

# PATCHING AND THE $p$ -ADIC LOCAL LANGLANDS CORRESPONDENCE

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ABSTRACT. We use the patching method of Taylor–Wiles and Kisin to construct a candidate for the  $p$ -adic local Langlands correspondence for  $\mathrm{GL}_n(F)$ ,  $F$  a finite extension of  $\mathbb{Q}_p$ . We use our construction to prove many new cases of the Breuil–Schneider conjecture.

## CONTENTS

1. Introduction	1
2. The patching argument	7
3. Hecke algebras and types	19
4. Local-global compatibility	30
5. The Breuil–Schneider conjecture	42
6. Relationship with a hypothetical $p$ -adic local Langlands correspondence	45
References	46

## 1. INTRODUCTION

Our goal in this paper is to use global methods (specifically, the Taylor–Wiles–Kisin patching method) to construct a candidate for the  $p$ -adic local Langlands correspondence for  $\mathrm{GL}_n(F)$ , where  $F$  is an arbitrary finite extension of  $\mathbb{Q}_p$ , and  $p \nmid 2n$ . At present, the existence of such a correspondence is only known for  $\mathrm{GL}_1(F)$  (where it is given by local class field theory), and for  $\mathrm{GL}_2(\mathbb{Q}_p)$  (*cf.* [Col10, Paš13]). We do not prove that our construction gives a purely local correspondence (and it would perhaps be premature to conjecture that it should), but we are able to say enough about our construction to prove many new cases of the Breuil–Schneider conjecture, and to reduce the general case of the Breuil–Schneider conjecture (under some mild technical hypotheses) to standard conjectures related to automorphy lifting theorems.

The idea that global methods could be used to construct the correspondence is a natural one; the only proof at present of the classical local Langlands correspondence [HT01, Hen00] is by global means, and indeed the first proofs of local class

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field theory were global. The basic idea is to embed a local situation into a global one, apply a global correspondence (for example, the association of Galois representations to certain automorphic forms), and then to prove that the construction is independent of the choice of global situation. In this paper, we carry out the first half of this idea (although, in contrast to the constructions of [HT01], the direction of the correspondence we construct is from representations of the local Galois group  $G_F$  to representations of  $\mathrm{GL}_n(F)$ ). We intend to return to the second half (investigating the question of independence of the global situation) in subsequent work. (See Section 6 for a discussion of the relationship of our construction to conjectural extensions of the existing  $p$ -adic Langlands correspondence for  $\mathrm{GL}_2(\mathbb{Q}_p)$ .)

**1.1. The Breuil–Schneider conjecture.** Before giving an overview of our construction, we discuss an application of it to the Breuil–Schneider conjecture [BS07]; this conjecture predicts that locally algebraic representations of  $\mathrm{GL}_n(F)$  admit invariant norms (and thus nonzero completions to unitary Banach representations) if and only if they arise from regular de Rham Galois representations by applying (a generic version of) the classical local Langlands correspondence to the corresponding Weil–Deligne representations. (This conjecture is motivated by the case of  $\mathrm{GL}_2(\mathbb{Q}_p)$ , where it is an immediate consequence of known properties of the  $p$ -adic Langlands correspondence.) In one direction, Hu [Hu09] showed that if such a norm exists, the locally algebraic representation necessarily comes from a regular de Rham representation.

The converse direction is largely open. We recall the conjecture in more detail in Section 5 below, to which the reader should refer for any unfamiliar notation or terminology. Given a de Rham representation  $r : G_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$  of regular weight, in [BS07] there is associated to  $r$  a locally algebraic  $\overline{\mathbb{Q}}_p$ -representation  $\mathrm{BS}(r)$  of  $\mathrm{GL}_n(F)$ . The following is [BS07, Conjecture 4.3] (in the open direction).

**A. Conjecture.** *If  $r : G_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$  is de Rham and has regular weight, then  $\mathrm{BS}(r)$  admits a nonzero unitary Banach completion.*

In fact, the known properties of the  $p$ -adic local Langlands correspondence for  $\mathrm{GL}_2(\mathbb{Q}_p)$  suggest that there should even be a nonzero admissible completion. Conjecture A was proved in the case that  $\pi_{\mathrm{sm}}(r)$  is supercuspidal in [BS07, Theorem 5.2], and in the more general case that  $\mathrm{WD}(r)$  is indecomposable in [Sor13]. The argument of [Sor13] is global. It makes use of a strategy of one of us (M.E.) who observed that if  $r$  arises as the local Galois representation coming from an automorphic representation, then one can obtain an admissible completion from the completed cohomology of [Eme06b], *cf.* Proposition 4.6 of [Eme05] (and also [Sor12]). However, as there are only countably many automorphic representations, it is not possible to say anything about most principal series representations in this way; indeed, as was already remarked in [BS07] (see the discussion before Remark 5.7), the principal series case seems to be the deepest case of the conjecture.

Other than for  $\mathrm{GL}_2(\mathbb{Q}_p)$ , the only previous results in the general principal series case that we are aware of are those of [AKdS13, Ies12, KdS12, Vig08], which prove the conjecture for certain principal series cases for  $\mathrm{GL}_2(F)$ , under additional restrictions on the Hodge filtration of  $r$ . The methods of these papers do not seem to shed any light on the stronger question of the existence of admissible completions. Under the assumption that  $p \nmid 2n$ , which we make from now on, the construction explained below associates an admissible unitary Banach representation  $V(r)$  to

any continuous representation  $r : G_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ . In order to prove Conjecture A, it would be enough to establish that  $V(r)$  contains a copy of  $\mathrm{BS}(r)$  when  $r$  is de Rham of regular weight. We expect this to be true in general, and we are able to show that it is equivalent to proving a certain automorphy lifting theorem. The following is our main result in this direction. (See sections 2 and 5 for any unfamiliar terminology; note in particular that the hypothesis that  $r$  lies on an automorphic component does not imply that  $r$  arises from the Galois representation associated to an automorphic representation, but is rather the much weaker condition that it lies on the same component of a local deformation ring as some such representation. It is a folklore conjecture (closely related to the problem of deducing the Fontaine–Mazur conjecture from generalisations of Serre’s conjecture via automorphy lifting theorems) that every de Rham representation of regular weight satisfies this condition.)

**B. Theorem** (Theorem 5.3 and Remark 4.19). *Suppose that  $p \nmid 2n$ , that  $r : G_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$  is potentially crystalline of regular weight, and that  $r$  is generic. Suppose that  $r$  lies on an automorphic component of the corresponding potentially crystalline deformation ring. Then  $\mathrm{BS}(r)$  admits a nonzero unitary admissible Banach completion.*

By taking known automorphy lifting theorems, in particular those of [BLGGT13], we are able to deduce new cases of Conjecture A. In particular, we deduce the following result (Corollary 5.5).

**C. Theorem.** *Suppose that  $p > 2$ , that  $r : G_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$  is de Rham of regular weight, and that  $r$  is generic. Suppose further that either*

- (1)  $n = 2$ , and  $r$  is potentially Barsotti–Tate, or
- (2)  $F/\mathbb{Q}_p$  is unramified and  $r$  is crystalline with Hodge–Tate weights in the extended Fontaine–Laffaille range, and  $n \neq p$ .

*Then  $\mathrm{BS}(r)$  admits a nonzero unitary admissible Banach completion.*

Actually, we prove a more general result (Corollary 5.4) which establishes the conjecture for potentially diagonalizable representations; conjecturally, every potentially crystalline representation is potentially diagonalizable. We remark that while we expect these results to extend to potentially semistable (rather than just potentially crystalline) representations, and to non-generic representations, we have restricted to the potentially crystalline case for two reasons: we can use the main theorems of [BLGGT13] without modification, and we do not have to consider issues related to the possible reducibility of  $\mathrm{BS}(r)$ .

**1.2. An overview of our construction.** In the proof of the classical local Langlands correspondence [HT01, Hen00], the globalisation argument uses a reduction to the supercuspidal case (via the classification of irreducible smooth representations of  $\mathrm{GL}_n(F)$  given in [BZ77, Zel80]), and then uses trace formula methods to realise supercuspidal representations as the local components of cuspidal automorphic representations. No such argument is possible in our setting; there is at present no analogue of the theory of [BZ77, Zel80] in the setting of admissible  $p$ -adic Banach space representations, and even in the supercuspidal case, there are uncountably many  $p$ -adic Galois representations up to twist, but only countably many automorphic representations.

It is natural to hope that in the  $p$ -adic setting, one could carry out an analogous globalisation using “ $p$ -adic automorphic representations”, such as those arising from the completed cohomology of [Eme06b]. However, since the locally algebraic vectors in completed cohomology are computed by classical automorphic representations, one cannot expect to see any regular de Rham Galois representations in completed cohomology other than those arising from classical automorphic representations.

The globalisation argument in the proof of classical local Langlands is effectively a result showing the Zariski-density of automorphic points in the Bernstein spectrum; the analogous result for  $p$ -adic local Langlands (or rather, for the part of it pertaining to regular de Rham representations) would be a Zariski-density result for automorphic points in the corresponding local Galois deformation rings. This is not known in general, but strong results in this direction follow from the Taylor–Wiles–Kisin patching method, which provides a Zariski-density result for a non-empty collection of components of a local deformation ring (and in general shows that each component either contains no automorphic points, or a Zariski-dense set of points; as mentioned above, the problem of showing that each component contains an automorphic point is closely related to the problem of deducing the Fontaine–Mazur conjecture from generalisations of Serre’s conjecture, *cf.* Remark 5.5.3 of [EG13]).

The Taylor–Wiles–Kisin method patches together spaces of automorphic forms with varying tame level. Usually, the weight and the  $p$ -part of the level of these forms is fixed, and one obtains a patched module for a certain universal local deformation ring corresponding to de Rham representations of fixed Hodge–Tate weights, and a fixed inertial type. Instead, we vary, in addition, over all weights and levels at  $p$ , obtaining a module over the unrestricted local deformation ring. By construction, this module naturally has an action of  $\mathrm{GL}_n(\mathcal{O}_F)$ ; by keeping track of the action of the Hecke operators at  $p$ , we are able to promote this to an action of  $\mathrm{GL}_n(F)$ . Taking the fiber of this patched module at the point corresponding to a particular Galois representation  $r$ , and inverting  $p$ , gives the unitary admissible Banach representation  $V(r)$  that we seek. The condition that  $p \nmid 2n$  is needed to employ the Taylor–Wiles–Kisin method (for example, this condition is necessary in order to be able to appeal to various results from [BLGGT13]), but we suspect that it is not ultimately needed to carry out variants of these constructions.

We do not know whether it is reasonable to expect that our construction is purely local, and thus defines a  $p$ -adic local Langlands correspondence; this amounts to the problem of showing that the patched modules that we construct are purely local objects. For some weak evidence in this direction, see [EGS13], which proves a related result for lattices corresponding to certain 2-dimensional tamely potentially Barsotti–Tate representations. It can also be shown that our construction recovers the known correspondence for  $\mathrm{GL}_2(\mathbb{Q}_p)$ ; we will report on this in a future paper. We refer to Section 6 below for a slightly more detailed discussion of how the construction of this paper might relate to a hypothetical  $p$ -adic local Langlands correspondence.

**1.3. Outline of the paper.** In Section 2, we carry out our patching construction. Section 3 contains an introduction to the results of Bernstein–Zelevinsky, Bushnell–Kutzko, Schneider–Zink and Dat on types and the local Langlands correspondence for  $\mathrm{GL}_n$ . We then prove some refinements of some of these results, and explain how to descend them from algebraically closed coefficient fields to finite extensions of  $\mathbb{Q}_p$ . In Section 4 we begin by proving a result of independent interest, showing

that the classical local Langlands correspondence interpolates over a (potentially semistable) local deformation ring, and then apply this to show local-global compatibility for our patched modules. Finally, in Section 5 we combine this local-global compatibility result with automorphy lifting theorems to prove our results on the Breuil–Schneider conjecture.

**1.4. Acknowledgements.** The constructions made in this paper originate in work carried out at a focused research group on “The  $p$ -adic Langlands program for non-split groups” at the Banff Centre in August 2012; we would like to thank BIRS for providing an excellent working atmosphere, and for its financial support. We would also like to thank AIM for providing financial support and an excellent working atmosphere towards the end of this project. The idea that patching could be used to prove the Breuil–Schneider conjecture goes back to conversations between two of us (M.E. and T.G.) at the Harvard Eigen-semester in 2006, and we would like to thank the mathematics department of Harvard University for its hospitality. Finally, we thank Brian Conrad and Florian Herzig for helpful remarks on an earlier draft of this paper.

**1.5. Notation.** We fix a prime  $p$ , and an algebraic closure  $\overline{\mathbb{Q}}_p$  of  $\mathbb{Q}_p$ . Throughout the paper we work with a finite extension  $E/\mathbb{Q}_p$  in  $\overline{\mathbb{Q}}_p$ , which will be our coefficient field. We write  $\mathcal{O} = \mathcal{O}_E$  for the ring of integers in  $E$ ,  $\varpi = \varpi_E$  for a uniformiser, and  $\mathbb{F} := \mathcal{O}/\varpi$  for the residue field. At any particular moment  $E$  is fixed, but we allow ourselves to modify  $E$  (typically via an extension of scalars) during the course of our arguments. Furthermore, we will often assume without further comment that  $E$  and  $\mathbb{F}$  are sufficiently large, and in particular that if we are working with representations of the absolute Galois group of a  $p$ -adic field  $F$ , then the images of all embeddings  $F \hookrightarrow \overline{E}$  are contained in  $E$ .

If  $F$  is a field, we let  $G_F$  denote its absolute Galois group. Let  $\varepsilon$  denote the  $p$ -adic cyclotomic character, and  $\overline{\varepsilon}$  the mod  $p$  cyclotomic character. If  $F$  is a finite extension of  $\mathbb{Q}_p$  for some  $p$ , we write  $I_F$  for the inertia subgroup of  $G_F$ , and  $\varpi_F$  for a uniformiser of the ring of integers  $\mathcal{O}_F$  of  $F$ . If  $\tilde{F}$  is a number field and  $v$  is a finite place of  $\tilde{F}$  then we let  $\text{Frob}_v$  denote a geometric Frobenius element of  $G_{\tilde{F}_v}$ .

If  $F$  is a  $p$ -adic field,  $W$  is a de Rham representation of  $G_F$  over  $E$ , and  $\kappa : F \hookrightarrow E$ , then we will write  $\text{HT}_\kappa(W)$  for the multiset of Hodge–Tate weights of  $W$  with respect to  $\kappa$ . By definition, the multiset  $\text{HT}_\kappa(W)$  contains  $i$  with multiplicity  $\dim_E(W \otimes_{\kappa, F} \widehat{F}(i))^{G_F}$ . Thus for example  $\text{HT}_\kappa(\varepsilon) = \{-1\}$ .

