Bounds for Multiplicities of Unitary Representations of Cohomological Type in Spaces of Cusp Forms

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1 Introduction

Let G_∞ be a semisimple real Lie group with unitary dual $\widehat{G}_\infty$. The goal of this note is to produce new upper bounds for the multiplicities with which representations $\pi \in \widehat{G}_\infty$ of cohomological type appear in certain spaces of cusp forms on G_∞.

More precisely, we suppose that $G_\infty := \mathbf{G}(\mathbf{R} \otimes_{\mathbf{Q}} F)$ for some connected semisimple linear algebraic group $\mathbf{G}$ over a number field F. Let K_∞ be a maximal compact subgroup of G_∞. We fix an embedding $\mathbf{G} \hookrightarrow \mathrm{GL}_N$ for some N, and for any ideal $\mathfrak{q}$ of $\mathcal{O}_F$, we let $G(\mathfrak{q})$ denote the intersection of G_∞ with the congruence subgroup of $\mathrm{GL}_N(\mathcal{O}_F)$ of full level $\mathfrak{q}$. We also fix an arithmetic lattice Γ in G_∞ (i.e. a subgroup commensurable with the congruence subgroups $G(\mathfrak{q})$) and write $\Gamma(\mathfrak{q}) := \Gamma \bigcap G(\mathfrak{q})$. For any $\pi \in \widehat{G}_\infty$, let $m(\pi, \Gamma(\mathfrak{q}))$ denote the multiplicity with which π occurs in the decomposition of the regular representation of G_∞ on $L^2_{\mathrm{cusp}}(\Gamma(\mathfrak{q})\backslash G_\infty)$. Let $V(\mathfrak{q})$ denote the volume of the arithmetic quotient $\Gamma(\mathfrak{q})\backslash G_\infty$.

In terms of this notation, we may state our main results.

Theorem 1.1. *Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_F$. Let $\pi \in \widehat{G}_\infty$ be of cohomological type. Suppose either that G_∞ does not admit discrete series, or, if G_∞ admits discrete series, that π contributes to cohomology in degrees other than $\frac{1}{2}\dim(G_\infty/K_\infty)$. Then*

$$m(\pi, \Gamma(\mathfrak{p}^k)) \ll V(\mathfrak{p}^k)^{1-1/\dim(G_\infty)}$$

as $k \to \infty$, with the implied constant depending on Γ and $\mathfrak{p}$.

Theorem 1.2. *Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_F$. Let W be a finite-dimensional representation of G_∞, and let $\mathcal{V}_{W,k}$ denote the local system on $\Gamma(\mathfrak{p}^k)\backslash G_\infty$ induced by W (assuming that k is taken large enough for $\Gamma(\mathfrak{p}^k)$ to be torsion free). Let $n \geq 0$, and if G_∞ admits discrete series, then suppose furthermore that $n \neq \frac{1}{2}\dim(G_\infty/K_\infty)$. Then*

$$\dim H^n\big(\Gamma(\mathfrak{p}^k)\backslash G_\infty, \mathcal{V}_{W,k}\big) \ll V(\mathfrak{p}^k)^{1-1/\dim(G_\infty)}$$

as $k \to \infty$, with the implied constant depending on Γ and $\mathfrak{p}$.

These two theorems are evidently closely related, in light of the results of [5], which show that $H^n\big(\Gamma(\mathfrak{p}^k)\backslash G_\infty, \mathcal{V}_{W,k}\big)$ may be computed in terms of automorphic forms.

In the remainder of this introduction, we discuss the relation of Theorem 1.1 to prior results in this direction before briefly describing the main ingredients in the proof of the two theorems.

