T \times_{\mathcal{X}_{i'}} \mathcal{X}_i \xrightarrow{\sim} T \times_{\mathcal{X}} \mathcal{X}_i$ of algebraic stacks over T. By assumption the projection of the source onto T has property P, and hence so does the projection of the target onto T. Thus (taking into account Remark 3.12), we find that (4.6) has property P, as required. Conversely, suppose that each of these latter morphisms has property P, and let $\mathcal{X}_i \to \mathcal{X}_{i'}$ be a transition morphism. Since (4.6) is a monomorphism, in the 2-commutative diagram

$$\begin{tikzcd} \mathcal{X}_i \times_{\mathcal{X}} \mathcal{X}_{i'} \arrow[d] \arrow[dr] & \\ \mathcal{X}_i \arrow[r] & \mathcal{X}_{i'} \end{tikzcd}$$

we find that vertical morphism is an isomorphism. The diagonal morphism has property P, being the base-change to $\mathcal{X}_{i'}$ of (4.6) (which has property P by assumption), and thus so does the horizontal arrow. $\square$

We refer to [Stacks, Tag 07XL], for the definition of a stack over S being *limit preserving.*

Lemma 4.7. *Suppose that $\{\mathcal{X}_i\}_{i\in\mathcal{I}}$ is a 2-directed system of algebraic stacks, each locally of finite presentation over S. If $\mathcal{X} \xrightarrow{\sim} \varinjlim_i \mathcal{X}_i$, then $\mathcal{X}$ is limit preserving.*

Proof. For each of the algebraic stacks $\mathcal{X}_i$ over S, being locally of finite presentation is equivalent to being limit preserving [EG19b, Lem. 2.1.9]. Since the property of being limit preserving is tested on affine S-schemes (by definition), and affine schemes are in particular quasi-compact, the present lemma follows from Lemma 4.4. $\square$

Lemma 4.8. *Suppose that we are given stacks $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ over S, and morphisms $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$, latter being representable by algebraic stacks. Suppose further that $\mathcal{X}$ is an Ind-algebraic stack, and let $\{\mathcal{X}_i\}_{i\in I}$ be a 2-directed system of algebraic* *stacks for which there is an isomorphism* $\varinjlim_i \mathcal{X}_i \xrightarrow{\sim} \mathcal{X}$*. Then* $\{\mathcal{X}_i \times_{\mathcal{Y}} \mathcal{Z}\}_{i\in I}$ *again forms a 2-directed system of algebraic stacks, and there is an isomorphism*

$$\varinjlim_i \mathcal{X}_i \times_{\mathcal{Y}} \mathcal{Z} \xrightarrow{\sim} \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}.$$

In particular, $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ *is again an Ind-algebraic stack.*

Proof. Since $\mathcal{Z} \to \mathcal{Y}$ is representable by algebraic stacks (by assumption), the 2-fibre products $\mathcal{X}_i \times_{\mathcal{Y}} \mathcal{Z}$ are indeed algebraic stacks. Each of the morphisms $\mathcal{X}_i \to \mathcal{X}$ induces a morphism $\mathcal{X}_i \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$, which are compatible (in a 2-categorical sense) as i varies. Thus we obtain a morphism $\varinjlim_i \mathcal{X}_i \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$, which we must show is an isomorphism of stacks. To check this, it suffices to show that it induces an equivalence of groupoids after evaluating on any affine S-scheme. This is easily verified using Lemma 2.50 □

Remark 4.9. A useful special case of the preceding lemma comes by setting $\mathcal{X}$ equal to $\mathcal{Y}$; namely, we see that if $\mathcal{Z} \to \mathcal{Y}$ is a morphism of stacks over S that is representable by algebraic stacks, and if $\mathcal{Y}$ is an Ind-algebraic stack, then so is $\mathcal{Z}$. More precisely, if $\{\mathcal{Y}_i\}_{i\in I}$ is a 2-directed system of algebraic stacks for which there is an isomorphism $\varinjlim \mathcal{Y}_i \xrightarrow{\sim} \mathcal{Y}$, then each of 2-fibre products $\mathcal{Z}_i := \mathcal{Y}_i \times_{\mathcal{Y}} \mathcal{Z}$ is an algebraic stack, which fit into a 2-directed system $\{\mathcal{Z}_i\}$, and there is an isomorphism $\varinjlim \mathcal{Z}_i \xrightarrow{\sim} \mathcal{Z}$.

We now state and prove a lemma that provides a supplement to the preceding remark, but before doing so, we introduce some notation and conventions. Namely, suppose that $\{\mathcal{X}_i\}_{i\in\mathcal{I}}$ and $\{\mathcal{Y}_i\}_{i\in\mathcal{I}}$ are two 2-directed systems of algebraic stacks, with the same indexing category $\mathcal{I}$. We note that a 1-morphism between these 2-directed systems (in the sense of Definition 2.4) consists of giving a morphism of stacks $f_i : \mathcal{X}_i \to \mathcal{Y}_i$,for each object i of $\mathcal{I}$, such that for each $i' \geq i$ the square

$$\begin{array}{ccc} \mathcal{X}_i & \longrightarrow & \mathcal{X}_{i'} \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i'}} \\ \mathcal{Y}_i & \longrightarrow & \mathcal{Y}_{i'} \end{array} \tag{4.10}$$

(the horizontal arrows being the transition morphisms in the respective 2-directed systems) is 2-commutative. In fact, the 2-commutativity constraints of these diagrams are part of the data of the 1-morphism, and there is an additional condition on the 2-commutativity constraints (conditions (4) and (5) of Definition 2.4); however, in practice we won't introduce notation for these constraints, and will denote the 1-morphism simply by $\{f_i\}_{i\in\mathcal{I}}$.

Lemma 2.41 shows that the 1-morphism $\{f_i\}$ induces a limiting morphism

$$\varinjlim_i f_i : \varinjlim_i \mathcal{X}_i \to \varinjlim_i \mathcal{Y}_i.$$

Lemma 4.11. *Suppose that* $\{f_i\} : \{\mathcal{X}_i\}_{i\in I} \to \{\mathcal{Y}_i\}$ *is a* 1*-morphism between two* 2*-directed systems of algebraic stacks* (*with the same indexing category* $\mathcal{I}$)*. Let* $f : \mathcal{X} := \varinjlim_i \mathcal{X}_i \to \varinjlim_i \mathcal{Y}_i =: \mathcal{Y}$ *denote the induced morphism* $\varinjlim_i f_i$*.*

(1) If each of the morphisms f_i *is a monomorphism* (*i.e. fully faithful*)*, then the morphism* f *is a monomorphism.*

(2) If each of the squares (4.10) *is* 2*-Cartesian, then the morphism f is representable by algebraic stacks, and for any $i \in I$, the canonical morphism $\mathcal{X}_i \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_i$ is an isomorphism.*

(3) Suppose that each of the squares (4.10) *is* 2*-Cartesian, and let P be a property of morphisms of algebraic stacks which is preserved by base-change and fppf local on the target. Then f has property P (in the sense of Definition* 3.11*) if and only if each f_i has property P.*

Proof. The claim of (1) may be verified by evaluating on affine schemes (which are in particular quasi-compact and quasi-separated); its proof is then a routine application of Lemma 4.4.

We turn to proving (2), and thus assume that each of the squares (4.10) is 2-Cartesian. Let

$$T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_i \tag{4.12}$$

be a morphism whose source is an affine scheme. By Lemma 4.4, we may factor the induced morphism $T \to \mathcal{X}$ (up to 2-isomorphism) through a morphism $T \to \mathcal{X}_{i'}$, for some $i' \geq i$. The induced morphism $T \to \mathcal{Y}$ then factors (up to 2-isomorphism) through a morphism $T \to \mathcal{Y}_{i'}$. Since the composites $T \to \mathcal{Y}_{i'} \to \mathcal{Y}$ and $T \to \mathcal{Y}_i \to \mathcal{Y}$ coincide (up to 2-isomorphism), we find, again by Lemma 4.4, that the composites $T \to \mathcal{Y}_i \to \mathcal{Y}_{i''}$ and $T \to \mathcal{Y}_{i'} \to \mathcal{Y}_{i''}$ are 2-isomorphic for some $i'' \geq i$. Thus the given morphism (4.12) factors (up to 2-isomorphism) through the 2-fibre product $\mathcal{X}_{i''} \times_{\mathcal{Y}_{i''}} \mathcal{Y}_i$. By assumption the canonical map $\mathcal{X}_i \to \mathcal{X}_{i''} \times_{\mathcal{Y}_{i''}} \mathcal{Y}_i$ in an isomorphism, which is to say, an equivalence of categories, and thus the given morphism factors (up to 2-isomorphism) through $\mathcal{X}_i$. Thus the canonical morphism of groupoids $\mathcal{X}_i(T) \to (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_i)(T)$ is essentially surjective. Similar arguments show that it is full and faithful, and hence an equivalence. This proves the claimed isomorphism of (2).

Now suppose that $T \to \mathcal{Y}$ is a morphism whose source is an affine scheme. By Lemma 4.4, we may factor this morphism (up to 2-isomorphism) through a morphism $T \to \mathcal{Y}_i$ for some i. We then find that

$$\mathcal{X} \times_{\mathcal{Y}} T \cong (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_i) \times_{\mathcal{Y}_i} T \cong \mathcal{X}_i \times_{\mathcal{Y}_i} T,$$

so that $\mathcal{X} \times_{\mathcal{Y}} T$ is an algebraic stack (being the 2-fibre product of algebraic stacks over an algebraic stack). Taking into account Remark 3.2, this shows that the morphism $\mathcal{X} \to \mathcal{Y}$ of (1) is representable by algebraic stacks, and completes the proof of (2).

Finally, suppose given a property P as in (3). If f has property P, then, taking into account (2), we find that each of the morphisms $f_i : \mathcal{X}_i \to \mathcal{Y}_i$ has property P. Conversely, suppose that each f_i does have property P. If $T \to \mathcal{Y}$ is morphism whose source is an affine scheme, then Lemma 4.4 again shows that this morphism may be factored (up to 2-isomorphism) through $\mathcal{Y}_i$ for some $i \in I$. Then we compute that

$$\mathcal{X} \times_{\mathcal{Y}} T \cong (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_i) \times_{\mathcal{Y}_i} T \cong \mathcal{X}_i \times_{\mathcal{Y}_i} T,$$

the last isomorphism holding by (2). Since $\mathcal{X}_i \to \mathcal{Y}_i$ has property P by assumption, and since P is preserved by base-change, we see that $\mathcal{X} \times_{\mathcal{Y}} T$ also has property P, as required. (Here we are taking into account Remark 3.12 to restrict to affine schemes T as test objects.) $\square$

The preceding lemma can be applied to deduce properties of the diagonal of an Ind-algebraic stack in certain contexts.

Lemma 4.13. *Let P be a property of morphisms of algebraic spaces which is preserved by base-change and fppf local on the target. Let $\{\mathcal{X}_i\}_{i\in\mathcal{I}}$ be a 2-directed system of algebraic stacks, in which the transition morphisms are monomorphisms, write $\varinjlim_i \mathcal{X}_i \xrightarrow{\sim} \mathcal{X}$, and suppose given a morphism $\mathcal{X}\to\mathcal{Y}$, where $\mathcal{Y}$ is a stack over S whose diagonal morphism $\mathcal{Y}\to\mathcal{Y}\times_S\mathcal{Y}$ is representable by algebraic spaces, and for which each of the diagonal morphisms $\Delta_{\mathcal{X}_i/\mathcal{Y}}:\mathcal{X}_i\to\mathcal{X}_i\times_{\mathcal{Y}}\mathcal{X}_i$ have property P. Then the diagonal morphism $\Delta_{\mathcal{X}/\mathcal{Y}}:\mathcal{X}\to\mathcal{X}\times_{\mathcal{Y}}\mathcal{X}$ is representable by algebraic spaces, and has property P.*

Proof. Form the 2-directed system $\{\mathcal{X}_i\times_{\mathcal{Y}}\mathcal{X}_i\}$. If $i\le i'$, then since $\mathcal{X}_i\to\mathcal{X}_{i'}$ is a monomorphism, the diagram

$$\begin{tikzcd}
\mathcal{X}_i \arrow[r] \arrow[d] & \mathcal{X}_{i'} \arrow[d] \\
\mathcal{X}_i\times_{\mathcal{Y}}\mathcal{X}_i \arrow[r] & \mathcal{X}_{i'}\times_{\mathcal{Y}}\mathcal{X}_{i'}
\end{tikzcd}$$

is 2-Cartesian. The present lemma thus follows from Lemma 4.11 (2) and (3). (Note that the diagonal morphism $\Delta_{\mathcal{X}/\mathcal{Y}}$ is always representable by sheaves, and hence is representable by algebraic spaces if it is representable by algebraic stacks.) $\square$

Remark 4.14. Taking $\mathcal{Y}=S$ in the preceding lemma, we find in particular that if $\mathcal{X}$ is an Ind-algebraic stack obtained by taking the 2-colimit of a 2-directed system of algebraic stacks whose transition morphisms are monomorphisms, then $\mathcal{X}\to\mathcal{X}\times_S\mathcal{X}$ is representable by algebraic spaces.

Example 4.15. Some properties P to which the preceding lemma applies are that of being a monomorphism, in which case the lemma shows that if each of the morphisms $\mathcal{X}_i\to\mathcal{Y}$ is representable by algebraic spaces, then so is the morphism $\mathcal{X}\to\mathcal{Y}$, and that of being quasi-compact and quasi-separated, in which case the lemma shows that if each of the morphisms $\mathcal{X}_i\to\mathcal{Y}$ is quasi-separated, then so is the morphism $\mathcal{X}\to\mathcal{Y}$.[6]

Lemma 4.16. *Suppose that $\mathcal{X}$ is an Ind-algebraic stack, and write $\varinjlim_i \mathcal{X}_i \xrightarrow{\sim} \mathcal{X}$, for some 2-directed system $\{\mathcal{X}_i\}_{i\in\mathcal{I}}$ of algebraic stacks. The induced morphism $\varinjlim_i(\mathcal{X}_i)_{\mathrm{red}}\to\mathcal{X}_{\mathrm{red}}$ is then an isomorphism, and so in particular $\mathcal{X}_{\mathrm{red}}$ is again an Ind-algebraic stack.*

Proof. Lemma 4.11 (2) shows that there is an induced monomorphism

$$\varinjlim_i(\mathcal{X}_i)_{\mathrm{red}}\hookrightarrow\varinjlim_i\mathcal{X}_i\xrightarrow{\sim}\mathcal{X},$$

whose essential image is then a substack of the target. We claim that this substack is precisely $\mathcal{X}_{\mathrm{red}}$. Remark 3.28 shows that this substack is contained in $\mathcal{X}_{\mathrm{red}}$. To prove the claim, then, it suffices to show that if $T\to\mathcal{X}$ is a morphism from a reduced affine scheme to $\mathcal{X}$, then we may factor this morphism (up to a 2-isomorphism) through $(\mathcal{X}_i)_{\mathrm{red}}$, for some i. This follows from Lemma 4.4, which shows that the

[6] Note that the morphism $\mathcal{X}\to\mathcal{Y}$ need not be representable by algebraic stacks, but the lemma shows that the diagonal $\mathcal{X}\to\mathcal{X}\times_{\mathcal{Y}}\mathcal{X}$ is representable by algebraic spaces, so that the notion of $\mathcal{X}\to\mathcal{Y}$ being quasi-separated makes sense.

morphism $T \to \mathcal{X}$ factors (up to a 2-isomorphism) through $\mathcal{X}_i$, for some i, and thus through $(\mathcal{X}_i)_{\mathrm{red}}$, since T is reduced. $\square$

Remark 4.17. In general, in the context of the preceding lemma, the monomorphism of Ind-algebraic stacks $\mathcal{X}_{\mathrm{red}} \hookrightarrow \mathcal{X}$ need not be representable by algebraic stacks/spaces.[7] Indeed, this is already the case with Ind-schemes, as the following example shows.

Example 4.18. Consider the Ind-scheme $X := \varinjlim_n X_n$, where

$$X_n := \operatorname{Spec} \mathbb{C}[x,y]\big(x(x-1)\cdots(x-n)y^2\big).$$

Applying Lemma 4.16, one finds that

$$X_{\mathrm{red}} = \varinjlim_n (X_n)_{\mathrm{red}} = \varinjlim_n \operatorname{Spec} \mathbb{C}[x,y]/\big(x(x-1)\cdots(x-n)y\big).$$

Applying Lemma 4.8, we then find that

$$\begin{aligned} X_0 \times_X X_{\mathrm{red}} &\xrightarrow{\sim} \varinjlim_n \big(X_0 \times_X (X_n)_{\mathrm{red}}\big) \\ &\xrightarrow{\sim} \varinjlim_n \operatorname{Spec} \mathbb{C}[x,y]/\big(xy^2, x(x-1)\cdots(x-n)y\big) \end{aligned}$$

which is an affine formal algebraic space (in the sense of [Stacks, Tag 0AI7]; see also Definition 5.1 below), but not an algebraic space.

There is one context, closely related to that of formal algebraic stacks (as we will see in the discussion of Section 6) in which we can obtain more precise control over the morphism $\mathcal{X}_{\mathrm{red}} \to \mathcal{X}$.

Lemma 4.19. *Suppose that $\mathcal{X}$ is an Ind-algebraic stack, and that we can write $\varinjlim_i \mathcal{X}_i \xrightarrow{\sim} \mathcal{X}$, for some 2-directed system $\{\mathcal{X}_i\}_{i \in \mathcal{I}}$ of algebraic stacks for which the transition morphisms are thickenings. Then $\mathcal{X}_{\mathrm{red}}$ is an algebraic stack, as well as a closed substack of $\mathcal{X}$, each of the induced morphisms $(\mathcal{X}_i)_{\mathrm{red}} \to \mathcal{X}_{\mathrm{red}}$ is an isomorphism, and the morphism $\mathcal{X}_{\mathrm{red}} \hookrightarrow \mathcal{X}$ is a thickening.*

Proof. Since the transition morphisms in the 2-directed system $\{\mathcal{X}_i\}$ are thickenings, they induce isomorphisms on underlying reduced substacks, i.e. the transition morphisms in the 2-directed system $\{(\mathcal{X}_i)_{\mathrm{red}}\}$ are isomorphisms. Taking into account Lemma 4.16, we find that indeed each of the morphisms $(\mathcal{X}_i)_{\mathrm{red}} \to \mathcal{X}_{\mathrm{red}}$ is an isomorphism, and in particular, that $\mathcal{X}_{\mathrm{red}}$ is an algebraic stack.

If $i' \geq i$, then the transition morphism $\mathcal{X}_i \to \mathcal{X}_{i'}$ is a thickening, and so in particular is a monomorphism. It thus follows from Lemma 4.5 that each of the morphisms $\mathcal{X}_i \to \mathcal{X}$ is a monomorphism. This explains why the second arrow in the following sequence of morphisms is an isomorphism:

$$(\mathcal{X}_i)_{\mathrm{red}} \xrightarrow{\sim} \mathcal{X}_i \times_{\mathcal{X}_i} (\mathcal{X}_i)_{\mathrm{red}} \xrightarrow{\sim} \mathcal{X}_i \times_{\mathcal{X}} (\mathcal{X}_i)_{\mathrm{red}} \xrightarrow{\sim} \mathcal{X}_i \times_{\mathcal{X}} \mathcal{X}_{\mathrm{red}}.$$

Clearly the first arrow is also an isomorphism, and in the preceding paragraph we showed that the third arrow is an isomorphism as well. Consequently, the composite $(\mathcal{X}_i)_{\mathrm{red}} \to \mathcal{X}_i \times_{\mathcal{X}} \mathcal{X}_{\mathrm{red}}$ is an isomorphism, and thus the morphism

$$\mathcal{X}_i \times_{\mathcal{X}} \mathcal{X}_{\mathrm{red}} \to \mathcal{X}_i \tag{4.20}$$

[7] Recall that for monomorphism of stacks, being representable by algebraic stacks is equivalent to being representable by algebraic spaces.

is a thickening.

Lemma 4.4 shows that if $T \to \mathcal{X}$ is an morphism whose source is an affine scheme, then this morphism factors (up to 2-isomorphism) through some $\mathcal{X}_i$. Thus the morphism $T \times_{\mathcal{X}} \mathcal{X}_{\mathrm{red}} \to T$ is obtained (up to 2-isomorphism) as the base-change of the morphism (4.20), and so is a thickening. This completes the proof of the lemma. $\square$

Lemma 4.21. *(1) If $\mathcal{X}$ is an Ind-algebraic stack, and if $\mathcal{U} \to \mathcal{X}$ is an object of* $\mathrm{Et}_{\mathcal{X}}$*, then $\mathcal{U}$ is again an Ind-algebraic stack.*

(2) Suppose furthermore that there is an isomorphism $\varinjlim_i \mathcal{X}_i \xrightarrow{\sim} \mathcal{X}$, where $\{\mathcal{X}_i\}_{i\in\mathcal{I}}$ is a 2-directed system of algebraic stacks for which the transition morphisms are thickenings. Then the strict pseudo-functor $\mathcal{U} \mapsto \mathcal{U}_{\mathrm{red}}$ induces 2-equivalences (in the sense of Definition 2.16 (2)) $\mathrm{Zar}_{\mathcal{X}} \to \mathrm{Zar}_{\mathcal{X}_{\mathrm{red}}}$ *and* $\mathrm{Et}_{\mathcal{X}} \to \mathrm{Et}_{\mathcal{X}_{\mathrm{red}}}$.

Proof. If $\mathcal{U} \to \mathcal{X}$ is DM, then it is in particular representable by algebraic stacks. Thus if $\mathcal{X}$ is an Ind-algebraic stack, then so is $\mathcal{U}$, by Remark 4.9. This proves (1).

Suppose now that we are in the context of (2). Lemma 3.37 shows that $\mathcal{U}_{\mathrm{red}} \xrightarrow{\sim} \mathcal{U} \times_{\mathcal{X}} \mathcal{X}_{\mathrm{red}}$, while Lemma 4.19 shows that $\mathcal{X}_{\mathrm{red}} \hookrightarrow \mathcal{X}$ is a thickening. The claimed 2-equivalences thus follow from Lemma 3.41. $\square$

Lemma 4.22. *If $\mathcal{X}$ is an Ind-algebraic stack over a locally Noetherian scheme S, then $\mathcal{X}$ satisfies the condition* (RS*).

Proof. Since the diagram appearing in the definition of (RS*) (i.e. Definition 3.8) involves only affine schemes, which are in particular quasi-separated, we see that this follows from Lemma 4.4, together with [Stacks, Tag 0CXP], which shows that algebraic stacks satisfy (RS*). $\square$

5. Formal algebraic stacks

In this section, we begin by briefly recalling the definition of a formal algebraic space from [Stacks], and then proceed to give the definition of, and establish some basic results regarding, formal algebraic stacks.

We fix a base scheme S. The following definitions follow [Stacks, Tag 0AI7] (for affine formal algebraic spaces) and [Stacks, Tag 0AIL] (for formal algebraic spaces in general).

Definition 5.1. (1) An affine formal algebraic space over S is a sheaf X on the *fppf* site of S which admits a description as an Ind-scheme $X \xrightarrow{\sim} \varinjlim_i X_i$, where the X_i are affine schemes and the transition morphisms are thickenings (in the sense of [Stacks, Tag 04EX]).

(2) A formal algebraic space over S is a sheaf X on the *fppf* site of S which receives a morphism $\coprod U_i \to X$ which is representable (by schemes), étale, and surjective, and whose source is a disjoint union of affine formal algebraic spaces U_i.

Remark 5.2. We remark that the notion of affine formal algebraic space used in [Stacks], and recalled above, is more general than the notion of affine formal scheme defined in [EGA1, §10] or other references. Similarly, the notion of formal algebraic space used in [Stacks] is more general than the notion developed in e.g. Knutson's original treatment [Knu71, Ch. 5], where only the separated Noetherian case is treated. (And in our intended applications, it is actually crucial that we work in a framework that allows formal algebraic spaces and formal algebraic stacks that are not *a priori* Noetherian.)

The following definition is the evident analogue of Definition 5.1 in the context of stacks.

Definition 5.3. Let $\mathcal{X}$ be a stack in groupoids[8] on the *fppf* site of a scheme S. We say that $\mathcal{X}$ is a *formal algebraic stack* if there is a morphism $U \to \mathcal{X}$, whose domain U is a formal algebraic space, and which is representable by algebraic spaces, smooth and surjective.

Remark 5.4. This definition should be compared with the characterization of algebraic stacks as stacks which receive a morphism whose source is an algebraic space, and which is representable, smooth, and surjective [Stacks, Tag 05UL].

Remark 5.5. If we take into account the definition of a formal algebraic space, we find that we may equivalently define a formal algebraic stack to be a stack $\mathcal{X}$ which receives a morphism $\coprod X_i \to \mathcal{X}$ which is representable by algebraic spaces, smooth, and surjective, and whose source is a disjoint union of affine formal algebraic spaces.

Remark 5.6. If, in Definition 5.3, we ask that the morphism $U \to \mathcal{X}$ be étale, rather than merely smooth, we obtain the notion of a *formal Deligne–Mumford stack*. Since our interest is primarily in the case of formal algebraic stacks that are not necessarily Deligne–Mumford, we won't say much about this special case of the definition in these notes.

Example 5.7. A formal algebraic space over S may be regarded as a formal algebraic stack in a natural way, by reinterpreting the underlying sheaf as a category fibred in setoidsover the *fppf* site of S. We will prove in Lemma 6.11 that, conversely, if $\mathcal{X}$ is a formal algebraic stack which has trivial inertia stack (i.e. corresponds to a category fibred in setoids, rather than just in groupoids), then $\mathcal{X}$ is in fact (isomorphic to) a formal algebraic space. (In the same result, we also show that it suffices to check the triviality of the inertia stack after pulling back to geometric points.)

Example 5.8. Since any algebraic space is also a formal algebraic space, an algebraic stack over S is also a formal algebraic stack over S.

Example 5.9. Suppose that $\mathcal{X}$ is an algebraic stack over S, and that $T \subseteq |\mathcal{X}|$ is a locally closed subset of the topological space underlying $\mathcal{X}$. We may then define $\widehat{\mathcal{X}}_{|T}$, the *completion of* $\mathcal{X}$ *along* T, to be the substack of $\mathcal{X}$ defined as

$$\widehat{\mathcal{X}}_{|T}(U) := \{f : U \to \mathcal{X} \mid f(|U|) \subseteq T\}.$$

(For the analogous construction in the case of algebraic spaces, see [Stacks, Tag 0AIX].[9])

To see that $\widehat{\mathcal{X}}_{|T}$ is a formal algebraic stack, choose a smooth surjection $V \to \mathcal{X}$, where V is an algebraic space. If we let T' denote the preimage of T in $|V|$, then

[8] Since from now on we only consider stacks in groupoids, we will typically write "stack" rather than "stack in groupoids".

[9] In that discussion, the subset T is required to be closed. But note that if T is locally closed in $|\mathcal{X}|$, then there is an open substack $\mathcal{U}$ of $\mathcal{X}$ so that T is closed in $|\mathcal{U}|$. One sees immediately from the definition that the open immersion $\mathcal{U} \to \mathcal{X}$ induces an isomorphism $\widehat{\mathcal{U}}_{|T} \xrightarrow{\sim} \widehat{\mathcal{X}}_{|T}$, and so, after replacing $\mathcal{X}$ by $\mathcal{U}$, we may reduce ourselves to the case when T is closed.

the fibre product $\widehat{\mathcal{X}}_{|T} \times_{\mathcal{X}} V$ is naturally isomorphic to $\widehat{V}_{|T'}$, the completion of V along T'. Since $\widehat{V}_{|T'}$ is a formal algebraic space, by [Stacks, Tag 0AIZ], and since the projection $\widehat{\mathcal{X}}_{|T} \times_{\mathcal{X}} V \to \widehat{\mathcal{X}}_{|T}$ is representable by algebraic spaces, smooth, and surjective (being the base-change of a morphism with these properties), we see that $\widehat{\mathcal{X}}_{|T}$ is indeed a formal algebraic stack.

Remark 5.10. In some other treatments of formal algebraic spaces and stacks, such as that of [Har05, App. A], these notions are defined by considering sheaves or stacks on a site Nilp_X, where X is a formal scheme. However, in the framework of the Stacks Project, a formal scheme X (over some base scheme S) is simply a sheaf on the *fppf* site of S, and Nilp_X can then be thought of as the category of S-morphisms $T \to X$ (with T an S-scheme). Giving a sheaf, or stack, on Nilp_X, is then equivalent to giving a sheaf, or stack, on the *fppf* site of S, equipped with a morphism (of sheaves, or stacks) to X. In particular, giving a formal algebraic space, or formal algebraic stack, on Nilp_X, say in the sense of [Har05, App. A], will then amount to giving a formal algebraic space, in the sense of the Stacks Project, or a formal algebraic stack, in the sense defined above, equipped with a morphism to X; thus our approach to the subject, while differing in its foundational set-up from that of [Har05, App. A], is nevertheless compatible with that approach. (See e.g. [Stacks, Tag 00XZ] and [Stacks, Tag 04WV] for a discussion of the relevant sheaf and stack-theoretic background, as well as [Stacks, Tag 03I3] for an explanation of how it applies in the context of algebraic spaces, and [Stacks, Tag 04X4] for the corresponding explanation in the context of algebraic stacks.)

(In fact, there are some other minor differences between our treatment and that of [Har05, App. A], which we will briefly discuss. One is that, as already noted, we work with the *fppf* topology, rather than the étale topology. This is harmless in practice. Another is that the definition of a formal algebraic space in the Stacks Project is more general than that of [Har05, App. A]: for example, non-quasi-separated objects are allowed, and the underlying notion of formal scheme is also more general than that of [EGA1, §10] or other references. This means that our notion of formal algebraic stack is correspondingly more general.)