We say that  $W$  has *regular* Hodge–Tate weights if for each  $\kappa$ , the elements of  $\text{HT}_\kappa(W)$  are pairwise distinct. Let  $\mathbb{Z}_+^n$  denote the set of tuples  $(\xi_1, \dots, \xi_n)$  of integers with  $\xi_1 \geq \xi_2 \geq \dots \geq \xi_n$ . Then if  $W$  has regular Hodge–Tate weights, there is a  $\xi = (\xi_{\kappa, i}) \in (\mathbb{Z}_+^n)^{\text{Hom}_{\mathbb{Q}_p}(F, E)}$  such that for each  $\kappa : F \hookrightarrow E$ ,

$$\text{HT}_\kappa(W) = \{\xi_{\kappa, 1} + n - 1, \xi_{\kappa, 2} + n - 2, \dots, \xi_{\kappa, n}\},$$

and we say that  $W$  is *regular of weight*  $\xi$ . For any  $\xi \in \mathbb{Z}_+^n$ , view  $\xi$  as a dominant weight (with respect to the upper triangular Borel subgroup) of the algebraic group  $\text{GL}_n$  in the usual way, and let  $M'_\xi$  be the algebraic  $\mathcal{O}_F$ -representation of  $\text{GL}_n$  given by

$$M'_\xi := \text{Ind}_{B_n}^{\text{GL}_n}(w_0\xi)_{/\mathcal{O}_F}$$

where  $B_n$  is the Borel subgroup of upper-triangular matrices of  $\text{GL}_n$ , and  $w_0$  is the longest element of the Weyl group (see [Jan03] for more details of these notions,

and note that  $M'_\xi$  has highest weight  $\xi$ ). Write  $M_\xi$  for the  $\mathcal{O}_F$ -representation of  $\mathrm{GL}_n(\mathcal{O}_F)$  obtained by evaluating  $M'_\xi$  on  $\mathcal{O}_F$ . For any  $\xi \in (\mathbb{Z}_+^n)^{\mathrm{Hom}_{\mathbb{Q}_p}(F, E)}$  we write  $L_\xi$  for the  $\mathcal{O}$ -representation of  $\mathrm{GL}_n(\mathcal{O}_F)$  defined by

$$L_\xi := \otimes_{\kappa: F \hookrightarrow E} M_{\xi_\kappa} \otimes_{\mathcal{O}_F, \kappa} \mathcal{O}.$$

If  $F$  is a  $p$ -adic field, then an *inertial type* is a representation  $\tau : I_F \rightarrow \mathrm{GL}_n(\overline{\mathbb{Q}_p})$  with open kernel which extends to the Weil group  $W_F$ .

Then we say that a de Rham representation  $\rho : G_F \rightarrow \mathrm{GL}_n(E)$  has inertial type  $\tau$  if the restriction to  $I_F$  of the Weil–Deligne representation  $\mathrm{WD}(\rho)$  associated to  $\rho$  is equivalent to  $\tau$ . Given an inertial type  $\tau$ , there is a finite-dimensional smooth irreducible  $\overline{\mathbb{Q}_p}$ -representation  $\sigma(\tau)$  of  $\mathrm{GL}_n(\mathcal{O}_F)$  associated to  $\tau$  by the “inertial local Langlands correspondence”; see Theorem 3.7 below. (Note that by the results of Section 3.12 below, we will be able to replace  $\sigma(\tau)$  by a model defined over a finite extension of  $E$  in our main arguments.)

Let  $F$  be a finite extension of  $\mathbb{Q}_p$ , and let  $\mathrm{rec}$  denote the local Langlands correspondence from isomorphism classes of irreducible smooth representations of  $\mathrm{GL}_n(F)$  over  $\mathbb{C}$  to isomorphism classes of  $n$ -dimensional Frobenius semisimple Weil–Deligne representations of  $W_F$  as in the introduction to [HT01]. Fix once and for all an isomorphism  $\iota : \overline{\mathbb{Q}_p} \xrightarrow{\sim} \mathbb{C}$ . We define the local Langlands correspondence  $\mathrm{rec}_p$  over  $\overline{\mathbb{Q}_p}$  by  $\iota \circ \mathrm{rec}_p = \mathrm{rec} \circ \iota$ . This depends only on  $\iota^{-1}(\sqrt{p})$ , and if we define  $r_p(\pi) := \mathrm{rec}_p(\pi \otimes |\det|^{(1-n)/2})$ , then  $r_p$  is independent of the choice of  $\iota$ . Furthermore, if  $V$  is a Frobenius semisimple Weil–Deligne representation of  $W_F$  over  $E$ , then  $r_p^{-1}(V)$  is also defined over  $E$  by [Clo90, Prop 3.2] and the fact that  $r_p$  commutes with automorphisms of  $\mathbb{C}$ , cf. Section 4 of [BS07].

If  $A$  is a linear topological  $\mathcal{O}$ -module, we write  $A^\vee$  for its Pontrjagin dual  $\mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(A, E/\mathcal{O})$ , where  $E/\mathcal{O}$  has the discrete topology, and we give  $A^\vee$  the compact open topology.

By the proof of Theorem 1.2 of [ST02], the functor  $A \mapsto A^d := \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(A, \mathcal{O})$  induces an anti-equivalence of categories between the category of compact,  $\mathcal{O}$ -torsion-free linear-topological  $\mathcal{O}$ -modules  $A$  and the category of  $\varpi$ -adically complete and separated  $\mathcal{O}$ -torsion-free  $\mathcal{O}$ -modules. A quasi-inverse is given by  $B \mapsto B^d := \mathrm{Hom}_{\mathcal{O}}(B, \mathcal{O})$ , where the target is given the weak topology of pointwise convergence. We refer to this duality as Schikhof duality. Note that if  $A$  is an  $\mathcal{O}$ -torsion free profinite linear-topological  $\mathcal{O}$ -module, then  $A^d$  is the unit ball in the  $E$ -Banach space  $\mathrm{Hom}_{\mathcal{O}}(A, E)$ .

If  $r : G_F \rightarrow \mathrm{GL}_n(E)$  is de Rham of regular weight  $a$ , then we write  $\pi_{\mathrm{alg}}(r) := L_a^d \otimes_{\mathcal{O}} E$ , and  $\pi_{\mathrm{sm}}(r) := r_p^{-1}(\mathrm{WD}(r)^{F\text{-ss}})$ , both of which are  $E$ -representations of  $\mathrm{GL}_n(F)$ . (The  $\mathrm{GL}_n(\mathcal{O}_F)$ -action on  $L_a^d$  extends linearly to a  $\mathrm{GL}_n(F)$ -action on  $\pi_{\mathrm{alg}}(r)$ .) As the names suggest,  $\pi_{\mathrm{alg}}(r)$  is an algebraic representation, and  $\pi_{\mathrm{sm}}(r)$  is a smooth representation. Note that  $\pi_{\mathrm{alg}}(r) = L_\xi \otimes_{\mathcal{O}} E$  for  $\xi_{\kappa, i} := -a_{\kappa, n+1-i}$ .

We let  $\mathrm{Art}_F : F^\times \xrightarrow{\sim} W_F^{\mathrm{ab}}$  be the isomorphism provided by local class field theory, which we normalise so that uniformisers correspond to geometric Frobenius elements.

We write all matrix transposes on the left; so  ${}^t g$  is the transpose of  $g$ . We let  $\mathcal{G}_n$  denote the group scheme over  $\mathbb{Z}$  defined to be the semidirect product of  $\mathrm{GL}_n \times \mathrm{GL}_1$  by the group  $\{1, j\}$ , which acts on  $\mathrm{GL}_n \times \mathrm{GL}_1$  by

$$j(g, \mu)j^{-1} = (\mu \cdot {}^t g^{-1}, \mu).$$

We have a homomorphism  $\nu : \mathcal{G}_n \rightarrow \mathrm{GL}_1$ , sending  $(g, \mu)$  to  $\mu$  and  $j$  to  $-1$ .

Further notation is introduced in the course of our arguments; we mention just some of it here, for the reader's convenience.

From Subsection 2.6 on, we will have fixed a particular finite extension  $F$  of  $\mathbb{Q}_p$ , with ring of integers  $\mathcal{O}_F$  and uniformiser  $\varpi_F$ . To ease notation we will typically write  $G := \mathrm{GL}_n(F)$ ,  $K := \mathrm{GL}_n(\mathcal{O}_F)$ , and  $Z := Z(G)$ . For each  $m \geq 0$ , we write  $\Gamma_m = \mathrm{GL}_n(\mathcal{O}_F/\varpi_F^m)$  and  $K_m := \ker(\mathrm{GL}_n(\mathcal{O}_F) \rightarrow \mathrm{GL}_n(\mathcal{O}_F/\varpi_F^m))$ , so that  $K/K_m \xrightarrow{\sim} \Gamma_m$ .

Furthermore, throughout Section 2, a large amount of notation is introduced related to automorphic forms on a definite unitary group, and Taylor–Wiles–Kisin patching. Here we merely signal that the main construction of this section, and our major object of study in the paper, is a patched  $G$ -representation that we denote by  $M_\infty$ . Beginning in Section 4, we will also write

$$M_\infty(\sigma^\circ) := \left( \mathrm{Hom}_{\mathcal{O}[[K]]}^{\mathrm{cont}}(M_\infty, (\sigma^\circ)^d) \right)^d,$$

when  $\sigma^\circ$  is a  $K$ -invariant  $\mathcal{O}_E$ -lattice in a  $K$ -representation  $\sigma$  of finite dimension over  $E$ .

We use  $\mathrm{Ind}$  to denote induction, and  $\mathrm{c}\text{-Ind}$  to denote induction with compact supports. In Sections 3 and 4, we use  $i_P^G$  to denote normalised parabolic induction.

If  $\sigma$  is a representation of  $K$ , then we will write  $\mathcal{H}(\sigma) := \mathrm{End}(\mathrm{c}\text{-Ind}_K^G \sigma)$  to denote the Hecke algebra of  $G$  with respect to  $\sigma$ . Sometimes, when it is helpful to emphasise the role of  $G$ , we will write  $\mathcal{H}(G, \sigma)$  instead. We also use obvious variants with  $G$  and  $K$  replaced by another  $p$ -adic group and compact open subgroup.

## 2. THE PATCHING ARGUMENT

In this section we will carry out our patching argument on definite unitary groups. The key difference between the construction presented here and previous patching constructions is that the object we end up with is not simply a module over a certain Galois deformation ring, but rather a  $\mathrm{GL}_n(F)$ -representation over that ring; we refer to the discussion at the beginning of Subsection 2.6 below for a more detailed discussion of this difference.

Our construction uses the same general framework as in section 5 of [EG13] (which in turn is based on the approach taken in [CHT08], [BLGG11] and [Tho12]); we recall the key elements of this framework in the first several subsections that follow. We follow the notation of [EG13] as closely as possible, and we indicate explicitly where we deviate from it.

The construction itself is the subject of Subsection 2.6, and the key fact that it actually produces a  $\mathrm{GL}_n(F)$ -representation is verified in Proposition 2.8. In Subsection 2.10 we explain how our patched  $\mathrm{GL}_n(F)$ -representation gives rise to admissible unitary Banach representations attached to local Galois representations.

**2.1. Globalisation.** Let  $F/\mathbb{Q}_p$  be a finite extension, and fix a continuous representation  $\bar{r} : G_F \rightarrow \mathrm{GL}_n(\mathbb{F})$ . Our goal in this subsection is to give a criterion for  $\bar{r}$  to be obtained as the restriction of a global Galois representation that is automorphic, in a suitable sense (and satisfies some additional convenient properties).

We will assume that the following hypotheses are satisfied:

- $p \nmid 2n$ , and

- $\bar{r}$  admits a potentially crystalline lift of regular weight, which is potentially diagonalizable in the sense of [BLGGT13].

Conjecturally, the second hypothesis is always satisfied; this is Conjecture A.3 of [EG13]. In this direction, we note the following result.

**2.2. Lemma.** *After possibly making a finite extension of scalars, the second hypothesis is satisfied if either  $n = 2$  or  $\bar{r}$  is semisimple.*

*Proof.* If  $n = 2$ , this is Remark A.4 of [EG13]. If  $\bar{r}$  is semisimple, then after extending scalars, we may write it as a sum of inductions of characters, and it is easy to see that by lifting these characters to crystalline characters, we can find a potentially crystalline lift which has regular weight, and is a sum of inductions of characters. Such a lift is obviously potentially diagonalizable (indeed, after restriction to some finite extension, it is a sum of crystalline characters).  $\square$

Having assumed these hypotheses, Corollary A.7 of [EG13] (with  $K$  our  $F$ , and  $F$  our  $\tilde{F}$ ) provides us with an imaginary CM field  $\tilde{F}$  with maximal totally real subfield  $\tilde{F}^+$ , and a continuous irreducible representation  $\bar{\rho} : G_{\tilde{F}^+} \rightarrow \mathcal{G}_n(\mathbb{F})$  such that  $\bar{\rho}$  is a suitable globalisation of  $\bar{r}$  in the sense of Section 5.1 of [EG13]. In particular,  $\bar{\rho}$  is automorphic, it is unramified outside of  $p$ , and it globalises  $\bar{r}$  in the sense that:

- each place  $v \mid p$  of  $\tilde{F}^+$  splits in  $\tilde{F}$ , and has  $\tilde{F}_v^+ \cong F$ ; we fix a choice of such isomorphisms.
- For each place  $v \mid p$  of  $\tilde{F}^+$ , there is a place  $\tilde{v}$  of  $\tilde{F}$  lying over  $v$  with  $\bar{\rho}|_{G_{\tilde{F}_{\tilde{v}}}}$  isomorphic to  $\bar{r}$ .

In fact, in order to arrange that our patched modules have a certain multiplicity one property, we will also demand that:

- $\bar{\rho}(G_{\tilde{F}}) = \mathrm{GL}_n(\mathbb{F}')$  for some subfield  $\mathbb{F}' \subseteq \mathbb{F}$  with  $\#\mathbb{F}' > 3n$ , and
- $\tilde{F}^{\overline{\ker \mathrm{ad} \bar{\rho}|_{G_{\tilde{F}}}}}$  does not contain  $\tilde{F}(\zeta_p)$ .

The second of these properties is already part of the definition of a suitable globalisation. To see that we can arrange that the first property holds, note that the proof of Proposition A.2 of [EG13] allows us to arrange that  $\bar{\rho}(G_{\tilde{F}}) = \mathrm{GL}_n(\mathbb{F}_{p^m})$  for any sufficiently large  $m$ .

**2.3. Unitary groups.** We now use the globalisation  $\bar{\rho}$  of our local Galois representation  $\bar{r}$  to carry out the Taylor–Wiles–Kisin patching argument as in Section 5 of [EG13]. The definitions of Hecke algebras, the choices of auxiliary primes and so on are essentially identical to the arguments made in [EG13], and rather than repeating them verbatim, we often refer the reader to [EG13] for the details of these definitions, indicating only the differences in our construction.