DeGeorge and Wallach [3] established general upper bounds for $m(\pi, \Gamma)$ in the case where Γ is cocompact. In particular (ibid, Corollary 3.2), they showed that

$$m(\pi, \Gamma) \leq \left(\int_B |\phi(g)|^2 dg \right)^{-1} \mathrm{vol}(\Gamma \backslash G_\infty), \tag{1.1}$$

where $\phi(g) = \langle \pi(g)v, v \rangle$ is a matrix coefficient, and B is the preimage in $\Gamma\backslash G_\infty$ of a ball in $\Gamma\backslash G_\infty/K_\infty$ of radius equal to the injectivity radius of $\Gamma\backslash G_\infty/K_\infty$. Suppose, however, that π is *not* a discrete series. In particular, the corresponding matrix coefficients of π are then not square integrable. If $\Gamma(\mathfrak{q})$ denotes the mod $\mathfrak{q}$ congruence subgroup of Γ, then $\mathrm{inj.rad}(\Gamma(\mathfrak{q})\backslash G_\infty) \to \infty$ as $N_{F/\mathbf{Q}}(\mathfrak{q}) \to \infty$, and thus the formula of DeGeorge–Wallach implies that

$$\lim_{N_{F/\mathbf{Q}}(\mathfrak{q}) \to \infty} V(\mathfrak{q})^{-1} \cdot m(\pi, \Gamma(\mathfrak{q})) = 0. \tag{1.2}$$

For non-cocompact Γ, an analogous result was established by Savin [10].

It is natural to try to improve this result so as to obtain an estimate on the rate of decay in (1.2) as $N_{F/\mathbf{Q}}(\mathfrak{q}) \to \infty$. If π is non-tempered, then (1.1) itself implies an estimate of the form

$$m(\pi, \Gamma(\mathfrak{q})) \ll V(\mathfrak{q})^{1-\mu} \tag{1.3}$$

for some $\mu > 0$. (See [9], Lemma 1 and displayed equation (6).) In fact Sarnak and Xue in [9] have conjectured an inequality of the following form (in the case of cocompact Γ):

Conjecture 1.3 (Sarnak–Xue). *For $\pi \in \widehat{G}_\infty$ fixed,*

$$m(\pi, \Gamma(\mathfrak{q})) \ll V(\mathfrak{q})^{(2/p(\pi))+\epsilon}, \qquad \text{for all } \epsilon > 0,$$

where $p(\pi)$ is the infimum over $p \geq 2$ such that the K-finite matrix coefficients of π are in $L^p(G_\infty)$.

Sarnak and Xue proved their conjecture for arithmetic lattices in $\mathrm{SL}_2(\mathbf{R})$ and $\mathrm{SL}_2(\mathbf{C})$, obtaining partial results in the direction of this conjecture for $\mathrm{SU}(2,1)$. Note, however, that their conjecture is non-trivial only for *non-tempered* representations, since, for tempered representations, $p(\pi) = 2$. In particular, in the tempered but non-discrete series case, Conjecture 1.3 is weaker than the known result (1.2).

In Theorem 1.1, we restrict our attention to congruence covers of the form $\Gamma(\mathfrak{p}^k)$ for the fixed prime $\mathfrak{p}$. For such covers we obtain a quantitative improvement of (1.2) even in the case of tempered representations (at least for those of cohomological type; note that non-discrete series tempered representations of cohomological type exist precisely when G_∞ admits no discrete series – see [1], Thm. 5.1, p. 101). For such representations, our result provides the first general bound of the form (1.3) for any $\mu > 0$.

As we already noted, our two main theorems are closely related. Indeed, Theorem 1.1 is an easy corollary of Theorem 1.2 (see the end of Section 3 below), and most of our efforts will be concentrated on establishing the latter result.

When studying the Betti numbers of arithmetic quotients of symmetric spaces, it is natural to try to use tools such as Euler characteristics and the Lefschetz trace formula. When applied to analyzing contributions from the discrete series, such methods tend to be very powerful; for example, the $(\mathfrak{g}, \mathfrak{k})$-cohomology of a discrete series representation is concentrated in a single dimension [1], and so no cancellations occur when taking alternating sums. However, in other situations, these methods can be useless. For example, if π is tempered but not discrete series, then the Euler characteristic of its $(\mathfrak{g}, \mathfrak{k})$-cohomology vanishes [1]. Similarly, in situations where the symmetric space is a real manifold of odd dimension n, Poincare duality leads to cancellations in the natural sum $\sum_{k=0}^{n}(-1)^k \dim(H^k)$. One is thus forced to find different techniques. The proof of Theorem 1.2 takes as input the inequality (1.2) of [3] and [10] and a spectral sequence from [4], proceeding via a bootstrapping argument relying on non-commutative Iwasawa theory.