We begin our theoretical development by proving the representability by algebraic spaces of the diagonal morphism of a formal algebraic stack. (The corresponding result for formal algebraic spaces is [Stacks, Tag 0AIP].) Before giving the proof, we recall a lemma which provides a general strategy for studying properties of diagonal morphisms (which is useful in part for its notational convenience).

Lemma 5.11. *If $\mathcal{X}$ is a category fibred in groupoids over the scheme S, and if P is a property of morphisms of sheaves (on the fppf site of S) which is preserved under arbitrary base-change, then the following are equivalent:*

(1) If $T \to \mathcal{X} \times_S \mathcal{X}$ is any morphism whose source is a scheme, then the base-changed morphism
$$T \times_{\mathcal{X} \times_S \mathcal{X}} \mathcal{X} \to T$$
has the property P.

(2) If $T_1 \to \mathcal{X}$ and $T_2 \to \mathcal{X}$ are arbitrary morphisms to $\mathcal{X}$ whose sources are schemes, then the base-changed morphism
$$T_1 \times_{\mathcal{X}} T_2 \to T_1 \times_S T_2$$
has the property P.

Proof. This is well-known, but we briefly recall the proof. The key point is that we there is an isomorphism $T_1 \times_{\mathcal{X}} T_2 \xrightarrow{\sim} (T_1 \times_S T_2) \times_{\mathcal{X} \times_S \mathcal{X}} \mathcal{X}$, which allows us to rewrite the morphism of (2) in the form

$$(T_1 \times_S T_2) \times_{\mathcal{X} \times_S \mathcal{X}} \mathcal{X} \to T_1 \times_S T_2.$$

Taking this into account, we see that if (1) holds, then we may deduce (2) by applying (1) to the morphism $T := T_1 \times_S T_2 \to \mathcal{X} \times_S \mathcal{X}$. Conversely, if (2) holds, and if we are in the context of (1), the given morphism $T \to \mathcal{X} \times_S \mathcal{X}$ corresponds to a pair of morphisms $T \rightrightarrows \mathcal{X}$. Applying (2) to the product of these morphisms, which is a morphism $T \times_S T \to \mathcal{X} \times_S \mathcal{X}$, we find that the base-changed morphism $(T \times_S T) \times_{\mathcal{X} \times_S \mathcal{X}} \mathcal{X} \to T \times_S T$ has the property P. Pulling back this morphism along the diagonal morphism $T \to T \times_S T$ shows that (1) holds. □

Lemma 5.12. *The diagonal of a formal algebraic stack is representable by algebraic spaces, and locally of finite type.*

Proof. Taking into account Lemma 5.11 (and taking P to be the property of being representable by algebraic spaces), we consider a pair of morphisms $T_1, T_2 \to \mathcal{X}$, each of whose source is an S-scheme. If we choose a morphism $U \to \mathcal{X}$ as in the definition of a formal algebraic stack (so U is a formal algebraic space, and the morphism is representable by algebraic spaces, smooth, and surjective), then we may pull back this morphism along the morphism of sheaves $T_1 \times_{\mathcal{X}} T_2 \to \mathcal{X}$ to obtain a morphism of sheaves

$$(T_1 \times_{\mathcal{X}} U) \times_U (T_2 \times_{\mathcal{X}} U) \to T_1 \times_{\mathcal{X}} T_2$$

which is representable by algebraic spaces, smooth, and surjective. Each fibre product $T_i \times_{\mathcal{X}} U$ is an algebraic space (since $U \to \mathcal{X}$ is representable by algebraic spaces), and so their fibre product over the formal algebraic space U is again an algebraic space (since the diagonal of U is representable). Thus $T_1 \times_{\mathcal{X}} T_2$ is a sheaf receiving a morphism whose source is an algebraic space, and which is representable, smooth, and surjective. It follows that $T_1 \times_{\mathcal{X}} T_2$ is itself an algebraic space [Stacks, Tag 04S6]. This shows that the diagonal of $\mathcal{X}$ is representable by algebraic spaces.

To see that the diagonal is furthermore locally of finite type, we must show that the natural morphism $T_1 \times_{\mathcal{X}} T_2 \to T_1 \times_S T_2$ is locally of finite type. This may be checked smooth locally on the source and target, and so it suffices to show that the natural morphism

$$(T_1 \times_{\mathcal{X}} U) \times_U (T_2 \times_{\mathcal{X}} U) \to (T_1 \times_{\mathcal{X}} U) \times_S (T_2 \times_{\mathcal{X}} U)$$

is locally of finite type. This is a base-change of the diagonal morphism $U \to U \times_S U$, which is indeed locally of finite type [Stacks, Tag 0AIP]. □

We recall another general result related to diagonals (see also [EG19b, Prop. 2.3.16, 2.3.18], where essentially the same result is proved).

Lemma 5.13. *(1) If $\mathcal{X}$ is a category fibred in groupoids over S whose diagonal $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable by algebraic spaces, then the double diagonal $\Delta_\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X}$ is also representable by algebraic spaces. If the diagonal Δ is furthermore locally of finite type, then the double diagonal Δ_Δ is locally of finite presentation.*

(2) If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of categories fibred in groupoids over S which is representable by algebraic spaces, and if the diagonal morphism $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$

and the double diagonal morphism $\Delta_{\Delta_{\mathcal{Y}}} : \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{Y}\times_S\mathcal{Y}} \mathcal{Y}$ are each also representable by algebraic spaces (*by (1), the last hypothesis holds if $\Delta_{\mathcal{Y}}$ is representable by algebraic spaces*), *then the diagonal morphism $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is also representable by algebraic spaces. If furthermore each of $\Delta_{\mathcal{X}}$ and $\Delta_{\Delta_{\mathcal{Y}}}$ are locally of finite type, then so is Δ_f.* (*By (1), this hypothesis on $\Delta_{\Delta_{\mathcal{Y}}}$ holds if $\Delta_{\mathcal{Y}}$ is locally of finite type.*)

Proof. We begin by proving (1). If T is an S-scheme, then the groupoid of T-valued points of $\mathcal{X} \times_{\mathcal{X}\times_S\mathcal{X}} \mathcal{X}$ is equivalent to the groupoid classifying pairs $(\xi \in \mathcal{X}(T), \alpha \in \mathrm{Aut}(\xi))$, and given an object (ξ, α) of this groupoid, the groupoid (which is in fact a setoid) of T'-valued points of the corresponding fibre product $T \times_{\mathcal{X},\Delta_\Delta} \mathcal{X} \times_{\mathcal{X}\times_S\mathcal{X}} \mathcal{X}$ is equivalent to the sheaf that classifies morphisms $T' \to T$ such that $\alpha_{T'} = \mathrm{id}_{\xi_{T'}}$. This sheaf can be described as the fibre product

$$T \times_{\alpha\times\mathrm{id},\mathrm{Aut}(\xi)\times_T\mathrm{Aut}(\xi),\Delta} \mathrm{Aut}(\xi).$$

If Δ is representable by algebraic spaces, then $\mathrm{Aut}(\xi)$ is an algebraic space, and hence so is this fibre product. If Δ is furthermore locally of finite type, then $\mathrm{Aut}(\xi)$ is locally of finite type over T, and hence its diagonal is locally of finite presentation [Stacks, Tag 084P]. Thus this fibre product is locally of finite presentation over T.

Claim (2) follows by the usual graph argument: we factor the diagonal morphism $\Delta_{\mathcal{X}}$ as the composite

$$\mathcal{X} \xrightarrow{\Delta_f} \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}.$$

This composite is representable by algebraic spaces by assumption, and since the second morphism is a base-change of the diagonal $\Delta_{\mathcal{Y}}$ of $\mathcal{Y}$, its diagonal is representable by algebraic spaces by assumption. Factoring Δ_f as

$$\mathcal{X} \xrightarrow{\Gamma_{\Delta_f}} \mathcal{X} \times_{\mathcal{X}\times_S\mathcal{X}} (\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}) \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X},$$

we see that it is a composite of morphisms that are representable by algebraic spaces, and so is representable by algebraic spaces itself. The claim regarding Δ_f being locally of finite type (under the appropriate hypotheses) follows in the same way. ☐

Lemma 5.14. *(1) If $\mathcal{X}$ is a formal algebraic stack, then the double diagonal $\Delta_\Delta : \mathcal{X} \to \mathcal{X}_{\mathcal{X}\times_S\mathcal{X}}\mathcal{X}$ is representable by algebraic spaces, and locally of finite presentation.*

(2) If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of formal algebraic stacks, then $\Delta_f : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is representable by algebraic spaces, and locally of finite type.

Proof. This follows immediately from Lemmas 5.12 and 5.13. ☐

Lemma 5.12 shows that the discussion of Example 3.16 applies to morphisms from algebraic stacks to formal algebraic stacks. For example, it makes sense to speak of a morphism from an algebraic stack to a formal algebraic stack being flat and surjective, and so the statement of the following lemma makes sense. (The lemma may be thought of as showing that the property of a formal algebraic stack actually being an algebraic stack can be checked flat locally.)

Lemma 5.15. *If $\mathcal{X} \to \mathcal{Y}$ is a morphism from an algebraic stack to a formal algebraic stack which is flat and surjective* (*in the sense explained in Example* 3.16), *then $\mathcal{Y}$ is itself an algebraic stack.*

Proof. It suffices to show that if $W \to \mathcal{Y}$ is a morphism from an affine formal algebraic space to $\mathcal{Y}$ which is representable by algebraic spaces, smooth, and surjective, then W is actually an affine scheme. (Indeed, from this, and the assumption that $\mathcal{Y}$ is a formal algebraic stack, it follows that $\mathcal{Y}$ admits a morphism $\coprod W_i \to \mathcal{Y}$ whose source is a disjoint union of affine schemes which is representable by algebraic spaces, smooth, and surjective, and thus — by Remark 5.4 — that $\mathcal{Y}$ is an algebraic stack.) Given such a morphism, we find that $\mathcal{X} \times_{\mathcal{Y}} W \to W$ is flat and surjective, with its source being an algebraic stack. We may then find a morphism $U \to \mathcal{X} \times_{\mathcal{Y}} W$ whose source is a scheme, and which is smooth and surjective, and the composite morphism $U \to W$ is then representable by algebraic stacks, flat, and surjective. Since its source is a scheme and its target has a diagonal which is representable (by schemes) [Stacks, Tag 0AIP]), it is in fact representable, flat, and surjective. Since W is an affine formal algebraic space, we may find an affine scheme U', and a morphism $U' \to U$ which is locally on the source an open immersion, such that the composite $U' \to U \to W$ is surjective. Replacing U by U', we may thus assume that U and W are both affine. The morphism $U \to W$ then factors as $U \to V \hookrightarrow W$, where V is an affine scheme and $V \hookrightarrow W$ is a thickening. Thus if $T \to V$ is any morphism whose source is a scheme, we obtain morphisms

$$T \times_W U \to T \times_W V \to T,$$

with the composite being faithfully flat, and the second arrow being a thickening. Since faithfully flat morphisms are scheme-theoretically dominant, we find that the second arrow is necessarily an isomorphism, and thus that $V \to W$ is itself an isomorphism (since $T \to W$ was arbitrary). Thus W is an affine scheme, as claimed. □

We recall that in the Stacks Project, a formal algebraic space is defined to be *locally countably indexed*, resp. *locally adic**, if it admits an étale surjection from a union of affine formal algebraic spaces that are *countably indexed*, resp. adic* [Stacks, Tag 0AKY]. (An affine formal algebraic space is *countably indexed* if it may be written as an inductive limit of a sequence of thickenings of affine schemes [Stacks, Tag 0AII]; an affine formal algebraic space is *adic** if it is of the form $\operatorname{Spf} A$ for an adic* topological ring A, i.e. a topological ring which is complete,[10] and which admits a finitely generated ideal — called an *ideal of definition* of A — whose powers are all open, and form a basis of open neighbourhoods of zero [Stacks, Tag 0AID].) We now extend these notions to the context of formal algebraic stacks.

Definition 5.16. We say that a formal algebraic stack $\mathcal{X}$ is *countably indexed*, resp. *locally adic**, if we may find a representable morphism $U \to \mathcal{X}$ which is smooth and surjective, and whose source is a countably indexed, resp. locally adic*, formal algebraic space.

Remark 5.17. Note that it follows from [Stacks, Tag 0AKS], resp. [Stacks, Tag 0AKT], that if a formal algebraic space (thought of as a formal algebraic stack) is locally countably indexed, resp. locally adic*, in the sense of this definition, then it is in fact locally countably indexed, resp. locally adic*, in the sense already defined in the Stacks Project.

[10] By convention, *complete* here means *complete and separated.*

Example 5.18. If $\mathcal{X}$ is an algebraic stack over S, and if T is a closed subset of $|\mathcal{X}|$ for which the open immersion $\mathcal{X} \setminus T \to \mathcal{X}$ is quasi-compact, then the completion $\widehat{\mathcal{X}}_{|T}$ is locally adic*. To see this, we use a fact which was noted in the discussion of Example 5.9, namely that if $V \to \mathcal{X}$ is a smooth surjection whose source is an algebraic space, and if T' denotes the preimage of T in $|V|$, then $\widehat{V}_{|T'}$ admits a morphism to $\widehat{\mathcal{X}}_{|T}$ which is representable by algebraic spaces, smooth, and surjective. If $\mathcal{X} \setminus T \to \mathcal{X}$ is quasi-compact, then so is the base-changed morphism $V \setminus T' \to V$, and so it follows from [Stacks, Tag 0AQ1] that $\widehat{V}_{|T'}$ is locally adic*.

We next discuss various base-change results in the context of formal algebraic stacks.

Lemma 5.19. *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of stacks over S which is representable by algebraic spaces.*

1. *If $T \to \mathcal{Y}$ is a morphism from a formal algebraic space to $\mathcal{Y}$, then $\mathcal{X} \times_{\mathcal{Y}} T$ is again a formal algebraic space over S. If T is locally countably indexed, resp. locally adic*, then so is $\mathcal{X} \times_{\mathcal{Y}} T$.*
2. *If $\mathcal{Z} \to \mathcal{Y}$ is a morphism from a formal algebraic stack to $\mathcal{Y}$, then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is again a formal algebraic stack over S. If $\mathcal{Z}$ is locally countably indexed, resp. locally adic*, then so is $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.*

Proof. We begin with (1). By definition, we may find a family $\{T_i\}$ of affine formal algebraic spaces and a morphism $\coprod T_i \to T$ which is representable (by schemes), étale and surjective. The base-changed morphism $\coprod \mathcal{X} \times_{\mathcal{Y}} T_i \to \mathcal{X} \times_{\mathcal{Y}} T$ is then representable, étale and surjective. Thus it suffices to show that each of the products $\mathcal{X} \times_{\mathcal{Y}} T_i$ is a formal algebraic space.

Since T_i is an affine formal algebraic space, we may write $T_i \cong \varinjlim_{\lambda} T_{i,\lambda}$ as an inductive limit of schemes, with the transition morphisms being thickenings. Lemma 4.8 shows that we then have an induced isomorphism

$$\mathcal{X} \times_{\mathcal{Y}} T_i \cong \varinjlim_{\lambda} \mathcal{X} \times_{\mathcal{Y}} T_{i,\lambda},$$

expressing $\mathcal{X} \times_{\mathcal{Y}} T_i$ as an inductive limit of algebraic spaces, with the transition morphisms being thickenings. It follows from [Stacks, Tag 0AIU] that $\mathcal{X} \times_{\mathcal{Y}} T_i$ is indeed a formal algebraic space.

Now suppose that T is furthermore locally countably indexed, resp. locally adic*. The projection $\mathcal{X} \times_{\mathcal{Y}} T \to T$ is a morphism representable by algebraic spaces, whose source is a formal algebraic space (by what we have already proved). To simplify notation, denote the source of this projection by X; we wish to show that X is locally countably indexed, resp. locally adic*. Since T is locally countably indexed, resp. locally adic*, we may find a morphism $\coprod_i T_i \to T$ which is representable, étale, and surjective, with each T_i being a countably indexed, resp. adic*, affine formal algebraic space. Base-changing over X yields a morphism $\coprod X \times_T T_i \to X$ which is again representable, étale, and surjective.

In order to conclude that X is locally countably indexed, resp. locally adic*, it suffices to prove that each of the fibre products $X \times_T T_i$, which *a priori* is a formal algebraic space, is furthermore locally countably indexed, resp. locally adic*. Let $U \to X \times_T T_i$ be an étale representable morphism whose source is an affine formal algebraic space. Since the composite $U \to X \times_T T_i \to T_i$ is representable by algebraic spaces and T_i is countably indexed, resp. adic*, it follows from [Stacks, Tag 0AKQ] that U is countably indexed, resp. adic*, as required.

We now turn to proving (2). Choose a morphism $T \to \mathcal{Z}$ which is representable by algebraic spaces, smooth, and surjective, and whose source T is a formal algebraic space; furthermore, choose T to be locally countably indexed or locally adic* if $\mathcal{Z}$ is. Taking the fibre product of this morphism with $\mathcal{X}$ over $\mathcal{Y}$ yields a morphism $\mathcal{X} \times_{\mathcal{Y}} T \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ which is again representable by algebraic spaces, smooth, and surjective. By (1), the source of this morphism is a formal algebraic space, which is locally countably indexed or locally adic* if T is, and thus the target is indeed a formal algebraic stack, which is locally countably index or locally adic* if T is, and thus (by our choice of T), if $\mathcal{Z}$ is. ☐

Remark 5.20. Let us note some particular consequences of the preceding result, namely: if $X \to Y$ is a morphism of sheaves over S which is representable by algebraic spaces, and if Y is furthermore a formal algebraic space (resp. a locally countably indexed formal algebraic space, resp. a locally adic* formal algebraic space), then the same is true of X (set $T = \mathcal{Y} = Y$ in (1)); and similarly, if $\mathcal{X} \to \mathcal{Y}$ is a morphism of stacks over S which is representable by algebraic spaces, and if $\mathcal{Y}$ is furthermore a formal algebraic stack (resp. a locally countably indexed formal algebraic stack, resp. a locally adic* formal algebraic stack), then the same is true of $\mathcal{X}$ (set $\mathcal{Z} = \mathcal{Y}$ in (2)).

Lemma 5.21. *If $\mathcal{Y}$ is a stack over S whose diagonal is representable by algebraic spaces, and if $U \to \mathcal{Y}$ and $V \to \mathcal{Y}$ are morphisms each of whose domains is a formal algebraic space, then $U \times_{\mathcal{Y}} V$ is again a formal algebraic space.*

Proof. If we interpret $U \times_{\mathcal{Y}} V$ as the pull-back of $U \times_S V$ over the diagonal of $\mathcal{Y}$, then this follows from Lemma 5.19 (1). ☐

Lemma 5.22. *If $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{Z} \to \mathcal{Y}$ are morphism of stacks over S, with $\mathcal{X}$ and $\mathcal{Z}$ being formal algebraic stacks, and $\mathcal{Y}$ having its diagonal being representable by algebraic spaces (e.g., by Lemma 5.12, one could take $\mathcal{Y}$ itself to be a formal algebraic stack), then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is again a formal algebraic stack.*

Proof. Choose morphisms $U \to \mathcal{X}$ and $V \to \mathcal{Z}$ that are representable by algebraic spaces, surjective, and smooth, whose domains are formal algebraic spaces. Then $U \times_{\mathcal{Y}} V \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is again representable by algebraic spaces, surjective and smooth, and its source is again a formal algebraic space, by Lemma 5.21 (since $\Delta_{\mathcal{Y}/S}$ is representable by algebraic spaces by assumption). Thus $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is indeed a formal algebraic stack. ☐

We next establish an analogue of Lemma 5.19 for morphisms representable by algebraic stacks. Because we don't have a complete analogue of [Stacks, Tag 0AIU] (to which we appealed in the proof of that lemma) in the context of formal algebraic stacks, we are not able to prove a complete analogue of Lemma 5.19, but instead make do with the following slightly weaker variant.

Lemma 5.23. *Let $\mathcal{X} \to \mathcal{Y}$ be a morphism of stacks over S which is representable by algebraic stacks, and consider a morphism $\mathcal{Z} \to \mathcal{Y}$ whose source is a formal algebraic stack.*

(1) If $\mathcal{X} \to \mathcal{Y}$ is DM, then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is a formal algebraic stack.

(2) *If $\mathcal{X}$ is a formal algebraic stack, then the fibre product $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is a formal algebraic stack.*

(3) *If $\mathcal{Z}$ is locally countably indexed, resp. locally adic*, then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is a locally countably indexed, resp. locally adic*, formal algebraic stack.*

Proof. Choose (as we may) a morphism $\coprod_i T_i \to \mathcal{Z}$ which is representable by algebraic spaces, smooth, and surjective, and whose source is the disjoint union of affine formal algebraic spaces. The base-changed morphism $\coprod_i \mathcal{X} \times_{\mathcal{Y}} T_i \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is again representable by algebraic spaces, smooth, and surjective, and so if we can show that each of the fibre products $\mathcal{X} \times_{\mathcal{Y}} T_i$ appearing in its source is a formal algebraic stack, the same is true of its target. In summary, we are reduced to the case when $\mathcal{Z}$ is in fact an affine formal algebraic space T (we omit the subscripts i from now on, since they play no further role). If $\mathcal{Z}$ is furthermore locally countable indexed, resp. locally adic*, then we may also choose T to be countably indexed, resp. adic*.

We begin by proving (1). For this, we follow the proof of Lemma 5.19 (1). Namely, since T is an affine formal algebraic space, we may write $T \cong \varinjlim_\lambda T_\lambda$ as an inductive limit of schemes, with the transition morphisms being thickenings, and so by Lemma 4.8 we have an induced isomorphism

$$\mathcal{X} \times_{\mathcal{Y}} T \cong \varinjlim_{\lambda} \mathcal{X} \times_{\mathcal{Y}} T_\lambda,$$

expressing $\mathcal{X} \times_{\mathcal{Y}} T$ as a 2-colimit limit of a 2-directed system of Deligne–Mumford stacks, with the transition morphisms being thickenings. It follows from Lemma 6.1 below[11] that $\mathcal{X} \times_{\mathcal{Y}} T$ is indeed a formal algebraic stack.

In the context of (2), we may additionally choose a formal algebraic space U and a morphism $U \to \mathcal{X}$ which is representable by algebraic spaces, smooth, and surjective. The base-changed morphism $U \times_{\mathcal{Y}} T \to \mathcal{X} \times_{\mathcal{Y}} T$ is then representable by algebraic spaces, smooth, and surjective, and so to prove that $\mathcal{X} \times_{\mathcal{Y}} T$ is a formal algebraic stack, it suffices to prove that $U \times_{\mathcal{Y}} T$ is a formal algebraic space. The composite $U \to \mathcal{X} \to \mathcal{Y}$ is representable by algebraic stacks, and so by Lemma 3.5 it is in fact representable by algebraic spaces; thus Lemma 5.19 (1) shows that indeed $U \times_{\mathcal{Y}} T$ is a formal algebraic space.

Suppose now that we are in the context of (3). As we already noted above, since $\mathcal{Z}$ is locally countably indexed, we may assume that T is countably indexed (and even adic*, if $\mathcal{Z}$ is locally adic*). Thus we may write $T \cong \varinjlim T_n$, where $\{T_n\}$ is a directed sequence of affine schemes related by thickenings. This induces an isomorphism

$$\mathcal{X} \times_{\mathcal{Y}} T \cong \varinjlim_{n} \mathcal{X} \times_{\mathcal{Y}} T_n,$$

realizing $\mathcal{X} \times_{\mathcal{Y}} T$ as the 2-colimit of a directed sequence of algebraic stacks related by thickenings. (Here we have used the fact that $\mathcal{X} \to \mathcal{Y}$ is representable by algebraic stacks.) Lemma 6.3 below[12] then shows that $\mathcal{X} \times_{\mathcal{Y}} T$ is a countably indexed formal algebraic stack, as required.

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[11]The reader will verify that the proof of this lemma doesn't require any of the results from the theory of formal algebraic stacks that are developed in this note; it is a direct consequence of the topological invariance of the small étale site for Deligne–Mumford stacks. Thus although its proof is postponed until the next section, there is no circularity in invoking it here.

[12]The reader can easily verify that the proof of that result is independent of present lemma.

Choose a morphism $U \to \mathcal{X} \times_{\mathcal{Y}} T$ which is representable by algebraic spaces, smooth, and surjective. Arguing as in the proof of (2), we find that the composite $U \to \mathcal{X} \times_{\mathcal{Y}} T \to T$ is representable by algebraic spaces, and thus Lemma 5.19 (1) shows that U is locally countably indexed, resp. locally adic*, if T is. Thus, by definition, the same is true of $\mathcal{X} \times_{\mathcal{Y}} T$. $\square$

Remark 5.24. As a useful consequence of the preceding lemma, we note that if $\mathcal{X} \to \mathcal{Y}$ is a morphism of stacks over S which is representable by algebraic stacks, with $\mathcal{Y}$ furthermore being a formal algebraic stack, then if either the morphism is in fact DM, or if $\mathcal{Y}$ is locally countably (resp. locally adic*), then $\mathcal{X}$ is a formal algebraic stack, which is locally countable (resp. locally adic*) if $\mathcal{Y}$ is. (To see this, set $\mathcal{Z} = \mathcal{Y}$ in (1) or (3), as appropriate).

It can be useful on occasion to view a formal algebraic stack as a quotient stack. Suppose, then, that $\mathcal{X}$ is a formal algebraic stack over S, and that $U \to \mathcal{X}$ is a morphism whose source is a formal algebraic space, and which satisfies the conditions of Definition 5.3. As in the discussion preceding Lemma 3.18, we may then form the fibre product $R := U \times_{\mathcal{X}} U$ which is a formal algebraic space (by Lemmas 5.12 and 5.21) having the structure of a groupoid in sheaves over U, and for which each of the projections are representable by algebraic spaces and smooth.

Lemma 5.25. *In the context of the preceding discussion, the morphism $U \to \mathcal{X}$ induces an isomorphism of stacks $[U/R] \to \mathcal{X}$.*

Proof. This follows from Lemma 3.18. $\square$

We now discuss the properties of the underlying reduced substack of a formal algebraic stack (the definition of which is provided by Definition 3.27).

Lemma 5.26. *If $\mathcal{X}$ is is a formal algebraic stack over S, then $\mathcal{X}_{\mathrm{red}}$ is a closed and reduced algebraic substack of $\mathcal{X}$, and the inclusion $\mathcal{X}_{\mathrm{red}} \hookrightarrow \mathcal{X}$ is a thickening (in the sense that its base-change over any algebraic space induces a thickening of algebraic spaces). Furthermore, any morphism $\mathcal{Y} \to \mathcal{X}$ with $\mathcal{Y}$ a reduced algebraic stack factors through $\mathcal{X}_{\mathrm{red}}$.*

Remark 5.27. Although the proof of this lemma that follows is slightly involved in its details, it can be summarized as follows: we choose a morphism $U \to \mathcal{X}$ whose source is a formal algebraic space, and which is representable by algebraic spaces, smooth, and surjective, and then, setting $R := U \times_{\mathcal{X}} U$, and using Lemma 5.25 to write $[U/R] \xrightarrow{\sim} \mathcal{X}$, we show that there is an induced isomorphism $[U_{\mathrm{red}}/R_{\mathrm{red}}] \xrightarrow{\sim} \mathcal{X}_{\mathrm{red}}$.

Proof of Lemma 5.26. Choose a morphism $U \to \mathcal{X}$ whose source is a formal algebraic space, and which is representable by algebraic spaces, smooth, and surjective. Let U_{red} denote the underlying reduced algebraic space of U, in the sense of [Stacks, Tag 0AIN]. The natural morphism $U_{\mathrm{red}} \hookrightarrow U$ is a thickening (in the sense that its base-change over any morphism $T \to U$, whose source is a scheme, is a thickening), and thus so is its base-change

$$U_{\mathrm{red}} \times_{\mathcal{X}} U_{\mathrm{red}} \hookrightarrow U_{\mathrm{red}} \times_{\mathcal{X}} U. \tag{5.28}$$

We first claim that this base-changed thickening is an isomorphism. First, we note that source and target are algebraic spaces, since U_{red} is, and since $U \to \mathcal{X}$ is representable by algebraic spaces, as is the composite $U_{\mathrm{red}} \to U \to \mathcal{X}$. Secondly, since $U \to \mathcal{X}$ is smooth, so is the base-changed morphism of algebraic spaces $U_{\mathrm{red}} \times_{\mathcal{X}} U \to U_{\mathrm{red}}$. Since U_{red} is reduced, we deduce that $U_{\mathrm{red}} \times_{\mathcal{X}} U$ is also reduced, and hence the projection onto U factors through U_{red}. Thus we obtain a morphism

$$U_{\mathrm{red}} \times_{\mathcal{X}} U \to U_{\mathrm{red}} \times_{\mathcal{X}} U_{\mathrm{red}},$$

which is easily verified to be an inverse to (5.28). Thus (5.28) is indeed an isomorphism.