As in Sections 5.2 and 5.3 of [EG13], we fix a certain definite unitary group  $\tilde{G}/\tilde{F}^+$  together with a model (which we will also denote by  $\tilde{G}$ ) over  $\mathcal{O}_{\tilde{F}^+}$ . (The group  $\tilde{G}$  is denoted  $G$  in [EG13], but we will later use  $G$  to denote  $\mathrm{GL}_n(F)$ .) This model has the property that for each place  $v$  of  $\tilde{F}^+$  which splits as  $uw^c$  in  $\tilde{F}$ , there is an isomorphism  $\iota_w : \tilde{G}(\mathcal{O}_{\tilde{F}_v^+}) \xrightarrow{\sim} \mathrm{GL}_n(\mathcal{O}_{\tilde{F}_w})$ ; we fix a choice of such isomorphisms. We also choose a finite place  $v_1$  of  $\tilde{F}^+$  which is prime to  $p$ , with the properties that

- $v_1$  splits in  $\tilde{F}$ , say as  $v_1 = \tilde{v}_1 \tilde{v}_1^c$ ,
- $v_1$  does not split completely in  $\tilde{F}(\zeta_p)$ , and

- $\bar{\rho}(\text{Frob}_{\tilde{F}_{v_1}})$  has distinct  $\mathbb{F}$ -rational eigenvalues, no two of which have ratio  $(\mathbf{N}v_1)^{\pm 1}$ .

(It is possible to find such a place  $v_1$  by our assumptions that  $\bar{\rho}(G_{\tilde{F}}) = \text{GL}_n(\mathbb{F}')$  with  $\#\mathbb{F}' > 3n$ , and that  $\overline{\ker \text{ad } \bar{\rho}|_{G_{\tilde{F}}}}$  does not contain  $\tilde{F}(\zeta_p)$ . Note that this differs slightly from the choice of place  $v_1$  in the first paragraph of Section 5.3 of [EG13], but that it is still the case that any deformation of  $\bar{\rho}|_{G_{\tilde{F}_{v_1}}}$  is unramified. We have made this choice in order to be able to arrange that our patched modules satisfy multiplicity one.)

Let  $S_p$  denote the set of primes of  $\tilde{F}^+$  dividing  $p$ . We now fix a place  $\mathfrak{p} \mid p$  of  $\tilde{F}^+$ , and for each integer  $m \geq 0$  we consider the compact open subgroup  $U_m = \prod_v U_{m,v}$  of  $\tilde{G}(\mathbb{A}_{\tilde{F}^+}^\infty)$ , where

- $U_{m,v} = \tilde{G}(\mathcal{O}_{\tilde{F}_v^+})$  for all  $v$  which split in  $\tilde{F}$  other than  $v_1$  and  $\mathfrak{p}$ ;
- $U_{m,v_1}$  is the preimage of the upper triangular matrices under

$$\tilde{G}(\mathcal{O}_{\tilde{F}_{v_1}^+}) \rightarrow \tilde{G}(k_{v_1}) \xrightarrow{\iota_{v_1}} \text{GL}_n(k_{v_1})$$

- $U_{m,\mathfrak{p}}$  is the kernel of the map  $\tilde{G}(\mathcal{O}_{\tilde{F}_{\mathfrak{p}}^+}) \rightarrow \tilde{G}(\mathcal{O}_{\tilde{F}_{\mathfrak{p}}^+}/\varpi_{\tilde{F}_{\mathfrak{p}}^+}^m)$ ;
- $U_{m,v}$  is a hyperspecial maximal compact subgroup of  $\tilde{G}(\tilde{F}_v^+)$  if  $v$  is inert in  $\tilde{F}$ .

By the choice of  $v_1$  and  $U_{m,v_1}$  we see that  $U_m$  is sufficiently small. Write  $U := U_0$ . In order to make the patching argument, we will need to consider certain compact open subgroups of the  $U_m$  corresponding to choices of sets of auxiliary primes  $Q_N$  that will be introduced in Section 2.5. Specifically, for each integer  $N \geq 1$ , we will have a finite set of primes  $Q_N$  of  $\tilde{F}^+$  disjoint from  $S_p \cup \{v_1\}$  as well as open compact subgroups  $U_i(Q_N)_v$  of  $\tilde{G}(\mathcal{O}_{\tilde{F}_v^+})$  for each  $v \in Q_N$  and  $i = 0, 1$ . We then define subgroups  $U_i(Q_N)_m = \prod_v U_i(Q_N)_{m,v} \subset U_m$ , for  $i = 0, 1$  by setting  $U_0(Q_N)_{m,v} = U_1(Q_N)_{m,v} = U_{m,v}$  for  $v \notin Q_N$ , and  $U_i(Q_N)_{m,v} = U_i(Q_N)_v$  for  $v \in Q_N$ .

By assumption,  $\bar{r}$  has a potentially diagonalizable lift of regular weight, say  $r_{\text{pot.diag}} : G_F \rightarrow \text{GL}_n(\mathcal{O})$ . Suppose that  $r_{\text{pot.diag}}$  has weight  $\xi$  and inertial type  $\tau$  (in the sense of Section 1.5). Extending  $E$  if necessary, we may assume that the  $\text{GL}_n(\mathcal{O}_F)$ -representation  $\sigma(\tau)$  is defined over  $E$ . Then we have two representations  $L_\xi$  and  $L_\tau$  of  $\text{GL}_n(\mathcal{O}_F)$  on finite free  $\mathcal{O}$ -modules in the following way: the representation  $L_\xi$  is the one defined in the notation section, and  $L_\tau$  is a choice of  $\text{GL}_n(\mathcal{O}_F)$ -stable lattice in  $\sigma(\tau)$ . Set  $L_{\xi,\tau} := L_\tau \otimes_{\mathcal{O}} L_\xi$ , a finite free  $\mathcal{O}$ -module with an action of  $\text{GL}_n(\mathcal{O}_F)$ .

Returning to our global situation, let  $W_{\xi,\tau}$  denote the finite free  $\mathcal{O}$ -module with an action of  $\prod_{v \in S_p \setminus \{\mathfrak{p}\}} U_{m,v}$  given by  $W_{\xi,\tau} = \otimes_{v \in S_p \setminus \{\mathfrak{p}\}, \mathcal{O}} L_{\xi,\tau}$  where  $U_{m,v}$  acts on the factor corresponding to  $v$  via  $U_{m,v} = \tilde{G}(\mathcal{O}_{\tilde{F}_v^+}) \xrightarrow{\iota_v} \text{GL}_n(\mathcal{O}_{\tilde{F}_v^+}) \xrightarrow{\sim} \text{GL}_n(\mathcal{O}_F)$ . In order to avoid duplication of definition, we allow  $Q_N = \emptyset$  in the definitions we now make. For any finite  $\mathcal{O}$ -module  $V$  with a continuous action of  $U_{m,\mathfrak{p}}$ , we have spaces of algebraic modular forms  $S_{\xi,\tau}(U_i(Q_N)_m, V)$ ; these are just the functions

$$f : \tilde{G}(\tilde{F}^+) \backslash \tilde{G}(\mathbb{A}_{\tilde{F}^+}^\infty) \rightarrow W_{\xi,\tau} \otimes_{\mathcal{O}} V$$

with the property that if  $g \in \widetilde{G}(\mathbb{A}_{\widetilde{F}^+}^\infty)$  and  $u \in U_i(Q_N)_m$  then  $f(gu) = u^{-1}f(g)$ , where  $U_i(Q_N)_m$  acts on  $W_{\xi,\tau} \otimes_{\mathcal{O}} V$  via projection to  $\prod_{v \in S_p} U_{m,v}$ . (For example: when  $V = \mathcal{O}$  is the trivial representation, then after extending scalars from  $\mathcal{O}$  to  $\mathbb{C}$  via  $\mathcal{O} \subset \overline{\mathbb{Q}}_p \xrightarrow{\iota} \mathbb{C}$ , this space corresponds to classical automorphic forms of fixed type  $\sigma(\tau)$  at the places in  $S_p \setminus \{\mathfrak{p}\}$ , full level  $\mathfrak{p}^m$  at  $\mathfrak{p}$ , and whose weight (via our fixed isomorphism  $\iota : \overline{\mathbb{Q}}_p \rightarrow \mathbb{C}$ ) is 0 at places above  $\mathfrak{p}$ , and given by  $\xi$  at each of the places in  $S_p \setminus \{\mathfrak{p}\}$ .)

We let  $\mathbb{T}^{S_p \cup Q_N, \text{univ}}$  be the commutative  $\mathcal{O}$ -polynomial algebra generated by formal variables  $T_w^{(j)}$  for all  $1 \leq j \leq n$ ,  $w$  a place of  $\widetilde{F}$  lying over a place  $v$  of  $\widetilde{F}^+$  which splits in  $\widetilde{F}$  and is not contained in  $S_p \cup Q_N \cup \{v_1\}$ , together with formal variables  $T_{\widetilde{v}_1}^{(j)}$  for  $1 \leq j \leq n$ . The algebra  $\mathbb{T}^{S_p \cup Q_N, \text{univ}}$  acts on  $S_{\xi,\tau}(U_i(Q_N)_m, V)$  via the Hecke operators

$$T_w^{(j)} := \left[ U_{m,w} \iota_w^{-1} \begin{pmatrix} \varpi_w 1_j & 0 \\ 0 & 1_{n-j} \end{pmatrix} U_{m,w} \right]$$

where  $\varpi_w$  is a fixed uniformiser in  $\mathcal{O}_{\widetilde{F}_w}$ .

Choose an ordering  $\delta_1, \dots, \delta_n$  of the (distinct) eigenvalues of  $\bar{\rho}(\text{Frob}_{\widetilde{v}_1})$ . Since  $\bar{\rho}$  is a suitable globalisation of  $\bar{r}$ , it is in particular automorphic in the sense of Definition 5.3.1 of [EG13], and we let  $\mathfrak{m}_{Q_N}$  be the maximal ideal of  $\mathbb{T}^{S_p \cup Q_N, \text{univ}}$  corresponding to  $\bar{\rho}$ , and containing  $(T_{\widetilde{v}_1}^{(j)} - (\mathbf{N}v_1)^{j(1-j)/2}(\delta_1 \cdots \delta_j))$  for each  $1 \leq j \leq n$ .

We will write  $\mathfrak{m}$  for  $\mathfrak{m}_{\emptyset}$ .

**2.4. Galois deformations.** Let  $S$  be a set of places of  $\widetilde{F}^+$  which split in  $\widetilde{F}$ , with  $S_p \subseteq S$ . As in [CHT08], we will write  $\widetilde{F}(S)$  for the maximal extension of  $\widetilde{F}$  unramified outside  $S$ , and from now on we will write  $G_{\widetilde{F}^+, S}$  for  $\text{Gal}(\widetilde{F}(S)/\widetilde{F}^+)$ . We will freely make use of the terminology (of liftings, framed liftings etc.) of Section 2 of [CHT08].

Let  $T = S_p \cup \{v_1\}$ . For each  $v \in S_p$ , we let  $\widetilde{v}$  be a choice of a place of  $\widetilde{F}$  lying over  $v$ , with the property that  $\bar{\rho}|_{G_{\widetilde{F}_{\widetilde{v}}}} \cong \bar{r}$ . (Such a choice is possible by our assumption that  $\bar{\rho}$  is a suitable globalisation of  $\bar{r}$ .) We let  $\widetilde{T}$  denote the set of places  $\widetilde{v}$ ,  $v \in T$ . For each  $v \in T$ , we let  $R_{\widetilde{v}}^{\square}$  denote the maximal reduced and  $p$ -torsion free quotient of the universal  $\mathcal{O}$ -lifting ring of  $\bar{\rho}|_{G_{\widetilde{F}_{\widetilde{v}}}}$ . For each  $v \in S_p \setminus \{\mathfrak{p}\}$ , we write  $R_{\widetilde{v}}^{\square, \xi, \tau}$  for the reduced and  $p$ -torsion free quotient of  $R_{\widetilde{v}}^{\square}$  corresponding to potentially crystalline lifts of weight  $\xi$  and inertial type  $\tau$ .

Consider (in the terminology of [CHT08]) the deformation problem

$$\mathcal{S} := (\widetilde{F}/\widetilde{F}^+, T, \widetilde{T}, \mathcal{O}, \bar{\rho}, \varepsilon^{1-n} \delta_{\widetilde{F}/\widetilde{F}^+}^n, \{R_{\widetilde{v}_1}^{\square}\} \cup \{R_{\mathfrak{p}}^{\square}\} \cup \{R_{\widetilde{v}}^{\square, \xi, \tau}\}_{v \in S_p \setminus \{\mathfrak{p}\}}).$$

There is a corresponding universal deformation  $\rho_{\mathcal{S}}^{\text{univ}} : G_{\widetilde{F}^+, T} \rightarrow \mathcal{G}_n(R_{\mathcal{S}}^{\text{univ}})$  of  $\bar{\rho}$ . In addition, there is a universal  $T$ -framed deformation ring  $R_{\mathcal{S}}^{\square, T}$  in the sense of Definition 1.2.1 of [CHT08], which parameterises deformations of  $\bar{\rho}$  of type  $\mathcal{S}$  together with particular local liftings for each  $\widetilde{v} \in \widetilde{T}$ .

**2.5. Auxiliary primes.** As in Section 5.5 of [EG13], the version of the Taylor–Wiles patching argument given in [Tho12] allows us to choose an integer  $q \geq \lceil \widetilde{F}^+ : \mathbb{Q} \rceil n(n-1)/2$  and for each  $N \geq 1$  sets of primes  $Q_N, \widetilde{Q}_N$  with the following properties:

- $Q_N$  is a finite set of finite places of  $\tilde{F}^+$  of cardinality  $q$  which is disjoint from  $T$  and consists of places which split in  $\tilde{F}$ ;
- $\tilde{Q}_N$  consists of a single place  $\tilde{v}$  of  $\tilde{F}$  above each place  $v$  of  $Q_N$ ;
- $\mathbf{N}v \equiv 1 \pmod{p^N}$  for  $v \in Q_N$ ;
- for each  $v \in Q_N$ ,  $\bar{\rho}|_{G_{\tilde{F}_v}} \cong \bar{s}_v \oplus \bar{\psi}_v$  where  $\bar{\psi}_v$  is an eigenspace of Frobenius on which Frobenius acts semisimply.

For each  $N$ ,  $v \in Q_N$  and  $i = 0, 1$ , we let  $U_i(Q_N)_v \subset \tilde{G}(\mathcal{O}_{\tilde{F}_v^+})$  denote the parahoric open compact subgroups defined in Section 5.5 of [EG13], following [Tho12].