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2 Iwasawa Theory

Let $G \subseteq \mathrm{GL}_N(\mathbf{Z}_p)$ be an analytic pro-p group. Let $G_k = G \cap (1 + p^k M_N(\mathbf{Z}_p)) \subseteq \mathrm{GL}_N(\mathbf{Z}_p)$. The subgroups G_k form a fundamental set of open neighbourhoods of the identity in G, and moreover, for large k, there exists a constant c such that $[G : G_k] = c \cdot p^{dk}$, where $d = \dim(G)$.

Fix a finite extension E of $\mathbf{Q}_p$ with ring of integers $\mathcal{O}_E$. Write $\Lambda = \mathcal{O}_E[[G]]$ and $\Lambda_E = E \otimes_{\mathcal{O}_E} \Lambda$. The module theory of Λ falls under the rubric of Iwasawa theory. A fundamental result of Lazard [7] states that Λ is Noetherian; the same is thus true of the ring Λ_E. The rings Λ and Λ_E are non-commutative domains admitting a common field of fractions which we will denote by $\mathscr{L}$. Thus, $\mathscr{L}$ is a division ring which contains Λ and Λ_E and is flat over each of them (on both sides). If M is a finitely generated left Λ-module (resp. Λ_E-module), then $\mathscr{L} \otimes_\Lambda M$ (resp. $\mathscr{L} \otimes_{\Lambda_E} M$) is a finite-dimensional left $\mathscr{L}$-vector space; we define the *rank* of M to be the $\mathscr{L}$-dimension of this vector space. Note that rank is additive in short exact sequences of finitely generated Λ-modules (resp. Λ_E-modules), by virtue of the flatness of $\mathscr{L}$ over Λ and Λ_E.

Recall that a continuous representation of G on an E-Banach space V is called *admissible* if its topological E-dual V' (which is naturally a Λ_E-module) is finitely generated over Λ_E. (See [11]; a key point is that since Λ_E is Noetherian, the category of admissible continuous G-representations is abelian. Indeed, passing to topological duals yields an anti-equivalence with the abelian category of finitely generated Λ_E-modules.) We define the *corank* of an admissible G-representation to be the rank of the finitely generated Λ_E-module V'.

A coadmissible G-representation V is not determined by the collection of subspaces of invariants V^{G_k} $(r \geq 1)$. However, the following result (Theorem 1.10 of Harris [6]) shows that its corank is so determined.

Theorem 2.1 (Harris). *Let V be an E-Banach space equipped with an admissible continuous G-representation and let $d = \dim(G)$. Then as $k \to \infty$,*

$$\dim_E V^{G_k} = r \cdot [G : G_k] + O(p^{(d-1)k}) = r \cdot c \cdot p^{dk} + O(p^{(d-1)k}),$$

where r is the corank of V and c depends only on G.

Using this result, we may obtain bounds on the dimensions of the continuous cohomology groups $H^i(G_k, V)$ in terms of k for admissible continuous G-representations V. (Let us remark that the continuous G_k-cohomology on the category of admissible continuous G-representations may also be computed as the right derived functors of the functor of G_k-invariants; see Prop. 1.1.3 of [4].)

Lemma 2.2. *Let V be an admissible continuous G-representation. For each $i \geq 1$,*

$$\dim_E H^i(G_k, V) \ll p^{(d-1)k},$$

as $k \to \infty$.

Proof. Let $\mathcal{C} := \mathcal{C}(G, E)$ denote the Banach space of continuous E-valued functions on G, equipped with the right regular G-action. The module $\mathcal{C}$ has corank one (indeed, it is cofree – its topological dual is free of rank one over Λ_E). Moreover, $\mathcal{C}$ is injective in the abelian category of admissible G-representations and is therefore acyclic. If V is an admissible continuous G-representation, then there exists an exact sequence

$$0 \longrightarrow V \longrightarrow \mathcal{C}^n \longrightarrow W \longrightarrow 0$$