We write $R_{\mathrm{red}} := U_{\mathrm{red}} \times_{\mathcal{X}} U_{\mathrm{red}} \rightrightarrows U_{\mathrm{red}}$. (The motivation for this notation is that, if we write $R := U \times_{\mathcal{X}} U$, then arguments similar to those of the preceding paragraph show that R_{red} is the underlying reduced algebraic space of the formal algebraic space R.) Identifying R_{red} with $U_{\mathrm{red}} \times_{\mathcal{X}} U$ (as the preceding paragraph allows us to do), and then also with $U \times_{\mathcal{X}} U_{\mathrm{red}}$ (by symmetry), we find that each of the projections $R_{\mathrm{red}} \rightrightarrows U_{\mathrm{red}}$ is a smooth morphism of algebraic spaces. Since R_{red} is thus smooth over U_{red} (by either projection), we see that $R_{\mathrm{red}} \times_{U_{\mathrm{red}}} R_{\mathrm{red}}$ (the fibre product being formed with respect to any of the projections to U_{red}) is reduced, and hence is the underlying reduced algebraic space of $R \times_U R$. Thus the groupoid structure on R over U (it is a groupoid in formal algebraic spaces over the formal algebraic space U) induces a groupoid structure on R_{red} over U_{red}, realizing it as a smooth groupoid in algebraic spaces over U_{red}. The definition of R_{red} as a fibre product over $\mathcal{X}$ shows that the composite

$$U_{\mathrm{red}} \hookrightarrow U \to \mathcal{X}$$

induces a morphism

$$[U_{\mathrm{red}}/R_{\mathrm{red}}] \to \mathcal{X}. \tag{5.29}$$

We claim that this latter morphism is representable by algebraic spaces, and is a thickening (and so in particular a monomorphism), whose essential image is equal to $\mathcal{X}_{\mathrm{red}}$. Note that this description of $\mathcal{X}_{\mathrm{red}}$ (namely that it is isomorphic to $[U_{\mathrm{red}}/R_{\mathrm{red}}]$) shows that $\mathcal{X}_{\mathrm{red}}$ is a reduced algebraic stack.

Since the properties of being representable by algebraic spaces and a thickening can be checked smooth locally on the target, in order to show that (5.29) is representable by algebraic spaces, and a thickening, it suffices to verify this after pulling back along the morphism $U \to \mathcal{X}$. The morphisms $U_{\mathrm{red}} \to [U_{\mathrm{red}}/R_{\mathrm{red}}]$ and $U_{\mathrm{red}} \hookrightarrow U$ induce a morphism

$$U_{\mathrm{red}} \to [U_{\mathrm{red}}/R_{\mathrm{red}}] \times_{\mathcal{X}} U,$$

which is in fact an isomorphism: indeed, to check that this morphism is an isomorphism, we can do so after pulling back over the smooth surjective morphism $U_{\mathrm{red}} \times_{\mathcal{X}} U \to [U_{\mathrm{red}}/R_{\mathrm{red}}] \times_{\mathcal{X}} U$; but the pulled back morphism is the morphism

$$R_{\mathrm{red}} \to U_{\mathrm{red}} \times_{\mathcal{X}} U,$$

which we have already shown to be an isomorphism. We conclude that (5.29) is indeed representable by algebraic spaces and a thickening, since $U_{\mathrm{red}} \hookrightarrow U$ is.

The final claim of the lemma follows from Remarks 3.28 and 3.29, and so to complete the proof of the lemma, it suffices to show that the essential image of (5.29) is equal to $\mathcal{X}_{\mathrm{red}}$. Choose a scheme V equipped with a surjective étale morphism $V \to U_{\mathrm{red}}$. (This is possible, since U_{red} is an algebraic space.) Since U_{red} is reduced, so is V. Thus the morphism $V \to \mathcal{X}$ is in fact an object of $(\mathcal{X}_{\mathrm{red}})'$, and hence of $\mathcal{X}_{\mathrm{red}}$. Since $V \to U_{\mathrm{red}} \to [U_{\mathrm{red}}/R_{\mathrm{red}}]$ is a smooth (and so *fppf*) cover of the target, we then find that the thickening (5.29) factors through $\mathcal{X}_{\mathrm{red}}$. To show that $\mathcal{X}_{\mathrm{red}}$ is equal to the essential image of this thickening, it suffices to show that any object of $(\mathcal{X}_{\mathrm{red}})'$ lies in the essential image of this thickening. This amounts to showing that if $T \to \mathcal{X}$ is a morphism whose source is a reduced scheme, then this morphism factors through $[U_{\mathrm{red}}/R_{\mathrm{red}}]$. Since $[U_{\mathrm{red}}/R_{\mathrm{red}}]$ is an *fppf* stack, it suffices to check this *fppf* locally on T. We may thus replace T by $T \times_{\mathcal{X}} U$, a reduced algebraic space admitting a smooth surjection onto T. The morphism of this algebraic space to $\mathcal{X}$ then factors through U by construction, and hence through U_{red}, and hence through $[U_{\mathrm{red}}/R_{\mathrm{red}}]$, as required. □

Example 5.30. If $\mathcal{X}$ is an algebraic stack and T is a locally closed subset of $|\mathcal{X}|$, then the reduced algebraic stack underlying the completion $\widehat{\mathcal{X}}_{|T}$ is the locally closed substack of $\mathcal{X}$ obtained by equipping T with its reduced induced structure [Stacks, Tag 050C].

We note the following lemma.

Lemma 5.31. *If $\mathcal{Y} \to \mathcal{X}$ is a morphism of formal algebraic stacks that is representable by algebraic stacks, then the induced morphism $\mathcal{Y}_{\mathrm{red}} \to \mathcal{X}_{\mathrm{red}}$ is also representable by algebraic spaces, and is surjective if and only if the original morphism is.*

Proof. The morphism $\mathcal{Y}_{\mathrm{red}} \to \mathcal{Y}$ is the inclusion of a closed substack, and hence even representable by schemes, so that the composite $\mathcal{Y}_{\mathrm{red}} \to \mathcal{Y} \to \mathcal{X}$ is representable by algebraic stacks. Since $\mathcal{X}_{\mathrm{red}} \to \mathcal{X}$ is a monomorphism, we see that the same is true of the induced morphism $\mathcal{Y}_{\mathrm{red}} \to \mathcal{X}_{\mathrm{red}}$. The second claim of the lemma follows from the first, together with Lemma 3.34. □

We now prove a version of the topological invariance of the small Zariski and étale sites for a formal algebraic stack. (We also state the case of algebraic stacks, since the argument applies just as well in that case.)

Lemma 5.32. *If $\mathcal{X}$ is an algebraic stack (resp. a formal algebraic stack), and if $\mathcal{U} \to \mathcal{X}$ is an object of $\mathrm{Et}_{\mathcal{X}}$, then $\mathcal{U}$ is again an algebraic stack (resp. a formal algebraic stack). Furthermore, the strict pseudo-functor $\mathcal{U} \mapsto \mathcal{U}_{\mathrm{red}}$ induces an isomorphism of categories $\mathrm{Zar}_{\mathcal{X}} \to \mathrm{Zar}_{\mathcal{X}_{\mathrm{red}}}$, as well as a 2-equivalence (in the sense of Definition 2.16 (2)) of 2-categories $\mathrm{Et}_{\mathcal{X}} \to \mathrm{Et}_{\mathcal{X}_{\mathrm{red}}}$.*

Proof. If $\mathcal{U} \to \mathcal{X}$ is DM, then it is in particular representable by algebraic stacks. Thus if $\mathcal{X}$ is an algebraic stack, then $\mathcal{U}$ certainly is, while if $\mathcal{X}$ is a formal algebraic stack, then so is $\mathcal{U}$, by Lemma 5.23 (1).[13] (See also Remark 5.24.) Lemma 3.37 shows that $\mathcal{U}_{\mathrm{red}} \xrightarrow{\sim} \mathcal{U} \times_{\mathcal{X}} \mathcal{X}_{\mathrm{red}}$, while Lemma 5.26 shows that $\mathcal{X}_{\mathrm{red}} \hookrightarrow \mathcal{X}$ is a thickening. The claimed equivalence and 2-equivalence thus follow from Lemma 3.41. □

We briefly develop the notion of *quasi-compact* (and also of quasi-separated) formal algebraic stacks, and of *quasi-compact morphisms* of formal algebraic stacks. We refer to [Stacks, Tag 0AJ8] for the corresponding definitions in the context of formal algebraic spaces.

Lemma 5.33. *If $\mathcal{X}$ is a formal algebraic stack, then the following are equivalent:*

[13] The proof of this latter result relies on Lemma 6.1, whose proof in turn relies on the topological invariance of the small étale site for algebraic stacks. Since that case of the present lemma does not involve any application of Lemmas 5.23 or 6.1, there is no circularity in the argument.

(1) $\mathcal{X}_{\text{red}}$ *is a quasi-compact algebraic stack.*
(2) There exists a quasi-compact formal algebraic space U *and a morphism* $U \to \mathcal{X}$ *which is representable by algebraic spaces, smooth, and surjective.*
(3) There exists an affine formal algebraic space U *and a morphism* $U \to \mathcal{X}$ *which is representable by algebraic spaces, smooth, and surjective.*

Proof. Clearly condition (3) implies condition (2). Conversely, condition (2) implies condition (3), by [Stacks, Tag 0AJ9]. If $U \to \mathcal{X}$ is a morphism as in condition (2), then $U_{\text{red}} \to \mathcal{X}_{\text{red}}$ (which Lemma 3.37 shows is obtained by pulling back the given morphism $U \to \mathcal{X}$ along the thickening $\mathcal{X}_{\text{red}} \hookrightarrow \mathcal{X}$) is a smooth surjective morphism from an algebraic space to an algebraic stack whose source is quasi-compact; it follows that the target is also quasi-compact, and so (2) implies (1).

Conversely, suppose that (1) holds, and choose a morphism $U \to \mathcal{X}$ which is representable by algebraic spaces, smooth, and surjective, and whose source is a formal algebraic space. Invoking Lemma 3.37 again, we see that $U_{\text{red}} := U \times_{\mathcal{X}} \mathcal{X}_{\text{red}}$ is the underlying reduced subspace of U, and that the morphism $U_{\text{red}} \to \mathcal{X}_{\text{red}}$ is smooth and surjective. Since $\mathcal{X}_{\text{red}}$ is quasi-compact by assumption, we may also find a quasi-compact scheme V and a morphism $V \to \mathcal{X}_{\text{red}}$ which is smooth and surjective. If we form the fibre product $U_{\text{red}} \times_{\mathcal{X}_{\text{red}}} V$, then its projection onto each of U_{red} and V is smooth, and hence open, as well as surjective. Since V is quasi-compact, we may then find a quasi-compact open subspace W of $U_{\text{red}} \times_{\mathcal{X}_{\text{red}}} V$ which surjects onto V. The image W' of W under the projection to U_{red} is then a quasi-compact open subspace subspace of U_{red} which surjects onto $\mathcal{X}_{\text{red}}$. If we let W'' denote the corresponding open subspace of U, then W'' is quasi-compact, and $W'' \to \mathcal{X}$ is representable by algebraic spaces, smooth, and surjective (this last property following from Lemma 5.31 and the fact that $W''_{\text{red}} = W' \to \mathcal{X}_{\text{red}}$ is surjective). Thus (2) holds. □

Definition 5.34. We say that a formal algebraic stack is *quasi-compact* if it satisfies the equivalent conditions of Lemma 5.33.

We say that a formal algebraic stack is *quasi-separated* if its diagonal morphism (which Lemma 5.12 shows is representable by algebraic spaces) is quasi-compact and quasi-separated.

Lemma 5.35. *For a morphism* $\mathcal{X} \to \mathcal{Y}$ *of formal algebraic stacks, the following are equivalent.*

(1) The induced morphism $\mathcal{X}_{\text{red}} \to \mathcal{Y}_{\text{red}}$ *is a quasi-compact morphism of algebraic stacks.*
(2) For every morphism $\mathcal{Z} \to \mathcal{Y}$ *whose source is a quasi-compact formal algebraic stack, the fibre product formal algebraic stack* $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ *is quasi-compact.*
(2") For every morphism $Z \to \mathcal{Y}$ *whose source is a quasi-compact formal algebraic space, the fibre product formal algebraic stack* $\mathcal{X} \times_{\mathcal{Y}} Z$ *is quasi-compact.*
(2") For every morphism $Z \to \mathcal{Y}$ *whose source is an affine formal algebraic space, the fibre product formal algebraic stack* $\mathcal{X} \times_{\mathcal{Y}} Z$ *is quasi-compact.*
(3) For every morphism $\mathcal{Z} \to \mathcal{Y}$ *whose source is a quasi-compact algebraic stack, the fibre product formal algebraic stack* $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ *is quasi-compact.*
(3') For every morphism $Z \to \mathcal{Y}$ *whose source is a quasi-compact scheme, the fibre product formal algebraic stack* $\mathcal{X} \times_{\mathcal{Y}} Z$ *is quasi-compact.*
(3") For every morphism $Z \to \mathcal{Y}$ *whose source is an affine scheme, the fibre product formal algebraic stack* $\mathcal{X} \times_{\mathcal{Y}} Z$ *is quasi-compact.*

(4) *There exists a morphism* $\coprod Y_i \to \mathcal{Y}$ *whose source is a disjoint union of affine formal algebraic spaces, and which is representable by algebraic spaces, smooth, and surjective, such that each of the fibre product formal algebraic stacks* $\mathcal{X} \times_{\mathcal{Y}} Y_i$ *are quasi-compact.*

Proof. The argument is routine, and we leave it to the reader. □

Definition 5.36. We say that a morphism of formal algebraic stacks is *quasi-compact* if it satisfies the equivalent conditions of Lemma 5.35.

6. Ind-algebraic stacks and formal algebraic stacks

As recalled in Definition 5.1, an affine formal algebraic space X is, by definition [Stacks, Tag 0AI7], a sheaf on the *fppf* site of S that can be written as an Ind-scheme

$$X \cong \varinjlim_i X_i,$$

where the X_i are affine schemes and the transition morphisms are thickenings. More generally, if X is a formal algebraic space which is quasi-compact and quasi-separated, then [Stacks, Tag 0AJE] shows that it may written as an Ind-algebraic space

$$X \cong \varinjlim_i X_i,$$

where the X_i are algebraic spaces and the transition morphisms are thickenings. Conversely, any Ind-algebraic space of this form *is* a formal algebraic space [Stacks, Tag 0AIU]. The proof of this latter result depends on the topological invariance of the étale site. Thus, it carries over directly to the context of Deligne–Mumford stacks: any 2-colimit of a 2-directed system of Deligne–Mumford stacks with respect to transition morphisms that are thickenings is necessarily a formal Deligne–Mumford stack.

Here is a formal statement and proof of this result.

Lemma 6.1. *If* $\{\mathcal{X}_i\}_i$ *is a* 2*-directed system of Deligne–Mumford stacks, with each of the transition morphisms being a thickening, then* $\mathcal{X} := \varinjlim_i \mathcal{X}_i$ *is a formal algebraic stack, and there is a morphism* $U \to \mathcal{X}$ *whose source is a formal algebraic space, and which is representable by algebraic spaces, étale, and surjective.*

Proof. Fix an index i_0, and choose an étale morphism $U_{i_0} \to \mathcal{X}_{i_0}$ whose source is a scheme. For each $i \geq i_0$, it follows from Lemma 3.41 that we may find an étale morphism $U_i \to \mathcal{X}_i$ which is a thickening of $U_{i_0} \to \mathcal{X}_{i_0}$. (We note that U_i is indeed a scheme, as the notation suggests, by [Stacks, Tag 0BPW].) The transition morphisms $\mathcal{X}_i \to \mathcal{X}_{i'}$ induce compatible transition morphisms $U_i \to U_{i'}$, which make $\{U_i\}$ into a directed system of schemes, whose transition morphisms are thickenings. If we set $U := \varinjlim_{i \geq i_0} U_i$, then U is a formal algebraic space, by [Stacks, Tag 0AIU].) By Lemma 2.34, the morphisms $U_i \to \mathcal{X}_i$ induce a morphism

$$U \to \varinjlim_{i \geq i_0} \mathcal{X}_i \xrightarrow{\sim} \varinjlim_i \mathcal{X}_i := \mathcal{X},$$

which, by Lemma 4.11, is representable by algebraic spaces, étale, and surjective, as required; note that the claimed isomorphism is provided by Lemma 2.36. □

In the general case of formal algebraic stacks that are not necessarily Deligne–Mumford, less seems to be known about the possibility of relating them to 2-colimits of thickenings. However, we do have the following results, the first of which is a direct analogue for formal algebraic stacks of [Stacks, Tag 0AJE] (and whose proof follows closely the proof of that result).

Lemma 6.2. *If $\mathcal{X}$ is a quasi-compact and quasi-separated formal algebraic stack, then $\mathcal{X} \cong \varinjlim_i \mathcal{X}_i$ for a 2-directed system $\{\mathcal{X}_i\}_{i\in I}$ of algebraic stacks in which the transition morphisms are thickenings. If $\mathcal{X}$ is furthermore locally countably indexed, then we may choose the indexing category to be countable, and so write $\mathcal{X} \cong \varinjlim_n \mathcal{X}_n$ as the 2-colimit of a directed sequence of algebraic stacks with respect to transition morphisms that are thickenings.*

Proof. Choose a morphism $V \to \mathcal{X}$ which is representable by algebraic spaces, smooth, and surjective, and whose source is an affine formal algebraic space (as we may, since $\mathcal{X}$ is quasi-compact), and write $V = \varinjlim_{i\in\mathcal{I}} V_i$ as the inductive limit of affine schemes via thickenings.

If $i \in \mathcal{I}$, then $V \times_{\mathcal{X}} V_i$ is an algebraic space, which is quasi-compact, since V_i and V are quasi-compact (being an affine scheme and a formal affine algebraic space, respectively), and $\mathcal{X}$ is quasi-separated. Thus the projection $V \times_{\mathcal{X}} V_i \to V$ factors through V_j, for some $j \geq i$. If we let Z_i denote the scheme-theoretic image in V_j of this projection, then Z_i (thought of as a closed subsheaf of V) is independent of the choice of j, and so we may regard it as being the scheme-theoretic image of the morphism $V \times_{\mathcal{X}} V_i \to V$. Since the projections $V \times_{\mathcal{X}} V_i \to V$ and $V_i \times_{\mathcal{X}} V \to V$ are isomorphic, we may equally well regard Z_i as the scheme-theoretic image of the latter morphism.

We now compute that $Z_i \times_{\mathcal{X}} V$ and $V \times_{\mathcal{X}} Z_i$ coincide as subsheaves of $V \times_{\mathcal{X}} V$; indeed, since the formation of scheme-theoretic images of quasi-compact morphisms is compatible with flat base-change, the first may be described, using the first description of Z_i, as the scheme-theoretic image of the morphism $V \times_{\mathcal{X}} V_i \times_{\mathcal{X}} V \to V \times_{\mathcal{X}} V$ (projection onto the first and third factors), while the second may be described, using the second description of Z_i, as the scheme-theoretic image of the morphism $V \times_{\mathcal{X}} V_i \times_{\mathcal{X}} V \to V \times_{\mathcal{X}} V$ (where the morphism is again projection onto the first and third factors). Thus each of these fibre products also coincides with $Z_i \times_{\mathcal{X}} Z_i$. We denote these three (equal) fibre products by R_i. The last description of R_i gives it a natural structure of a groupoid in algebraic spaces over Z_i, while the first two descriptions show that it is a smooth groupoid (because the projections $Z_i \times_{\mathcal{X}} Z_i \rightrightarrows Z_i$ are obtained by base-change from the smooth morphism $V \to \mathcal{X}$). Thus we may form the algebraic stack $\mathcal{X}_i := [Z_i/R_i]$, and the composite $Z_i \to V \to \mathcal{X}$ factors through $\mathcal{X}_i$, i.e. induces a morphism $\mathcal{X}_i \to \mathcal{X}$. Essentially by definition, the base-change of this morphism to V over $\mathcal{X}$ recovers the closed immersion $Z_i \to \mathcal{X}$, and thus this morphism itself is a closed immersion.

Since Z_i contains V_i (when both are thought of as closed subsheaves of V), we see that $\varinjlim_i Z_i \cong V$. Thus the natural morphism $\varinjlim_i \mathcal{X}_i \to \mathcal{X}$ is also an isomorphism, and we have proved the first statement of the lemma.

If $\mathcal{X}$ is furthermore locally countably indexed, then $\mathcal{V}$ is countably indexed, and so the indexing category $\mathcal{I}$ can be taken to be countable. This proves the second statement of the lemma. $\square$

The next result gives a partial analogue of [Stacks, Tag 0AIU], or, in other words, a partial converse to the preceding result.

Lemma 6.3. *If $\mathcal{X}_1 \hookrightarrow \mathcal{X}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{X}_n \hookrightarrow \cdots$ is a sequence of finite order thickenings of algebraic stacks, then $\varinjlim_n \mathcal{X}_n$ is a locally countably indexed formal algebraic stack.*

Proof. Choose (as we may) a morphism $\coprod U_{1,i} \to \mathcal{X}_1$ which is smooth and surjective, and whose source is a disjoint union of affine schemes. Lemma 6.4 below allows us to successively lift each of the morphisms $U_{1,i} \to \mathcal{X}_1$ to a morphism $\mathcal{U}_{n,i} \to \mathcal{X}_n$, so that $\mathcal{U}_{n-1,i}$ is identified with the fibre product $\mathcal{X}_{n-1} \times_{\mathcal{X}_n} U_{n,i}$. If we write $U_i := \varinjlim_n U_{i,n}$, then U_i is an inductive limit of a sequence of affine schemes with respect to thickenings, and so is a countably indexed affine formal algebraic space. Furthermore, there is an evident morphism $U_i \to \varinjlim_n \mathcal{X}_n$, which by construction is representable by algebraic spaces and smooth. The induced morphism $\coprod U_i \to \mathcal{X}$ is furthermore surjective, and thus realizes $\mathcal{X}$ as a locally countably indexed formal algebraic stack. $\square$

Lemma 6.4. *If $\mathcal{X} \hookrightarrow \mathcal{X}'$ is a finite order thickening of algebraic stacks, and if $U \to \mathcal{X}$ is a smooth morphism whose source is an affine scheme, then we may find a smooth morphism $U' \to \mathcal{X}'$ whose source is an affine scheme, and which fits into a Cartesian diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U' \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X}' \end{tikzcd}$$

Proof. This is proved in the stated generality as [Stacks, Tag 0CKI].

Here we present our original argument, which requires the addition assumption that each of $\mathcal{X}$ and $\mathcal{X}'$ has a quasi-compact diagonal.

An evident induction reduces us to the case when $\mathcal{X} \hookrightarrow \mathcal{X}'$ is a first order thickening. In this case we argue using the deformation theoretic formalism developed in [Ols06] (and we require the assumptions on the diagonals of $\mathcal{X}$ and $\mathcal{X}'$ that we do because it is a running hypothesis in that paper). Namely, Theorem 1.4 of that paper shows that the obstruction to deforming $U \to \mathcal{X}$ to a flat morphism $U' \to \mathcal{X}'$ is given by an element of $\mathrm{Ext}^2_{\mathcal{O}_U}(L_{U/\mathcal{X}}, \mathcal{I})$, where $L_{U/\mathcal{X}}$ denotes the cotangent complex of U over $\mathcal{X}$, and $\mathcal{I}$ denotes the pull-back to U of the ideal sheaf on $\mathcal{X}'$ that cuts out $\mathcal{X}$. Since $U \to \mathcal{X}$ is smooth, the cotangent complex $L_{U/\mathcal{X}}$ is precisely the space of differentials $\Omega^1_{U/\mathcal{X}}$, which is a locally free sheaf of finite rank on U. (We refer to [LMB00; Ols07] for the general theory of the cotangent complex on algebraic stacks, and to [LMB00, Lem. 17.5.8] for this particular fact. More precisely, it is proved there that the cotangent complex of a morphism that is representable by algebraic spaces and smooth collapses to the sheaf Ω^1 supported in degree zero. The formation of Ω^1 is compatible with smooth base-change, and so is easily seen to be locally free, by reduction to the case of a smooth morphism of schemes.) Thus the Ext^2 above may be rewritten as $H^2\big(U, \mathcal{I} \otimes (\Omega^1_{U/\mathcal{X}})^\vee\big)$, which vanishes, since U is affine.

Finally, the following lemma (Lemma 6.5) shows that the flat morphism $U' \to \mathcal{X}'$ is in fact smooth, since the base-changed morphism $U \to \mathcal{X}$ is. $\square$

Lemma 6.5. *Suppose that*

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] & \mathcal{X}' \arrow[d] \\ \mathcal{Y} \arrow[r] & \mathcal{Y}' \end{tikzcd}$$

is a Cartesian diagram of morphisms of algebraic stacks, in which the horizontal arrows are finite order thickenings, and the vertical arrows are flat.

(1) If the left-hand vertical arrow is locally of finite presentation, so is the right-hand vertical arrow.

(2) If the left-hand vertical arrow is smooth, so is the right-hand vertical arrow.

Proof. The properties of being locally of finite presentation, or of being smooth, may be checked smooth locally. Thus if we choose a smooth surjective morphism $V' \to \mathcal{Y}'$ whose source is a scheme and pull-back the diagram over V', we may assume that $\mathcal{Y} = V$ is a scheme. If we then choose a smooth surjective morphism $U' \to \mathcal{X}'$, and write $U = \mathcal{X} \times_{\mathcal{X}'} U'$, then in order to prove the lemma, it suffices to prove the corresponding statement for the Cartesian diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U' \arrow[d] \\ V \arrow[r] & V' \end{tikzcd}$$

in which all the objects appearing are schemes. Since the properties to be checked are local on the source, we reduce to the case where all the schemes involved are affine, in which the case the problem may be rephrased as follows: if $A \to A/I$ is a surjection of rings with I a nilpotent ideal, if B is a flat A-algebra, and if B/IB is of finite presentation (resp. smooth) over A/I, then B is of finite presentation (resp. smooth) over A.

If $f : A[x_1, \ldots, x_n] \to B$ is a morphism for which the induced morphism $\overline{f} : (A/I)[x_1, \ldots, x_n] \to B/I$ is surjective, then Nakayama's Lemma shows that the morphism f itself is surjective. If J denotes its kernel, then since B is flat over A, we see that J/IJ is identified with the kernel of $\overline{f}$. Nakayama's Lemma again shows that if J/IJ is finitely generated over $(A/I)[x_1, \ldots, x_n]$, then J is finitely generated over $A[x_1, \ldots, x_n]$. Thus we have shown that if B/IB is finitely presented over A/I, then B is finitely presented over A.

If furthermore B/IB is smooth over A/I, then we find that B is a finitely presented A-algebra, for which the induced morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ has smooth fibres. The fibrewise criterion for smoothness [Stacks, Tag 00TF] then shows that B is a smooth A-algebra, as required. $\square$

We have the following corollary to Lemma 6.3.

Corollary 6.6. *Suppose that $\mathcal{X}$ is an Ind-algebraic stack that can be written as the 2-colimit $\mathcal{X} \xrightarrow{\sim} \varinjlim \mathcal{X}_n$ of a directed sequence $(\mathcal{X}_n)_{n \geq 1}$ in which the $\mathcal{X}_n$ are algebraic stacks, and the transition morphisms are closed immersions. If $\mathcal{X}_{\mathrm{red}}$ is a quasi-compact algebraic stack, then $\mathcal{X}$ is a locally countably indexed formal algebraic stack.*

Proof. By assumption, $\mathcal{X}_{\mathrm{red}}$ is a quasi-compact algebraic stack, and Lemma 4.4 shows that the induced isomorphism $\mathcal{X}_{\mathrm{red}} \xrightarrow{\sim} \varinjlim_n \mathcal{Y}_{n,\mathrm{red}}$ (given by Lemma 4.16) necessarily factors through $\mathcal{Y}_{n,\mathrm{red}}$ for some sufficiently large value of n. This shows that the sequence $(\mathcal{Y}_{n,\mathrm{red}})_{n\geq 1}$ stabilizes when n becomes sufficiently large, and thus that the sequence $(\mathcal{Y}_n)_{n\geq 1}$ becomes a sequence of thickenings for sufficiently large values of n. Thus Lemma 6.3 shows that $\mathcal{X}$ is indeed a locally countably indexed formal algebraic stack. □

We now turn to establishing the characterization of formal algebraic spaces discussed in Example 5.7 above. Lemma 6.2 plays a crucial role in the argument.

Lemma 6.7. *If $\mathcal{X}$ is a formal algebraic stack, then the following are equivalent:*

(1) $\mathcal{X}$ is an affine formal algebraic space.
(2) $\mathcal{X}_{\mathrm{red}}$ is an affine scheme.

Proof. That (1) implies (2) is clear, and we focus on proving the converse implication. Thus we assume that $\mathcal{X}_{\mathrm{red}}$ is an affine scheme. Then $\mathcal{X}_{\mathrm{red}}$ is quasi-compact and quasi-separated, and hence the same is true of $\mathcal{X}$. Lemma 6.2 allows us to write $\mathcal{X} \cong \varinjlim_i \mathcal{X}_i$ as the 2-colimit of a 2-directed system of algebraic stacks having thickenings for transition morphisms. Since $\mathcal{X}_{\mathrm{red}} \cong (\mathcal{X}_i)_{\mathrm{red}}$ for each i, we conclude from [Stacks, Tag 0BPW] that each $\mathcal{X}_i$ is an affine scheme. Thus $\mathcal{X}$ is indeed an affine formal algebraic space. □

We can now show that formal algebraic spaces are precisely the formal algebraic stacks that have trivial inertia stacks. This is the analogue, for formal algebraic stacks, of [Stacks, Tag 04SZ], and we begin by recalling that statement, as well as a useful variant statement [Con07, Thm. 2.5.1 (1)], which reduces one to checking triviality of the inertia stack over field-valued, or even geometric, points. (Since the paper [Con07] imposes they blanket hypothesis that algebraic stacks be quasi-separated, in the sense of [LMB00], we give a proof here of its Thm. 2.2.5 (1) which applies without any such assumption.)