For each  $v \in Q_N$ , a quotient  $R_{\tilde{v}}^{\bar{\psi}_v}$  of  $R_{\tilde{v}}^\square$  is defined in section 5.5 of [EG13] (following [Tho12]). We let  $\mathcal{S}_{Q_N}$  denote the deformation problem

$$\mathcal{S}_{Q_N} := (\tilde{F}/\tilde{F}^+, T \cup Q_N, \tilde{T} \cup \tilde{Q}_N, \mathcal{O}, \bar{\rho}, \varepsilon^{1-n} \delta_{\tilde{F}/\tilde{F}^+}^n, \\ \{R_{\tilde{v}_1}^\square\} \cup \{R_{\mathfrak{p}}^\square\} \cup \{R_{\tilde{v}}^{\square, \xi, \tau}\}_{v \in S_p \setminus \{\mathfrak{p}\}} \cup \{R_{\tilde{v}}^{\bar{\psi}_v}\}_{v \in Q_N}).$$

We let  $R_{\mathcal{S}_{Q_N}}^{\text{univ}}$  denote the corresponding universal deformation ring, and we let  $R_{\mathcal{S}_{Q_N}}^{\square T}$  denote the corresponding universal  $T$ -framed deformation ring. We define

$$R^{\text{loc}} := R_{\mathfrak{p}}^\square \hat{\otimes} \left( \hat{\otimes}_{v \in S_p \setminus \{\mathfrak{p}\}} R_{\tilde{v}}^{\square, \xi, \tau} \right) \hat{\otimes} R_{\tilde{v}_1}^\square$$

where all completed tensor products are taken over  $\mathcal{O}$ . By the choice of the sets of primes  $Q_N$ , we also know that

- the ring  $R_{\mathcal{S}_{Q_N}}^{\square T}$  can be topologically generated over  $R^{\text{loc}}$  by  $q - [\tilde{F}^+ : \mathbb{Q}]n(n-1)/2$  elements.

For each  $v \in Q_N$  we choose a uniformiser  $\varpi_{\tilde{v}} \in \mathcal{O}_{\tilde{F}_v}$ , so that we have the projection operator  $\text{pr}_{\varpi_{\tilde{v}}} \in \text{End}_{\mathcal{O}}(S_{\xi, \tau}(U_i(Q_N)_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}_{Q_N}})$  defined as in Proposition 5.9 of [Tho12]. We define  $\text{pr}$  to be the composite of the projections  $\text{pr}_{\varpi_{\tilde{v}}}$ . (These projections commute among themselves, and so it doesn't matter in which order we compose them. Whenever we use  $\text{pr}$  it will be clear from the context what the underlying set  $Q_N$  is.) Then, as in Section 5.5 of [EG13]:

(1) The map

$$\text{pr} : S_{\xi, \tau}(U_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}} \rightarrow \text{pr} \left( S_{\xi, \tau}(U_0(Q_N)_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}_{Q_N}} \right)$$

is an isomorphism. Moreover, since  $\text{pr}$  is defined using Hecke operators at places in  $Q_N$ , it commutes with the action of  $\tilde{G}(\tilde{F}_{\mathfrak{p}}^+)$  on the spaces of algebraic automorphic forms. More precisely, if  $g \in \tilde{G}(\tilde{F}_{\mathfrak{p}}^+)$  satisfies  $g^{-1}U_{m', \mathfrak{p}}g \subseteq U_{m, \mathfrak{p}}$  for some positive integers  $m \leq m'$ , then we have a commutative diagram

$$\begin{array}{ccc} S_{\xi, \tau}(U_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}} & \xrightarrow{\text{pr}} & \text{pr} \left( S_{\xi, \tau}(U_0(Q_N)_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}_{Q_N}} \right) \\ \downarrow g & & \downarrow g \\ S_{\xi, \tau}(U_{m'}, \mathcal{O}/\varpi^r)_{\mathfrak{m}} & \xrightarrow{\text{pr}} & \text{pr} \left( S_{\xi, \tau}(U_0(Q_N)_{m'}, \mathcal{O}/\varpi^r)_{\mathfrak{m}_{Q_N}} \right) \end{array}$$

where both vertical arrows are induced by the action of  $g \in \tilde{G}(\tilde{F}_{\mathfrak{p}}^+)$  (namely:  $(gf)(g') = f(g'g)$ ).

- (2) Let  $\Gamma_m = \mathrm{GL}_n(\mathcal{O}_F/\varpi_F^m) \cong U/U_m$  and  $\Delta_{Q_N} = \prod_{v \in Q_N} U_0(Q_N)_v/U_1(Q_N)_v$ . Then  $U_0(Q_N)_0$  acts on  $S_{\xi, \tau}(U_1(Q_N)_m, \mathcal{O}/\varpi^r)$  via  $(gf)(g') = g_p f(g'g)$ , and this action factors through  $\Delta_{Q_N} \times \Gamma_m$ . With respect to this action,  $\mathrm{pr}\left(S_{\xi, \tau}(U_1(Q_N)_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}_{Q_N}}\right)$  is a projective  $(\mathcal{O}/\varpi^r)[\Delta_{Q_N}][\Gamma_m]$ -module, and there is a natural  $\Gamma_m$ -equivariant isomorphism

$$\mathrm{pr}\left(S_{\xi, \tau}(U_1(Q_N)_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}_{Q_N}}\right)^{\Delta_{Q_N}} \xrightarrow{\sim} S_{\xi, \tau}(U_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}}.$$

(The projectivity follows from the proof of Lemma 3.3.1 of [CHT08]. The fact that there is a  $\Gamma_m$ -equivariant isomorphism follows immediately from point (1) and the definitions. We shall not need the analogous statement about coinvariants which is recalled in [EG13] and proved in [Tho12]; see Remark 2.7 below for an indication as to why not.)

- (3) Let  $\mathbb{T}_{\xi, \tau}^{S_p \cup Q_N}(U_i(Q_N)_m, \mathcal{O}/\varpi^r)$  be the image of  $\mathbb{T}^{S_p \cup Q_N, \mathrm{univ}}$  in the ring  $\mathrm{End}_{\mathcal{O}}(\mathrm{pr}(S_{\xi, \tau}(U_i(Q_N)_m, \mathcal{O}/\varpi^r)))$ . Then there exists a deformation

$$G_{\tilde{F}^+, T \cup Q_N} \rightarrow \mathcal{G}_n\left(\mathbb{T}_{\xi, \tau}^{S_p \cup Q_N}(U_1(Q_N)_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}_{Q_N}}\right)$$

of  $\bar{\rho}$  which is of type  $\mathcal{S}_{Q_N}$ . In particular,  $\mathrm{pr}\left(S_{\xi, \tau}(U_i(Q_N)_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}_{Q_N}}\right)$  is a finite  $R_{\mathcal{S}_{Q_N}}^{\mathrm{univ}}$ -module.

As in Section 5.5 of [EG13], there is a homomorphism  $\Delta_{Q_N} \rightarrow (R_{\mathcal{S}_{Q_N}}^{\mathrm{univ}})^{\times}$  obtained by identifying  $\Delta_{Q_N}$  with the product of the inertia subgroups in the maximal abelian  $p$ -power order quotient of  $\prod_{v \in Q_N} G_{\tilde{F}_v}$ , and thus a homomorphism  $\mathcal{O}[\Delta_{Q_N}] \rightarrow R_{\mathcal{S}_{Q_N}}^{\mathrm{univ}}$ . The  $R_{\mathcal{S}_{Q_N}}^{\mathrm{univ}}$ -module structure on  $\mathrm{pr}\left(S_{\xi, \tau}(U_1(Q_N)_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}_{Q_N}}\right)$  thus induces an action of  $\mathcal{O}[\Delta_{Q_N}]$  on  $\mathrm{pr}\left(S_{\xi, \tau}(U_1(Q_N)_m, \mathcal{O}/\varpi^r)_{\mathfrak{m}_{Q_N}}\right)$ , which agrees with the one in (2) above.

**2.6. Patching.** We now make our patching construction, by applying the Taylor–Wiles–Kisin method. Before doing so, we provide a brief comparison and contrast with the patching constructions in some previous papers, such as [EG13] and [EGS13]. In the latter paper, we employ Taylor–Wiles–Kisin patching to construct what we call *patching functors*, which are (essentially) certain exact functors from the category of continuous  $\mathrm{GL}_n(\mathcal{O}_F)$ -representations on finitely generated  $\mathcal{O}$ -modules to the category of coherent sheaves on an appropriate deformation space of local Galois representations (perhaps with some auxiliary patching variables added). Although this is not discussed in [EGS13], such a functor can be (pro-)represented by an object  $M_{\infty}$ , which is a continuous  $\mathrm{GL}_n(\mathcal{O}_F)$ -representation over the local deformation ring (again, perhaps with patching variables added). More precisely, in terms of such a  $\mathrm{GL}_n(\mathcal{O}_F)$ -representation  $M_{\infty}$ , the patching functor can be defined as  $\mathrm{Hom}_{\mathcal{O}[[\mathrm{GL}_n(\mathcal{O}_F)]]}^{\mathrm{cont}}(M_{\infty}, V^{\vee})^{\vee}$ , if  $V$  is a continuous representation of  $\mathrm{GL}_n(\mathcal{O}_F)$  on a finitely generated  $\mathcal{O}$ -module. The exactness of the patching functor can be encoded in the requirement that  $M_{\infty}$  be a projective  $\mathcal{O}[[\mathrm{GL}_n(\mathcal{O}_F)]]$ -module.

In this paper our approach is to construct the representing object  $M_{\infty}$  directly, and (most importantly) to promote it from being merely a  $\mathrm{GL}_n(\mathcal{O}_F)$ -representation to being a representation of the full  $p$ -adic group  $\mathrm{GL}_n(F)$ . (In terms of patching functors, one can somewhat loosely think of this as extending the patching functor from the category of  $\mathrm{GL}_n(\mathcal{O}_F)$ -representations to a category that we

might call the *Hecke category*, whose objects are the same, but in which the morphisms between any two  $\mathrm{GL}_n(\mathcal{O}_F)$ -representations  $U$  and  $V$  are defined to be  $\mathrm{Hom}_{\mathrm{GL}_n(F)}(\mathrm{c}\text{-Ind}_{\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(F)} U, \mathrm{c}\text{-Ind}_{\mathrm{GL}_n(\mathcal{O}_F)}^{\mathrm{GL}_n(F)} V)$ .

From now on, to ease notation we write  $K = \mathrm{GL}_n(\mathcal{O}_{\tilde{F}_p}) = \mathrm{GL}_n(\mathcal{O}_F)$ ,  $G = \mathrm{GL}_n(\tilde{F}_p) = \mathrm{GL}_n(F)$ , and  $Z = Z(G)$ . For each integer  $N \geq 0$ , we set  $K_N := \ker(\mathrm{GL}_n(\mathcal{O}_F) \rightarrow \mathrm{GL}_n(\mathcal{O}_F/\varpi_F^N))$ , so that  $K/K_N \xrightarrow{\sim} \Gamma_N$ . We have the Cartan decomposition  $G = KAK$ , where  $A$  is the set of diagonal matrices whose diagonal entries are powers of the uniformiser  $\varpi_F$ , and we let  $A_N$  be the subset of  $A$  consisting of matrices with the property that the ratio of any two diagonal entries is of the form  $\varpi_F^r$  with  $|r| \leq N$ , and set  $G_N = KA_NK$ . Note that  $G_N$  is not a subgroup of  $G$  unless  $N = 0$ , but that each  $K \backslash G_N / KZ$  is finite, and  $G = \cup_{N \geq 0} G_N$ .

If  $(\sigma, W)$  is a representation of  $KZ$ , then we write  $\mathrm{Ind}_{KZ}^{G_N} \sigma$  for the space of functions  $f : G_N \rightarrow W$  with  $f(kg) = \sigma(k)f(g)$  for all  $g \in G_N$ ,  $k \in KZ$ ; this is naturally a  $KZ$ -representation via  $(kf)(g) := f(gk)$ . We define  $\mathrm{Ind}_{KZ}^G \sigma$  in the same way; then  $\mathrm{Ind}_{KZ}^G \sigma$  is a representation of  $G$  via  $(gf)(g') := f(g'g)$ .

For each  $N$ , we set

$$M_{i,Q_N} := \mathrm{pr}(S_{\xi,\tau}(U_i(Q_N)_{2N}, \mathcal{O}/\varpi^N)_{\mathfrak{m}_{Q_N}})^\vee.$$

Note that  $M_{i,Q_N}$  depends on the integer  $N$  as well as on the set of primes  $Q_N$  (it could happen that  $Q_M = Q_N$  for  $M \neq N$ ), but we will only include  $Q_N$  in the notation for the sake of simplicity. Note also that we could have equivalently defined

$$M_{i,Q_N} := \mathrm{pr}^\vee(S_{\xi,\tau}(U_i(Q_N)_{2N}, \mathcal{O}/\varpi^N)_{\mathfrak{m}_{Q_N}}^\vee),$$

since  $\mathrm{pr}$  is an endomorphism of  $S_{\xi,\tau}(U_i(Q_N)_{2N}, \mathcal{O}/\varpi^N)_{\mathfrak{m}_{Q_N}}$  and Pontrjagin duality is an exact contravariant functor.

Let  $\Delta_{Q_N}$  be as above; it is of  $p$ -power order by the definitions of the  $U_i(Q_N)_{2N}$ . It follows from point (2) in the previous section that  $M_{1,Q_N}$  is a finite projective  $(\mathcal{O}/\varpi^N)[\Delta_{Q_N}][\Gamma_{2N}]$ -module. Since  $Z$  centralises  $U_1(Q_N)_{2N}$ , there is also a natural action of  $Z$  on  $M_{1,Q_N}$ .

*2.7. Remark.* The reason for including a Pontrjagin dual in the definition of  $M_{i,Q_N}$  is that  $S_{\xi,\tau}(U_i(Q_N)_{2N}, \mathcal{O}/\varpi^N)$  is a space of automorphic forms, and so is most naturally thought of as being contravariant in the level, while patching is a process that involves passing to a projective limit over the level (rather than a direct limit). Now since  $S_{\xi,\tau}(U_i(Q_N)_{2N}, \mathcal{O}/\varpi^N)$  is a space of automorphic forms on the definite unitary group  $\tilde{G}$ , it is a space of functions on a finite set, and so has a natural self-duality. Thus, by exploiting this self-duality to convert its contravariant functoriality into a covariant functoriality, we could omit the Pontrjagin dual in the preceding definition, and indeed it *is* traditionally omitted (see e.g. [CHT08], [BLGG11], [Tho12], and [EG13]). However, we have found it conceptually clearer to include this duality in our definitions and constructions.