of admissible continuous G-representations for some integer $n \geq 0$. Since $\mathcal{C}$ is acyclic, from the long exact sequence of cohomology we obtain the following:

$$0 \longrightarrow V^{G_k} \longrightarrow (\mathcal{C}^{G_k})^n \longrightarrow W^{G_k} \longrightarrow H^1(G_k, V) \longrightarrow 0, \tag{2.1}$$

$$H^i(G_k, V) \simeq H^{i-1}(G_k, W), \ i \geq 2. \tag{2.2}$$

The lemma for $i = 1$ follows from a consideration of (2.1), taking into account Theorem 2.1 and the fact that corank of W is equal to n minus the corank of V (since corank is additive in short exact sequences). We now proceed by induction on i. Assume the result for $i \leq m$ and all admissible continuous representations, in particular for W. The result for $i = m+1$ then follows directly from the isomorphism (2.2). This completes the proof. □

3 Cohomology of Arithmetic Quotients of Symmetric Spaces

We now return to the situation considered in the introduction and use the notation introduced there. In particular, we fix a connected semisimple linear group $\mathbf{G}$ over F, an embedding $\mathbf{G} \hookrightarrow \mathrm{GL}_N$ over F, an arithmetic lattice Γ of the associated real group G_∞, and a prime $\mathfrak{p}$ of F.

If we write $G := \varprojlim_k \Gamma/\Gamma(\mathfrak{p}^k)$, then G is a compact open subgroup of the p-adic Lie group $\mathbf{G}(F_\mathfrak{p})$ (where $F_\mathfrak{p}$ denotes the completion of F at $\mathfrak{p}$); alternatively, we may define G to be the closure of Γ in $\mathbf{G}(F_\mathfrak{p})$. If we replace Γ by $\Gamma(\mathfrak{p}^k)$ for some sufficiently large value of k (i.e. discarding finitely many initial terms in the descending sequence of lattices $\Gamma(\mathfrak{p}^k)$), then G will be pro-p and, hence, will be an analytic pro-p-group. Note that Γ is a dense subgroup of G. Let e and f denote respectively the ramification and inertial indices of $\mathfrak{p}$ in F (so that $[F_\mathfrak{p} : \mathbf{Q}_p] = ef$). For each $k \geq 0$, write G_k to denote the closure of $\Gamma(\mathfrak{p}^{ek})$ in G. Alternatively, if we consider the embedding

$$\mathbf{G}(F_\mathfrak{p}) \hookrightarrow \mathrm{GL}_N(F_\mathfrak{p}) \hookrightarrow \mathrm{GL}_{efN}(\mathbf{Q}_p),$$

then $G_k = G \cap (1 + p^k M_{efN}(\mathbf{Z}_p))$; thus, our notation is compatible with that of the preceding section. We let d denote the dimension of G; note that $d = \frac{ef}{[F:\mathbf{Q}]} \cdot \dim(G_\infty)$.

For each $k \geq 0$, we write

$$Y_k := \Gamma(\mathfrak{p}^{ek})\backslash G_\infty / K_\infty.$$

There is a natural action of G on Y_k through its quotient G/G_k ($\cong \Gamma/\Gamma(\mathfrak{p}^{ek})$), which is compatible with the projections $Y_k \to Y_{k'}$ for $0 \leq k' \leq k$.

Fix a finite-dimensional representation W of $\mathbf{G}$ over E, and let W_0 denote a G-invariant $\mathcal{O}_E$-lattice in W. Let $\mathcal{V}_k$ denote the local system of free finite rank $\mathcal{O}_E$-modules on Y_k associated to W_0. If $0 \leq k' \leq k$, then the sheaf $\mathcal{V}_k$ on Y_k is naturally isomorphic to the pull-back of the sheaf $\mathcal{V}_{k'}$ on $Y_{k'}$ under the projection $Y_k \to Y_{k'}$. In particular the sheaf $\mathcal{V}_k$ is G/G_k-equivariant.