Lemma 6.8. *If $\mathcal{X}$ is an algebraic stack, then the following are equivalent:*

(1) $\mathcal{X}$ is a stack in setoids.
(2) The canonical morphism $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is an equivalence (where $\mathcal{I}_{\mathcal{X}}$ denotes the inertia stack of $\mathcal{X}$).
(3) $\mathcal{X}$ is an algebraic space.
(4) For any morphism $x : \operatorname{Spec} k \to \mathcal{X}$ with k a field, the automorphism group algebraic space of x is trivial.
(5) For any morphism $x : \operatorname{Spec} k \to \mathcal{X}$ with k an algebraically closed field, the automorphism group algebraic space of x is trivial.

Proof. The equivalence of (1), (2), and (3) is precisely the statement of [Stacks, Tag 04SZ]. Clearly (4) implies (5), and since the formation of the automorphism group algebraic space of an object of $\mathcal{X}$ is compatible with the formation of flat base-change, we see that in fact (4) and (5) are equivalent, and are implied by (1).

Now (1) is equivalent to the statement that for any morphism $T \to \mathcal{X}$ with source a scheme, the automorphism group algebraic space of this T-valued point of $\mathcal{X}$ is trivial. If we assume (5), then we find that the fibre of this group algebraic space over each geometric point of T is trivial. It follows from Lemma 6.9 that this group algebraic space itself is trivial (note that this group algebraic space admits the identity section, so that all the hypotheses of that lemma are satisfied); thus (5) implies (1). □

The lemma that follows isolates the key argument from the proof of [Con07, Thm. 2.2.5 (1)].

Lemma 6.9. *Let $X \to Y$ be a locally finite type morphism of algebraic spaces with the property that, for any morphism $\operatorname{Spec} k \to Y$, with k being an algebraically closed field, the base-changed morphism $X \times_Y \operatorname{Spec} k \to \operatorname{Spec} k$ is an isomorphism. If the morphism $X \to Y$ admits a section, then it itself is an isomorphism.*

Proof. The property of being an isomorphism may be checked étale locally on the target, and so, pulling back over an étale surjection $V \to Y$ with V a scheme, we may assume that Y is a scheme.

We first claim that the morphism $X \to Y$ is unramified. This may be checked étale locally on the source, and so composing with an étale surjection $U \to X$ with U a scheme, we are reduced to showing that a morphism of schemes $U \to Y$ which is locally of finite type, and which satisfies the additional property that for each morphism $\operatorname{Spec} k \to Y$ with k an algebraically closed field, the base-changed morphism $U \times_Y \operatorname{Spec} k \to \operatorname{Spec} k$ is étale, is unramified. Since the property of a morphism being étale is *fppf* local on the target, we see that in fact $U_y \to \operatorname{Spec} \kappa(y)$ is étale for each $y \in Y$. The fact that $U \to Y$ is unramified now follows from [Stacks, Tag 02G8].

Since $X \to Y$ is unramified, the diagonal $\Delta_{X/Y}$ is an open immersion [Stacks, Tag 05W1]. Thus the section $Y \to X$ (which we assumed to exist) is an open immersion. It is also surjective (by virtue of our assumption regarding geometric points), and thus is an isomorphism. This completes the proof of the lemma. □

Remark 6.10. In the preceding lemma, the hypothesis that $X \to Y$ admits a section is crucial. For example, if Z is a closed subscheme of the scheme X, and $U := X \setminus Z$, then the natural morphism $Z \amalg U \to X$ satisfies the hypothesis on geometric points from the lemma, but typically is not an isomorphism.

We now return to the context of formal algebraic stacks.

Lemma 6.11. *Let $\mathcal{X}$ be a formal algebraic stack. The following are equivalent:*

1. *$\mathcal{X}$ is a stack in setoids.*
2. *The canonical morphism $\mathcal{I}_{\mathcal{X}} \to \mathcal{X}$ is an equivalence (where $\mathcal{I}_{\mathcal{X}}$ denotes the inertia stack of $\mathcal{X}$).*
3. *$\mathcal{X}$ is a formal algebraic space.*
4. *$\mathcal{X}_{\mathrm{red}}$ is an algebraic space.*
5. *For any morphism $x : \operatorname{Spec} k \to \mathcal{X}$ with k a field, the automorphism group of x is trivial.*
6. *For any morphism $x : \operatorname{Spec} k \to \mathcal{X}$ with k an algebraically closed field, the automorphism group of x is trivial.*

Proof. The equivalence of (1) and (2) is a general fact about stacks; see e.g. [Stacks, Tag 04ZM], and the fact that (3) implies (2) was observed in Example 5.7. Suppose now that (1) holds. Since $\mathcal{X}_{\mathrm{red}}$ is a substack of $\mathcal{X}$, it is also a category fibred in setoids (rather than just in groupoids), and so it follows from [Stacks, Tag 04SZ] that $\mathcal{X}_{\mathrm{red}}$ is an algebraic space, so (1) implies (4).

Since $\operatorname{Spec} k$ is reduced for any field k, the groupoid of morphisms from $\operatorname{Spec} k$ to each of $\mathcal{X}$ and $\mathcal{X}_{\mathrm{red}}$ coincide. Furthermore, if $x : \operatorname{Spec} k \to \mathcal{X}_{\mathrm{red}}$ is such a morphism, then because $\mathcal{X}_{\mathrm{red}} \to \mathcal{X}$ is a monomorphism, the automorphisms of x, thought of either as a $\operatorname{Spec} k$-valued point of either $\mathcal{X}_{\mathrm{red}}$ or of $\mathcal{X}$, coincide [Stacks, Tag 06R5].

Thus, conditions (5) and (6) are equivalent to the analogous conditions for $\mathcal{X}_{\mathrm{red}}$, which in turn are equivalent to (4), by Lemma 6.8.

It remains to show that (4) implies (3). In fact, we first show that (4) implies (1) and (2); equivalently, we show that if $\mathcal{X}_{\mathrm{red}}$ is a formal algebraic space, then the diagonal morphism $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is a monomorphism. Since $\mathcal{X}_{\mathrm{red}} \to \mathcal{X}$ is a monomorphism, the diagram

$$\begin{tikzcd} \mathcal{X}_{\mathrm{red}} \arrow[r] \arrow[d] & \mathcal{X}_{\mathrm{red}} \times_S \mathcal{X}_{\mathrm{red}} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times_S \mathcal{X} \end{tikzcd}$$

whose horizontal arrows are the evident diagonal morphisms, is 2-Cartesian. Its horizontal arrows are representable by algebraic spaces, and locally of finite type (Lemma 5.12), while its vertical arrows are representable by algebraic spaces, and thickenings. Since $\mathcal{X}_{\mathrm{red}}$ is an algebraic space, by assumption, the upper horizontal arrow is furthermore a monomorphism. It follows from [Stacks, Tag 0BPJ] that the lower horizontal arrow is also a monomorphism, as required.

Thus, assuming that $\mathcal{X}_{\mathrm{red}}$ is an algebraic space, we have shown that $\mathcal{X}$ is a stack in setoids. We now show that it is in fact a formal algebraic space. To this end, choose (as we may) a morphism $\coprod U_{i,\mathrm{red}} \to \mathcal{X}_{\mathrm{red}}$ which is representable by algebraic spaces and étale, and whose source is a disjoint union of affine schemes $U_{i,\mathrm{red}}$. Lemmas 5.32 and 6.7 allow us to promote each $U_{i,\mathrm{red}}$ to an affine formal algebraic space U_i, equipped with a morphism $U_i \to \mathcal{X}$ which is representable by algebraic spaces and étale. The induced morphism $\coprod_i U_i \to \mathcal{X}$ is then representable by algebraic spaces, étale, and surjective, so realizes $\mathcal{X}$ as a formal algebraic space. This completes the proof that (4) implies (3), and so also completes the proof of the lemma. $\square$

Remark 6.12. One can establish a characterization of formal Deligne–Mumford stacks analogous to the characterization of formal algebraic spaces given by the preceding lemma, using very similar arguments. Namely, a formal algebraic stack $\mathcal{X}$ is formal Deligne–Mumford if and only if $\mathcal{X}_{\mathrm{red}}$ is a Deligne–Mumford stack, if and only if $\Delta_{\mathcal{X}}$ is unramified, if and only if the automorphism group schemes of points of $\mathcal{X}$ are étale, if and only if this last condition holds for field-valued, or even merely algebraically closed field-valued, points of $\mathcal{X}$.

We may use the results developed in this section to define the notion of a *scheme-theoretically dominant* morphism between formal algebraic stacks that are quasi-compact and quasi-separated. Namely, suppose that $\mathcal{X} \to \mathcal{Y}$ is a morphism of formal algebraic stacks, each of which is quasi-compact and quasi-separated. The usual diagonal argument shows that $\mathcal{X} \to \mathcal{Y}$ is then also quasi-compact and quasi-separated. Lemma 6.2 allows us to write $\mathcal{X} \cong \varinjlim \mathcal{X}_\lambda$, with the transition morphisms being thickenings of algebraic stacks. Similarly, we may write $\mathcal{Y} \cong \varinjlim \mathcal{Y}_\mu$. For each index λ, the composite morphism $\mathcal{X}_\lambda \to \mathcal{X} \to \mathcal{Y}$ factors through one of the $\mathcal{Y}_\mu$ (since $\mathcal{X}_\lambda$ is quasi-compact and quasi-separated). The resulting morphism $\mathcal{X}_\lambda \to \mathcal{Y}_\mu$ is quasi-compact, and so has a scheme-theoretic image (see e.g. [EG19b, §3.1], or Example 9.9 below), which we denote by $\mathcal{Z}_\lambda$. One easily checks that $\mathcal{Z}_\lambda$, thought of as a closed substack of $\mathcal{Y}$, is independent of the particular choice of the index μ used in its definition, and we may regard $\mathcal{Z}_\lambda$ as the scheme-theoretic image of

$\mathcal{X}_\lambda$ in $\mathcal{Y}$. Evidently $\mathcal{Z}_\lambda$ is a closed substack of $\mathcal{Z}_{\lambda'}$ if $\lambda \leq \lambda'$, with the resulting morphism $\mathcal{Z}_\lambda \hookrightarrow \mathcal{Z}_{\lambda'}$ being a thickening. In particular, we may form the 2-colimit $\varinjlim \mathcal{Z}_\lambda$, which is an Ind-algebraic stack.There is a natural morphism $\varinjlim \mathcal{Z}_\lambda \to \mathcal{Y}$.

Definition 6.13. A morphism $\mathcal{X} \to \mathcal{Y}$ of quasi-compact and quasi-separated formal algebraic stacks is called *scheme-theoretically dominant* if, in the notation of the preceding discussion, the morphism $\varinjlim \mathcal{Z}_\lambda \to \mathcal{Y}$ is an isomorphism.

Remark 6.14. One easily verifies that the condition of the preceding definition is independent of the particular representation of $\mathcal{X}$ as an inductive limit $\mathcal{X} \cong \varinjlim \mathcal{X}_\lambda$.

The property of being scheme-theoretically dominant is preserved by a suitable flat base-change.

Lemma 6.15. *If $\mathcal{X} \to \mathcal{Y}$ is a scheme-theoretically dominant morphism between quasi-compact and quasi-separated formal algebraic stacks, and if $\mathcal{Y}' \to \mathcal{Y}$ is a morphism whose source is again a quasi-compact and quasi-separated algebraic stack, and which is furthermore representable by algebraic stacks and flat, then the base-changed morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}' \to \mathcal{Y}'$ is scheme-theoretically dominant.*

Proof. As in the discussion preceding Definition 6.13, write $\mathcal{X} \cong \varinjlim \mathcal{X}_\lambda$; since $\mathcal{X} \to \mathcal{Y}$ is scheme-theoretically dominant by assumption, we obtain the corresponding description $\mathcal{Y} \cong \varinjlim \mathcal{Y}_\lambda$, where the composite $\mathcal{X}_\lambda \to \mathcal{X} \to \mathcal{Y}$ factors through a scheme-theoretically dominant morphism $\mathcal{X}_\lambda \to \mathcal{Y}_\lambda$, for each index λ. Let $\mathcal{X}' := \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}'$. If we write $\mathcal{X}'_\lambda := \mathcal{X}_\lambda \times_{\mathcal{Y}} \mathcal{Y}'$, then $\mathcal{X}'_\lambda$ is an algebraic stack, the morphism $\mathcal{X}'_\lambda \hookrightarrow \mathcal{X}'$ is a closed immersion, the $\mathcal{X}'_\lambda$ form a 2-directed system of algebraic stacks with thickenings as transition morphisms, and the natural morphism $\varinjlim \mathcal{X}'_\lambda \to \mathcal{X}'$ is an isomorphism. Similarly, if we write $\mathcal{Y}'_\lambda := \mathcal{Y}_\lambda \times_{\mathcal{Y}} \mathcal{Y}'$, then $\mathcal{Y}'_\lambda$ is an algebraic stack, the morphism $\mathcal{Y}'_\lambda \hookrightarrow \mathcal{Y}'$ is a closed immersion, the $\mathcal{Y}'_\lambda$ form a 2-directed system having thickenings as transition morphisms, and the natural morphism $\varinjlim \mathcal{Y}'_\lambda \to \mathcal{Y}'$ is an isomorphism.

Since $\mathcal{X}_\lambda \to \mathcal{Y}_\lambda$ is a scheme-theoretically dominant quasi-compact morphism of algebraic stacks, since the formation of scheme-theoretic images of quasi-compact morphisms is compatible with flat base-change, and since $\mathcal{Y}_{\lambda'} \to \mathcal{Y}_\lambda$ is flat, we find that $\mathcal{X}'_\lambda \to \mathcal{Y}'_\lambda$ is again scheme-theoretically dominant. Thus by definition $\mathcal{X}' \to \mathcal{Y}'$ is scheme-theoretically dominant, as claimed. ☐

7. Adic morphisms

We now discuss the concept of an adic morphism, beginning by recalling this notion in the context of formal algebraic spaces.

Definition 7.1. If $f : X \to Y$ is a morphism of sheaves over S with Y being a locally adic* formal algebraic space, then we say that f is *adic* if and only if it representable by algebraic spaces.

Remark 7.2. This is the definition of [Stacks, Tag 0AQ3]. (Note that in that definition, it is required that additionally X be a locally adic* formal algebraic space; however, Remark 5.20 shows that this holds automatically if the morphism is representable by algebraic spaces.)

We extend this definition to the context of formal algebraic stacks as follows.

Definition 7.3. We say that a morphism $\mathcal{X} \to \mathcal{Y}$ of stacks over S, whose target $\mathcal{Y}$ is a locally adic* formal algebraic stack, is *adic* if and only if it is representable by algebraic stacks.

Remark 7.4. In the context of Definition 7.3, Remark 5.24 shows that the source $\mathcal{X}$ of an adic morphism is necessarily a locally adic* formal algebraic stack.

Remark 7.5. It follows from Lemma 3.5 that a morphism of sheaves whose target is a locally adic* algebraic space, thought of as a morphism of stacks, is adic in the sense of Definition 7.3 if and only if it is adic in the sense of Definition 7.1.

Suppose now that A is an adic* topological ring. with finitely generated ideal of definition I. As usual, write $\operatorname{Spf} A := \varinjlim_n \operatorname{Spec} A/I^n$; then the natural morphism $\operatorname{Spf} A \to \operatorname{Spec} A$ is a monomorphism. In particular, a stack over $\operatorname{Spec} A$ admits at most one morphism to $\operatorname{Spf} A$.

Definition 7.6. We say that a formal algebraic stack $\mathcal{X}$ over $\operatorname{Spec} A$ is an *I-adic formal algebraic stack* if it admits a morphism $\mathcal{X} \to \operatorname{Spf} A$ which is furthermore adic.

Example 7.7. If $\mathcal{X}$ is an algebraic stack over $\operatorname{Spec} A$, then the I-adic completion $\widehat{\mathcal{X}}$ of $\mathcal{X}$ gives an example of an I-adic formal algebraic stack. We may define the I-adic completion in two ways: as the fibre product $\operatorname{Spf} A \times_{\operatorname{Spec} A} \mathcal{X}$, or as the completion of $\mathcal{X}$ along its closed substack $\operatorname{Spec} A/I \times_{\operatorname{Spec} A} \mathcal{X}$. (The first definition makes it clear that $\widehat{\mathcal{X}}$ is I-adic, since its structure morphism to $\operatorname{Spf} A$ is the base-change of the morphism $\mathcal{X} \to \operatorname{Spec} A$, which is representable by algebraic stacks since $\mathcal{X}$ itself is an algebraic stack.)

Example 7.8. Suppose that $\{\mathcal{X}_n\}_{n\geq 1}$ is a directed sequence of algebraic stacks over $\operatorname{Spec} A$, for which each $\mathcal{X}_n$ in fact lies over $\operatorname{Spec} A/I^n$, such that each of the induced morphisms $\mathcal{X}_n \to \operatorname{Spec} A/I^n \times_{\operatorname{Spec} A/I^{n+1}} \mathcal{X}_{n+1}$ is an isomorphism. We may then form the Ind-algebraic stack $\mathcal{X} := \varinjlim_n \mathcal{X}_n$, and the obvious morphism $\mathcal{X} \to \operatorname{Spf} A$ is representable by algebraic stacks, or, equivalently, adic, so that $\mathcal{X}$ is an I-adic formal algebraic stack.

We have the following simple lemmas.

Lemma 7.9. *If A is an adic* topological ring, with finitely generated ideal of definition I, if $\mathcal{X} \to \mathcal{Y}$ is a morphism of stacks over* $\operatorname{Spec} A$ *which is representable by algebraic stacks, and if $\mathcal{Y}$ is furthermore an I-adic formal algebraic stack, then $\mathcal{X}$ is also an I-adic formal algebraic stack.*

Proof. Remark 7.4 shows that $\mathcal{X}$ is again a formal algebraic stack over $\operatorname{Spec} A$. The fact that it is I-adic follows from the fact that a composite of morphisms that are representable by algebraic stacks is again representable by algebraic stacks. □

Lemma 7.10. *If A is an adic* topological ring, with finitely generated ideal of definition I, then any morphism $\mathcal{X} \to \mathcal{Y}$ between I-adic formal algebraic stacks is necessarily adic.*

Proof. Since $\operatorname{Spf} A \to \operatorname{Spec} A$ is a monomorphism, we have that $\mathcal{Y} \times_{\operatorname{Spf} A} \mathcal{Y} \cong \mathcal{Y} \times_{\operatorname{Spec} A} \mathcal{Y}$, and hence Lemma 5.12 shows that the diagonal morphism $\mathcal{Y} \to \mathcal{Y} \times_{\operatorname{Spf} A} \mathcal{Y}$ is representable by algebraic spaces. The claim of the lemma then follows from Lemma 3.7. □

8. Properties of formal algebraic stacks

We fix a base-scheme S, and throughout, we work with objects (schemes, sheaves, stacks, etc.) over S. We will define certain properties of formal algebraic stacks (over S)[14] in terms of properties of formal algebraic spaces that are of a sufficiently local nature.

Definition 8.1. We say that a property P of formal algebraic spaces is *smooth local* (resp. *fppf local*) if it satisfies the following three conditions:

(1) If $X \to Y$ is a morphism of formal algebraic spaces which is representable by algebraic spaces and smooth (resp. flat and locally of finite presentation), and if Y satisfies P, then so does X.
(2) If $X \to Y$ is a morphism of formal algebraic spaces which is representable by algebraic spaces, smooth (resp. flat and locally of finite presentation), and surjective, and if X satisfies P, then so does Y.
(3) If $\{X_i\}_{i\in I}$ is a collection of formal algebraic spaces, each of which satisfies P, then the disjoint union $\coprod_i X_i$ also satisfies P.

Remark 8.2. Since smooth morphisms are in particular flat and locally of finite presentation, we see that an *fppf* local property is in particular smooth local.

Lemma 8.3. *If $\mathcal{X}$ is a formal algebraic stack over S, and if P is a property of formal algebraic spaces that is smooth local* (*resp. fppf local*) *then the following are equivalent:*

(1) For every morphism $U \to \mathcal{X}$ whose domain is a formal algebraic space, and which is representable by algebraic spaces and smooth (*resp. flat and locally of finite presentation*)*, the formal algebraic space U has property P.*
(2) For every morphism $U \to \mathcal{X}$ whose domain is an affine formal algebraic space, and which is representable by algebraic spaces and smooth (*resp. flat and locally of finite presentation*)*, the affine formal algebraic space U has property P.*
(3) There exists a formal algebraic space V having the property P, and a morphism $V \to \mathcal{X}$ which is representable by algebraic spaces, smooth (*resp. flat and locally of finite presentation*)*, and surjective.*
(4) There exists a family $\{U_i\}$ of affine formal algebraic spaces, each having the property P, and a morphism $\coprod U_i \to \mathcal{X}$ which is representable by algebraic spaces, smooth (*resp. flat and locally of finite presentation*)*, and surjective.*

Proof. Clearly condition (1) implies condition (2), and almost as clearly, condition (4) implies condition (3) (take the formal algebraic space V of (3) to be the disjoint union of the affine formal algebraic spaces U_i of (4)). Since $\mathcal{X}$ is a formal algebraic stack, there exists a morphism $\coprod U_i \to \mathcal{X}$, whose source is a disjoint union of affine formal algebraic spaces, which is representable by algebraic spaces, smooth, and surjective. Since smooth morphisms are in particular flat and locally of finite presentation, we see that condition (2) implies condition (4).

Finally, suppose that (3) holds, and choose a morphism $V \to \mathcal{X}$ as in the statement of (3). Now consider a morphism $U \to \mathcal{X}$ as in the statement of (1). The projection $U \times_{\mathcal{X}} V \to U$ is then representable by algebraic spaces, smooth (resp. flat

[14] As already indicated, we will work with formal algebraic spaces over S throughout, and so we won't continue to emphasize that all our objects and morphisms are over S.

and locally of finite presentation), and surjective, while the projection $U \times_{\mathcal{X}} V \to V$ is representable by algebraic spaces, and smooth (resp. flat and locally of finite presentation). Since V has property P, by assumption, we conclude first that the fibre product $U \times_{\mathcal{X}} V$ has property P, and then that U has property P, as required. □

Definition 8.4. If P is a smooth local property of formal algebraic spaces, then we say that a formal algebraic stack $\mathcal{X}$ is P if $\mathcal{X}$ satisfies the equivalent conditions of Lemma 8.3.

Remark 8.5. Since *fppf* local properties are in particular smooth local, this definition applies to such properties. Also, note that since formal algebraic spaces are particular types of formal algebraic stacks, this definition encompasses formal algebraic spaces. Similarly, algebraic stacks are particular types of formal algebraic stacks, and so this definition also encompasses algebraic stacks. In this case it coincides with the notion of an algebraic stack being P in the sense of [Stacks, Tag 04YG] (provided that the property P, as understood for formal algebraic spaces, coincides with the property P for schemes, when the latter are thought of as particular cases of the former).

Having made this definition, Lemma 8.3 admits the following extension.

Lemma 8.6. *If $\mathcal{X}$ is a formal algebraic stack over S, and if P is a property of formal algebraic spaces that is smooth local* (*resp. fppf local*) *then the following are equivalent:*

(1) $\mathcal{X}$ has property P.
(2) For every morphism $\mathcal{U} \to \mathcal{X}$ whose domain is a formal algebraic stack, and which is representable by algebraic stacks and smooth (*resp. flat and locally of finite presentation*)*, the formal algebraic stack $\mathcal{U}$ has property P.*
(3) There exists a formal algebraic stack $\mathcal{U}$ that has property P, and a morphism $\mathcal{U} \to \mathcal{X}$ which is representable by algebraic stacks, smooth (*resp. flat and locally of finite presentation*)*, and surjective.*

Proof. Clearly (2) implies (1), while (1) implies (3).

We next show that (1) implies (2). To this end, suppose that $\mathcal{X}$ has property P, and that we are given a morphism $\mathcal{U} \to \mathcal{X}$ as in (2). Since $\mathcal{U}$ is a formal algebraic stack, we may find a disjoint union $\coprod U_i$ of formal algebraic spaces, and a morphism $\coprod U_i \to \mathcal{U}$ which is representable by algebraic spaces, smooth, and surjective. The composite $\coprod U_i \to \mathcal{U} \to \mathcal{X}$ is then representable by algebraic spaces (here we are applying Lemma 3.5) and smooth (resp. flat and locally of finite presentation), and (by definition of $\mathcal{X}$ satisfying P and Lemma 8.3) U_i satisfies P. Thus $\mathcal{U}$ satisfies P (again by definition), as required.

Finally, we show that (3) implies (1). Thus, we suppose now that $\mathcal{X}$ satisfies (3), and let $\mathcal{U} \to \mathcal{X}$ be a morphism as in the statement of (3). As in the preceding paragraph, we may find a disjoint union $\coprod U_i$ of affine formal algebraic spaces, and a morphism $\coprod U_i \to \mathcal{U}$ which is representable by algebraic spaces, smooth, and surjective. Since $\mathcal{U}$ furthermore satisfies P, each U_i satisfies P. The composite $\coprod U_i \to \mathcal{U} \to \mathcal{X}$ is then representable by algebraic spaces (here we are applying Lemma 3.5), smooth (resp. flat and locally of finite presentation), and surjective, and (by definition of $\mathcal{X}$ satisfying P, and Lemma 8.3) we see that $\mathcal{X}$ satisfies P, as claimed. □

Remark 8.7. It is shown in [Stacks, Tag 0AKS], resp. [Stacks, Tag 0AKT], that the property of being countably indexed, resp. adic*, is an *fppf* local property of affine formal algebraic spaces. It follows immediately that the property of a formal algebraic space being locally countably indexed or locally adic* (in the sense of [Stacks, Tag 0AKY]) is again *fppf* local. If we apply the preceding framework to the property P of being locally countably indexed, resp. locally adic*, then we recover the notion of a locally countably indexed, resp. locally adic*, formal algebraic stack which we introduced above.

The following notion provides another instance of the preceding framework.

Maybe there is better terminology, or established terminology, for this notion?

Definition 8.8. We say that a formal algebraic stack $\mathcal{X}$ is *residually Jacobson* if $\mathcal{X}_{\text{red}}$ is a Jacobson algebraic stack, in the sense of [Stacks, Tag 04YH].

Remark 8.9. Since the property of a scheme being Jacobson depends only on its underlying topological space, and hence only on its underlying reduced subscheme, and since this property is also local for the *fppf* topology on schemes [Stacks, Tag 0368], we see that the property of an affine formal algebraic space being residually Jacobson (in the sense of Definition 8.8) is *fppf* local. Thus we may apply the preceding framework to the property P of being residually Jacobson, and we recover the notion of Definition 8.8.

Remark 8.10. An affine formal algebraic space is defined to be *Noetherian* if it is of the form $\operatorname{Spf} A$, where A is a Noetherian adic topological ring [Stacks, Tag 0AID]. It is shown in [Stacks, Tag 0AKW] that the property of being Noetherian is *fppf* local. It follows immediately that the property of an algebraic space being *locally Noetherian* (in the sense of [Stacks, Tag 0AKY]) is again *fppf* local. Thus the above framework allows us to define the notion of a *locally Noetherian* formal algebraic stack. This definition is sufficiently important that we record it separately (and also take the opportunity to define the notion of *Noetherian* formal algebraic stacks).

Definition 8.11. Let $\mathcal{X}$ be a formal algebraic stack.

(1) We say that $\mathcal{X}$ is *locally Noetherian* if it so in the sense of Definition 8.4 (taking P to be the property of a formal algebraic space being locally Noetherian).
(2) We say that $\mathcal{X}$ is *Noetherian* if it is locally Noetherian, quasi-compact, and quasi-separated.

Remark 8.12. The definition of Noetherian formal algebraic stacks as being locally Noetherian algebraic stacks that are furthermore quasi-compact and quasi-separated is modelled on the analogous definition for algebraic stacks [Stacks, Tag 0510] (and coincides with that definition when applied to an algebraic stack, thought of as a formal algebraic stack).

When we apply Definition 8.11 (1) to formal algebraic spaces (thought of as particular types of formal algebraic stacks), we recover the notion of locally Noetherian formal algebraic spaces, as defined in [Stacks, Tag 0AKY]. A consideration of the definitions also shows that a formal algebraic stack $\mathcal{X}$ is locally Noetherian if and only if it admits a morphism $U \to \mathcal{X}$ which is representable by algebraic spaces, smooth, and surjective, and whose source is a locally Noetherian formal algebraic space.

Lemma 8.13. *If $\mathcal{Y}$ is a locally Noetherian formal algebraic stack, and if $\mathcal{X} \to \mathcal{Y}$ is a morphism of formal algebraic stacks that is representable by algebraic stacks and locally of finite type, then $\mathcal{X}$ is also locally Noetherian.*

Proof. If $V \to \mathcal{Y}$ is representable by algebraic spaces, smooth, and surjective, with V being a locally Noetherian algebraic space, then the fibre product $\mathcal{X} \times_{\mathcal{Y}} V$ is a formal algebraic stack, by Lemma 5.22, and the projection $\mathcal{X} \times_{\mathcal{Y}} V \to \mathcal{X}$ is again representable by algebraic spaces, smooth, and surjective. Thus it suffices to show that $\mathcal{X} \times_{\mathcal{Y}} V$ is locally Noetherian. The projection $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is representable by algebraic stacks and locally of finite type. If we let $U \to \mathcal{X} \times_{\mathcal{Y}} V$ be a morphism that is representable by algebraic spaces, smooth, and surjective, then the composite $U \to \mathcal{X} \times_{\mathcal{Y}} V \to V$ is representable by algebraic spaces (here we have used Lemma 3.5) and locally of finite type. Thus its source is locally Noetherian, since its target is [Stacks, Tag 0AQ7]. Thus $\mathcal{X} \times_{\mathcal{Y}} V$ is locally Noetherian, and hence so is $\mathcal{X}$, as claimed. □

Recall that a Noetherian ring is said to be *Cohen–Macaulay* if its localization at each prime ideal is a Cohen–Macaulay local ring [Stacks, Tag 00NC]. It in fact suffices to check this property for the localizations at maximal ideals [Stacks, Tag 00NB], and it is equivalent to ask that the completions at maximal ideals are Cohen–Macaulay [Stacks, Tag 07NX].