We now define a  $KZ$ -equivariant map

$$\alpha_N : M_{1,Q_N} \rightarrow \mathrm{Ind}_{KZ}^{G_N}((M_{1,Q_N})_{K_N})$$

(where  $(M_{1,Q_N})_{K_N}$  denotes the  $K_N$ -coinvariants in  $M_{1,Q_N}$ ) in the following way. Note firstly that there is a natural identification

$$(M_{1,Q_N})_{K_N} = \mathrm{pr}^\vee(S_{\xi,\tau}(U_1(Q_N)_N, \mathcal{O}/\varpi^N)_{\mathfrak{m}_{Q_N}}^\vee),$$

so it suffices to define a  $KZ$ -equivariant map

$$\alpha_N : S_{\xi, \tau}(U_1(Q_N)_{2N}, \mathcal{O}/\varpi^N)^\vee \rightarrow \text{Ind}_{KZ}^{G_N} S_{\xi, \tau}(U_1(Q_N)_N, \mathcal{O}/\varpi^N)^\vee.$$

Now, given  $g \in G_N$ , we have  $g^{-1}K_{2N}g \subseteq K_N$ , so that there is a natural map

$$g^* : S_{\xi, \tau}(U_1(Q_N)_N, \mathcal{O}/\varpi^N) \rightarrow S_{\xi, \tau}(U_1(Q_N)_{2N}, \mathcal{O}/\varpi^N)$$

given by  $(g^* \cdot f)(x) := f(xg)$ , and a map

$$g_* := ((g^{-1})^*)^\vee : S_{\xi, \tau}(U_1(Q_N)_{2N}, \mathcal{O}/\varpi^N)^\vee \rightarrow S_{\xi, \tau}(U_1(Q_N)_N, \mathcal{O}/\varpi^N)^\vee.$$

(The latter is well defined since  $G_N$  is stable under taking inverses. We note that  $g^*$  (resp.  $g_*$ ) may be interpreted as the natural pullback (resp. pushforward) map on cohomology (resp. homology) under the natural right (resp. left) action of  $\tilde{G}(\mathbb{A}^\infty)$  on the tower of arithmetic quotients of  $\tilde{G}$ .) We have  $(gh)^* = g^* \circ h^*$  and hence  $(gh)_* = g_* \circ h_*$ , whenever all are defined. Then we define  $\alpha_N$  by

$$(\alpha_N(x))(g) := g_*(x).$$

In order to check that this is  $KZ$ -equivariant, we must check that for all  $k \in KZ$  we have  $(\alpha_N(kx))(g) = (k\alpha_N(x))(g)$ ; this is equivalent to checking that  $g_*(kx) = (gk)_*(x)$ , which is immediate from the definition.

Now set  $M_{1, Q_N}^\square := M_{1, Q_N} \otimes_{R_{S_{Q_N}}^{\text{univ}}} R_{S_{Q_N}}^{\square T}$ . We have an induced  $KZ$ -equivariant map  $\alpha_N : M_{1, Q_N}^\square \rightarrow \text{Ind}_{KZ}^{G_N}(M_{1, Q_N}^\square)_{K_N}$ . Define

$$R_\infty := R^{\text{loc}}[[x_1, \dots, x_{q - [\bar{F}^+ : \mathbb{Q}]n(n-1)/2}],$$

$$S_\infty := \mathcal{O}[[z_1, \dots, z_{n^2 \# T}, y_1, \dots, y_q]],$$

for formal variables  $x_1, \dots, x_{q - [\bar{F}^+ : \mathbb{Q}]n(n-1)/2}$ ,  $y_1, \dots, y_q$  and  $z_1, \dots, z_{n^2 \# T}$ . For each  $N$ , we fix a surjection  $R_\infty \twoheadrightarrow R_{S_{Q_N}}^{\square T}$  of  $R^{\text{loc}}$ -algebras (which, as recalled in Section 2.5, is possible by the choice of the sets  $Q_N$ ). These choices allow us to regard each  $M_{1, Q_N}^\square$  as an  $R_\infty$ -module. Also, we fix choices of lifts representing the universal deformations over  $R_S^{\text{univ}}$  and each  $R_{S_{Q_N}}^{\text{univ}}$  such that our chosen lift over each  $R_{S_{Q_N}}^{\text{univ}}$  reduces to our chosen lift over  $R_S^{\text{univ}}$ . These choices give rise to isomorphisms  $R_{S_{Q_N}}^{\square T} \xrightarrow{\sim} R_{S_{Q_N}}^{\text{univ}} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[z_1, \dots, z_{n^2 \# T}]]$  compatible with a fixed isomorphism  $R_S^{\square T} \xrightarrow{\sim} R_S^{\text{univ}} \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[z_1, \dots, z_{n^2 \# T}]]$ ; they also allow us to regard each  $M_{1, Q_N}^\square$  as an  $\mathcal{O}[[z_1, \dots, z_{n^2 \# T}]]$ -module. Finally, for each  $N$ , choose a surjection  $\mathcal{O}[[y_1, \dots, y_q]] \twoheadrightarrow \mathcal{O}[\Delta_{Q_N}]$ . This gives each  $M_{1, Q_N}^\square$  the structure of an  $S_\infty$ -module and hence the structure of an  $S_\infty[[K]]$ -module (where the action of  $K$  factors through  $\Gamma_{2N}$ ).

We now apply the Taylor–Wiles method in the usual way to pass to a subsequence, and patch the modules  $M_{1, Q_N}^\square$  together with the maps  $\alpha_N : M_{1, Q_N}^\square \rightarrow \text{Ind}_{KZ}^{G_N}(M_{1, Q_N}^\square)_{K_N}$ . More precisely, for each  $N \geq 1$ , let  $\mathfrak{b}_N$  denote the ideal of  $S_\infty$  generated by  $\varpi^N$ ,  $z_i^N$  and  $(1 + y_i)^{p^N} - 1$ . Let  $\mathfrak{a}$  denote the ideal of  $S_\infty$  generated by the  $z_i$  and the  $y_i$ . Fix a sequence  $(\mathfrak{d}_N)_{N \geq 1}$  of open ideals of  $R_S^{\text{univ}}$  such that

- $\varpi^N R_S^{\text{univ}} \subset \mathfrak{d}_N \subset \text{Ann}_{R_S^{\text{univ}}}(S_{\xi, \tau}(U_{2N}, \mathcal{O}/\varpi^N)_{\mathfrak{m}}^\vee)$ ;
- $\mathfrak{d}_N \supset \mathfrak{d}_{N+1}$  for all  $N$ ;
- $\bigcap_{N \geq 1} \mathfrak{d}_N = (0)$ .

(For example, we may take  $\mathfrak{d}_N = \mathfrak{m}_{R_S^{\text{univ}}}^N \cap \text{Ann}_{R_S^{\text{univ}}}(S_{\xi,\tau}(U_{2N}, \mathcal{O}/\varpi^N)_{\mathfrak{m}}^\vee)$ .)

At level  $N$ , we consider tuples consisting of

- a surjective homomorphism of  $R^{\text{loc}}$ -algebras  $\phi : R_\infty \twoheadrightarrow R_S^{\text{univ}}/\mathfrak{d}_N$ ;
- a finite projective  $(S_\infty/\mathfrak{b}_N)[\Gamma_{2N}]$ -module  $M^\square$  which carries a commuting action of  $R_\infty \otimes_{\mathcal{O}} \mathcal{O}[Z]$  such that the action of  $S_\infty$  can be factored through that of  $R_\infty$ ;
- a  $\Gamma_{2N}$  and  $Z$ -equivariant isomorphism  $\psi : (M^\square/\mathfrak{a}M^\square) \xrightarrow{\sim} S_{\xi,\tau}(U_{2N}, \mathcal{O}/\varpi^N)_{\mathfrak{m}}^\vee$  compatible with  $\phi$ ;
- a  $KZ$ -equivariant and  $R_\infty \otimes_{\mathcal{O}} S_\infty$ -linear map  $\alpha : M^\square \rightarrow \text{Ind}_{KZ}^{G_N}[(M^\square)_{K_N}]$ , whose reduction modulo  $\mathfrak{a}$  is compatible with the map

$$S_{\xi,\tau}(U_{2N}, \mathcal{O}/\varpi^N)_{\mathfrak{m}}^\vee \rightarrow \text{Ind}_{KZ}^{G_N}[S_{\xi,\tau}(U_N, \mathcal{O}/\varpi^N)_{\mathfrak{m}}^\vee]$$

induced by the action of  $G_N$  on the spaces of algebraic modular forms.

We consider two such tuples  $(\phi, M^\square, \psi, \alpha)$  and  $(\phi', M^{\square'}, \psi', \alpha')$  to be equivalent if  $\phi = \phi'$  and if there is an isomorphism of  $(S_\infty/\mathfrak{b}_N)[\Gamma_{2N}] \otimes_{\mathcal{O}} R_\infty \otimes_{\mathcal{O}} \mathcal{O}[Z]$ -modules  $M^\square \xrightarrow{\sim} M^{\square'}$  which identifies  $\psi$  with  $\psi'$  and  $\alpha$  with  $\alpha'$ . Note that there are at most finitely many equivalence classes of such tuples. (For a given  $M^\square$  there are only finitely many  $KZ$ -equivariant homomorphisms  $\alpha : M^\square \rightarrow \text{Ind}_{KZ}^{G_N}[(M^\square)_{K_N}]$ , because  $M^\square$  and  $K \backslash G_N / KZ$  are finite.) Note also that a tuple  $(\phi, M^\square, \psi, \alpha)$  of level  $N$  gives rise to a tuple  $(\phi', M^{\square'}, \psi', \alpha')$  of level  $(N-1)$  by setting  $\phi' := \phi \bmod \mathfrak{d}_{N-1}$ ,  $M^{\square'} := (M^\square/\mathfrak{b}_{N-1})_{K_{2(N-1)}}$  and  $\psi' := \psi \bmod \varpi^{N-1}$ . The map  $\alpha'$  is defined by the formula  $\alpha'(\overline{m})(g) = \overline{\alpha(m)(g)}$ . Here  $\overline{m}$  denotes the image of  $m \in M^\square$  in  $M^{\square'}$  and  $\overline{\alpha(m)(g)}$  denotes the image of  $\alpha(m)(g)$  in  $(M^{\square'})_{K_{N-1}}$ . Note that for any  $m \in M^\square$ ,  $\gamma \in K_{2(N-1)}$  and  $g \in G_{N-1}$ , we have

$$\alpha((\gamma-1)m)(g) = \alpha(m)(g(\gamma-1)) = \alpha(m)((g\gamma g^{-1}-1)g) = (g\gamma g^{-1}-1)\alpha(m)(g),$$

by the  $K$ -equivariance of  $\alpha$ . Using the fact that  $g\gamma g^{-1} \in K_{N-1}$  it is straightforward to see that  $\alpha'$  is well-defined.

For each pair of integers  $N' \geq N \geq 1$ , we define a tuple  $(\phi, M^\square, \psi, \alpha)$  of level  $N$  as follows: we set  $\phi$  equal to  $R_\infty \twoheadrightarrow R_{S_{Q_{N'}}}^{\square_T} \twoheadrightarrow R_S^{\text{univ}}/\mathfrak{d}_N$  and we set  $M^\square = (M_{1,Q_{N'}}^\square \otimes_{S_\infty} S_\infty/\mathfrak{b}_N)_{K_{2N}}$ . The map  $\psi$  comes from points (1) and (2) in the previous section. The map  $\alpha$  comes from  $\alpha_{N'}$  defined above (in the same way that  $\alpha'$  is defined in terms of  $\alpha$  in the previous paragraph) and the compatibility it is required to satisfy comes from the commutative diagram in point (1) of the previous section. Since there are only finitely many isomorphism classes of tuples at each level  $N$ , but  $N'$  is allowed to be arbitrarily large, we can apply a diagonal argument to find a subsequence of pairs  $(N'(N), N)_{N \geq 1}$  indexed by  $N$  such that for each  $N \geq 2$ , the tuple indexed by  $N$  reduced to level  $(N-1)$  is isomorphic to the tuple indexed by  $(N-1)$ . For each  $N \geq 2$ , we fix a choice of such an isomorphism.

We now define

$$M_\infty := \varprojlim_N (M_{1,Q_{N'(N)}}^\square \otimes_{S_\infty} S_\infty/\mathfrak{b}_N)_{K_{2N}},$$

where the transition maps are induced by the isomorphisms fixed in the previous paragraph. (We drop the square from the notation here in order to avoid notational overload in later sections.) Each of the terms in the projective limit is a (literally) finite  $\mathcal{O}$ -module, endowed with commuting actions of  $S_\infty[[K]]$  and  $R_\infty \otimes_{\mathcal{O}} \mathcal{O}[Z]$ ,

and by construction the transition maps in the projective limit respect these actions. Thus  $M_\infty$  is naturally a profinite topological  $S_\infty[[K]]$ -module which carries a commuting action of  $R_\infty \otimes_{\mathcal{O}} \mathcal{O}[Z]$ , the topology on  $M_\infty$  being the projective limit topology (where each of the terms in the projective limit is endowed with the discrete topology). Moreover, the action of  $S_\infty$  on  $M_\infty$  can be factored through a map  $S_\infty \rightarrow R_\infty$  (since the analogous statement holds at each level  $N$ ).

The module  $M_\infty$  is the key construction of the paper; the remainder of this section is devoted to recording some additional properties that it enjoys. Firstly, since the transition maps in the projective limit are given simply by reducing from level  $N$  to level  $N - 1$ , it is easily verified that the natural map induces an isomorphism

$$(M_\infty/\mathfrak{b}_N)_{K_{2N}} \xrightarrow{\sim} (M_{1,Q_{N'(N)}}^\square/\mathfrak{b}_N)_{K_{2N}}.$$

Next, from this, it follows from the topological form of Nakayama's lemma that  $M_\infty$  is in fact a finite  $S_\infty[[K]]$ -module. It follows that the topology on  $M_\infty$  coincides with the quotient topology obtained by writing it as a quotient of  $S_\infty[[K]]^r$ , where  $S_\infty[[K]]$  is endowed with its natural profinite topology. Crucially, there is a  $KZ$ -equivariant and  $R_\infty$ -linear map

$$\alpha_\infty : M_\infty \rightarrow \mathrm{Ind}_{KZ}^G M_\infty$$

by taking the projective limit of the maps

$$(M_{1,Q_{N'(N)}}^\square \otimes_{S_\infty} S_\infty/\mathfrak{b}_N)_{K_{2N}} \rightarrow \mathrm{Ind}_{KZ}^{G_N} \left( (M_{1,Q_{N'(N)}}^\square \otimes_{S_\infty} S_\infty/\mathfrak{b}_N)_{K_N} \right).$$

induced by  $\alpha_{N'(N)} : M_{1,Q_{N'(N)}} \rightarrow \mathrm{Ind}_{KZ}^{G_{N'(N)}} \left( (M_{1,Q_{N'(N)}})_{K_{N'(N)}} \right)$ . We denote this induced map by  $\bar{\alpha}_{N'(N)}$ .

The following proposition establishes the additional key properties of the patched module  $M_\infty$  that we will need.