Recall the following definitions from from [4], p. 21:

$$\widetilde{H}^n(\mathcal{V}) := \varprojlim_{l} \varinjlim_{k} H^n(Y_k, \mathcal{V}_k/p^l), \qquad \widetilde{H}^n(\mathcal{V})_E := E \otimes_{\mathcal{O}_E} \widetilde{H}^n(\mathcal{V}).$$

Each $\widetilde{H}^n(\mathcal{V})$ is a p-adically complete $\mathcal{O}_E$-module, equipped with a left G-action in a natural way, and hence, each $\widetilde{H}^n(\mathcal{V})_E$ has a natural structure of E-Banach space and is equipped with a continuous left G-action. In fact, they are admissible continuous representations of G ([4], Thm. 2.1.5 (i)), and in particular, Theorem 2.1 and Lemma 2.2 apply to them. (Note that the results of [4] are stated in the adèlic language. We leave it to the reader to make the easy translation to the more classical language we are using in this paper.)

The following result, which is Theorem 2.1.5 (ii) of [4], p. 22, is a "control theorem" relating G_k invariants in $\widetilde{H}^n(\mathcal{V})_E^{G_k}$ to the classical cohomology classes $H^n(Y_k, \mathcal{V}_k) \otimes E$.

Theorem 3.1. *Fix an integer k. There is a spectral sequence*

$$E_2^{i,j}(Y_k) = H^i(G_k, \widetilde{H}^j(\mathcal{V})_E) \Longrightarrow H^{i+j}(Y_k, \mathcal{V}_k)_E.$$

One should view this spectral sequence as a version of the Hochschild-Serre spectral sequence "compatible in the G-tower."

Theorem 3.2. *For any $n \geq 0$, if r_n denotes the corank of $\widetilde{H}^n(\mathcal{V})$, then*

$$\dim_E H^n(Y_k, \mathcal{V}_k)_E = r_n \cdot c \cdot p^{dk} + O(p^{(d-1)k})$$

as $k \to \infty$. (Here c denotes the constant appearing in the statement of Theorem 2.1; it depends only on G.)

Proof. For each $i, j \geq 0$ and $l \geq 2$, let $E_l^{i,j}(Y_k)$ denote the terms in the spectral sequence of Theorem 3.1. Since $\widetilde{H}^j(\mathcal{V})$ is admissible, Lemma 2.2 implies that $\dim_E H^i(G_k, \widetilde{H}^j(\mathcal{V})_E) \ll p^{(d-1)k}$ if $i > 0$, and thus that $\dim_E E_l^{i,j}(Y_k) \ll p^{(d-1)k}$ as $k \to \infty$, if $i > 0$ (since $E_l^{i,j}(Y_k)$ is a subquotient of $E_2^{i,j}(Y_k) := H^i(G_k, \widetilde{H}^j(\mathcal{V})_E)$ for $l \geq 2$). Theorem 2.1 shows that

$$\dim_E E_2^{0,n}(Y_k) = \dim_E \widetilde{H}^n(\mathcal{V})_E^{G_k} = r_n \cdot c \cdot p^{dk} + O(p^{(d-1)k}).$$

On the other hand, since the spectral sequence of Theorem 3.1 is an upper right quadrant exact sequence, $E_\infty^{0,n}$ is obtained by taking finitely many successive kernels of differentials d_l to $E_l^{i+l,j-l+1}$, which all have order $\ll p^{(d-1)k}$ by the first part of our argument. Thus,

$$\dim_E E_\infty^{0,n} = r_n \cdot c \cdot p^{dk} + O(p^{(d-1)k}).$$

Since $H^n(Y_k, \mathcal{V}_k)_E$ admits a finite length filtration whose associated graded pieces are isomorphic to $E_\infty^{i,j}$ for $i + j = n$, we conclude that $\dim_E H^n(Y_k, \mathcal{V}_k)_E = r_n \cdot c \cdot p^{dk} + O(p^{(d-1)k})$, as claimed. □

The following lemma quantifies the precise relationship between multiplicities and the dimensions of cohomology groups that we will require to deduce Theorem 1.1 from Theorem 1.2.