Similarly, a Noetherian ring is said to be *regular* if its localization at each prime ideal is a regular local ring [Stacks, Tag 00OD]. It in fact suffices to check this property for the localizations at maximal ideals [Stacks, Tag 0AFS], and it is equivalent to ask that the completions at maximal ideals are regular [Stacks, Tag 07NY].

A ring is said to be *normal* if its localization at each prime ideal is a normal local domain [Stacks, Tag 00GV]. It in fact suffices to check this property for the localizations at maximal ideals [Stacks, Tag 030B]. Unfortunately (and as is well-known), even in the Noetherian case, it is not equivalent to require that the completions at maximal ideals are normal domains; this latter condition is a stronger condition in general.

The property of being reduced behaves similarly to that of being normal. Namely, a ring is reduced if and only if all its localizations are reduced if and only if all its localizations at maximal ideals are reduced. In the Noetherian case, since completions are then faithfully flat, we also see that it suffices for the completion at every maximal ideal to be reduced. However, in general this last condition is stronger than that of being reduced.

The preceding discussion of normality and reducedness prompts us to make the following definition.

Definition 8.13.1. A Noetherian ring A is defined to be *analytically unramified* if the completion of A at each maximal ideal of A is reduced, and to be *analytically normal* if the completion of A at each maximal ideal of A is a normal local domain.

Is this terminology standard, or in conflict with some other standard notion? Note that analytically unramified *is* used in the Stacks Project, but only for local rings (as far as I could tell).

As remarked on above, we have the following lemma.

Lemma 8.14. *If A is an analytically unramified* (*resp. analytically normal*) *Noetherian ring, then A is reduced* (*resp. normal*).

Proof. As noted above, it suffices to show that the localization $A_{\mathfrak{m}}$ of A at each of its maximal ideals $\mathfrak{m}$ is reduced (resp. a normal domain). This follows from the fact that the morphism $A_{\mathfrak{m}} \to \widehat{A_{\mathfrak{m}}}$ from this local ring to its completion is faithfully flat (as A is Noetherian), and the assumption that $\widehat{A_{\mathfrak{m}}}$ is reduced (resp. normal). (See [Stacks, Tag 033F, Tag 033G].) □

The following technical lemma will be useful below.

Lemma 8.15. *If $A \to C$ is a smooth morphism of Noetherian rings, if $\mathfrak{n}$ is a maximal ideal of C lying over a maximal ideal $\mathfrak{m}$ of A, and if $\widehat{A_{\mathfrak{m}}}$ is reduced (resp. normal), then the same is true of $\widehat{C_{\mathfrak{n}}}$.*

Proof. If we write $C' := \widehat{A_{\mathfrak{m}}} \otimes_A C$, then the base-changed morphism $\widehat{A_{\mathfrak{m}}} \to C'$ is again smooth, and the maximal ideal $\mathfrak{n}$ of C corresponds to a maximal ideal $\mathfrak{n}'$ of C' for which there is an isomorphism $\widehat{C_{\mathfrak{n}}} \cong \widehat{C'_{\mathfrak{n}'}}$. Thus we may assume that A is a reduced (resp. normal) complete Noetherian local ring, and in particular that A is excellent [Stacks, Tag 07QW].

Since C is smooth over the reduced (resp. normal) and excellent ring A, it is again reduced (resp. normal) and excellent, as is its localization $C_{\mathfrak{n}}$. (See [Stacks, Tag 033C] and [Stacks, Tag 07QU].) An excellent reduced (resp. normal) local ring is analytically unramified (resp. analytically normal), as required (see e.g. [Stacks, Tag 0C21, Tag 0C22]). □

The next lemma records the fact that the properties of being analytically normal, Cohen–Macaulay, or regular are preserved under completion.

Lemma 8.16. *Let A be a Noetherian ring, and let I be an ideal of A. If A is analytically normal, analytically unramified, Cohen–Macaulay, or regular, then so is the I-adic completion $\widehat{A}$ of A.*

Proof. By definition in the analytically normal and analytically unramified cases, and as discussed above in the case of Cohen–Macaulayness or regularity, it suffices to show that the completion of $\widehat{A}$ at any of its maximal ideals is normal, reduced, Cohen–Macaulay, or regular, as the case may be. This follows from the fact that the maximal ideals of $\widehat{A}$ necessarily contain $I\widehat{A}$, and so are in bijection with the maximal ideals of A containing I, and that if $\mathfrak{m}$ is such a maximal ideal, then $A_{\mathfrak{m}}$ and $(\widehat{A})_{\mathfrak{m}}$ have naturally isomorphic completions. □

Definition 8.17. We say that a Noetherian affine formal algebraic space is *analytically normal*, *analytically unramified*, *Cohen–Macaulay*, or *regular*, if, writing it in the form $\operatorname{Spf} A$ for a Noetherian adic topological ring A, the ring A is analytically normal, Cohen–Macaulay, or regular.

We would like to apply the framework of Definition 8.4 to the properties of Definition 8.17, which would require showing that they are smooth local properties of affine formal algebraic spaces. We are able to do this in the case of Cohen–Macaulayness and regularity. In the case of analytically normal and analytically unramified rings, we are able to do this after imposing the additional hypothesis of being residually Jacobson.

We begin by recalling that if $\operatorname{Spf} B \to \operatorname{Spf} A$ is a morphism of Noetherian affine formal algebraic spaces, then there is a corresponding morphism $A \to B$ inducing the given morphism on formal spectra [Stacks, Tag 0AN0]. If this morphism is representable by algebraic spaces (or, equivalently, adic), and if I is an ideal of definition of A, then the topology on B is again the I-adic topology (which is the reason for introducing the terminology *adic* to describe this type of morphism); see [Stacks, Tag 0AN5] and [Stacks, Tag 0APU].

We next note the following lemma.

Lemma 8.18. *A morphism of Noetherian affine formal algebraic spaces* $\operatorname{Spf} B \to \operatorname{Spf} A$ *which is representable by algebraic spaces is* (*faithfully*) *flat if and only if the corresponding morphism* $A \to B$ *is* (*faithfully*) *flat.*

Proof. If A is equipped with I-adic topology, then the condition that $\operatorname{Spf} B \to \operatorname{Spf} A$ be representable by algebraic spaces is equivalent to the condition that B be equipped with the IB-adic topology. The condition that this morphism furthermore be (faithfully) flat is equivalent to the condition that the morphism $A/I^i \to B/I^i$ be (faithfully) flat for each $i \geq 0$. If $A \to B$ is (faithfully) flat, then certainly each of these morphisms is (faithfully) flat, and so the if the direction of the lemma follows.

Suppose, conversely, that each of the morphisms $A/I^i \to B/I^i$ is flat. Let $J \to A$ be an inclusion of an ideal into A. Write $J_i := J \cap I^i$; the Artin–Rees lemma shows that the filtration $\{J_i\}$ of J is cofinal with the I-adic filtration. Since B is Noetherian, the Artin–Rees lemma also shows that $J \otimes_A B$ is I-adically complete, and so the morphism $J \otimes_A B \to \varprojlim (J/J_i) \otimes_{A/I^i} B/I^i$ is an isomorphism. Since B/I^i is flat over A/I^i, and since each of the morphisms $J/J_i \to A/I^i$ is injective, we see that $(J/J_i) \otimes_{A/I^i} B/I^i \to B/I^i$ is injective, and so, passing to the projective limit, we find that $J \otimes_A B \to B$ is injective. As J was arbitrary, we conclude that $A \to B$ is flat, as claimed.

Suppose in addition that $A/I \to B/I$ is faithfully flat, and let M be a non-zero finitely generated A-module. Artin–Rees implies that M is I-adically complete, and hence that $M/IM \neq 0$. Thus $(M/IM) \otimes_{A/I} B/I \neq 0$, and so in particular $M \otimes_A B \neq 0$. Thus in this case $A \to B$ is faithfully flat. □

Lemma 8.19. *Suppose that* $X \to Y$ *is a morphism of Noetherian affine formal algebraic spaces which is representable by algebraic spaces, flat, and surjective. If* X *is furthermore analytically normal, analytically unramified, Cohen–Macaulay, or regular, then the same is true of* Y*.*

Proof. If we write $X = \operatorname{Spf} B$ and $Y = \operatorname{Spf} A$, then Lemma 8.18 shows that $A \to B$ is faithfully flat, and so the Cohen–Macaulayness or regularity of B implies that of A [Stacks, Tag 0352, Tag 0353].

Suppose then that B is analytically normal (resp. analytically unramified). Since A is I-adically complete, any maximal ideal $\mathfrak{m}$ of A contains I. Since the morphism $A/IA \to B/IB$ is faithfully flat, we may then find a maximal ideal $\mathfrak{n}$ of B which pulls back to $\mathfrak{m}$. The morphism $\widehat{A_{\mathfrak{m}}} \to \widehat{B_{\mathfrak{n}}}$ is flat and local, and hence faithfully flat. Since the target is normal (resp. reduced), so is the source [Stacks, Tag 033G, Tag 033F]. Thus A is analytically normal (resp. analytically unramified), as claimed. □

Lemma 8.20. *If* $X \to Y$ *is a morphism of affine formal algebraic spaces which is representable by algeraic spaces and smooth, and if* Y *is Noetherian and either residually Jacobson and analytically normal, residually Jacobson and analytically unramified, Cohen–Macaulay, or regular, then the same is true of* X*.*

Proof. Recall Remark 8.9, which shows that the property of being residually Jacobson is *fppf* local. Write $X = \operatorname{Spf} B$ and $Y = \operatorname{Spf} A$; Lemma 8.18 shows that the morphism $A \to B$ is flat. Since A is I-adically complete, a collection of elements $f_1, \ldots, f_r \in A$ generates the unit ideal in A if and only if it generates the unit ideal in A/I^i for each $i > 0$. If these equivalent conditions hold, then each $\operatorname{Spec} A/I^i$ is the union of the affine open subsets $\operatorname{Spec}(A/I^i)_{f_j}$, and $\operatorname{Spf} A/I^i$ is the union of the open formal subspaces $\operatorname{Spf} \widehat{A_{f_j}}$ (where $\widehat{A_{f_j}}$ denotes the I-adic completion of the localization A_{f_j}). Similarly, we may write $\operatorname{Spf} B$ as the union of the open formal subspaces $\operatorname{Spf} \widehat{B_{f_j}}$.

If A satisfies one of the properties under consideration, then so does each of its localizations A_{f_j}, and hence so does each of the completions $\widehat{A_{f_j}}$ (by Lemma 8.16). If we can show that $\widehat{B_{f_j}}$ satisfies the same property, then by applying Lemma 8.19 to the morphism $\coprod \operatorname{Spf} \widehat{B_{f_j}} \to \operatorname{Spf} B$, we find that $\operatorname{Spf} B$ also satisfies this property. Thus, we are free to prove the lemma after localizing on $\operatorname{Spf} A$, in the above sense.

By assumption the morphism $\operatorname{Spec} B/I \to \operatorname{Spec} A/I$ is smooth. Thus we may find a cover by distinguished affine opens $\operatorname{Spec} A/I = \bigcup \operatorname{Spec}(A/I)_{g_j}$ so that each of the morphisms $\operatorname{Spec}(B/I)_{g_j} \to \operatorname{Spec}(A/I)_{g_j}$ is in fact standard smooth, in the sense of [Stacks, Tag 00T6]. (See [Stacks, Tag 00TA].) Thus, replacing A by $\widehat{A_{g_j}}$ and B by $\widehat{B_{g_j}}$, we may assume that B/I admits a presentation as an (A/I)-algebra of the form

$$B/I = (A/I)[x_1, \ldots, x_n]/(\overline{f}_1, \ldots, \overline{f}_c),$$

with the determinant $\overline{\Delta} := \det\big(\dfrac{\partial \overline{f}_i}{\partial x_j}\big)_{1 \le i,j \le c}$ being invertible. If we choose lifts of the $\overline{f}_j$ to elements $f_j \in A[x_1, \ldots, x_n]$, and write $C := A[x_1, \ldots, x_n]/(f_1, \ldots, f_c)$ and $\Delta := \det\big(\dfrac{\partial f_i}{\partial x_j}\big)_{1 \le i,j \le c}$, then $C[1/\Delta]$ is a smooth A-algebra. Since $\overline{\Delta}$ is invertible in C/I, we note that $C/I^i = C[1/\Delta]/I^i$ for each i, and thus that the I-adic completions of C and of $C[1/\Delta]$ coincide; we denote this common I-adic completion by $\widehat{C}$. Repeated application of the infinitesimal lifting property for smooth morphisms gives a morphism of A-algebras $C[1/\Delta] \to B$, inducing an isomorphism $C/I \cong B/I$, and hence a surjection $C/I^i \to B/I^i$ for each i. Let J_i denote the kernel of this surjection. Since B/I^i is flat over A/I^i, we find that $J_i/IJ_i \cong J_1 = 0$, and thus that $J_i = IJ_i = \cdots = I^i J_i = 0$. Thus we obtain an isomorphism $\widehat{C} \cong B$.

Since $C[1/\Delta]$ is smooth over A, it inherits Cohen–Macaulayness or regularity from A (see [Stacks, Tag 0339] and [Stacks, Tag 033A]). Thus, by Lemma 8.16, so does its I-adic completion $\widehat{C}$.

It remains to consider the case when A is residually Jacobson and either analytically normal or analytically unramified. If $\mathfrak{n}$ is a maximal ideal of $\widehat{C}$ then it contains I, and so corresponds to a maximal ideal of C/IC. The morphism $A/IA \to C/IC$ is then a finite type morphism of Jacobson rings, and so we see that $\mathfrak{n}$ lies over a maximal ideal $\mathfrak{m}$ of A. By assumption $\widehat{A_{\mathfrak{m}}}$ is either normal or reduced. Since $C[1/\Delta]$ is smooth over A, we may apply Lemma 8.15 to find that $\widehat{\widehat{C}_{\mathfrak{n}}}$ (which is naturally isomorphic to the completion of C at $\mathfrak{n} \cap C$) is either normal or reduced. Thus $\widehat{C}$ is analytically normal, or analytically unramified, as required. (We have already noted that it is residually Jacobson.) □

Lemma 8.21. *If X is a formal algebraic space, then the following properties are equivalent:*

(1) *If $U \to X$ is a morphism whose source is an affine formal algebraic space and which is representable by algebraic spaces and smooth, then U is Noetherian and furthermore is residually Jacobson and analytically normal (resp. residually Jacobonson and analytically unramified, resp. Cohen–Macaulay, resp. regular).*

(2) *There exists a morphism $\coprod_i U_i \to X$, which is representable by algebraic spaces and smooth, with each U_i being an affine formal algebraic space which is Noetherian and is residually Jacobson and analytically normal (resp. residually Jacobonson and analytically unramified, resp. Cohen–Macaulay, resp. regular).*

Proof. This follows from Lemmas 8.19 and 8.20 by standard manipulations. □

Definition 8.22. We say that a formal algebraic space is *residually Jacobson and analytically normal*, *residually Jacobson and analytically unramified*, *Cohen–Macaulay*, or *regular*, if it satisfies the equivalent conditions of Lemma 8.21 for whichever of these four conditions is under consideration.

Remark 8.23. Lemma 8.21 allows us to follow the prescription of Definition 8.4 and then define the notion of a formal algebraic stack being *residually Jacobson and analytically normal*, *residually Jacobson and analytically unramified*, *Cohen–Macaulay*, or *regular.*

We recall the following theorem of Gabber [RL14, Thm. 9.2].

Theorem 8.24. *If A is a quasi-excellent Noetherian ring, then the completion of A at any ideal is again quasi-excellent.*

Corollary 8.25. *Suppose that A is a quasi-excellent Noetherian ring which is I-adically complete for some ideal I. If $\operatorname{Spf} B \to \operatorname{Spf} A$ is a finite type adic morphism of affine formal algebraic spaces, then B is reduced if and only if B is analytically unramified.*

Proof. Since $\operatorname{Spf} B \to \operatorname{Spf} A$ is adic, the topology on B is the I-adic topology, and so $\operatorname{Spf} B := \varinjlim \operatorname{Spec} B/I^i B$. Since $\operatorname{Spf} B$ is furthermore of finite type over $\operatorname{Spf} A$, we see that B/I is of finite type over A/I. Choose a surjection $A[x_1, \ldots, x_n] \to B/I$ (for some sufficiently large value of n). We may lift this to a morphism $A[x_1, \ldots, x_n] \to B$. If C denotes the I-adic completion of the domain of this morphism, then there is an induced morphism $C \to B$. The image of this morphism is I-adically dense in B, and also I-adically complete. Thus this morphism is surjective, and so B is a quotient of C. Theorem 8.24 shows that C is quasi-excellent, and thus so is its quotient B. We have already observed that analytically unramified implies reduced in general; on the other hand, since B is quasi-excellent, it is analytically unramified if it is reduced. □

We now discuss a slightly different class of examples of smooth local properties.

Definition 8.26. If P is a property of S-algebraic stacks that is inherited by closed substacks, then we say that a formal algebraic stack $\mathcal{X}$ is *Ind-P* if we may write $\mathcal{X} \xrightarrow{\sim} \varinjlim_i \mathcal{X}_i$ as an inductive limit of algebraic spaces having property P with respect to transition morphisms that are thickenings.

Lemma 8.27. *If P is a property of S-algebraic stacks that is inherited by closed substacks, if $\mathcal{X}$ is an Ind-P formal algebraic stack, and if $\mathcal{Y} \hookrightarrow \mathcal{X}$ is a closed immersion, then $\mathcal{Y}$ is also Ind-P.*

Proof. If we write $\mathcal{X} \xrightarrow{\sim} \varinjlim \mathcal{X}_i$ as in Definition 8.26, thesn taking the fibre product with $\mathcal{Y}$ over $\mathcal{X}$ yields an isomorphism $\mathcal{Y} \xrightarrow{\sim} \varinjlim \mathcal{X}_i \times_{\mathcal{X}} \mathcal{Y}$. Each morphism $\mathcal{Y}_i := \mathcal{X}_i \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{X}_i$ is a closed immersion, and since $\mathcal{X}_i$ has property P by assumption, we find that the same is true of $\mathcal{Y}_i$. Thus $\mathcal{Y}$ also satisfies the requirement of Definition 8.26, and hence is Ind-P. □

Lemma 8.28. *If P is a property of S-algebraic stacks that is inherited by closed substacks, and if $\mathcal{X}$ is an Ind-P formal algebraic stack, then $\mathcal{X}_{\mathrm{red}}$ is an algebraic stack satisfying P.*

Proof. If $\mathcal{X} \xrightarrow{\sim} \varinjlim \mathcal{X}_i$ as in Definition 8.26, then $\mathcal{X}_{\mathrm{red}} \xrightarrow{\sim} (\mathcal{X}_i)_{\mathrm{red}}$ for any choice of i. Since $(\mathcal{X}_i)_{\mathrm{red}}$ is a closed substack of $\mathcal{X}_i$, which satisfies P by assumption, our hypothesis on P implies that $(\mathcal{X}_i)_{\mathrm{red}}$, and hence also $\mathcal{X}_{\mathrm{red}}$, satisfies P. □

Lemma 8.29. *If $\mathcal{X}$ is a quasi-compact and quasi-separated formal algebraic stack, then the following are equivalent:*

(1) $\mathcal{X}$ is Ind-P;

(1') We may write $\mathcal{X} \xrightarrow{\sim} \varinjlim_i \mathcal{X}_i$ as an inductive limit of algebraic stacks having property P with respect to transition morphisms that are closed immersions;

(2) If $\mathcal{Z} \hookrightarrow \mathcal{X}$ is a closed immersion whose source is an algebraic stack then $\mathcal{Z}$ satisfies P;

(3) Any morphism $T \to \mathcal{X}$, whose source is an affine scheme, may be factored as $T \to \mathcal{Z} \hookrightarrow \mathcal{X}$, where Z is an algebraic stack satisfying P and the second arrow is a closed immersion;

(3') Any morphism $\mathcal{T} \to \mathcal{X}$, whose source is a quasi-compact algebraic stack, may be factored as $\mathcal{T} \to \mathcal{Z} \hookrightarrow \mathcal{X}$, where $\mathcal{Z}$ is an algebraic stack satisfying P, and the second arrow is a closed immersion.

Proof. It is clear that (3') implies (3). Suppose that (3) holds, and consider a closed immersion $\mathcal{Z} \hookrightarrow \mathcal{X}$ whose source is an algebraic stack. Since $\mathcal{X}$ is quasi-compact, so is $\mathcal{Z}$, and so we may find a morphism $T \to \mathcal{Z}$ whose source is an affine scheme, and which is smooth and surjective. By assumption this morphism factors as $T \to \mathcal{Z}' \hookrightarrow \mathcal{X}$, with the second morphism being a closed immersion whose source $\mathcal{Z}'$ is an algebraic stack satisying P. Thus we find that $\mathcal{Z}$ is a closed substack of $\mathcal{Z}'$, and so our assumption on P implies that $\mathcal{Z}$ also satisfies P. Hence (3) implies (2).

Suppose now that (2) holds. By Lemma 6.2 we may write $\varinjlim_i \mathcal{X}_i \xrightarrow{\sim} \mathcal{X}$ for some directed system of algebraic spaces $\{X_i\}$, with the transition morphisms being thickenings. Applying (2) to each of the various closed embedding $\mathcal{X}_i \hookrightarrow \mathcal{X}$ shows that each $\mathcal{X}_i$ then satisfies P, so that (1) holds.

Obviously (1) implies (1'). Finally, suppose that (1') holds, and write $\mathcal{X} \cong \varinjlim \mathcal{X}_i$ with the X_i satisfying P, and the morphisms being closed immersions. Any morphism $T \to X$ as in (3') then factors through one of the $\mathcal{X}_i$ and so (3') holds, as claimed. □

Lemma 8.30. *If P is a property of S-algebraic stacks that is inherited by closed substacks, and if $\mathcal{X}$ is a quasi-compact and quasi-separated algebraic stack that is Ind-P when regarded as a formal algebraic stack, then $\mathcal{X}$ satisfies P.*

Proof. This follows by applying condition (2) of Lemma 8.29 (which by that lemma follows from the fact that $\mathcal{X}$ is Ind-P) to the identity morphism $\mathcal{X} \xrightarrow{\sim} \mathcal{X}$. □

Lemma 8.31. *Let P be a property of S-algebraic stacks that is inherited by closed subschemes, and which is also smooth local (respectively fppf local). If $\mathcal{X} \to \mathcal{Y}$ is a morphism of formal algebraic stacks over S that is representable by algebraic spaces and smooth (respectively fppf), and if $\mathcal{Y}$ is Ind-P, then $\mathcal{X}$ is Ind-P. If the given morphism is furthermore surjective, and if $\mathcal{Y}$ is quasi-compact and quasi-separated, then the converse also holds: namely, if $\mathcal{X}$ is Ind-P then so is $\mathcal{Y}$.*

Proof. If $\mathcal{Z} \hookrightarrow \mathcal{Y}$ is a closed immersion whose source is an algebraic stack, then $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \hookrightarrow \mathcal{X}$ is a closed immersion whose source is an algebraic space, and $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ is smooth (respectively flat and locally of finite presentation), and is furthermore surjective if $\mathcal{X} \to \mathcal{Y}$ is surjective. Thus if $\mathcal{Y}$ is Ind-P, and if we write $\varinjlim_i \mathcal{Y}_i \xrightarrow{\sim} \mathcal{Y}$, where each $\mathcal{Y}_i$ is an algebraic stack satisfying P and the transition morphisms are thickenings, then we obtain an isomorphism $\varinjlim_i \mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_i \xrightarrow{\sim} \mathcal{X}$. Since P is smooth (respectively *fppf*) local, we find that each $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Y}_i$ satisfies P, and thus that $\mathcal{X}$ is Ind-P.

Conversely, if $\mathcal{X}$ is Ind-P — so that it satisfies condition (2) of 8.29 — and if $\mathcal{X} \to \mathcal{Y}$ is surjective, then we find that $\mathcal{Y}$ also satisfies condition (2) of Lemma 8.29, and thus is Ind-P, provided that it is also quasi-compact and quasi-separated (so that that lemma applies to $\mathcal{Y}$). □

Lemma 8.32. *Let P be a property of S-algebraic stacks that is inherited by closed substacks, and which is also smooth local (respectively fppf local).*

If $\mathcal{X}$ is a formal algebraic stack over S, then the following properties are equivalent:

1. *If $\mathcal{U}$ is a quasi-compact and quasi-separated formal algebraic stack equipped with a morphism $\mathcal{U} \to \mathcal{X}$ which is representable by algebraic spaces and smooth (respectively fppf), then U is Ind-P.*
2. *We may find a morphism $\coprod U_i \to \mathcal{X}$ which is representable by algebraic spaces, smooth (respectively fppf), and surjective, with each U_i being a formal affine algebraic space which is Ind-P.*
3. *We may find a morphism $\coprod \mathcal{U}_i \to \mathcal{X}$ which is representable by algebraic spaces, smooth (respectively fppf), and surjective, with each $\mathcal{U}_i$ being a formal algebraic stack which is Ind-P.*

Proof. Since $\mathcal{X}$ is a formal algebraic space, by definition we may find a morphism $\coprod U_i \to X$ which is representable by algebraic spaces, smooth, and surjective. Thus if (1) holds, each of the U_i is Ind-P, and so (2) holds as well. Clearly (2) implies (3).

Suppose then that condition (3) holds, and fix a morphism $\coprod \mathcal{U}_i \to \mathcal{X}$ as in its statement. Let $\mathcal{U} \to \mathcal{X}$ be a morphism as in (1). Then $\coprod \mathcal{U} \times_{\mathcal{X}} \mathcal{U}_i \to \mathcal{U}$ is representable by algebraic spaces, smooth (respectively *fppf*), and surjective. Each fibre product $\mathcal{U} \times_{\mathcal{X}} \mathcal{U}_i$ is again a formal algebraic stack, and so we may find a morphism $\coprod V_{i,j} \to U \times_{\mathcal{X}} U_i$ which is representable by algebraic spaces, smooth, and surjective, with each $V_{i,j}$ an affine formal algebraic space. Then the composite

$$\coprod_{i,j} V_{i,j} \to \coprod_i \mathcal{U} \times_{\mathcal{X}} \mathcal{U}_i \to \mathcal{U} \tag{8.33}$$

is representable by algebraic spaces, smooth (respectively *fppf*), and surjective.

Lemma 8.31, applied first to the various morphisms $V_{i,j} \to \mathcal{U}_i$, and then to the morphism (8.33) (taking into account that $\mathcal{U}$ is quasi-compact and quasi-separated by assumption), implies that U is Ind-P. □

Definition 8.34. Let P be a property of S-algebraic stacks that is inherited by closed substacks, and which is also smooth local. We say that a formal algebraic stack $\mathcal{X}$ over S is *locally Ind-P* if it satisfies the equivalent conditions of Lemma 8.32. It is evident from this lemma that this property that the property of being locally Ind-P, when restricted to formal algebraic spaces, is smooth local in the sense of Definition 8.1, so that we may also regard ourselves as being in the context of Definition 8.4.

Lemma 8.35. *Let P be a property of S-algebraic stacks that is inherited by closed substacks, and which is also smooth local* (*so that Definition* 8.34 *applies*), *and let $\mathcal{X}$ be a locally Ind-P formal algebraic stack. If $\mathcal{Z} \to \mathcal{X}$ is a morphism whose source $\mathcal{Z}$ is an algebraic stack and which is unramified,*[15] *then $\mathcal{Z}$ satisfies P.*

Proof. By condition (2) of Lemma 8.32, we may choose a morphism $\coprod U_i \to \mathcal{X}$ whose source is a disjoint union of affine formal algebraic spaces, and which is representable by algebraic spaces, smooth, and surjective, for which each U_i is Ind-P. Pulling-back over $\mathcal{Z}$, we obtain a morphism $\coprod U_i \times_{\mathcal{X}} \mathcal{Z} \to \mathcal{Z}$, which is again representable by algebraic spaces, smooth, and surjective. Thus, to conclude that $\mathcal{Z}$ satisfies P, it suffices to show (by Lemma 8.6) that each of the fibre products $U_i \times_{\mathcal{X}} \mathcal{Z}$ is an algebraic stack satisfying P.

Since $\mathcal{Z}$ is an algebraic stack, and since $U_i \times_{\mathcal{X}} \mathcal{Z} \to \mathcal{Z}$ is representable by algebraic spaces, we see that the fibre product $U_i \times_{\mathcal{X}} \mathcal{Z}$ is itself an algebraic stack. The morphism $U_i \times_{\mathcal{X}} \mathcal{Z} \to U_i$ is furthermore unramified, being the base-change of an unramified morphism. In order to show that $U_i \times_{\mathcal{X}} \mathcal{Z}$ satisfies P, it suffices to show that for any smooth morphism $V \to U_i \times_{\mathcal{X}} \mathcal{Z}$ whose source is an affine scheme, the affine scheme V satisfies P, and this is equivalent to showing the image of V under this morphism, which is an open substack $\mathcal{V}$ of $U_i \times_{\mathcal{X}} \mathcal{Z}$, satisfies P. Consider such a morphism; since U_i is Ind-P by assumption, the composite $V \to U_i \times_{\mathcal{X}} \mathcal{Z} \to U_i$ factors through a closed affine subscheme T_i of U_i which satisfies P (by Lemma 8.29). Also, the morphism $\mathcal{V} \to T_i$ is unramified (since the composite $\mathcal{V} \to T_i \to U_i$ is, being the composite of an unramified morphism and an open immersion, and since the diagonal of U_i is also unramified, being a monomorphism that is locally of finite type [Stacks, Tag 0AIP]). Thus it has étale, and hence unramified, diagonal, and so is a DM morphism [Stacks, Tag 04YW]. As is explained in that reference, since T_i is a scheme, the algebraic stack $\mathcal{V}$ is in fact a Deligne–Mumford stack, and so we may find a surjective étale morphism $V' \to \mathcal{V}$ with V' a scheme. Consequently $V' \to \mathcal{V} \to T_i$ is unramified, and thus, taking into account our assumptions on P and the étale local structure of unramified morphisms [Stacks, Tag 04HH], we find that each affine open subset of V' satisfies P. Thus V' satisfies P, and hence so does $\mathcal{V}$. This completes the proof. □

Lemma 8.36. *Let P be a property of S-algebraic stacks that is inherited by closed substacks, and which is also smooth local. If $\mathcal{X}$ is a locally Ind-P formal algebraic stack, then $\mathcal{X}_{\mathrm{red}}$ is an algebraic stack satisfying P.*

Proof. This follows from Lemma 8.35, applied to the closed immersion $\mathcal{X}_{\mathrm{red}} \hookrightarrow \mathcal{X}$. □

[15] The notion of a morphism $\mathcal{Z} \to \mathcal{X}$ being unramified is defined, thanks to Lemma 5.12, Definition 3.11 and Example 3.16.