**2.8. Proposition.**  *$M_\infty$  is projective over  $S_\infty[[K]]$ . Furthermore,  $\alpha_\infty$  is injective, and its image is  $G$ -stable, so that  $\alpha_\infty$  induces an action of  $G$  on  $M_\infty$ .*

*Proof.* Since  $M_\infty$  is finite over  $S_\infty[[K]]$ , we may choose a presentation  $S_\infty[[K]]^r \rightarrow M_\infty$  for some  $r \geq 1$ . In order to check that  $M_\infty$  is a projective  $S_\infty[[K]]$ -module, it is enough to check that this surjection splits. Since each  $M_{1,Q_N}$  is a projective  $(\mathcal{O}/\varpi^N)[\Delta_{Q_N}][\Gamma_{2N}]$ -module, we see that

$$(M_\infty/\mathfrak{b}_N)_{K_{2N}} \cong (M_{1,Q_{N'(N)}}^\square/\mathfrak{b}_N)_{K_{2N}}$$

is a projective  $S_\infty/\mathfrak{b}_N[\Gamma_{2N}]$ -module, so we have a cofinal system of ideals (namely, the ideals generated by  $\mathfrak{b}_N + \ker(\mathcal{O}[[K]] \rightarrow \mathcal{O}[\Gamma_{2N}])$ ) defining the topology of  $S_\infty[[K]]$  modulo which the surjection splits. The sets of possible splittings at these finite levels then give us a projective system of non-empty finite sets, and an element of the projective limit of this projective system gives the required splitting.

We now check that  $\alpha_\infty$  is injective. Note that by definition, for each  $\alpha_N : M_{1,Q_N} \rightarrow \mathrm{Ind}_{KZ}^{G_N} \left( (M_{1,Q_N})_{K_N} \right)$  we have  $(\alpha_N(m))(1) = 1_*(m) = \bar{m}$ , where  $\bar{m}$  denotes the image of  $m$  in  $(M_{1,Q_N})_{K_N}$ . From this (with  $N$  replaced by  $N'(N)$ ) we deduce that  $(\bar{\alpha}_{N'(N)}(m))(1) = \bar{m}$  where  $\bar{m}$  is the image of  $m \in (M_{1,Q_{N'(N)}}^\square/\mathfrak{b}_N)_{K_{2N}}$  in  $(M_{1,Q_{N'(N)}}^\square/\mathfrak{b}_N)_{K_N}$ . This then implies that  $(\alpha_\infty(m))(1) = m$  for each  $m \in M_\infty$ , and  $\alpha_\infty$  is certainly injective.

In order to show that the image of  $\alpha_\infty$  is  $G$ -stable, we will show that for all  $g \in G$ ,  $m \in M_\infty$  we have

$$g(\alpha_\infty(m)) = \alpha_\infty((\alpha_\infty(m))(g)).$$

In other words, we will show that for all  $g, h \in G$  and  $m \in M_\infty$ , we have

$$(\alpha_\infty(m))(hg) = (\alpha_\infty((\alpha_\infty(m))(g)))(h).$$

Let  $m$  be an element of  $M_\infty$  and let  $N$  be any integer large enough so that  $g, h, gh \in G_N$ . This certainly means that  $g, h, gh \in G_{2N}$  as well. Since  $N$  can be arbitrarily large, it is enough to show that both sides of the equation above become equal in  $(M_\infty/\mathfrak{b}_N)_{K_N}$  and we do this by explicit computation.

We let  $\pi_N : M_\infty \rightarrow (M_\infty/\mathfrak{b}_N)_{K_{2N}}$  and  $\sigma_N : M_\infty \rightarrow (M_\infty/\mathfrak{b}_N)_{K_N}$  denote the projection maps. Then by definition, we have

$$\sigma_N(\alpha_\infty(m)(hg)) = \bar{\alpha}_{N'(N)}(\pi_N(m))(hg)$$

and

$$\begin{aligned} \sigma_N(\alpha_\infty(\alpha_\infty(m)(g))(h)) &= \bar{\alpha}_{N'(N)}(\pi_N(\alpha_\infty(m)(g)))(h) \\ &= \bar{\alpha}_{N'(N)}(\sigma_{2N}(\alpha_\infty(m)(g)) \bmod \mathfrak{b}_N)(h) \\ &= \bar{\alpha}_{N'(N)}(\bar{\alpha}_{N'(2N)}(\pi_{2N}(m))(g) \bmod \mathfrak{b}_N)(h). \end{aligned}$$

Now, for integers  $N, \tilde{N}, N'' \geq 1$  with  $\tilde{N} \leq N'(N)$ , we let  $U_1(Q_{N'(N)}, \tilde{N})_{N''}$  be the open compact subgroup intermediate to  $U_1(Q_{N'(N)})_{N''}$  and  $U_0(Q_{N'(N)})_{N''}$  such that  $U_0(Q_{N'(N)})_{N''}/U_1(Q_{N'(N)}, \tilde{N})_{N''} \cong (\mathbb{Z}/p^{\tilde{N}}\mathbb{Z})^q$ . Then we have a commutative diagram

$$\begin{array}{ccc} S_{\xi, \tau}(U_1(Q_{N'(2N)}, 2N)_{4N}, \mathcal{O}/\varpi^{2N})^\vee & \xrightarrow{g_*} & S_{\xi, \tau}(U_1(Q_{N'(2N)}, 2N)_{2N}, \mathcal{O}/\varpi^{2N})^\vee \\ \parallel & & \downarrow \text{nat} \\ S_{\xi, \tau}(U_1(Q_{N'(2N)}, 2N)_{4N}, \mathcal{O}/\varpi^{2N})^\vee & & S_{\xi, \tau}(U_1(Q_{N'(2N)}, N)_{2N}, \mathcal{O}/\varpi^N)^\vee \\ \downarrow \text{nat} & & \downarrow h_* \\ S_{\xi, \tau}(U_1(Q_{N'(2N)}, N)_{2N}, \mathcal{O}/\varpi^N)^\vee & \xrightarrow{(hg)_*} & S_{\xi, \tau}(U_1(Q_{N'(2N)}, N)_N, \mathcal{O}/\varpi^N)^\vee. \end{array}$$

Localising at  $\mathfrak{m}_{Q_{N'(2N)}}$ , applying the projector  $\text{pr}^\vee$ , tensoring over  $R_{S_{Q_{N'(N)}}}^{\text{univ}}$  with  $R_{S_{Q_{N'(N)}}}^{\square\tau}$  and reducing modulo  $\mathfrak{b}_{2N}$  or  $\mathfrak{b}_N$  as appropriate, we obtain a commutative diagram:

$$\begin{array}{ccc} (M_\infty/\mathfrak{b}_{2N})_{K_{4N}} & \xrightarrow{g_*} & (M_\infty/\mathfrak{b}_{2N})_{K_{2N}} \\ \parallel & & \downarrow \text{nat} \\ (M_\infty/\mathfrak{b}_{2N})_{K_{4N}} & & (M_\infty/\mathfrak{b}_N)_{K_{2N}} \\ \downarrow \text{nat} & & \downarrow h_* \\ (M_\infty/\mathfrak{b}_N)_{K_{2N}} & \xrightarrow{(hg)_*} & (M_\infty/\mathfrak{b}_N)_{K_N}. \end{array}$$

Now, on the one hand, the element  $\pi_{2N}(m)$  lies in the space in the upper left corner of this diagram. By definition, we have

$$\bar{\alpha}_{N'(N)}(\bar{\alpha}_{N'(2N)}(\pi_{2N}(m))(g) \bmod \mathfrak{b}_N)(h) = h_* \circ \text{nat} \circ g_*(\pi_{2N}(m))$$

On the other hand,  $\pi_N(m)$  lies in the space in the lower left corner of the diagram, and we have

$$\bar{\alpha}_{N'(N)}(\pi_N(m))(hg) = (hg)_*(\pi_N(m)) = (hg)_* \circ \text{nat}(\pi_{2N}(m)).$$

The desired equality now follows from the commutativity of the above diagrams.  $\square$

Note that for each positive integer  $m$ , the compact open subgroups  $U_m$  have the same level away from  $\mathfrak{p}$ . Let  $U^{\mathfrak{p}} \subset \tilde{G}(\mathbb{A}_{\tilde{F}^+}^{\infty, \mathfrak{p}})$  denote that common level. Define the  $\varpi$ -adically completed cohomology space

$$\tilde{S}_{\xi, \tau}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}} := \varprojlim_s (\varinjlim_m S_{\xi, \tau}(U_m, \mathcal{O}/\varpi^s)_{\mathfrak{m}}).$$

The space  $\tilde{S}_{\xi, \tau}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}}$  is equipped with a natural  $G$ -action, induced from the action of  $G$  on algebraic automorphic forms. Moreover, the space  $\tilde{S}_{\xi, \tau}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}}$  is equipped with a natural action of the Hecke algebra

$$\mathbb{T}_{\xi, \tau}^{S_p}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}} := \varprojlim_m \mathbb{T}_{\xi, \tau}^{S_p}(U_m, \mathcal{O})_{\mathfrak{m}}.$$

By taking the inverse limit of the  $\mathbb{T}_{\xi, \tau}^{S_p}(U_m, \mathcal{O})_{\mathfrak{m}}$ -valued deformations of  $\bar{\rho}$  of type  $\mathcal{S}$  we obtain a map  $R_{\mathcal{S}}^{\text{univ}} \rightarrow \mathbb{T}_{\xi, \tau}^{S_p}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}}$ . Therefore,  $\tilde{S}_{\xi, \tau}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}}$  is also equipped with an action of the local deformation ring  $R_{\mathfrak{p}}^{\square}$  via the composition  $R_{\mathfrak{p}}^{\square} \rightarrow R_{\mathcal{S}}^{\text{univ}} \rightarrow \mathbb{T}_{\xi, \tau}^{S_p}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}}$ .

**2.9. Corollary.** *There is a  $G$ -equivariant isomorphism*

$$(M_{\infty}/\mathfrak{a}M_{\infty}) \xrightarrow{\sim} \tilde{S}_{\xi, \tau}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}}^d,$$

which is compatible with the  $R_{\mathfrak{p}}^{\square}$ -action on both sides.

*Proof.* We have a  $KZ$ -equivariant isomorphism  $\psi_{\infty} : (M_{\infty}/\mathfrak{a}M_{\infty}) \xrightarrow{\sim} \tilde{S}_{\xi, \tau}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}}^d$  obtained by taking the projective limit of the  $\Gamma_{2N}$ - and  $Z$ -equivariant isomorphisms  $\psi$  at level  $N$ .

We first show that  $\psi_{\infty}$  is compatible with the  $R_{\mathfrak{p}}^{\square}$ -action on both sides. The  $R_{\mathfrak{p}}^{\square}$ -action on the left hand side is via the map  $R_{\mathfrak{p}}^{\square} \rightarrow R_{\infty}$  and each  $\psi$  at level  $N$  is compatible with  $\phi : R_{\infty} \rightarrow R_{\mathcal{S}}^{\text{univ}}/\mathfrak{d}_N$ . On the other hand, the action of  $R_{\mathfrak{p}}^{\square}$  on the right hand side is via the map  $R_{\mathfrak{p}}^{\square} \rightarrow R_{\mathcal{S}}^{\text{univ}}$  and  $\cap_{N \geq 1} \mathfrak{d}_N = (0)$ . The desired compatibility follows.

Moreover,  $\psi_{\infty}$  fits into a commutative diagram

$$\begin{array}{ccc} (M_{\infty}/\mathfrak{a}M_{\infty}) & \xrightarrow{\psi_{\infty}} & \tilde{S}_{\xi, \tau}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}}^d \\ \downarrow g & & \downarrow g \\ \text{Ind}_{KZ}^G(M_{\infty}/\mathfrak{a}M_{\infty}) & \xrightarrow{\psi_{\infty}} & \text{Ind}_{KZ}^G(\tilde{S}_{\xi, \tau}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}}^d) \end{array}$$

where both vertical maps are induced by the action of  $g \in G$ . The  $G$ -equivariance of  $\psi_{\infty}$  now follows from Proposition 2.8.  $\square$

**2.10. Admissible unitary Banach representations.** Fix a lifting  $r : G_F \rightarrow \mathrm{GL}_n(\mathcal{O})$  of  $\bar{r}$ . We now explain how  $M_\infty$  allows us to associate an admissible unitary Banach representation  $V(r)$  of  $\mathrm{GL}_n(F)$  to  $r$ .

As above, we identify  $F$  with  $\tilde{F}_{\mathfrak{p}}$ . By definition,  $r$  comes from a homomorphism of  $\mathcal{O}$ -algebras  $x : R_{\mathfrak{p}}^{\square} \rightarrow \mathcal{O}$ . We extend this to a homomorphism of  $\mathcal{O}$ -algebras

$$x' : R_{\mathfrak{p}}^{\square} \widehat{\otimes} \left( \widehat{\otimes}_{v \in S_p \setminus \{\mathfrak{p}\}} R_{\bar{v}}^{\square, \xi, \tau} \right) \rightarrow \mathcal{O}$$

by using the homomorphisms  $R_{\bar{v}}^{\square, \xi, \tau} \rightarrow \mathcal{O}$  corresponding to our given potentially diagonalizable representation  $r_{\mathrm{pot.diag}}$ , and then extend  $x'$  arbitrarily to a homomorphism of  $\mathcal{O}$ -algebras  $y : R_\infty \rightarrow \mathcal{O}$ . We set  $V(r) := (M_\infty \otimes_{R_\infty, y} \mathcal{O})^d[1/p]$ .

**2.11. Proposition.** *The representation  $V(r)$  is an admissible unitary Banach representation of  $\mathrm{GL}_n(F)$ .*

**2.12. Remark.** Note that we do not know if  $V(r)$  is independent of either the global setting or the choice of  $y$ . We also do not know that  $V(r)$  is necessarily nonzero, although we will prove that  $V(r) \neq 0$  for many regular de Rham representations  $r$  in Section 5 below, as a consequence of the stronger result that (for the particular choice of  $r$  under consideration) the subspace of locally algebraic vectors in  $V(r)$  is nonzero.

*Proof of Proposition 2.11.* The image of  $(M_\infty \otimes_{R_\infty, y} \mathcal{O})^d$  in  $V(r)$  is a  $\mathrm{GL}_n(F)$ -stable unit ball, so  $V(r)$  is a unitary representation.

In order to see that  $V(r)$  is admissible, we must show that for each  $N \geq 0$ , the  $\mathbb{F}$ -vector space  $((M_\infty \otimes_{R_\infty, y} \mathcal{O})^d \otimes \mathbb{F})^{K_N}$  is finite-dimensional. Writing  $\bar{y}$  for the composite  $y : R_\infty \rightarrow \mathcal{O} \rightarrow \mathbb{F}$ , we must check that  $(M_\infty \otimes_{R_\infty, \bar{y}} \mathbb{F})_{K_N}$  is finite-dimensional. Since  $M_\infty$  is a finite  $S_\infty[[K]]$ -module, and  $\ker \bar{y}$  induces an open ideal of  $S_\infty$ , this is immediate.  $\square$

**2.13. Remark.** While we have assumed throughout this section that  $\bar{r}$  has a potentially diagonalizable lift with regular Hodge–Tate weights, this hypothesis is not needed for our main results, which concern representations  $r : G_F \rightarrow \mathrm{GL}_n(E)$ . Indeed, possibly after making a ramified extension of  $E$ , it is easy to see that any such representation can be conjugated to a representation  $r'$  which factors through  $\mathrm{GL}_n(\mathcal{O})$  and whose reduction  $\bar{r}'$  is semisimple, so that  $\bar{r}'$  has a lift of the required kind (possibly after further extending  $E$ ) by Lemma 2.2.