Lemma 3.3. *Fix a cohomological degree n, and let S denote the set of isomorphism classes $[\pi]$ of $\pi \in \widehat{G}_\infty$ that contribute to cohomology with coefficients in $\mathcal{V}$ in degree n. Then*

$$\sum_{[\pi]\in S} m(\pi, G_k) \asymp \dim_E H^n_{\mathrm{cusp}}(Y_k, \mathcal{V}_k)_E.$$

Proof. Since the set S is finite, there is an integer $d \geq 1$ so that

$$1 \leq \dim H^n(\mathfrak{g}, \mathfrak{k}; \pi \otimes W) \leq d$$

for each isomorphism class $[\pi] \in S$. This implies that

$$\sum_{[\pi]\in S} m(\pi, G_k) \leq \dim_E H^n_{\mathrm{cusp}}(Y_k, \mathcal{V}_k)_E \leq d\big(\sum_{[\pi]\in S} m(\pi, G_k)\big)$$

for each $k \geq 0$. □

We can now prove our main result.

Theorem 3.4. *Let $n \geq 0$, and suppose either that G_∞ does not admit discrete series, or that $n \neq \frac{1}{2}\dim(G_\infty/K_\infty)$. Then*

$$\dim_E H^n(Y_k, \mathcal{V}_k)_E \ll p^{(d-1)k}$$

as $k \to \infty$, for all $n \geq 0$.

Proof. In the case when G_∞ admits discrete series, recall that these contribute to cohomology only in the dimension $\frac{1}{2}\dim(G_\infty/K_\infty)$ ([1], Thm. 5.1, p. 101). Thus, under the assumptions of the theorem, there is no contribution from the discrete series to $H^n_{\mathrm{cusp}}(Y_k, \mathcal{V}_k)$. The inequality (1.2) of [3] and [10], together with Lemma 3.3 and the main result of [8] (which states that $\dim_E\big(H^n(Y_k, \mathcal{V}_k)/H^n_{\mathrm{cusp}}(Y_k, \mathcal{V}_k)\big) = o(p^{dk})$), thus shows that $\dim_E H^n(Y_k, \mathcal{V}_k)_E = o(p^{dk})$ as $k \to \infty$, for all $n \geq 0$. From Theorem 3.2, we then infer that each $\widetilde{H}^n(\mathcal{V})$ has corank 0. Another application of the same theorem now gives our result. □

Note that $V(\mathfrak{p}^{ek}) \sim [G : G_k] \sim c \cdot p^{dk}$; thus Theorem 3.4 implies Theorem 1.2 since

$$\dim(G) \leq \dim(G_\infty). \tag{3.1}$$

Theorem 1.2 and Lemma 3.3 together imply Theorem 1.1.

Remark 3.5. We have equality in (3.1) precisely when $\mathfrak{p}$ is the unique prime lying over p in $\mathcal{O}_F$. If there is more than one prime lying over p, then $\dim(G)$ is strictly less than $\dim(G_\infty)$, and we obtain a corresponding improvement in the bounds of Theorems 1.1 and 1.2, namely (in the notation of their statements), that

$$m(\pi, \Gamma(\mathfrak{p}^k)) \quad \text{and} \quad \dim_E H^i(Y(\mathfrak{p}^k), \mathcal{V}_{W,k})_E \ \ll\ V(\mathfrak{p}^n)^{1-1/\dim(G)}$$

(where, as we noted above, $\dim(G) = \frac{ef}{[F:\mathbf{Q}]} \cdot \dim(G_\infty)$, with e and f being the ramification and inertial index of $\mathfrak{p}$ respectively).

Example/Question 3.6. Let $F/\mathbf{Q}$ be an imaginary quadratic field, and let $\mathbf{G} = \mathrm{SL}_{2/F}$. The corresponding symmetric space $G_\infty/K_\infty = \mathrm{SL}_2(\mathbf{C})/\mathrm{SU}(2)$ is a real hyperbolic three space $\mathcal{H}$, and the quotients Y are commensurable with the Bianchi manifolds $\mathcal{H}/\mathrm{PGL}_2(\mathcal{O}_K)$. Choose a local system $\mathcal{V}_0$ associated to some finite-dimensional representation W of $G_\infty = \mathrm{GL}_2(\mathbf{C})$ and a congruence subgroup Γ. Assume that $p = \mathfrak{p}\overline{\mathfrak{p}}$ splits in $\mathcal{O}_F$, and apply Theorem 3.4 to the $\mathfrak{p}$-power tower. We obtain the inequality

$$H^1_{\mathrm{cusp}}(Y_k, \mathcal{V}_k) \ll p^{2k}$$

as $k \to \infty$. It is natural to ask how tight this inequality is.