Lemma 8.37. *Let P be a property of S-algebraic stacks that is inherited by closed substacks, and which is also smooth local. If $\mathcal{X}$ is an algebraic stack which is locally Ind-P when regarded as an algebraic stack, then $\mathcal{X}$ in fact satisfies P.*

Proof. This follows from Lemma 8.35, applied to the identity morphism $\mathcal{X} \xrightarrow{\sim} \mathcal{X}$. ☐

The following lemma shows that being locally Ind-P is not so far from being Ind-P itself.

Lemma 8.38. *Let P be a property of S-algebraic stacks that is inherited by closed substacks, and which is also smooth local. If $\mathcal{X}$ is a formal algebraic stack that is Ind-P, then $\mathcal{X}$ is locally Ind-P. Conversely, if $\mathcal{X}$ is locally Ind-P and is also quasi-compact and quasi-separated, then $\mathcal{X}$ is Ind-P.*

Proof. If $\mathcal{X}$ is Ind-P, then the identity morphism $\mathcal{X} \to \mathcal{X}$ satisfies condition (3) of Lemma 8.32, showing that $\mathcal{X}$ is locally Ind-P. Conversely, suppose that $\mathcal{X}$ is locally Ind-P and also quasi-compact and quasi-separated. If $\mathcal{Z} \hookrightarrow \mathcal{X}$ is a closed immersion whose source is an algebraic stack, then Lemma 8.35 implies that $\mathcal{Z}$ satisfies P. Thus $\mathcal{X}$ satisfies condition (2) of Lemma 8.29, and so $\mathcal{X}$ is Ind-P by that same lemma. ☐

Remark 8.39. Two examples of a property P to which we may apply Definitions 8.26 and 8.34 are being locally of finite type over S, and being locally Noetherian. When S is locally Noetherian, the former condition implies the latter, so we see that a formal algebraic stack that is (locally) Ind-locally of finite type over a locally Noetherian scheme S is in particular (locally) Ind-locally Noetherian.

We also emphasize that while any (locally) Noetherian formal algebraic stack is (locally) Ind-locally Noetherian, the condition of being (locally) Ind-locally Noetherian is considerably weaker.

Example 8.40. A typical example of an Ind-locally Noetherian formal algebraic space X which is not Noetherian is as follows: for each $n \geq 0$, set

$$A_n := \mathbb{C}[x, y]/(x(x-1)\cdots(x-n)y, y^2),$$

and set $X := \varinjlim \operatorname{Spec} A_n$, with respect to the evident transition morphisms.

In Section 11 below, we will give a criterion for a locally Ind-locally of finite type formal algebraic stack over a locally Noetherian base scheme S to be locally Noetherian.

We close this section by mentioning that it would be natural, and desirable, to develop a framework for defining properties of morphisms between formal algebraic stacks, analogous to the corresponding framework for morphisms between formal algebraic spaces that is developed in [Stacks, Tag 0ANA]; namely, based on properties of continuous ring maps. In particular, this is a natural framework for defining properties of morphisms that are *not* representable by algebraic spaces. We don't do this in general here, but give one example, as an illustration. As in [Stacks, Tag 0ANA], it will be necessary to restrict the class of formal algebraic stacks for which we make the definition, ultimately because the theory of continuous ring maps is not terribly robust, and often requires auxiliary hypotheses to yield reasonable results.

Lemma 8.41. *If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism between locally Noetherian formal algebraic stacks, then the following properties are equivalent:*

(1) For every commutative diagram of morphisms

$$\begin{tikzcd} U \arrow[r] \arrow[d] & V \arrow[d] \\ \mathcal{X} \arrow[r, "f"] & \mathcal{Y} \end{tikzcd}$$

in which $U \cong \operatorname{Spf} A$ and $V \cong \operatorname{Spf} B$ are affine formal algebraic spaces and in which the vertical arrows are representable by algebraic spaces and flat (so that U and V are necessarily Noetherian affine formal algebraic spaces, by Remark 8.10 *and Lemma* 8.3*), the ring morphism $B \to A$ induced by the upper horizontal arrow is flat.*

(2) There exists a morphism $\coprod_j V_j \to \mathcal{Y}$, where each $V_j \cong \operatorname{Spf} B_j$ is an affine formal algebraic space and which is representable by algebraic spaces, flat, and surjective, and for each j a morphism $\coprod_i U_{j,i} \to \mathcal{X} \times_{\mathcal{Y}} V_j$, where each $U_{j,i} \cong \operatorname{Spf} A_{j,i}$ is an affine formal algebraic space and which is representable by algebraic spaces, flat, and surjective, such that each of the induced morphisms $B_j \to A_{i,j}$ is flat. (In order to make sense of this induced morphism, we again note that each $U_{j,i}$ and V_j is necessarily a Noetherian affine formal algebraic space, by Remark 8.10 *and Lemma* 8.3*).*

(3) There exists a morphism $\coprod_i U_i \to \mathcal{X}$, where each $U_i \cong \operatorname{Spf} A_i$ is an affine formal algebraic space which is representable by algebraic spaces, flat, and surjective, and a factorization of the induced morphism $U_i \to \mathcal{Y}$ as $U_i \to V_i \to \mathcal{Y}$, where $V_i \cong \operatorname{Spf} B_i$ is an affine formal algebraic space, such that the morphism $V_i \to \mathcal{Y}$ is representable by algebraic spaces and flat, and such the induced morphism $B_i \to A_i$ is flat. (Again, in order to make sense of this induced morphism, we note that each $U_{j,i}$ and V_j is necessarily a Noetherian affine formal algebraic space, by Remark 8.10 *and Lemma* 8.3*).*

If $f : \mathcal{X} \to \mathcal{Y}$ is furthermore adic, i.e. representable by algebraic stacks, then these conditions are equivalent to f being flat in the sense of Definition 3.11*.*

Proof. It is clear that (1) implies (2), and that (2) implies (3). By standard manipulations (see e.g. the proof of [Stacks, Tag 0ANG], on which the present argument is modelled), to see that (3) implies (1), it suffices to show the following: if

$$\begin{tikzcd} U' \cong \operatorname{Spf} A' \arrow[r] \arrow[d] & V' \cong \operatorname{Spf} B' \arrow[d] \\ U \cong \operatorname{Spf} B \arrow[r] & V \cong \operatorname{Spf} A \end{tikzcd}$$

is a commutative diagram of morphisms of Noetherian affine formal algebraic spaces, in which the vertical arrows are representable by algebraic spaces and flat, with the left-hand arrow being furthermore surjective, and if the upper horizontal arrow induces a flat morphism $B' \to A'$, then the lower horizontal morphism induces a flat morphism $B \to A$. To see this, consider the corresponding diagram of induced ring morphisms

$$\begin{tikzcd} B' \arrow[r] & A' \\ A \arrow[r] \arrow[u] & B \arrow[u] \end{tikzcd}$$

The left and right-hand arrows are flat, with the right-hand arrow even being faithfully flat, by Lemma 8.18. The upper horizontal arrow is flat by assumption. Since the composite of flat morphisms are flat, we find that the morphism $B \to A$ becomes flat after composition with the faithfully flat morphism $A' \to A'$. It follows that $B \to A$ is itself flat, as required.

It remains to be shown that if $\mathcal{X} \to \mathcal{Y}$ is representable by algebraic stacks, then conditions (1), (2), and (3) are equivalent to $\mathcal{X} \to \mathcal{Y}$ being flat in the previously defined sense, namely that for any morphism $\mathcal{Z} \to \mathcal{Y}$ whose source is an algebraic stack, or even simply a scheme, that the induced morphism $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} \to \mathcal{Z}$ is flat.

We begin with some preliminary constructions. Namely, we choose (as we may) a morphism $\coprod V_j \to \mathcal{Y}$ whose source is a disjoint union of Noetherian affine formal algebraic spaces, which is representable by algebraic spaces, smooth, and surjective. Then we obtain a base-changed morphism $\coprod \mathcal{X} \times_{\mathcal{Y}} V_j \to \mathcal{X}$, whose source is a disjoint union of formal algebraic stacks (by Lemma 5.22) Lemma 8.13), which is representable by algebraic spaces, smooth, and surjective, and such that each of the projections $\mathcal{X} \times_{\mathcal{Y}} V_j \to V_j$ is representable by algebraic stacks. Lemma 8.13 shows that each of the fibre products $\mathcal{X} \times_{\mathcal{Y}} V_j$ is furthermore locally Noetherian, and so we may (and do) choose, for each of them, a morphism $\coprod U_{j,i} \to \mathcal{X} \times_{\mathcal{Y}} V_j$ whose source is a disjoint union of Noetherian affine formal algebraic stacks, which is representable by algebraic spaces, smooth, and surjective. Each of the composite morphisms $U_{j,i} \to \mathcal{X} \times_{\mathcal{Y}} V_j \to V_j$ is then representable by algebraic stacks, and hence by algebraic spaces (by Lemma 3.5).

Now suppose that $\mathcal{X} \to \mathcal{Y}$ is flat, in the sense of Definition 3.11. Then each of the composite morphism $\coprod U_{j,i} \to \mathcal{X} \times_{\mathcal{Y}} V_j \to V_j$ is a composite of morphisms that are representable by algebraic stacks and flat in this sense, and hence this morphism (which we have seen is representable by algebraic spaces) is flat. Lemma 8.18 then implies that each of these morphisms is induced by a flat morphism of the corresponding topological rings. Thus (2) holds (and hence so do (1) and (3)).

Suppose instead now that condition (1) holds. Then we find that each of the morphisms $U_{j,i} \to V_j$ is induced by a flat morphism of topological rings, and hence by Lemma 8.18, is flat (when regarded as a morphism representable by algebraic spaces). Since $\coprod U_{j,i} \to \mathcal{X}$ are $\coprod V_j \to \mathcal{Y}$ are each morphisms representable by algebraic spaces, smooth, and surjective, standard arguments then show that $\mathcal{X} \to \mathcal{Y}$ is flat in the sense of Definition 3.11. □

Definition 8.42. We say that a morphism $\mathcal{X} \to \mathcal{Y}$ of locally Noetherian formal algebraic stacks is *flat* if it satisfies the equivalent conditions (1), (2), and (3) of Lemma 8.41. (The final statement of the lemma ensures that this definition is compatible with the existing definition in the case of adic morphisms.)

Our final result proves that the regularity properties discussed in Remark 8.23 are *flat local* in the sense of Definition 8.42.

Lemma 8.43. *If $\mathcal{X} \to \mathcal{Y}$ is a flat and surjective morphism of locally Noetherian formal algebraic stacks, and if $\mathcal{X}$ satisfies any of the conditions of Remark* 8.23*, then the same is true of $\mathcal{Y}$.*

Proof. This amounts to showing that if $A \to B$ is a faithfully flat morphism of adically complete Noetherian local rings, and if B is residually Jacobson and analytically normal, residually Jacobson and analytically unramified, Cohen-Macaulay, or regular, then the same is true of A. This follows by an argument analogous to that used to prove Lemma 8.19. □

9. Functorial closed substacks of formal algebraic stacks

In this section we explain how to extend various constructions of closed substacks from the context of affine formal algebraic spaces to the context of formal algebraic stacks. In order to maximize the flexibility of our construction, we put ourselves in the following situation: we fix a base formal algebraic stack $\mathcal{S}$, we let $\mathcal{D}$ denote the 2-category of formal algebraic stacks lying over $\mathcal{S}$, and then we fix a strong sub-2-category (in the sense of Definition 2.19) $\mathcal{C}$ of $\mathcal{D}$, which we assume has the following property: if $\mathcal{U} \to \mathcal{X}$ is a morphism that is representable by algebraic spaces and smooth, and if we are given a morphism $\mathcal{X} \to \mathcal{S}$ making $\mathcal{X}$ an object of $\mathcal{C}$, then $\mathcal{U}$ (equipped with the composite morphism $\mathcal{U} \to \mathcal{X} \to \mathcal{S}$) is also an object of $\mathcal{C}$.

We let $\mathrm{Aff}_{\mathcal{C}}$ denote the full subcategory of $\mathcal{C}$ whose objects are the affine formal algebraic spaces belonging to $\mathcal{C}$. (We note that since any object $\mathcal{X}$ of $\mathcal{C}$ receives a morphism $U \to \mathcal{X}$ which is representable by algebraic spaces, smooth, and surjective, and whose source is a disjoint union of affine formal algebraic spaces, $\mathrm{Aff}_{\mathcal{C}}$ contains plenty of objects. We note also that $\mathrm{Aff}_{\mathcal{C}}$ is (equivalent to) a usual category (i.e. a 1-category), because its objects are formal algebraic spaces rather than more general formal algebraic stacks.) Let $i : \mathrm{Aff}_{\mathcal{C}} \to \mathcal{D}$ denote the inclusion, and suppose that we are given a functor $F : \mathrm{Aff}_{\mathcal{C}} \to \mathcal{D}$, equipped with a natural transformation $F \hookrightarrow i$ which is a closed immersion, in the sense that for any object U of $\mathrm{Aff}_{\mathcal{C}}$, the corresponding morphism $F(U) \hookrightarrow U$ is representable by algebraic spaces, and a closed immersion (and so is in fact representable by schemes).

We assume that F satisfies the following axiom:

Axiom 9.1. If $U \to V$ is a morphism in $\mathrm{Aff}_{\mathcal{C}}$ which is representable by algebraic spaces and smooth, then the natural morphism $F(U) \to U \times_V F(V)$ is an isomorphism.

Lemma 9.2. *Let $\mathcal{X}$ be an object of $\mathcal{C}$, and let $T \to \mathcal{X}$ be a morphism whose source is a scheme. The following properties are equivalent:*

(1) For some morphism $\coprod V_i \to \mathcal{X}$ whose source is a disjoint union of affine formal algebraic spaces, and which is representable by algebraic spaces, smooth, and surjective, each of the induced morphisms $T \times_{\mathcal{X}} V_i \to V_i$ factors through $F(V_i)$.

(2) For any morphism $U \to \mathcal{X}$ which is representable by algebraic spaces and smooth, and whose source is an affine formal algebraic space, the base-changed morphism $T \times_{\mathcal{X}} U \to U$ factors through $F(U)$.

Furthermore, there is a closed substack $\mathcal{Z} \hookrightarrow \mathcal{X}$, characterized by the property that a morphism $T \to \mathcal{X}$, whose source is a scheme, factors through $\mathcal{Z}$ if and only if it satisfies the preceding two equivalent conditions.

Proof. Let $T \to \mathcal{X}$ be a morphism whose source is a scheme. Since $\mathcal{X}$ is a formal algebraic stack, we may choose a morphism $V \to \mathcal{X}$ which is representable by algebraic spaces, smooth, and surjective, and whose source may be written as a disjoint union $V = \coprod V_i$ of affine formal algebraic schemes. It is clear that if (1) holds, then (2) holds for any such choice. Suppose conversely that (2) holds for some such choice; we will show that (1) holds, i.e. that $T \times_{\mathcal{X}} U \to U$ factors through $F(U)$ for every $U \to \mathcal{X}$ as in the statement of (1).

Consider the base-changed morphism $\coprod U \times_{\mathcal{X}} V_i \to U$; its source is a formal algebraic space, its target is the affine formal algebraic space U, and it is representable by algebraic spaces, smooth, and surjective. Since U is quasi-compact, we may find a finite sequence $W_1, \ldots, W_n$ of affine algebraic spaces, such that each W_j is equipped with a morphism $W_j \to U \times_{\mathcal{X}} V_{i(j)}$ (where the index $i(j)$ depends on j) which is representable by algebraic spaces and étale, and such that the composite

$$\coprod_{j=1}^{n} W_j \to \coprod_{j=1}^{n} U \times_{\mathcal{X}} V_{i(j)} \to U$$

is (representable by algebraic spaces, smooth, and) surjective.

We are assuming that $T \times_{\mathcal{X}} V_i \to V_i$ factors through $F(V_i)$ for each index i. In particular, for each value of j, we have that $T \times_{\mathcal{X}} V_{i(j)} \to V_{i(j)}$ factors through $F(V_{i(j)})$. Since, for each value of j, the composite $W_j \to U \times_{\mathcal{X}} V_{i(j)} \to V_{i(j)}$ is representable by algebraic spaces and smooth, we find by Axiom 9.1 that

$$T \times_{\mathcal{X}} W_j \to W_j$$

factors through $F(W_j)$ for each value of j.

We want to show that $T \times_{\mathcal{X}} U \to U$ factors through $F(U)$. Since $\coprod W_j \to U$ is surjective, it suffices to show that $T \times_{\mathcal{X}} W_j$ factors through $F(U) \times_U W_j$ for each value of j. Taking into account Axiom 9.1, this is equivalent to showing that $T \times_{\mathcal{X}} W_j$ factors through $F(W_j)$ for each value of j, which condition has already been shown to hold. Thus (1) does indeed hold.

We now show that the condition that $T \times_{\mathcal{X}} V_i \to V_i$ factors through $F(V_i)$ cuts out a closed substack $\mathcal{Z}$ of $\mathcal{X}$. Since the property of a morphism factoring through a closed subscheme of the target can be checked after passing to a faithfully flat cover of the source, we see that $\mathcal{Z}$ is indeed a substack of $\mathcal{X}$. By descent for closed subschemes, it suffices to show that the base-changed morphism $\mathcal{Z} \times_{\mathcal{X}} V_i \to V_i$ is a closed immersion for each index i. Now $\mathcal{Z} \times_{\mathcal{X}} V_i$ classifies morphisms $T \to V_i$ for which $T \times_{V_i} (V_i \times_{\mathcal{X}} V_j) \to V_j$ factors through $F(V_j)$ for each index j, or equivalently, for which $T \times_{V_i} (V_i \times_{\mathcal{X}} V_j) \to V_i \times_{\mathcal{X}} V_j$ factors through $V_i \times_{\mathcal{X}} F(V_j)$ for each index j. We claim that $F(V_i) \times_{\mathcal{X}} V_j$ and $V_i \times_{\mathcal{X}} F(V_j)$ coincide, as closed subsheaves of $V_i \times_{\mathcal{X}} V_j$. Given this, we see that $\mathcal{Z} \times_{\mathcal{X}} V_i$ classifies $T \to V_i$ for which $T \times_{V_i} (V_i \times_{\mathcal{X}} V_j) \to V_i \times_{\mathcal{X}} V_j$ factors through $F(V_i) \times_{\mathcal{X}} V_j$ for each j. Since $\coprod_j V_i \times_{\mathcal{X}} V_j \to V_i$ is representable by algebraic spaces, smooth, and surjective, we see that $\mathcal{Z} \times_{\mathcal{X}} V_i$ in fact classifies $T \to V_i$ for which T factors through $F(V_i)$. In other words, the base-changed morphism $\mathcal{Z} \times_{\mathcal{X}} V_i \to V_i$ coincides with the closed immersion $F(V_i) \to V_i$; thus $\mathcal{Z} \to \mathcal{X}$ is a closed substack, as claimed.

It remains to show that $F(V_i) \times_{\mathcal{X}} V_j$ and $V_i \times_{\mathcal{X}} F(V_j)$ coincide as subsheaves of $V_i \times_{\mathcal{X}} V_j$. To this end, note that $V_i \times_{\mathcal{X}} V_j$ is a formal algebraic space, and so we may find a collection of morphisms $W \to V_i \times_{\mathcal{X}} V_j$ which are representable and étale, and whose coproduct surjects onto $V_i \times_{\mathcal{X}} V_j$. It suffices to check that $F(V_i) \times_{\mathcal{X}} V_j$ and $V_i \times_{\mathcal{X}} V_j$ pull back to the same subsheaf of W, for each such W. To see this, note that since each of the composites $W \to V_i \times_{\mathcal{X}} V_j \to V_i$ and $W \to V_i \times_{\mathcal{X}} V_j \to V_j$ is representable by algebraic spaces and smooth, Axiom 9.1 shows that both of these pull-backs coincide with the subsheaf $F(W)$ of W. □

Definition 9.3. If $\mathcal{X}$ is an object of $\mathcal{C}$, then we define $\widetilde{F}(\mathcal{X})$ to be the closed substack of $\mathcal{X}$ constructed in Lemma 9.2

Lemma 9.4. *If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism in $\mathcal{C}$ (so, more concretely, $\mathcal{X} \to \mathcal{Y}$ is a functor of categories fibred in groupoids over S), then f restricts to a morphism $\widetilde{F}(\mathcal{X}) \to \widetilde{F}(\mathcal{Y})$.*

Proof. Let $T \to \widetilde{F}(\mathcal{X})$ be a morphism whose source is a scheme, and let $V \to \mathcal{Y}$ be a morphism whose source is an affine formal algebraic space, and which is representable by algebraic spaces and smooth; we must show that the base-changed morphism

$$T \times_{\mathcal{Y}} V \to V \tag{9.5}$$

factors through $F(V)$.

The morphism $\mathcal{X} \times_{\mathcal{Y}} V \to V$ is representable by algebraic spaces and smooth, and Lemma 5.22 shows that its source is a formal algebraic stack. Thus we may find a morphism $\coprod U_i \to \mathcal{X} \times_{\mathcal{Y}} V$ whose source is a disjoint union of affine formal algebraic spaces, and which is representable by algebraic spaces, smooth, and surjective. The induced morphism

$$\coprod T \times_{\mathcal{X}} U_i \to T \times_{\mathcal{X}} (\mathcal{X} \times_{\mathcal{Y}} V) \xrightarrow{\sim} T \times_{\mathcal{Y}} V$$

is then also representable by algebraic spaces, smooth, and surjective, and so to show that (9.5) factors through $F(V)$, it suffices to show that the same is true of each of the induced morphisms

$$T \times_{\mathcal{X}} U_i \to V. \tag{9.6}$$

But by assumption, the morphism $T \to \mathcal{X}$ factors through $\widetilde{F}(\mathcal{X})$, and hence the projection $T \times_{\mathcal{X}} U_i \to U_i$ factors through $F(U_i)$. Since the morphism $U_i \to V$ restricts to a morphism $F(U_i) \to F(V)$ by assumption, we see that indeed each of the morphisms (9.6) factors through $F(V)$, as required. $\square$

Proposition 9.7. *Let $j : \mathcal{C} \to \mathcal{D}$ denote the inclusion. Then $\widetilde{F}$ induces a strict functor (in the sense of Definition 2.13) $\widetilde{F} : \mathcal{C} \to \mathcal{D}$, equipped with a natural transformation $\widetilde{F} \hookrightarrow j$, together with a natural isomorphism $F \xrightarrow{\sim} \widetilde{F}_{|\,\mathrm{Aff}_{\mathcal{C}}}$, which satisfies the following conditions:*

1. *For each object $\mathcal{X}$ of $\mathcal{C}$, the morphism $\widetilde{F}(\mathcal{X}) \hookrightarrow \mathcal{X}$ is representable by algebraic spaces and a closed immersion (and thus is also representable by schemes).*
2. *If $\mathcal{X} \to \mathcal{Y}$ is a morphism in $\mathcal{C}$ which is representable by algebraic stacks and smooth, then the natural transformation $\widetilde{F}(\mathcal{X}) \hookrightarrow \mathcal{X} \times_{\mathcal{Y}} \widetilde{F}(\mathcal{Y})$ is an isomorphism.*

Proof. By construction $\widetilde{F}(\mathcal{X})$ is a full subcategory fibred in groupoids of $\mathcal{X}$, and Lemma 9.4 shows that any morphism $\mathcal{X} \to \mathcal{Y}$ in $\mathcal{C}$ restricts to morphism $\widetilde{F}(\mathcal{X}) \to \widetilde{F}(\mathcal{Y})$. Thus $\widetilde{F}$ indeed induces a strict pseudo-functor $\mathcal{C} \to \mathcal{D}$. The natural transformation $\widetilde{F}(\mathcal{X}) \hookrightarrow \mathcal{X}$ is then given simply by the inclusion $\mathcal{Z} \hookrightarrow \mathcal{X}$, while if U is an object of $\mathrm{Aff}_{\mathcal{C}}$, the natural isomorphism $F(U) \xrightarrow{\sim} \widetilde{F}(U)$ is given by the isomorphism which identifies $F(U)$ with its image (thought of as a subsheaf of U) under the given natural transformation $F(U) \hookrightarrow U$.

Condition (1) is now satisfied by construction, and we turn to verifying condition (2). Lemma 9.4 shows that there is an inclusion of closed substacks $\widetilde{F}(\mathcal{X}) \to \mathcal{X} \times_{\mathcal{Y}} \widetilde{F}(\mathcal{Y})$, which we must show is an isomorphism, i.e. we must show that (the essential image in $\mathcal{X}$ of) the 2-fibre product $\mathcal{X} \times_{\mathcal{Y}} \widetilde{F}(\mathcal{Y})$ is contained in $\widetilde{F}(\mathcal{X})$. That is, we need to prove the following: (a) if $T \to \mathcal{X}$ is any morphism whose source is a scheme, and if for every morphism $U \to \mathcal{X}$ which is representable by algebraic spaces and smooth, and whose source is an affine formal stack, the base-changed morphism $T \times_{\mathcal{X}} U \to U$ factors through $F(U)$, then it factors through this fibre product. Applying the corresponding characterization of $\widetilde{F}(\mathcal{Y})$, what we know is: (b) the morphism $T \to \mathcal{X}$ factors through the fibre product $\mathcal{X} \times_{\mathcal{Y}} \widetilde{F}(\mathcal{Y})$ if and only if, for any morphism $V \to \mathcal{Y}$ which is representable by algebraic spaces and smooth, and whose source is an affine formal algebraic stack, the base-changed morphism $T \times_{\mathcal{Y}} V \to V$ factors through $F(V)$. If we rewrite this last fibre product as $T \times_{\mathcal{X}} (\mathcal{X} \times_{\mathcal{Y}} V)$, then we see that this is equivalent to asking that the projection $T \times_{\mathcal{X}} (\mathcal{X} \times_{\mathcal{Y}} V) \to \mathcal{X} \times_{\mathcal{Y}} V$ factor through $\mathcal{X} \times_{\mathcal{Y}} F(V)$. Our goal is to deduce (a) from (b).

In the context of (b), the fibre product $\mathcal{X} \times_{\mathcal{Y}} V$ is a formal algebraic stack (Lemma 5.23), and so we may find a morphism $\coprod U_i \to \mathcal{X} \times_{\mathcal{Y}} V$ which is representable by algebraic spaces, smooth, and surjective, with each U_i being an affine formal algebraic space. Since the property of factoring through a closed substack can be checked smooth locally, we find that $T \times_{\mathcal{X}} (\mathcal{X} \times_{\mathcal{Y}} V) \to \mathcal{X} \times_{\mathcal{Y}} V$ factors through $\mathcal{X} \times_{\mathcal{Y}} F(V)$ if and only if $T \times_{\mathcal{X}} U_i \to U_i$ factors through $U_i \times_V F(V)$, or equivalently (by Axiom 9.1; note that Lemma 3.5 ensures that the morphism $U_i \to V$, which *a priori* is representable by algebraic stacks, is in fact representable by algebraic spaces), through $F(U_i)$. Thus we are indeed able to deduce (a) from (b). □

Lemma 9.8. *Suppose that F is compatible with flat base-change, in the sense that if $U \to V$ is a morphism in $\mathrm{Aff}_{\mathcal{C}}$ which is representable by algebraic spaces and flat, then the natural morphism $F(U) \to U \times_V F(V)$ is an isomorphism. Then the same is true of $\widetilde{F}$: i.e. if $\mathcal{X} \to \mathcal{Y}$ is a morphism in $\mathcal{C}$ representable by algebraic stacks, then the natural morphism $\widetilde{F}(\mathcal{X}) \to \mathcal{X} \times_{\mathcal{Y}} \widetilde{F}(\mathcal{Y})$ is an isomorphism.*

Proof. The proof is identical to the proof of part (2) of Proposition 9.7, using the hypothesis on F in place of Axiom 9.1. □

We now present various applications of the preceding framework.

Example 9.9. We may use the preceding framework as a foundation for the theory of scheme-theoretic images for quasi-compact morphisms of algebraic stacks. (See e.g. [EG19b, §3.1] for a more ad hoc development of this theory.)