### 3. HECKE ALGEBRAS AND TYPES

In the following two sections we will use the local Langlands correspondence and the theory of types to establish local-global compatibility results for our patched modules. In particular, we will make use of the results of [BK99] and [SZ99] in order to study spaces of automorphic forms which correspond to fixed inertial types. As explained in the introduction, in some of our results we will for simplicity restrict ourselves to the case of Weil–Deligne representations with  $N = 0$ ; this means that we will limit ourselves to considering potentially crystalline Galois representations. However, some of our results are more naturally expressed in the more general context of representations with arbitrary monodromy, so we will make it clear when we impose this restriction. We begin by collecting and explaining various results from the literature that we will need.

Let  $F/\mathbb{Q}_p$  be a finite extension. Recall that we write  $K = \mathrm{GL}_n(\mathcal{O}_F)$ ,  $G = \mathrm{GL}_n(F)$ , and that  $W_F$  is the Weil group of  $F$ . Although several of our references work over  $\mathbb{C}$ , we work consistently over  $\overline{\mathbb{Q}_p}$  (except in Subsection 3.12, where we fix a single Bernstein component, and work over a finite extension of  $\mathbb{Q}_p$ ). The various results over  $\mathbb{C}$  are transferred to our context over  $\overline{\mathbb{Q}_p}$  via *the fixed choice* of  $\iota : \overline{\mathbb{Q}_p} \xrightarrow{\sim} \mathbb{C}$  everywhere so this is harmless. (The transfer depends on  $\iota$  since an automorphism of  $\mathbb{C}$  does not fix isomorphism classes of representations in general.)

As recalled in Section 1.5, the local Langlands correspondence  $\mathrm{rec}_p$  gives a bijection between the isomorphism classes of irreducible smooth representations of  $G$  over  $\overline{\mathbb{Q}_p}$ , and the  $n$ -dimensional Frobenius semisimple Weil–Deligne representations of  $W_F$ .

**3.1. Bernstein–Zelevinsky theory.** We now recall some details of the local Langlands correspondence and its relationship to Bernstein–Zelevinsky theory, following [Rod82].

Given a partition  $n = n_1 + \cdots + n_r$ , let  $P = MN$  be the corresponding standard parabolic subgroup of  $G$  with Levi subgroup  $M$  and unipotent radical  $N$  (standard with respect to the Borel subgroup of upper triangular matrices), so that  $M \cong \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F)$ . For any smooth representations  $\pi_i$  of  $\mathrm{GL}_{n_i}(F)$ , the tensor product  $\pi_1 \otimes \cdots \otimes \pi_r$  is naturally a representation of  $M$  and thus of  $P$ , and we denote by  $\pi_1 \times \cdots \times \pi_r$  the normalised induction  $i_P^G \pi_1 \otimes \cdots \otimes \pi_r$ . If  $\pi_1, \dots, \pi_r$  are irreducible supercuspidal representations then  $\pi_1 \times \cdots \times \pi_r$  has finite length by [Rod82, Proposition 4].

Any smooth irreducible representation  $\pi$  of  $G$  is necessarily a subrepresentation of some  $\pi_1 \times \cdots \times \pi_r$  for some  $P$  and some irreducible supercuspidal representations  $\pi_1, \dots, \pi_r$ . By [Rod82, Proposition 5],  $\pi$  determines the multiset  $\{\pi_1, \dots, \pi_r\}$  (and thus the corresponding partition of  $n$ , up to reordering) uniquely, and we refer to it as the *cuspidal support* of  $\pi$ . For any representation  $\pi_i$  of  $\mathrm{GL}_{n_i}(F)$ , and any integer  $s$ , we write  $\pi_i(s) := \pi_i \otimes |\det|^s$ . Then by [Rod82, Théorème 1] (and the remark which follows it),  $\pi_1 \times \cdots \times \pi_r$  is reducible if and only if there are some  $i, j$  with  $n_i = n_j$  and  $\pi_j \cong \pi_i(1)$ .

We define a *segment* to be a set of isomorphism classes of irreducible cuspidal representations  $\mathrm{GL}_{n_i}(F)$  of the form  $\Delta = \{\pi_i, \pi_i(1), \dots, \pi_i(r-1)\}$  for some  $r \geq 1$ , and we write  $\Delta = [\pi_i, \pi_i(r-1)]$ . We say that two segments  $\Delta_1, \Delta_2$  are *linked* if neither contains the other, and  $\Delta_1 \cup \Delta_2$  is also a segment. If  $\Delta_1 = [\pi_i, \pi_i']$  and  $\Delta_2 = [\pi_i'', \pi_i''']$  are linked, we say that  $\Delta_1$  *precedes*  $\Delta_2$  if  $\pi_i'' = \pi_i(r)$  for some  $r \geq 0$ .

If  $\Delta = [\pi_i, \pi_i(r-1)]$  is a segment, then  $\pi_i \times \cdots \times \pi_i(r-1)$  has a unique irreducible subrepresentation  $Z(\Delta)$  and a unique irreducible quotient  $L(\Delta)$ . By [Rod82, Théorème 2], if  $\Delta_1, \dots, \Delta_r$  are segments such that if  $i < j$ , then  $\Delta_i$  does not precede  $\Delta_j$ , then the representation  $Z(\Delta_1) \times \cdots \times Z(\Delta_r)$  has a unique irreducible subrepresentation, which we denote  $Z(\Delta_1 \times \cdots \times \Delta_r)$ . Every irreducible smooth representation of  $G$  is of the form  $Z(\Delta_1 \times \cdots \times \Delta_r)$  for some segments  $\Delta_1, \dots, \Delta_r$ , which are uniquely determined up to reordering. By [Rod82, Théorème 3], the analogous statement also holds for the  $L(\Delta_i)$ . By [Rod82, Théorème 5],  $Z(\Delta_1, \dots, \Delta_r)$  occurs with multiplicity one as a subquotient of  $Z(\Delta_1) \times \cdots \times Z(\Delta_r)$ .

The link with the local Langlands correspondence is as follows (see [Rod82, §4.4]). The representation  $\mathrm{rec}_p(\pi)$  is irreducible if and only if  $\pi$  is supercuspidal. For each integer  $r \geq 1$  there is an explicitly defined  $r$ -dimensional Weil–Deligne representation  $\mathrm{Sp}(r)$  whose restriction to the Weil group is unramified and whose

monodromy operator has rank  $r-1$ . Then if  $\Delta = [\pi_i, \pi_i(r-1)]$  is a segment, we have  $\text{rec}_p(L(\Delta)) = \text{rec}_p(\pi_i) \otimes \text{Sp}(r)$ , and more generally we have  $\text{rec}_p(L(\Delta_1, \dots, \Delta_r)) = \bigoplus_{i=1}^r \text{rec}_p(L(\Delta_i))$ .

**3.2. The Bernstein Centre.** We now briefly recall some of the results of [Ber84] (in the special case that the reductive group under consideration there is  $\text{GL}_n$ ) in a fashion adapted to our needs. The *Bernstein spectrum* is the set of  $G$ -orbits of pairs  $(M, \omega)$ , where  $M$  is a Levi subgroup of  $G$ ,  $\omega$  is an irreducible supercuspidal representation of  $M$ , and the action of  $G$  is via conjugation; note that up to conjugacy,  $(M, \omega)$  is of the form  $(\text{GL}_{n_1}(F) \times \dots \times \text{GL}_{n_r}(F), \pi_1 \otimes \dots \otimes \pi_r)$ , as in the preceding section. As we will explain below, the Bernstein spectrum naturally has the structure of an algebraic variety over  $\mathbb{Q}_p$  (with infinitely many connected components), the *Bernstein variety*. Given an irreducible smooth  $G$ -representation  $\pi$ , we obtain a point of the Bernstein spectrum by passing to the cuspidal support of  $\pi$ . This map is surjective; any point  $(M, \omega)$  of the Bernstein spectrum is equal to the cuspidal support of  $\pi$  for any Jordan–Hölder factor  $\pi$  of  $i_P^G \omega$  (and indeed these are precisely the  $\pi$  for which  $(M, \omega)$  arises as the cuspidal support; here  $P$  is any parabolic subgroup of  $G$  admitting  $M$  as a Levi quotient — the collection of Jordan–Hölder factors of  $i_P^G \omega$  is independent of the choice of  $P$ ).

The connected components of the Bernstein variety are as follows. Fix a pair  $(M, \omega)$  as above; then the component of the Bernstein variety containing  $(M, \omega)$  is the union of the  $G$ -orbits of the pairs  $(M, \alpha\omega)$ , where  $\alpha$  is an unramified quasicharacter of  $M$ . We say that two pairs  $(M, \omega)$  and  $(M', \omega')$  are *inertially equivalent* if they are in the same Bernstein component, and write  $[M, \omega]$  for the equivalence class. Fixing one such pair  $(M, \omega)$ , it is easy to see that there is a natural algebraic structure on the inertial equivalence class, because the set of unramified quasicharacters of  $M$  has a natural algebraic structure, and thus so does any quotient of it by a finite group; this gives the structure of the Bernstein variety. Given an irreducible smooth  $G$ -representation  $\pi$ , we will refer to the inertial equivalence class of its cuspidal support as the *inertial support* of  $\pi$ .

For any connected component  $\Omega$  of the Bernstein variety, we have a corresponding full subcategory of the category of smooth  $G$ -representations, whose objects are the smooth representations all of whose irreducible subquotients have cuspidal support in  $\Omega$ . Such a subcategory is called a *Bernstein component* of the category of smooth  $G$ -representations, and in fact the category of smooth  $G$ -representations is a direct product of the Bernstein components. Given a Bernstein component  $\Omega$ , the *centre*  $\mathfrak{Z}_\Omega$  of  $\Omega$  is the centre of the corresponding Bernstein component (that is, the endomorphism ring of the identity functor), so that  $\mathfrak{Z}_\Omega$  acts naturally on each irreducible smooth representation  $\pi \in \Omega$ . Since  $\pi$  is irreducible, each element of  $\mathfrak{Z}_\Omega$  will act on  $\pi$  through a scalar. In fact this scalar depends only on the cuspidal support of  $\pi$ , and in this way  $\mathfrak{Z}_\Omega$  is identified with the ring of regular functions on the connected component  $\Omega$  of the Bernstein variety.

The above notions extend in an obvious way to products of groups  $\text{GL}_{n_i}(F)$ , and we will make use of this extension below without further comment (in order to compare representations of  $G$  with representations of a Levi subgroup).

Finally, we note that from the link between the local Langlands correspondence and Bernstein–Zelevinsky theory explained in Section 3.1, it is immediate that two irreducible smooth representations  $\pi, \pi'$  of  $G$  lie in the same Bernstein component if and only if  $\text{rec}_p(\pi)|_{I_F} \cong \text{rec}_p(\pi')|_{I_F}$  (where we ignore the monodromy operators).

**3.3. Bushnell–Kutzko theory.** Fix a Bernstein component  $\Omega$ . In [BK99], there is the definition of a *semisimple Bushnell–Kutzko type*  $(J, \lambda)$  for  $\Omega$ , where  $J \subseteq K$  is a compact open subgroup, and  $\lambda$  is a smooth irreducible  $\overline{\mathbb{Q}}_p$ -representation of  $J$ . The pair  $(J, \lambda)$  has the property that if  $\pi$  is an irreducible smooth representation of  $G$ , then  $\mathrm{Hom}_J(\lambda, \pi) \neq 0$  if and only if  $\pi \in \Omega$ , and in fact the functor  $\mathrm{Hom}_J(\lambda, *)$  induces an equivalence of categories between  $\Omega$  and the category of left modules of the Hecke algebra  $\mathcal{H}(G, \lambda) := \mathcal{H}(G, J; \lambda) := \mathrm{End}_G(\mathrm{c}\text{-Ind}_J^G \lambda)$ , with the inverse functor being given by  $\mathrm{c}\text{-Ind}_J^G \lambda \otimes_{\mathcal{H}(G, \lambda)} (*)$ .

Let  $\pi$  be an irreducible element of  $\Omega$ , and let  $(M, \omega)$  be a representative for the inertial support of  $\pi$ . We can and do suppose that  $M$  is a standard Levi subgroup  $\prod_{i=1}^r \prod_{j=1}^{d_i} \mathrm{GL}_{e_i}(F)$ , and that  $\omega = \otimes_{i=1}^r \pi_i^{\otimes d_i}$ , where  $\pi_i$  and  $\pi_{i'}$  are not inertially equivalent (that is, do not differ by a twist by an unramified quasicharacter) if  $i \neq i'$ . Then by the construction of  $(J, \lambda)$  in [BK99], there is a pair  $(J \cap M, \lambda_M)$  which is a semisimple Bushnell–Kutzko type for the Bernstein component  $\Omega_M$  of  $M$  determined by  $\omega$ , in the sense that if  $\pi_M$  is an irreducible smooth representation of  $M$ , then  $\mathrm{Hom}_{J \cap M}(\lambda_M, \pi_M) \neq 0$  if and only if  $\pi_M \in \Omega_M$ , and the functor  $\mathrm{Hom}_{J \cap M}(\lambda_M, *)$  induces an equivalence of categories between  $\Omega_M$  and the category of left modules of  $\mathcal{H}(M, \lambda_M)$  in the same way as above.

Let  $P$  be a parabolic subgroup of  $G$  with Levi factor  $M$ . Then the normalised induction  $i_P^G$  restricts to a functor from  $\Omega_M$  to  $\Omega$ , and there is a unique injective algebra homomorphism  $t_P : \mathcal{H}(M, \lambda_M) \rightarrow \mathcal{H}(G, \lambda)$  (which is denoted  $j_{\mathcal{Q}}$  on page 55 of [BK99]) such that under the equivalences of categories explained above,  $i_P^G$  corresponds to the pushforward along  $t_P$ .

It will be useful to us to have a somewhat more explicit description of the pair  $(J \cap M, \lambda_M)$ . Recall that we may write  $M = \prod_{i=1}^r \prod_{j=1}^{d_i} \mathrm{GL}_{e_i}(F)$ , and that  $\omega = \otimes_{i=1}^r \pi_i^{\otimes d_i}$ . Then as is explained in [BK99, §1.4–5] (see also the paragraph after Lemma 7.6.3 of [BK93]), we may write  $J \cap M = \prod_{i=1}^r \prod_{j=1}^{d_i} J_i$ ,  $\lambda_M = \otimes_{i=1}^r \lambda_i^{\otimes d_i}$ , where  $(J_i, \lambda_i)$  is a maximal simple type occurring in  $\pi_i$  in the sense of [BK98]. There is a corresponding isomorphism  $\mathcal{H}(M, \lambda_M) \cong \otimes_{i=1}^r \mathcal{H}(\mathrm{GL}_{e_i}(F), \lambda_i)^{\otimes d_i}$ , and each  $\mathcal{H}(\mathrm{GL}_{e_i}(F), \lambda_i)$  is commutative, so that  $\mathcal{H}(M, \lambda_M)$  is commutative. The following somewhat technical lemma will be useful to us in Section 4. We remind the reader that the algebra  $\mathcal{H}(M, \lambda_M)$  is naturally isomorphic to the convolution algebra of compactly supported functions  $f : M \rightarrow \mathrm{End}_{\overline{\mathbb{Q}}_p}(\lambda_M)$  such that  $f(jmj') = j \circ f(m) \circ j'$  for all  $m \in M, j, j' \in J \cap M$  (see [BL94, §2.2]).