The main result of Calegari–Dunfield [2] shows that there exists at least one $(F, \Gamma, \mathfrak{p})$ for which

$$H^1_{\mathrm{cusp}}(Y_k, \mathbf{C}) = 0$$

for all k. On the other hand, if there exists at least one newform on $\Gamma(\mathfrak{p}^k)$ for some k, then a consideration of the associated oldforms shows that

$$H^1_{\mathrm{cusp}}(Y_k, \mathcal{V}_k) \gg p^k$$

as $k \to \infty$. Are there situations in which this lower bound gives the true rate of growth?

Remark 3.7. Our results are most interesting in the case when G_∞ does not admit any discrete series, since, as we noted in the introduction, in this case (and only in this case), G_∞ admits (non-discrete series) tempered representations of cohomological type.

On the other hand, Theorem 3.2 does have a consequence in the case when G_∞ admits discrete series which may be of some interest. Recall the following result from [10] (established in [3] in the cocompact case): if $\pi \in \widehat{G}_\infty$ lies in the discrete series, then

$$m(\pi, \Gamma(\mathfrak{p}^k)) = d(\pi)V(\mathfrak{p}^k) + o(V(\mathfrak{p}^k)) \tag{3.2}$$

as $k \to \infty$. Fix a finite-dimensional representation W of G_∞, and let $\widehat{G}_\infty(W)_d$ denote the subset of $\widehat{G}_\infty$ consisting of discrete series representations that contribute to cohomology with coefficients in W. Summing over all $\pi \in \widehat{G}_\infty(W)_d$, we obtain the formula

$$\frac{1}{|\widehat{G}_\infty(W)_d|} \sum_{\pi \in \widehat{G}_\infty(W)_d} m(\pi, \Gamma(\mathfrak{p}^k)) = d(\pi)V(\mathfrak{p}^k) + o(V(\mathfrak{p}^k)). \tag{3.3}$$

(A result first proved in [8].) The following result provides an improvement in the error term of (3.3).

Theorem 3.8. *There exists $\mu > 0$ such that*

$$\frac{1}{|\widehat{G}_\infty(W)_d|} \sum_{\pi \in \widehat{G}_\infty(W)_d} m(\pi, \Gamma(\mathfrak{p}^k)) = d(\pi)V(\mathfrak{p}^k) + O(V(\mathfrak{p}^k)^{1-\mu}).$$

Proof. Let $n = \dfrac{1}{2}\dim(G_\infty/K_\infty)$. As already noted, it follows from [1], Thm. 5.1, p. 101, that all non-discrete series contributions to $H^n_{\mathrm{cusp}}(Y_k, \mathcal{V}_k)$ are non-tempered. The same result shows that each discrete series has one-dimensional $(\mathfrak{g}, \mathfrak{k})$-cohomology in dimension n. As we recalled in the introduction, the multiplicity of any non-tempered representations is bounded by $V(\mathfrak{p}^k)^{1-\mu}$ for some $\mu > 0$ [9], and thus, Theorem 3.2 and (the proof of) Lemma 3.3 show that

$$\frac{1}{|\widehat{G}_\infty(W)_d|} \sum_{\pi \in \widehat{G}_\infty(W)_d} m(\pi, \Gamma(\mathfrak{p}^k)) = C \cdot V(\mathfrak{p}^k) + O(V(\mathfrak{p}^k)^{1-\mu}).$$

Comparing this formula with (3.3) yields the theorem. □

Question 3.9. Does the result of Theorem 3.8 hold term-by-term? That is, does (3.2) admit an improvement of the form

$$m(\pi, \Gamma(\mathfrak{p}^k)) \stackrel{?}{=} d(\pi) V(\mathfrak{p}^k) + O(V(\mathfrak{p}^k)^{1-\mu})$$

for some $\mu > 0$?

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