Suppose that $\mathcal{T} \to \mathcal{S}$ is a quasi-compact morphism of algebraic stacks, and take $\mathcal{C}$ to be the category of algebraic stacks lying over $\mathcal{S}$. The full subcategory $\mathrm{Aff}_{\mathcal{C}}$ then consists of the category of affine schemes equipped with a morphism to $\mathcal{S}$. If $\operatorname{Spec} A \to \mathcal{S}$ is an object of $\mathrm{Aff}_{\mathcal{C}}$, define $F(\operatorname{Spec} A)$ to be the scheme-theoretic image in A of the base-changed morphism $\operatorname{Spec} A \times_{\mathcal{S}} \mathcal{T} \to \operatorname{Spec} A$ (defined in an obvious way: consider a smooth surjection $U \to \operatorname{Spec} A \times_{\mathcal{S}} \mathcal{T}$ whose source is an affine scheme — which is possible, since $\operatorname{Spec} A \times_{\mathcal{S}} \mathcal{T}$ is quasi-compact — and define $F(\operatorname{Spec} A)$ to the be the scheme-theoretic image of the composite $U \to \operatorname{Spec} A \times_{\mathcal{S}} \mathcal{T} \to \operatorname{Spec} A$; well-definendness, i.e. independence of the choice of U, is easily checked). Axiom 9.1 is easily verified in this context, using the known base-change properties of scheme-theoretic images of quasi-compact morphisms of schemes [Stacks, Tag 081I], and thus F extends to a functor $\widetilde{F}$. For any morphism of algebraic stacks $\mathcal{X} \to \mathcal{S}$, we declare $\widetilde{F}(\mathcal{X})$ to be the *scheme-theoretic image* of the base-changed morphism $\mathcal{T} \times_{\mathcal{S}} \mathcal{X} \to \mathcal{X}$. In particular, $\widetilde{F}(\mathcal{S})$ is the scheme-theoretic image of the morphism $\mathcal{T} \to \mathcal{S}$ itself.

Example 9.10. Suppose that A is a quasi-excellent Noetherian ring which is I-adically complete with respect to an ideal I of A. Let $\mathcal{C}$ denote the category of formal algebraic stacks equipped with a morphism $\mathcal{X} \to \operatorname{Spf} A$ which is adic and locally of finite type. The full subcategory $\mathrm{Aff}_{\mathcal{C}}$ then consists of affine formal algebraic spaces $\operatorname{Spf} B$, endowed with a morphism $\operatorname{Spf} B \to \operatorname{Spf} A$ which is adic and of finite type. This morphism in turn corresponds to a morphism $A \to B$, with B being I-adically complete and B/I being finite type over A/I. Consider the reduced quotient B_{red} of B. Since B is Noetherian (by [Stacks, Tag 0AKP]; note that [Stacks, Tag 0AKN] ensures that for a morphism of affine formal algebraic spaces, being representable by algebraic spaces is equivalent to being representable by schemes), the A-algebra B_{red} is again I-adically complete, and B_{red}/I is also of finite type over A/I; thus $\operatorname{Spf} B_{\mathrm{red}}$ again lies in $\mathcal{C}$. We define $F(\operatorname{Spf} B) := \operatorname{Spf} B_{\mathrm{red}}$, equipped with the morphism $\operatorname{Spf} B_{\mathrm{red}} \hookrightarrow \operatorname{Spf} B$.

Corollary 8.25 shows that for any quotient of B, being reduced is equivalent to being analytically unramified. When combined with Lemma 8.20, this shows that the functor F satisfies Axiom 9.1. Thus we may extend F to a functor $\widetilde{F}$ on $\mathcal{C}$, which associates to any I-adic formal algebraic stack, locally of finite type over $\operatorname{Spf} A$, its *associated reduced formal algebraic substack*.[16]

Example 9.11. Suppose that A is a complete DVR, with maximal ideal I. Let $\mathcal{C}$ denote the category of locally Noetherian formal algebraic stacks equipped with a morphism $\mathcal{X} \to \operatorname{Spf} A$. The full subcategory $\mathrm{Aff}_{\mathcal{C}}$ then consists of Noetherian affine formal algebraic spaces $\operatorname{Spf} B$ endowed with a morphism $\operatorname{Spf} B \to \operatorname{Spf} A$, which in turn corresponds to a morphism $A \to B$, with B being a Noetherian A-algebra, complete with respect to an ideal J that contains IB. Let B_{fl} denote the maximal A-flat quotient of B, i.e. the quotient of B by its ideal of I-power torsion elements. Since B is Noetherian, the quotient B_{fl} is again J-adically complete, and we define $F(\operatorname{Spf} B) := \operatorname{Spf} B_{\mathrm{fl}}$, equipped with its canonical morphism $\operatorname{Spf} B_{\mathrm{fl}} \hookrightarrow \operatorname{Spf} B$.

Since smooth morphisms are in particular flat, it follows easily from Lemma 8.18 that the functor F satisfies Axiom 9.1, and so F extends to a functor $\widetilde{F}$ on $\mathcal{C}$, which associates to any locally Noetherian formal algebraic stack $\mathcal{X}$ lying over $\operatorname{Spf} A$ its *flat part* $\mathcal{X}_{\mathrm{fl}}$. By construction, the morphism $\mathcal{X}_{\mathrm{fl}} \to \operatorname{Spf} A$ is flat, in the sense of Definition 8.42, and it is easily seen to be the maximal substack of $\mathcal{X}$ with respect to this property, in that if $\mathcal{Y} \to \mathcal{X}$ is a morphism of locally Noetherian formal algebraic stacks for which the composite $\mathcal{Y} \to \operatorname{Spf} A$ is flat, then $\mathcal{Y} \to \mathcal{X}$ factors through $\mathcal{X}_{\mathrm{fl}}$.

The preceding construction applies in particular to $\mathcal{X}$ which are I-adic and locally of finite type over $\operatorname{Spf} A$ (since such formal algebraic stacks are in particular locally Noetherian). In this case, the morphism $\mathcal{X}_{\mathrm{fl}} \to \operatorname{Spf} A$ is again adic and locally of finite type (being the composite of such a morphism with a closed immersion).

Remark 9.12. Lemma 9.8 ensures that the constructions described in Examples 9.9 and 9.11 are compatible with base-change under morphisms in their respective categories $\mathcal{C}$ that are representable by algebraic spaces and flat. (In the case of

[16]We use this terminology to distinguish this construction from that of the underlying reduced algebraic substack of Lemma 5.26.

Example 9.9, the compatibility of the formation of scheme-theoretic images of quasi-compact morphisms with respect to flat base-change [Stacks, Tag 081I] shows that the hypothesis of Lemma 9.8 is satisfied. In the case of Example 9.11, the hypothesis of Lemma 9.8 is easily verified, using Lemma 8.18.)

10. Morphisms to affine formal algebraic spaces

Recall that if $\mathcal{X}$ is an algebraic stack, then we have a structure sheaf $\mathcal{O}_{\mathcal{X}}$ on $\mathcal{X}$. (We refer to [Stacks, Tag 06TF] for the foundations of sheaves on stacks; see [Stacks, Tag 06TU] in particular for a discussion of the structure sheaf.) If $B = \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, then there is a canonical morphism $\mathcal{X} \to \operatorname{Spec} B$, which is initial in the category of morphisms from $\mathcal{X}$ to affine schemes. We now explain how to extend this construction to the context of (certain) formal algebraic stacks.

To this end let $\mathcal{X}$ be a quasi-compact and quasi-separated formal algebraic stack, which is furthermore locally countably indexed. By Lemma 6.2 we may write

$$\mathcal{X} \cong \varinjlim_{n \geq 1} \mathcal{X}_n,$$

where each $\mathcal{X}_n$ is an algebraic stack, and the transition morphisms $\mathcal{X}_n \hookrightarrow \mathcal{X}_{n+1}$ are thickenings. For each $n \geq 1$, write $B'_n := \Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$. Each of the natural morphisms $B'_{n+1} \to B'_n$ has locally nilpotent kernel. We may form the projective limit $B := \varprojlim_n B'_n$; if we equip each B'_n with its discrete topology, and equip B with the projective limit topology, then B is a complete topological ring, which is easily seen to be independent (up to canonical isomorphism) of the choice of description of $\mathcal{X}$ as a 2-colimit. If we let J_n denote the kernel of the surjection $B \to B'_n$, then J_n is open in B. If B_n denotes the image of B in B'_n (so that $B_n \cong B/J_n$), then the transition morphisms $B_{n+1} \to B_n$ are surjective with locally nilpotent kernels (so that the corresponding morphisms $\operatorname{Spec} B_n \to \operatorname{Spec} B_{n+1}$ are thickenings), and we have an induced isomorphism of topological rings $B \xrightarrow{\sim} \varprojlim_n B_n$. Each of the ideals J_n is a weak ideal of definition in B, realizing B as a weakly admissible complete topological ring (in the sense of [Stacks, Tag 0AMV]).

Definition 10.1. If $\mathcal{X}$ is a quasi-compact, quasi-separated, locally countably indexed formal algebraic stack, then we write $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) := B$, where B is the weakly admissible linearly topologized ring attached to $\mathcal{X}$ as in the preceding discussion.

Lemma 10.2. *If $\mathcal{X}$ is a quasi-compact, quasi-separated, locally countably indexed formal algebraic stack, then there is a natural morphism of formal algebraic stacks $\mathcal{X} \to \operatorname{Spf} \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, which is initial in the category of morphisms from $\mathcal{X}$ to weakly admissible affine formal algebraic spaces.*

Proof. Let Y be a weakly admissible affine formal algebraic space, and write $Y \cong \varinjlim_I \operatorname{Spec} A/I$, where A is a weakly admissible topological ring and I runs over the weak ideals of definition of A. If $\mathcal{X} \to Y$ is a morphism, then the composite $\mathcal{X}_n \to \mathcal{X} \to Y$ factors through $\operatorname{Spec} A/I$ for some I. Since $\operatorname{Spec} A/I$ is affine, we obtain a further factorization of the form

$$\mathcal{X}_n \to \operatorname{Spec} B'_n \to \operatorname{Spec} A/I \to \operatorname{Spf} A.$$

(Here we are using the notation introduced in the discussion preceding Definition 10.1.) The induced morphism $A \to B'_n$ (which is continuous when B'_n is given its discrete topology) is independent of the particular weak ideal of definition I considered in this factorization, and we obtain a continuous morphism $A \to \varprojlim_n B'_n = B$, inducing the desired morphism $\operatorname{Spf} B \to \operatorname{Spf} A \cong Y$. $\square$

Suppose now that R is a Noetherian ring, complete with respect to a principal ideal $I = tR$. We regard R as a topological ring by equipping it with the I-adic, or equivalently t-adic, topology. We will refer to an I-adic formal algebraic stack (in the sense of Definition 7.6) as a *t-adic formal algebraic stack*. By definition, to give a formal algebraic stack $\mathcal{X}$ the structure of a t-adic formal algebraic stack structure is to equip it with a morphism $\mathcal{X} \to \operatorname{Spf} R$ which is representable by algebraic spaces. We then say that $\mathcal{X}$ is a t-adic formal algebraic stack *of finite presentation* if this morphism is of finite presentation.

If $\mathcal{X}$ is a t-adic formal algebraic stack of finite presentation, then it is both quasi-compact and quasi-separated, as well as being locally countably indexed. Thus Definition 10.1 applies, and allows us to define the weakly admissible linearly topologized ring $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Lemma 10.2 then shows that the structure morphism $\mathcal{X} \to \operatorname{Spf} R$ induces a natural morphism $\operatorname{Spf} \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \operatorname{Spf} R$, which corresponds in turn to a continuous ring morphism $R \to \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

Lemma 10.3. *If $\mathcal{X}$ is a t-adic formal algebraic stack of finite presentation, then the natural topology on $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ coincides with the t-adic topology; equivalently, the morphism $R \to \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is taut.*

Remark 10.4. The notion of a taut morphism of linearly topologized rings is defined in [Stacks, Tag 0AMX]. The equivalence claimed in the statement of the lemma follows from [Stacks, Tag 0APU].

Proof of Lemma 10.3. Since $\mathcal{X}$ is quasi-compact, we may choose a morphism $U \to \mathcal{X}$ whose source is an affine formal algebraic space, and which is representable by algebraic spaces, smooth, and surjective. Since $\mathcal{X}$ is furthermore quasi-separated, the fibre product $U \times_{\mathcal{X}} U$ is again quasi-compact, and so we may similarly find a morphism $V \to U \times_{\mathcal{X}} U$ whose source is an affine formal algebraic space, and which is representable by algebraic spaces, smooth, and surjective.

Write $\mathcal{X}_n := \operatorname{Spec} R/t^n \times_{\operatorname{Spf} R} \mathcal{X}$, and similarly write $U_n := \operatorname{Spec} R/t^n \times_{\operatorname{Spf} R} U$ and $V_n := \operatorname{Spec} R/t^n \times_{\operatorname{Spf} R} V$. We then have exact sequences of R-modules

$$0 \to \Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n}) \to \Gamma(U_n, \mathcal{O}_{U_n}) \to \Gamma(V_n, \mathcal{O}_{V_n})$$

(the first arrow being given by pull-back, and the second by the difference of the two pull-back maps), and (passing to the projective limit in n)

$$0 \to \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \Gamma(U, \mathcal{O}_U) \to \Gamma(V, \mathcal{O}_V).$$

We also have isomorphisms

$$\Gamma(U, \mathcal{O}_U)/t^n \Gamma(U, \mathcal{O}_U) \cong \Gamma(U_n, \mathcal{O}_{U_n})$$

and

$$\Gamma(V, \mathcal{O}_V)/t^n \Gamma(V, \mathcal{O}_V) \cong \Gamma(V_n, \mathcal{O}_{V_n}).$$

We let $J_n \subseteq \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ denote the kernel of the morphism $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$; since $\Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$ embeds into $\Gamma(U_n, \mathcal{O}_{U_n})$, we see that

$$J_n = t^n \Gamma(U, \mathcal{O}_U) \cap \Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n}).$$

Since each of the morphisms $U \to \operatorname{Spf} R$ and $V \to \operatorname{Spf} R$ is representable by algebraic spaces and of finite type, each of U and V is Noetherian [Stacks, Tag 0AQ7].

In particular, the ideal in $\Gamma(V, \mathcal{O}_V)$ consisting of the t-power torsion elements is finitely generated, and hence is annihilated by some uniformly bounded power of t, say t^N. It follows that

$$t^n\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \subseteq J_n = t^n\Gamma(U, \mathcal{O}_U) \cap \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \subseteq t^{n-N}\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$$

if $n \geq N$. Thus the topology on $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ induced by the ideals J_n, which is its natural topology, coincides with its t-adic topology, as claimed. $\square$

We give an application of the preceding construction and results which will be used in [Car+19] and [EG19a]. We continue to assume that R is a Noetherian ring, complete with respect to the t-adic topology, for some element $t \in R$. We furthermore assume given quasi-compact and quasi-separated formal algebraic stacks $\mathcal{X}$ and $\mathcal{Y}$, and a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[dr] & \mathcal{Y} \arrow[d] \\ & \operatorname{Spf} R \end{tikzcd}$$

in which the diagonal arrow makes $\mathcal{X}$ into a t-adic formal algebraic stack of finite presentation. The usual graph argument shows that the horizontal morphism $\mathcal{X} \to \mathcal{Y}$ is necessarily quasi-compact, quasi-separated, and representable by algebraic stacks. Suppose in addition that this morphism is proper, and scheme-theoretically dominant (in the sense of Definition 6.13). Finally, suppose that $\mathcal{Y}$ is Ind-locally of finite type over $\operatorname{Spec} R$ (in the sense of Remark 8.39).

Proposition 10.5. *In the preceding situation, and under the preceding hypotheses, the morphism $\mathcal{Y} \to \operatorname{Spf} R$ is adic, and hence realizes $\mathcal{Y}$ as a t-adic formal algebraic stack of finite presentation.*

Proof. Since $\mathcal{Y}$ is quasi-compact and quasi-separated by assumption, it suffices to show that it is a t-adic formal algebraic stack that is *locally of finite presentation* in the evident sense. Since R is Noetherian, this is equivalent to being *locally of finite type*, and it is this latter property that we will establish.

Since $\mathcal{Y}$ is quasi-compact, we may find a morphism $V \to \mathcal{Y}$ whose source is an affine formal algebraic space, and which is representable by algebraic spaces, smooth, and surjective. In order to show that $\mathcal{Y} \to \operatorname{Spf} R$ is adic (i.e. is representable by algebraic stacks) and (locally) of finite type, it suffices to show that the composite $V \to \mathcal{Y} \to \operatorname{Spf} R$ is representable by algebraic stacks and of finite type, or equivalently (by Lemma 3.5), representable by algebraic spaces and of finite type, or equivalently (by definition), adic and of finite type.

If we let $\mathcal{X}_n := \operatorname{Spec} R/t^n \times_{\operatorname{Spf} R} \mathcal{X}$, then $\mathcal{X} \cong \varinjlim \mathcal{X}_n$. Since $\mathcal{X} \to \mathcal{Y}$ is scheme-theoretically dominant, we may write $\mathcal{Y} \cong \varinjlim \mathcal{Y}_n$ as the 2-colimit of a directed sequence of algebraic stacks with thickenings as transition morphisms, such that the morphism $\mathcal{X} \to \mathcal{Y}$ induces a scheme-theoretically dominant morphism of algebraic stacks $\mathcal{X}_n \to \mathcal{Y}_n$ for each n. If we write $V_n := \mathcal{Y}_n \times_{\mathcal{Y}} V$, then each V_n is an algebraic space admitting a closed immersion $V_n \hookrightarrow V$, and hence is in fact an affine scheme, say $V_n \cong \operatorname{Spec} A_n$. The natural morphism $\varinjlim V_n \to V$ is an isomorphism, and so we have $V \cong \operatorname{Spf} A$, where $A \cong \varprojlim A_n$. Our goal, then, is to show that the projective limit topology on A coincides with the t-adic topology on A.

Since $\mathcal{Y}$ is Ind-locally of finite type over $\operatorname{Spf} R$, as well as being quasi-compact and quasi-separated, each of the closed algebraic substacks $\mathcal{Y}_n$ is locally of finite type over $\operatorname{Spec} R/t^n$ (by Lemma 8.29 (2)). Since $\mathcal{Y}$ is quasi-compact and quasi-separated, so is its closed substack $\mathcal{Y}_n$; thus in fact each $\mathcal{Y}_n$ is of finite type over $\operatorname{Spec} R/t^n$, and so V_n is also of finite type over $\operatorname{Spec} R/t^n$. Equivalently, the ring A_n is of finite type over R/t^n.

Write $B := \Gamma(\mathcal{X} \times_{\mathcal{Y}} V, \mathcal{O}_{\mathcal{X}\times_{\mathcal{Y}} V})$, and let B_n denote the image of the natural morphism $B \to \Gamma(\mathcal{X}_n \times_{\mathcal{Y}} V, \mathcal{O}_{\mathcal{X}_n \times_{\mathcal{Y}} V})$. Lemma 10.2 shows that the morphism $\mathcal{X} \times_{\mathcal{Y}} V \to \operatorname{Spf} A$ induces a morphism $A \to B$. As in the proof of Lemma 6.15, we find that each of induced morphisms $\mathcal{X}_n \times_{\mathcal{Y}} V \to V_n$ is scheme-theoretically dominant, which implies (indeed, is equivalent to) the fact that each A_n is equal to the image of the composite morphism $A \to B \to B_n$. Since $\mathcal{X}_n \times_{\mathcal{Y}} V \to V_n$ is furthermore proper, we see that B_n is a finite A_n-algebra for each n. (See e.g. [Ols05] for a proof of this finiteness statement for proper morphisms in the stacky context.) As already noted, we have an isomorphism of topological rings $A \cong \varprojlim_n A_n$, as well as an isomorphism of topological rings $B \cong \varprojlim_n B_n$. Since A_n embeds into B_n for each n, we see that the injection $A \hookrightarrow B$ is a topological embedding.

Since A is complete, and since the t-adic topology on A is finer than its given topology, we see that A is t-adically complete. Thus, if we fix a value of n, then since A_n if of finite type over R/t^n, we may find a morphism from $R\{x_1, \dots, x_r\}$ (restricted power series over R in some number of variables $x_1, \dots, x_r$) to A whose composite with the surjection $A \to A_n$ is again surjective. Since B_n is finite over A_n, it is thus also finite over $R\{x_1, \dots, x_r\}$.

Lemma 10.3 shows that the topology on B induced by writing it as the projective limit $B \cong \varprojlim_n B_n$ coincides with the t-adic topology. Thus B/tB is a quotient of B_n for some sufficiently large value of n, and so, applying the construction of the preceding paragraph with this value of n, we find that B/tB is finite as an $R\{x_1, \dots, x_r\}$-algebra. The topological version of Nakayama's Lemma then shows that B is finite as an $R\{x_1, \dots, x_r\}$-algebra.

Now $R\{x_1, \dots, x_n\}$ is Noetherian, and so the Artin–Rees Lemma shows that the t-adic topology on B induces the t-adic topology on its $R\{x_1, \dots, x_r\}$-submodule A. Taking into account Lemma 10.3 again, as well as the fact (noted above) that the given topology on B induces the given topology on A, we find that the given topology on A is the t-adic topology, as required. To see that $\operatorname{Spf} A \to \operatorname{Spf} R$ is of finite type, note that $A/t^m A$ is now known to be a discrete quotient of A, for any value of m, so that we have a closed immersion $\operatorname{Spec} A/t^m A \hookrightarrow \operatorname{Spec} A_n$ for some sufficiently large value of n. Since the target of this closed immersion is of finite type over R/t^n, so is the source. $\square$

11. A criterion for a formal algebraic stack to be locally Noetherian

Our main objective in this section is to provide a criterion for a locally Ind-locally of finite type formal algebraic stack $\mathcal{X}$ (in the sense discussed in Remark 8.39) lying over a locally Noetherian scheme S to in fact be locally Noetherian. Our argument, which can be thought of as a generalization of the proof that a pro-representing object for a formal deformation problem whose tangent space is finite dimensional is necessarily a complete Noetherian local ring, will be based on the existence of an obstruction theory satisfying certain properties, similar to those considered in [Stacks, Tag 07YV]. Before introducing the relevant notions, though, we first establish some simple criteria for affine formal algebraic spaces to be Noetherian.

Lemma 11.1. *Let* $\operatorname{Spf} A = \varinjlim_n \operatorname{Spec} A_n$ *be a countably indexed affine formal algebraic space of Ind-locally finite type over a Noetherian ring* Λ*; so each* A_n *is a finite type* Λ*-algebra, and the transition morphisms* $\operatorname{Spec} A_n \hookrightarrow \operatorname{Spec} A_{n+1}$ *are thickenings of* Λ*-schemes. Let* I_n *denote the nilradical of* A_n*, and suppose further that for each* $N \geq 1$*, the limit* $\varinjlim_n \operatorname{Spec} A_n/I_n^N$ *eventually stabilizes, i.e. that the morphism* $\operatorname{Spec} A_n/I_n^N \to \operatorname{Spec} A_{n+1}/I_{n+1}^N$ *is an isomorphism for each fixed value of* N *and for sufficiently large values of* n*. Then the topological ring* $A = \varprojlim_n A_n$ *is a Noetherian* Λ*-algebra, endowed with the* I*-adic topology, for some ideal* $I \subseteq A$ *with respect to which* A *is complete.*

Proof. For each value of $N \geq 0$, choose $n(N)$ large enough that each of the transition morphisms $A_m/I_m^{N+1} \to A_{n(N)}/I_{n(N)}^{N+1}$ is an isomorphism if $m \geq n(N)$, and write $B_N := A_{n(N)}/I_{n(N)}^{N+1}$. We may, and do, choose $n(N)$ so that it is a monotone increasing function of n, in which case there are natural transition morphisms $B_{N+1} \to B_N$, which are thickenings. Also, since each A_n is Noetherian, each of the ideals I_n is nilpotent, and so if N is chosen so large that $I_n^{N+1} = 0$, then there is a natural surjection $B_N \to A_n/I_n^{N+1} = A_n$. Thus $\{B_N\}$ is an inverse system of thickenings that is cofinal with the sequence $\{A_n\}$, and hence gives rise to the same affine formal algebraic space. Replacing the sequence $\{A_n\}$ by the sequence $\{B_N\}$, we may then assume that $I_n^{n+1} = 0$ for each n (so in particular, A_0 is reduced), and that $A_n/I_n^n \xrightarrow{\sim} A_{n-1}$.

Since A_0 is a finite type Λ-algebra, we may choose a surjection $\Lambda[x_1, \dots, x_r] \to A_0$, and lift this morphism to a morphism of Λ-algebras $\Lambda[x_1, \dots, x_r] \to A$. Also, if $n \geq 1$, then $I_n/I_n^2 \xrightarrow{\sim} I_1/I_1^2$ is a finitely generated A_0-module, and thus we may choose compatible surjections of $\Lambda[x_1, \dots, x_r]$-modules

$$\Lambda[x_1, \dots, x_r]^s \to I_n/I_n^2 \tag{11.2}$$

for some value of s.

Since each I_n is a nilpotent ideal in A_n, the morphisms (11.2) can be used to induce a compatible sequence of surjections $\Lambda[x_1, \dots, x_r][[y_1, \dots, y_s]] \to A_n$, which in turn give rise to a morphism

$$\Lambda[x_1, \dots, x_r][[y_1, \dots, y_s]] \to A.$$

If we equip the source with its $(y_1, \dots, y_s)$-adic topology, then this morphism is *taut* (in the sense of from [Stacks, Tag 0AMX]) with dense image, and so it follows from [Stacks, Tag 0APT] that this morphism is itself surjective and open, so that A is Noetherian and adic. □

Lemma 11.3. *If* X *is a countably indexed affine formal algebraic space of Ind-locally finite type over a Noetherian ring* Λ*, then the following conditions are equivalent:*

(1) X *is Noetherian.*

(2) Let $\operatorname{Spf} A := \varinjlim_n \operatorname{Spec} A_n$ *be any countably indexed affine formal algebraic space of Ind-locally finite type over* Λ *(so* $\{A_n\}$ *is an inverse system of finite type* Λ*-algebras with the transition morphisms* $\operatorname{Spec} A_n \hookrightarrow \operatorname{Spec} A_{n+1}$ *being thickenings), let* I_n *denote the nilradical of* A_n*, and suppose that there exists some* $N \geq 0$ *such that* $I_n^N = 0$ *for all* n*. Then any morphism* $\widehat{x} : \operatorname{Spf} A \to X$ *of affine formal algebraic*

spaces (*or equivalently, any compatible family of morphisms* $x_n : \operatorname{Spec} A_n \to X$) *is* effective, *i.e. there exists a morphism* $x : \operatorname{Spec} A \to X$ *inducing* $\widehat{x}$ (*or equivalently, inducing each of the* x_n).

(3) Let A be any finite type Λ-algebra, and let $\{A'_n\}$ be an inverse system of square-zero thickenings $0 \to J_n \to A'_n \to A \to 0$, with the property that each J_n is a finite type A-module that is furthermore annihilated by the nil-radical of A. Then, if $A' := \varinjlim_n \operatorname{Spec} A'_n$, any morphism $\widehat{x} : \operatorname{Spf} A' \to X$ is effective.

Proof. Suppose first that X is Noetherian, so that we may find an isomorphism $X \cong \operatorname{Spf} B$, where B is a Noetherian ring, endowed with its J-adic topology for some ideal J with respect to which B is complete. Let $\widehat{x} : \operatorname{Spf} A \to \operatorname{Spf} B$ be a morphism as in (2). Then the induced morphism $B \to A_n$ maps J into I_n, and hence contains J^N in its kernel. Thus we may pass to the inverse limit and obtain a morphism $B \to B/J^N \to A$, which in turn induces the desired morphism $x : \operatorname{Spec} A \to \operatorname{Spf} B$. Thus (1) implies (2).

Conversely, suppose that condition (2) holds. Choose an isomorphism $\operatorname{Spf} B \cong X$, where $B = \varprojlim_n B_n$ is an inverse limit of finite type Λ-algebras B_n with respect to transition maps that are thickenings, and let J_n be the nil-radical of B_n. Choose $N \geq 0$, and for each value of n, let $A_n := B_n/J_n^N$. By assumption the morphism $\varinjlim_n \operatorname{Spec} A_n \to \operatorname{Spf} B$ is effective, and thus factors through $\operatorname{Spec} B_n$ for some sufficiently large value of n. This implies that the inverse system B_n/J_n^N eventually stabilizes, and it follows from Lemma 11.1 that X is Noetherian, i.e. that (1) holds.

Condition (3) is a special case of condition (2). Indeed, the assumptions imply that each A'_n is of finite type over Λ. Let I denote the nilradical of A, and choose N so that $I^N = 0$. Then, if I'_n denotes the nilradical of A'_n, we see that $(I'_n)^N \subseteq J_n$, and thus that $(I'_n)^{N+1} = 0$.

Suppose now that (3) holds, and let $\widehat{x} : \operatorname{Spf} A \to X$ be a morphism as in the setting of (2). We will prove that $\widehat{x}$ is effective by induction on N. Consider the inverse system of short exact sequences (indexed by n)

$$0 \to I_n^{N-1} \to A_n \to A_n/I_n^{N-1} \to 0. \tag{11.4}$$

By induction, the morphism $\varinjlim_n \operatorname{Spec} A_n/I_n^{N-1} \to X$ is effective. Since X is Ind-locally of finite type, it factors through a morphism $y : \operatorname{Spec} C \to X$, where C is of finite type over Λ. Pulling back each of the short exact sequences (11.4) by the corresponding morphism $C \to A_n/I_n^{N-1}$, we obtain an inverse system of short exact sequences

$$0 \to I_n^{N-1} \to A'_n \to C \to 0. \tag{11.5}$$

We are now in the situation of (3), and thus the morphism $\operatorname{Spf} A' := \varinjlim \operatorname{Spec} A'_n \to X$ is effective. The natural morphism $A' \to A$ induces a corresponding morphism $\operatorname{Spec} A \to \operatorname{Spec} A'$, which, composed with the just-constructed morphism $\operatorname{Spec} A' \to X$, yields the desired morphism $x : \operatorname{Spec} A \to X$. □

We now introduce the obstruction theoretic notion that will be applied in our criterion for Noetherianness. We begin by recalling some material from [Stacks, Tag 07Y6].

Let $\mathcal{X}$ be a category fibred in groupoids over a locally Noetherian scheme S which satisfies the condition (RS*) of Definition 3.8. If $x : \operatorname{Spec} A \to \mathcal{X}$ is a morphism for which the composite morphism $\operatorname{Spec} A \to S$ factors through an affine open subset of S, then in [Stacks, Tag 07Y9] there is defined a functor T_x from the category of A-module to itself, whose formation is also functorial in the pair (x, A) [Stacks, Tag 07YA], and such that for any A-module M, there is a natural identification of $T_x(M)$ with the set of lifts of x to morphisms $x' : \operatorname{Spec} A[M] \to \mathcal{X}$ (where $A[M]$ denotes the square zero extension of A by M).

We now make the following key definition.

Definition 11.6. Let $\mathcal{X}$ be a category fibred in groupoids over a locally Noetherian scheme S, which satisfies (RS*). We say that $\mathcal{X}$ admits a *nice obstruction theory* if, for each finite type S-algebra A which lies over an affine open subscheme of S, equipped with a morphism $x : \operatorname{Spec} A \to \mathcal{X}$, there exists a complex of A-modules $K^\bullet_{(x,A)}$, as well as an auxiliary A-module $C_{(x,A)}$, such that the following conditions are satisfied:

(1) The complex $K^\bullet_{(x,A)}$ is bounded above and has finitely generated cohomology modules (or, equivalently, is isomorphic in the derived category to a bounded above complex of finitely generated A-modules), while $C_{(x,A)}$ is finitely generated and locally free.

(2) The formation of $K^\bullet_{(x,A)}$ is compatible (in the derived category) with pull-back, as is the formation of $C_{(x,A)}$. More precisely, if $f : \operatorname{Spec} B \to \operatorname{Spec} A$, inducing the morphism $y : \operatorname{Spec} B \to \operatorname{Spec} A \to \mathcal{X}$, then there is a natural isomorphism in the derived category $f^*K^\bullet_{(x,A)} \xrightarrow{\sim} K^\bullet_{(y,B)}$, as well as a natural isomorphism $f^*C_{(x,A)} \xrightarrow{\sim} C_{(y,B)}$.

(3) For any pair (x, A) as above, and for any A-module M, we have a short exact sequence

$$0 \to C_{(x,A)} \otimes M \to T_x(M) \to H^1(K^\bullet_{(x,A)} \otimes^{\mathbb{L}} M) \to 0$$

whose formation is functorial in M, as well as in the pair (x, A), in the sense that if we have a homomorphism $A \to B$, and let y denote the composite

$$\operatorname{Spec} B \to \operatorname{Spec} A \xrightarrow{x} \mathcal{X},$$

and if N is a B-module (regarded also as an A-module via the given morphism $A \to B$), then the diagram

$$\begin{tikzcd} 0 \arrow[r] & C_{(x,A)} \otimes N \arrow[r] \arrow[d] & T_x(N) \arrow[r] \arrow[d] & H^1(K^\bullet_{(x,A)} \otimes^{\mathbb{L}} N) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C_{(y,B)} \otimes N \arrow[r] & T_y(N) \arrow[r] & H^1(K^\bullet_{(y,B)} \otimes^{\mathbb{L}} N) \arrow[r] & 0 \end{tikzcd}$$

commutes; here the left-most and right-most vertical arrows are induced by the pull-back isomorphisms of condition (2) above, while the middle vertical arrow is the natural map arising from the functoriality of T, as discussed in [Stacks, Tag 07YA].

(4) For any pair (x, A) as above, and for any A-module M, the cohomology module $H^2(K^\bullet_{(x,A)} \otimes^{\mathbb{L}} M)$ serves as an obstruction module, so that for any deformation situation $(x, A' \to A)$, we have a functorial obstruction element $o_x(A') \in H^2(K^\bullet_{(x,A)} \otimes^{\mathbb{L}} I)$ (where I denotes the square-zero kernel of the surjection $A' \to A$).

Remark 11.7. In Definition 11.6, it might be more natural to dispense with the modules $C_{(x,A)}$, and in condition (3), to simply require the existence of a natural isomorphism between $T_x(M)$ and $H^1(K^\bullet_{(x,A)} \otimes^{\mathbb{L}} M)$. However the slightly more relaxed condition that we have imposed gives technical flexibility; for example, it makes it easy to prove Lemma 11.12 below.

Remark 11.8. We give an alternative formulation of condition (3) of Definition 11.6. Namely, suppose given a natural surjection $T_x(M) \to H^1(K^\bullet_{(x,A)} \otimes^{\mathbb{L}} M)$, and let $K_x(M)$ denote the kernel; the formation of $K_x(M)$ is then functorial in M, and in the pair (A, x).

Recall that a functor on A-modules is of the form $N \otimes -$ if and only if it is A-linear, right exact, and commutes with inductive limits. The functor $K_x(-)$ is certainly A-linear (since $T_x(-)$ is [Stacks, Tag 07Y9], as is $H^1(K^\bullet_{(x,A)} \otimes^{\mathbb{L}} -)$). Thus asking for a natural isomorphism $K_x(M) \xrightarrow{\sim} C_{(x,A)} \otimes M$, with $C_{(x,A)}$ finitely generated and locally free, is equivalent to asking that $K_x(-)$ be exact and commute with inductive limits, and that $K_x(A)$ be finitely generated.

Remark 11.9. We suppose given a nice obstruction theory in the sense of Definition 11.6. Let A be a finite type S-algebra that lies over an affine open subscheme of S, and let $x : \operatorname{Spec} A \to \mathcal{X}$ be a morphism. If M is an A-module, then there is a natural isomorphism $\varinjlim_i M_i \xrightarrow{\sim} M$, where M_i runs over the finitely generated submodules of M, inducing natural isomorphisms

$$\varinjlim_i H^\bullet(K^\bullet_{(x,A)} \otimes^{\mathbb{L}} M_i) \xrightarrow{\sim} H^\bullet(K^\bullet_{(x,A)} \otimes^{\mathbb{L}} M).$$

Since S is locally Noetherian, the finite type S-algebra A is Noetherian. If $0 \to I \to A' \to A \to 0$ is a square zero thickening, then we may write $\varinjlim_i A'_i \xrightarrow{\sim} A'$, where A'_i runs over the finite type S-subalgebras of A' that surject onto A. If we set $I_i := A'_i \cap I$, then I_i is a finite A-module, and there is a natural isomorphism $\varinjlim_i I_i \xrightarrow{\sim} I$, and so a corresponding natural isomorphism

$$\varinjlim_i H^\bullet(K^\bullet_{(x,A)} \otimes^{\mathbb{L}} I_i) \xrightarrow{\sim} H^\bullet(K^\bullet_{(x,A)} \otimes^{\mathbb{L}} I). \tag{11.10}$$

Since the formation of obstruction elements is functorial, we find that $o_x(A')$ (which lies in the target of this isomorphism), may be identified with the image of $o_x(A'_i)$ (for any choice of i; the images for the various different choices of i coincide) under this isomorphism. The isomorphism (11.10) then shows that $o_x(A') = 0$ if and only if $o_x(A'_i) = 0$ for some value (and hence for all sufficiently large values) of i.

If $\mathcal{X}$ is limit preserving, then there is an equivalence of groupoids

$$\varinjlim_i \mathcal{X}(A_i) \xrightarrow{\sim} \mathcal{X}(A)$$

(the left-hand side being the 2-colimit). Using this, we find that the morphism x lifts to A' if and only if it lifts to A'_i for some value of i. Similarly, we see that the formation of $T_x(M)$ is compatible with the formation of inductive limits.

In conclusion, we find that if $\mathcal{X}$ is limit preserving, then, in order to construct a nice obstruction theory, it suffices to consider the case of finitely generated A-modules M, and square zero extensions

$$0 \to I \to A' \to A \to 0$$

for which A' is of finite type over S (or, equivalently, for which I is a finite A-module).

Remark 11.11. For later reference, we briefly discuss a certain construction related to obstruction theory computations.

Suppose that
$$0 \to I \to A' \to A \to 0$$
is a square zero thickening, and that $\varphi : I \to J$ is a morphism of A-modules. We may pushforward the extension by φ to obtain a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] & \varphi_* A' \arrow[r] & A \arrow[r] & 0 \\ 0 \arrow[r] & I \arrow[r] \arrow[u] & A' \arrow[r] \arrow[u] & A \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

We note that $\varphi_* A'$ admits a natural ring structure, so that the preceding diagram becomes a morphism of square zero extensions. Namely, we may construct $\varphi_* A'$ as the quotient of the product $A' \times J$ by an anti-diagonally embedded copy of I. If we regard $A' \times J$ as the trivial square zero extension $A'[J]$, then the diagonally embedded copy of I is an ideal, and so the quotient $\varphi_* A'$ becomes a square zero extension of A by J, and the above diagram becomes a morphism of square zero thickenings.

If we are now given a morphism $x : \operatorname{Spec} A \to \mathcal{X}$ as in Definition 11.6, the functoriality of the formation of obstruction elements shows that $o_x(\varphi_* A') \in H^2(K^\bullet_{(x,A)} \otimes^{\mathbb{L}} J)$ is the image of $o_x(A') \in H^2(K^\bullet_{(x,A)} \otimes^{\mathbb{L}} I)$ under the morphism induced by φ.

Lemma 11.12. *If $\mathcal{U} \to \mathcal{X}$ is a morphism of categories fibred in groupoids over a locally Noetherian scheme S that is representable by algebraic spaces and smooth, and if $\mathcal{X}$ satisfies* (RS*) *and admits a nice obstruction theory* (*in the sense of Definition* 11.6)*, then the same is true of $\mathcal{U}$.*

Proof. It follows from Lemma 3.9 that $\mathcal{U}$ satisfies (RS*).

Suppose given a morphism $u : \operatorname{Spec} A \to \mathcal{U}$, where A is a finite type S-algebra lying over an affine open subscheme of S, and let $x : \operatorname{Spec} A \to \mathcal{X}$ denote the composite of f with the given morphism $\mathcal{U} \to \mathcal{X}$. We define $K^\bullet_{(u,A)} := K^\bullet_{(x,A)}$.

The infinitesimal lifting criterion for smooth morphisms (see e.g. [Ryd11, Appendix B]) shows that the natural morphism $T_u(M) \to T_x(M)$ is surjective. The composite
$$T_u(M) \to T_x(M) \to H^1(K^\bullet_{(x,A)} \otimes^{\mathbb{L}} M)$$
(the latter map being the surjection provided by the good obstruction theory for $\mathcal{X}$, which is assumed to exist) is then also surjective; we next compute its kernel. To this end, write $U := \operatorname{Spec} A \times_{\mathcal{X}} \mathcal{U}$; then U is an algebraic space smooth over $\operatorname{Spec} A$, and the morphism u corresponds to a section of U over $\operatorname{Spec} A$. The conormal bundle of this section to U is then a finitely generated locally free sheaf on $\operatorname{Spec} A$, whose global sections form a finitely generated locally free A-module P, and we have a natural short exact sequence
$$0 \to P \otimes M \to T_u(M) \to T_x(M) \to 0.$$

The kernel $K_u(M)$ of the surjection $T_u(M) \to H^1(K^\bullet_{(x,A)} \otimes^{\mathbb{L}} M)$ then sits in a short exact sequence
$$0 \to P \otimes M \to K_u(M) \to C_{(x,A)} \otimes M \to 0,$$

whose formation is functorial in M. The five-lemma then shows that $K_u(M)$ is an exact functor of M, and evidently $K_u(A)$ is finitely generated (being an extension of finitely generated A-modules). Remark 11.8 then shows that there is a natural isomorphism $K_u(M) \xrightarrow{\sim} C_{(u,A)} \otimes M$, for some finitely generated and locally free A-module $C_{(u,A)}$.

Suppose now that $0 \to I \to A' \to A \to 0$ is a square-zero thickening. We claim that if we define

$$o_u(A') := o_x(A') \in H^2(K^{\bullet}_{(x,A)} \otimes^{\mathbb{L}} I),$$

then we obtain functorial obstruction elements for extending morphism into $\mathcal{U}$ over square-zero thickenings.

Certainly if u extends to a morphism $u' : \operatorname{Spec} A' \to \mathcal{U}$, then the composite $x' : \operatorname{Spec} A' \to \mathcal{U} \to \mathcal{X}$ is an extension of x. Conversely, if such an extension x' of x exists, then the infinitesimal lifting criterion shows that we may lift x' to a morphism $u' : \operatorname{Spec} A' \to \mathcal{U}$ which extends x. This verifies the final condition for $K^{\bullet}_{(u,A)}$ to give a good obstruction theory, and so completes the proof of the lemma. $\square$

Theorem 11.13. *If $\mathcal{X}$ is a locally countably indexed and locally Ind-locally of finite type formal algebraic stack over the locally Noetherian scheme S which admits a nice obstruction theory, then $\mathcal{X}$ is locally Noetherian.*

Remark 11.14. Recall that, by Lemma 4.22, the Ind-algebraic stack $\mathcal{X}$ satisfies (RS*), so that it makes sense to speak of it admitting a nice obstruction theory.

Proof of Theorem 11.13. We may find a morphism $\coprod_i U_i \to \mathcal{X}$ whose source is a disjoint union of affine formal algebraic spaces, and which is representable by algebraic spaces, smooth, and surjective. We may furthermore assume that for each i, the composite morphism $U_i \to \mathcal{X} \to S$ factors through an open affine subset of S.

By assumption, each U_i is locally countably indexed and of Ind-locally finite type over S, and in order to prove the theorem, we must show that each U_i is in fact a Noetherian affine formal algebraic space. Lemma 11.12 shows that each U_i admits a nice obstruction theory, and thus, replacing $\mathcal{X}$ by U_i, and replacing S by an appropriately chosen affine open subscheme $\operatorname{Spec} \Lambda$, it suffices to prove the theorem in the case when $\mathcal{X} = X$ is a countably indexed affine formal algebraic space of Ind-locally finite type over $\operatorname{Spec} \Lambda$ for a Noetherian ring Λ.

We need to show that X is Noetherian; to do this, it is enough, by Lemma 11.3, to show that X satisfies condition (3) of that lemma. Thus we suppose given a sequence of square-zero thickenings

$$0 \to J_n \to A'_n \to A \to 0, \tag{11.15}$$

where A is a finite type Λ-algebra, and J_n is a finite type A_{red}-module, as well as a morphism $\widehat{x} : \operatorname{Spf} A' \to \mathcal{X}$, where

$$0 \to J \to A' \to A \to 0$$

is the inverse limit of the thickenings (11.15). We must show that $\widehat{x}$ is effective; to do this, we argue by induction on the dimension of $\operatorname{Spec} A_{\mathrm{red}}$.

Let $y : \operatorname{Spec} A \to X$ be the morphism induced by $\widehat{x}$, and let $y_{\mathrm{red}} : \operatorname{Spec} A_{\mathrm{red}} \to X$ denote the morphism induced by y. Each of the A_{red}-modules $H^i(K^{\bullet}_{(y_{\mathrm{red}},A_{\mathrm{red}})})$ is finitely generated, and furthermore vanishes if i is sufficiently large. Thus we may find an element $f \in A$ such that $\operatorname{Spec} A[1/f]$ is a dense open subset of $\operatorname{Spec} A$ (or, equivalently, such that $\operatorname{Spec} A_{\mathrm{red}}[1/f]$ is a dense open subset of $\operatorname{Spec} A_{\mathrm{red}}$), and such that each of the localizations $H^i(K^\bullet_{(y_{\mathrm{red}},A_{\mathrm{red}})})[1/f]$ is locally free for $i \geq 1$.

We recall two standard spectral sequences that we will require, namely

$$E_2^{p,q} := \operatorname{Tor}^{-p}_{A_{\mathrm{red}}}\big(H^q(K^\bullet_{(y_{\mathrm{red}},A_{\mathrm{red}})}), M\big) \implies H^{p+q}(K^\bullet_{(y_{\mathrm{red}},A_{\mathrm{red}})} \otimes^{\mathbb{L}} M) \tag{11.16}$$

(for any A_{red}-module M), and

$$E_2^{p,q} := R^p \varprojlim_n H^q(K^\bullet_{(y_{\mathrm{red}},A_{\mathrm{red}})} \otimes^{\mathbb{L}} J_n) \implies H^{p+q}(K^\bullet_{(y_{\mathrm{red}},A_{\mathrm{red}})} \otimes^{\mathbb{L}} J). \tag{11.17}$$

From the first of these spectral sequences, we obtain morphisms

$$H^1(K^\bullet_{(y_{\mathrm{red}},A_{\mathrm{red}})}) \otimes J_n \to H^1(K^\bullet_{(y_{\mathrm{red}},A_{\mathrm{red}})} \otimes^{\mathbb{L}} J_n). \tag{11.18}$$

If we let C_n denote the cokernels of these morphisms, then Lemma 11.22 below shows that each C_n is annihilated by some power of f which is bounded independently of n. Replacing f by this power, we may and do assume that these cokernels are in fact annihilated by f. Since the transition morphisms $J_{n+1} \to J_n$ are surjective, the same is true of the transition morphisms

$$H^1(K^\bullet_{(y_{\mathrm{red}},A_{\mathrm{red}})}) \otimes J_{n+1} \to H^1(K^\bullet_{(y_{\mathrm{red}},A_{\mathrm{red}})}) \otimes J_n,$$

so that $R^1 \varprojlim_n H^1(K^\bullet_{(y_{\mathrm{red}},A_{\mathrm{red}})}) \otimes J_n = 0$. If we let M_n denote the image of (11.18), then the right exactness of $R^1 \varprojlim_n$ shows that also $R^1 \varprojlim_n M_n = 0$ (alternatively, we could note that the transition morphisms $M_{n+1} \to M_n$ are again surjective), and hence that

$$R^1 \varprojlim_n H^1(K^\bullet_{(y_{\mathrm{red}},A_{\mathrm{red}})} \otimes^{\mathbb{L}} J_n) \xrightarrow{\sim} R^1 \varprojlim_n C_n$$

is annihilated by f. A consideration of (11.17) then shows that there is a surjection

$$H^2(K^\bullet_{(y_{\mathrm{red}},A_{\mathrm{red}})} \otimes^{\mathbb{L}} J) \to \varprojlim_n H^2(K^\bullet_{(y_{\mathrm{red}},A_{\mathrm{red}})} \otimes^{\mathbb{L}} J_n), \tag{11.19}$$

whose kernel is annihilated by f.

We are now ready to apply our obstruction theory. Note that since each of the A-modules J_n, as well as their inverse limit J, is in fact an A_{red}-module, and since the formation of the complex $K^\bullet$ is compatible with base-change, we have isomorphisms

$$H^i(K^\bullet_{(y,A)} \otimes^{\mathbb{L}} M) \cong H^i(K^\bullet_{(y_{\mathrm{red}},A_{\mathrm{red}})} \otimes^{\mathbb{L}} M)$$

when M is equal to one of the J_n or to J. In particular, we may view the obstruction classes $o_y(A'_n)$ as elements of $H^2(K^\bullet_{(y_{\mathrm{red}},A_{\mathrm{red}})} \otimes^{\mathbb{L}} J_n)$, and similarly, we may view the obstruction class $o_y(A')$ as an element of $H^2(K^\bullet_{(y_{\mathrm{red}},A_{\mathrm{red}})} \otimes^{\mathbb{L}} J)$. Since the formation of obstruction classes is functorial, we see that the obstruction class $o_y(A')$ maps to the compatible sequence of obstruction classes $o_y(A'_n)$ under the surjection (11.19). However, the existence of the morphisms $\operatorname{Spec} A'_n \to X$ shows that each of the obstruction classes $o_y(A'_n)$ vanishes, and hence that $o_y(A')$ lies in the kernel of (11.19). Thus this class is annihilated by f.

If A'' denotes the push-out of A' along the morphism $J \to J$ given by multiplication by f, then we see that $o_y(A'') = 0$. (See Remark 11.11 for a general discussion of pushouts and of the corresponding obstruction elements.) Thus there exists a morphism $x'' : \operatorname{Spec} A'' \to X$ prolonging y. If we let A''_n denote the pushout of A'_n along the morphism $J_n \to J_n$ given by multiplication by f (so that $A'' \xrightarrow{\sim} \varprojlim_n A''_n$), then the morphism $x''_n : \operatorname{Spec} A''_n \to X$ obtained by restricting x'' may not coincide with the composite

$$\operatorname{Spec} A''_n \to \operatorname{Spec} A'_n \xrightarrow{x_n} X, \tag{11.20}$$

but differs from this composite by an element $t_n \in T_{y_{\mathrm{red}}}(J_n)$. Since the morphisms x_n are compatible as n varies, so are the elements t_n. The morphism

$$H^1(K^\bullet_{(y_{\mathrm{red}},A_{\mathrm{red}})}, J) \to \varprojlim_n H^1(K^\bullet_{(y_{\mathrm{red}},A_{\mathrm{red}})}, J_n)$$

is surjective (as follows from a consideration of the spectral sequence (11.17)), and the morphism

$$C_{(y_{\mathrm{red}},A_{\mathrm{red}})} \otimes J \to \varprojlim_n C_{(y_{\mathrm{red}},A_{\mathrm{red}})} \otimes J_n$$

is even an isomorphism (since $C_{(y_{\mathrm{red}},A_{\mathrm{red}})}$ is finitely generated and locally free). Thus the morphism

$$T_{y_{\mathrm{red}}}(J) \to \varprojlim_n T_{y_{\mathrm{red}}}(J_n)$$

is also surjective, and hence we can find an element $t \in T_{y_{\mathrm{red}}}(J)$ which maps to the element $(t_n) \in \varprojlim_n T_{y_{\mathrm{red}}}(J_n)$. Modifying x'' by t, we obtain a morphism (which we continue to denote by x'') whose restriction x''_n to each $\operatorname{Spec} A''_n$ coincides with (11.20).

Let us take a moment to take stock of our situation. If we write $X \xrightarrow{\sim} \operatorname{Spec} B$, then the morphism $\widehat{x} : \operatorname{Spf} A' \to X$ corresponds to a continuous morphism $B \to A'$. The fact that the composite $\operatorname{Spf} A'' \to \operatorname{Spf} A \to X$ is induced by a morphism $\operatorname{Spec} A'' \to X$ is equivalent to the fact that the kernel of the composite

$$B \to A' \to A'' \tag{11.21}$$

is an open ideal in B. The kernel of the morphism $A' \to A''$ is the ideal $J[f]$ (the ideal of f-torsion elements in J), and the morphism $A' \to A''$ factors as the surjection $A' \to A'/J[f]$ followed by an embedding $A'/J[f] \hookrightarrow A''$. Thus the kernel of the composite (11.21) and the kernel of the composite $B \to A' \to A'/J[f]$ coincide, and hence the composite

$$\operatorname{Spf}(A'/J[f]) \to \operatorname{Spf} A' \to X$$

is effective, arising from a morphism $z : \operatorname{Spec}(A'/J[f]) \to X$.

We now consider the square zero thickening

$$0 \to J[f] \to A' \to A'/J[f] \to 0,$$

which may be written as an inverse limit of square zero thickenings

$$0 \to J_n[f] \to B_n \to A'/J[f] \to 0.$$

This latter square zero thickening is obtained as the pull-back, via the morphism $A'/J[f] \to A'_n/J_n[f]$, of the square zero thickening

$$0 \to J_n[f] \to A'_n \to A'_n/J_n[f] \to 0.$$

Since $\mathcal{X}$ satisfies (RS*), by Lemma 4.22, the compatible morphisms $x_n : \operatorname{Spec} A'_n \to X$ and $z : \operatorname{Spec}(A'/J[f]) \to X$ (compatible in the sense that they coincide when restricted to $\operatorname{Spec} A'_n/J_n[f]$) may then be glued to yield a morphism

$$u_n : \operatorname{Spec} B_n \to X.$$

These morphisms are compatible as n varies, and induce the given morphism $\widehat{x} : A' \to X$.

Since X is Ind-locally of finite type over Λ, we may find a finite type Λ-subalgebra C of $A'/J[f]$, such that the morphism $z : \operatorname{Spec}(A'/J[f]) \to X$ factors through $\operatorname{Spec} C$. Since A is of finite type over Λ, we may enlarge C if necessary so that $C \to A$ is surjective; we then let g denote an element of C which maps to the given element f of A. Since $(A'/J[f])_{\mathrm{red}} \xrightarrow{\sim} A_{\mathrm{red}}$, the inclusion of C into $A'/J[f]$ then induces an isomorphism of finite type Λ-algebras $C_{\mathrm{red}} \xrightarrow{\sim} A_{\mathrm{red}}$, mapping the image of g to the image of f. In particular, the Krull dimension of $C_{\mathrm{red}}/g\, C_{\mathrm{red}}$ is less than the Krull dimension of A_{red}.

If we pull-back the square zero thickening A' of $A'/J[f]$ over C, we obtain a square zero thickening

$$0 \to J[f] \to C' \to C \to 0,$$

which is the inverse limit of square zero thickenings

$$0 \to J_n[f] \to C'_n \to C \to 0$$

obtained by pulling back each of the thickenings B_n. Each of the morphisms $u_n : \operatorname{Spec} B_n \to X$ induces a morphism $v_n : \operatorname{Spec} C'_n \to X$. These morphisms are of course compatible as n varies, and so give rise to a morphism $\widehat{v} : \operatorname{Spf} C' \to X$. Since the morphism $\widehat{x} : \operatorname{Spf} A' \to X$ factors as the composite

$$\operatorname{Spf} A' \to \operatorname{Spf} C' \xrightarrow{\widehat{v}} X$$

by construction, we see that the effectivity of $\widehat{x}$ would follow from the effectivity of $\widehat{v}$. It is this latter effectivity that we now prove.

Since C is of finite type over Λ, it is Noetherian, and so its ideal of g-power torsion elements is annihilated by some fixed power of g, say g^N. It follows that each of the sequences

$$0 \to J_n[f] \to C'_n/g^{N+1}C'_n \to C/g^{N+1}C \to 0$$

is short exact, as is the sequence

$$0 \to J[f] \to C'/g^{N+1}C' \to C/g^{N+1}C \to 0$$

obtained as their inverse limit. Our inductive hypothesis then implies that the morphism $\operatorname{Spf} C'/g^{N+1}C' \to X$ obtained by restricting $\widehat{v}$ is effective. Since C' can be obtained as the fibre product

$$C' \xrightarrow{\sim} (C'/g^{N+1}C') \times_{C/g^{N+1}C} C,$$

another application of (RS*) yields a morphism $v : \operatorname{Spec} C \to X$, making $\widehat{v}$ effective, and so completing the proof of the theorem. $\square$

The following result is presumably well-known, but for lack of a reference, we include a proof.

Lemma 11.22. *Let A be a Noetherian ring, let $C^\bullet$ be a bounded above complex of A-modules all of whose cohomology modules are finitely generated, let $f \in A$, and suppose that the cohomology modules of $C^\bullet[1/f]$ are furthermore locally free $A[1/f]$-modules. Then for each index i, there is a power f^n of f such that the kernel and cokernel of the natural morphism*

$$H^i(C^\bullet) \otimes N \to H^i(C^\bullet \otimes^{\mathbb{L}} N) \tag{11.23}$$

are annihilated by f^n, for every A-module N (the power f^n being independent of the module N).

Proof. Consider the spectral sequence

$$E_2^{p,q} = \mathrm{Tor}^A_{-p}\big(H^q(C^\bullet), N\big) \implies H^{p+q}(C^\bullet \otimes^{\mathbb{L}} N),$$

the edge map of which induces the morphism (11.23) in the statement of the lemma. Since $C^\bullet$ is bounded above, for a fixed value of i (and an arbitrary A-module N), only finitely many terms of this spectral sequence contribute to the computation of the kernel and cokernel of (11.23). Thus, to prove the lemma, it suffices to show that if M is a finitely generated A-module for which $M[1/f]$ is locally free over $A[1/f]$, then for each positive degree i, there is a power f^n of f such that $\mathrm{Tor}_i^A(M, N)$ is annihilated by f^n (the power f^n being independent of the choice of N).

Let $P_\bullet \to M$ be a resolution of M by finitely generated projective A-modules. Since $M[1/f]$ is locally free, the complex

$$\ldots \to P_{i+1}[1/f] \to P_i[1/f] \to \ldots P_1[1/f] \to P_0[1/f] \to M[1/f] \to 0$$

is chain homotopic to zero. Write $P_{-1} := M$, and let $\tilde{s}_j : P_j[1/f] \to P_{j+1}[1/f]$ (for $j \geq -1$) denote this chain homotopy, so that

$$\partial_{j+1} \circ \tilde{s}_j - \tilde{s}_{j-1}\partial_j = \mathrm{id}_{P_j[1/f]}\,.$$

If we fix $i > 0$ and choose n large enough, we may then find morphisms $s_j : P_j \to P_{j+1}$, for $j = i$ and $i-1$, such that $s_j[1/f] = f^n s_j$, and such that

$$\partial_{i+1} \circ s_i - s_{i-1}\partial_i = f^n \,\mathrm{id}_{P_i}\,.$$

If we t_j (for $j = i$ and $i-1$) denote the morphism $P_j \otimes N \to P_{j+1} \otimes N$ induced by s_j, then we find that

$$\partial_{i+1} \circ t_i - t_{i-1}\partial_i = f^n \,\mathrm{id}_{P_i \otimes N}\,.$$

The existence of such morphisms t_j satisfying this identity evidently implies that $\mathrm{Tor}_i^A(M, N) := H_i(P_\bullet \otimes N)$ is annihilated by f^n, as required. □

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