**3.4. Lemma.** *There is an integer  $e \geq 1$  and a  $\overline{\mathbb{Q}}_p$ -basis for  $\mathcal{H}(M, \lambda_M)$  with the property that if  $\nu$  is an element of this basis, then the  $e$ -fold convolution of  $\nu$  with itself is supported on  $t_\nu(J \cap M)$  for some  $t_\nu \in Z(M)$ .*

*Proof.* By the above remarks, we need only prove the corresponding result for the Hecke algebra of a maximal simple type  $\mathcal{H}(\mathrm{GL}_{e_i}(F), \lambda_i)$ . In this case, the proof of Theorem 7.6.1 of [BK93] shows that we can take a basis given by Hecke operators supported on cosets of the form  $tJ_i$  where  $t \in \mathbf{D}(\mathfrak{B})$  in the notation of [BK93]. By Lemma 7.6.3 of [BK93], it suffices to show that there is a positive integer  $e$  such that if  $t \in \mathbf{D}(\mathfrak{B})$  then the  $e$ -fold composition of a Hecke operator  $\psi_t$  supported on  $tJ_i$  with itself is supported on  $sJ_i$ , where  $s$  is a scalar matrix. But  $\mathbf{D}(\mathfrak{B})$  is a cyclic group, generated by a uniformiser  $\varpi_E$  of some finite extension of fields  $E/F$  inside  $\mathrm{GL}_{e_i}(F)$ , so we can just take  $e$  to be the ramification degree  $e(E : F)$ . In that

case, the  $e$ -fold composition of a Hecke operator supported on  $\varpi_E J_i$  with itself is supported on  $\varpi_F J_i$ , as remarked on the bottom of page 201 of [BK93].  $\square$

3.5. *Remark.* If  $\lambda_i$  is a maximal simple type for  $\mathrm{GL}_{n_i}(F)$ , then the ramification degree  $e(E : F)$  is equal to  $n_i/f_i$ , where  $f_i$  is the number of unramified characters of  $F^\times$  which preserve a supercuspidal  $\mathrm{GL}_{n_i}(F)$ -representation containing  $\lambda_i$ . This follows from Lemma 6.2.5 of [BK93]. Therefore,  $\det(\varpi_E) \in F^\times$  has valuation  $f_i$ .

3.6. **Results of Schneider–Zink and Dat.** We will need a slight refinement of the Bernstein centre and of the theory of Bushnell–Kutzko, which is constructed in the paper [SZ99]. Note that there is not quite a bijection between irreducible smooth representations of  $G$  and characters of the Bernstein centre; as explained in Section 3.2, any two Jordan–Hölder constituents of a parabolic induction from an irreducible cuspidal representation correspond to the same character of the centre, so that for example the trivial representation and the Steinberg representation correspond to the same character. Furthermore, as recalled in Section 3.2, two irreducible representations lie in the same Bernstein component if and only if the corresponding Weil–Deligne representations agree on inertia, but have possibly differing monodromy operators, and it can be useful to have a finer decomposition. In particular, we wish to be able to consider only the representations with  $N = 0$ .

Let  $\Omega$  be a Bernstein component with corresponding semisimple Bushnell–Kutzko type  $(J, \lambda)$ . Let  $\mathrm{Irr}(\Omega)_0$  denote the irreducible elements  $\pi$  of  $\Omega$  with the property that  $N = 0$  on  $\mathrm{rec}_p(\pi)$ . In [SZ99, §2], the material recalled in Section 3.1 above is recast in terms of certain partition valued functions which are relevant for  $\Omega$  in the terminology of [BS07]. A partial ordering is defined on these functions, and there is a unique maximal element for this ordering, which we will denote by  $\mathcal{P}$  from now on. In [SZ99, §6] an irreducible direct summand  $\sigma_{\mathcal{P}}(\lambda)$  of  $\mathrm{Ind}_J^K \lambda$  is constructed, with the property that if  $\pi$  is an irreducible smooth  $G$ -representation, then  $\mathrm{Hom}_K(\sigma_{\mathcal{P}}(\lambda), \pi) \neq 0$  if and only if  $\pi \in \mathrm{Irr}(\Omega)_0$ , in which case  $\sigma_{\mathcal{P}}(\lambda)$  occurs in  $\pi|_K$  with multiplicity one. (This follows immediately from Proposition 6.2 of [SZ99], using the relationship between the partial ordering on partition valued functions in [SZ99] and the monodromy operators, which is explained in the proof of Proposition 6.5.3 of [BC09].)

If  $\tau$  is the inertial type corresponding to  $\Omega$ , then we write  $\sigma(\tau)$  for  $\sigma_{\mathcal{P}}(\lambda)$ . As remarked on page 201 of [SZ99], since  $\mathcal{P}$  is maximal,  $\sigma(\tau)$  coincides with the representation  $\rho_s = e_K \mathrm{Ind}_J^K \lambda$  constructed in Theorem 4.1 of [Dat99]. Here  $e_K$  is a certain idempotent in  $\mathcal{H}(G, \lambda)$  which is constructed in [Dat99, §4.2].

Let  $\mathfrak{Z}_\Omega$  be the centre of  $\Omega$ . By Theorem 4.1 of [Dat99], the action of  $\mathfrak{Z}_\Omega$  on  $\mathrm{c}\text{-Ind}_K^G \sigma(\tau)$  induces an isomorphism

$$\mathfrak{Z}_\Omega \cong \mathrm{End}_G(\mathrm{c}\text{-Ind}_K^G \sigma(\tau)) =: \mathcal{H}(G, \sigma(\tau)),$$

so in particular  $\mathcal{H}(G, \sigma(\tau))$  is commutative.

Let  $W_{[M, \omega]} := N_G([M, \omega])/M$  be the relative normaliser of the inertial equivalence class of  $(M, \omega)$ , with  $(M, \omega)$  as in Section 3.2. Then by Proposition 2.1 of [Dat99], the natural algebra homomorphism  $t_P : \mathcal{H}(M, \lambda_M) \rightarrow \mathcal{H}(G, \lambda)$  induces an isomorphism  $t_P : \mathcal{H}(M, \lambda_M)^{W_{[M, \omega]}} \xrightarrow{\sim} Z(\mathcal{H}(G, \lambda))$ . It follows from the discussion in [Dat99, §4.2] that there is an algebra homomorphism  $s_P : \mathcal{H}(G, \lambda) \rightarrow$

$\mathcal{H}(G, \sigma(\tau))$ , which makes  $\mathcal{H}(G, \sigma(\tau))$  a direct summand of  $\mathcal{H}(G, \lambda)$ , and the composite  $s_P \circ t_P$  induces an isomorphism  $\mathcal{H}(M, \lambda_M)^{W_{[M, \omega]}} \xrightarrow{\sim} \mathcal{H}(G, \sigma(\tau))$ . (The map  $s_P$  is just given by  $h \mapsto e_K * h * e_K$ , where  $e_K$  is the idempotent mentioned above.)

We summarise much of the preceding discussion as the following theorem.

**3.7. Theorem.** *If  $\tau$  is an inertial type, then there is a finite-dimensional smooth irreducible  $\overline{\mathbb{Q}}_p$ -representation  $\sigma(\tau)$  of  $K$  such that if  $\pi$  is any irreducible smooth  $\overline{\mathbb{Q}}_p$ -representation of  $G$ , then the restriction of  $\pi$  to  $K$  contains (an isomorphic copy of)  $\sigma(\tau)$  as a subrepresentation if and only if  $\text{rec}_p(\pi)|_{I_F} \sim \tau$  and  $N = 0$  on  $\text{rec}_p(\pi)$ . Furthermore, in this case the restriction of  $\pi$  to  $K$  contains a unique copy of  $\sigma(\tau)$ . The ring  $\mathcal{H}(G, \sigma(\tau)) := \text{End}_G(\text{c-Ind}_K^G \sigma(\tau))$  is commutative.*

**3.8. Proposition.** *Let  $\pi_i$  be an irreducible supercuspidal representation of  $\text{GL}_{n_i}(F)$  for  $1 \leq i \leq r$ , and  $n = n_1 + \dots + n_r$ . If  $\pi_j \not\cong \pi_i(1)$  (the condition being empty if  $n_i \neq n_j$ ) for every  $i < j$  then the  $G$ -socle and the  $G$ -cosocle of  $\pi_1 \times \dots \times \pi_r$  are irreducible. Moreover, the socle occurs as a subquotient with multiplicity one and is the only generic subquotient of  $\pi_1 \times \dots \times \pi_r$ .*

*Proof.* This is an exercise in Bernstein–Zelevinsky theory [BZ77, Zel80]. We will make use of the material recalled from [Rod82] at the beginning of this section. If we let  $\Delta_1 = \{\pi_1\}, \dots, \Delta_r = \{\pi_r\}$ , then by assumption the  $\Delta_i$  are segments such that  $\Delta_i$  does not precede  $\Delta_j$  for  $i < j$ . Since the segments are of length one, we have  $L(\Delta_i) \cong Z(\Delta_i) \cong \pi_i$  by definition, so that as recalled above  $\pi_1 \times \dots \times \pi_r$  has a unique irreducible subrepresentation  $Z(\Delta_1, \dots, \Delta_r)$  and a unique irreducible quotient  $L(\Delta_1, \dots, \Delta_r)$ . In addition,  $Z(\Delta_1, \dots, \Delta_r)$  occurs as a subquotient with multiplicity one, so we only need to show that it is the unique generic subquotient.

Let  $U_n$  be the subgroup of unipotent upper-triangular matrices in  $\text{GL}_n(F)$ , and let  $\theta_n : U_n \rightarrow \overline{\mathbb{Q}}_p^\times$  be the character  $(u_{ij}) \mapsto \psi(\sum_{i=1}^{n-1} u_{i, i+1})$ , where  $\psi : F \rightarrow \overline{\mathbb{Q}}_p^\times$  is a fixed smooth non-trivial character. If  $\pi$  is a representation of  $G$ , we let  $\pi_{\theta_n}$  be the largest quotient of  $\pi$  on which  $U_n$  acts by  $\theta_n$ . If  $\pi$  is an irreducible representation of  $G$  then the dimension of  $\pi_{\theta_n}$  is at most one, and is equal to one if and only if  $\pi$  is generic. Since  $U_n$  is equal to the union of its compact open subgroups, the functor  $\pi \mapsto \pi_{\theta_n}$  is exact. Thus it is enough to show that  $(\pi_1 \times \dots \times \pi_r)_{\theta_n}$  is one dimensional and  $Z(\Delta_1, \dots, \Delta_r)_{\theta_n}$  is non-zero.

In [BZ77, Zel80] the authors define a family of exact functors  $\pi \mapsto \pi^{(i)}$  for  $0 \leq i \leq n$  from the category of smooth representations of  $\text{GL}_n(F)$  to the category of smooth representations of  $\text{GL}_{n-i}(F)$ . The representation  $\pi^{(i)}$  is called the  $i$ -th derivative of  $\pi$ . We have  $\pi^{(0)} = \pi$  and  $\pi^{(n)} = \pi_{\theta_n}$ . If  $\pi$  is irreducible and supercuspidal then  $\pi^{(i)} = 0$  for  $0 < i < n$  and  $\pi^{(n)}$  is one dimensional, by [BZ77, Theorem 4.4]. By [BZ77, Corollary 4.6] we have that  $(\pi_1 \times \dots \times \pi_r)^{(n)} \cong (\pi_1)^{(n_1)} \otimes \dots \otimes (\pi_r)^{(n_r)}$ , which is one dimensional, since the  $\pi_i$  are supercuspidal representations of  $\text{GL}_{n_i}(F)$ . It follows from [Zel80, Theorem 6.2] that  $Z(\Delta_1, \dots, \Delta_r)^{(n)} \neq 0$ , as required.  $\square$

If  $\pi$  in  $\Omega$  is irreducible, then the action of  $\mathfrak{Z}_\Omega$  on  $\pi$  defines a maximal ideal  $\chi_\pi : \mathfrak{Z}_\Omega \rightarrow \text{End}_G(\pi) \cong \overline{\mathbb{Q}}_p$ . The following is a strengthening of Theorem 4.1 of [Dat99].

**3.9. Proposition.** *Let  $\pi$  be an irreducible smooth  $\overline{\mathbb{Q}}_p$ -representation of  $G$ , such that  $\text{rec}_p(\pi)|_{I_F} \sim \tau$  and  $N = 0$  on  $\text{rec}_p(\pi)$ . Then*

$$\text{c-Ind}_K^G \sigma(\tau) \otimes_{\mathfrak{Z}_\Omega, \chi_\pi} \overline{\mathbb{Q}}_p \cong \pi_1 \times \dots \times \pi_r,$$

where  $\pi_i$  is a supercuspidal representation of  $\mathrm{GL}_{n_i}(F)$ , such that if  $i < j$  then  $\pi_j \not\cong \pi_i(1)$  (the condition being empty if  $n_i \neq n_j$ ), and  $\mathrm{rec}_p(\pi) \cong \bigoplus_{i=1}^r \mathrm{rec}_p(\pi_i)$ . Moreover,  $\pi$  is the unique irreducible quotient of  $\mathrm{c}\text{-Ind}_K^G \sigma(\tau) \otimes_{\mathfrak{Z}_\Omega, \chi_\pi} \overline{\mathbb{Q}}_p$ .

*Proof.* Since  $N = 0$  on  $\mathrm{rec}_p(\pi)$ , we may write it as a direct sum of irreducible representations of the Weil group  $W_F$ , and there is a partition  $n = n_1 + \dots + n_r$  and supercuspidal representations  $\pi_i$  of  $\mathrm{GL}_{n_i}(F)$  such that  $\mathrm{rec}_p(\pi) \cong \bigoplus_{i=1}^r \mathrm{rec}_p(\pi_i)$ . After reordering we may assume that if  $i < j$  then  $\pi_j \not\cong \pi_i(1)$ . It follows from Proposition 3.8 that  $\pi_1 \times \dots \times \pi_r$  has a unique irreducible quotient  $\pi'$ .

Then  $\mathrm{rec}_p(\pi') \cong \bigoplus_{i=1}^r \mathrm{rec}_p(\pi_i)$  as representations of  $W_F$ , and  $N = 0$  on  $\mathrm{rec}_p(\pi')$  since all the segments have length one, as in the proof of Proposition 3.8. Thus  $\mathrm{rec}_p(\pi') \cong \mathrm{rec}_p(\pi)$ , which implies that  $\pi \cong \pi'$ . Since the socle of  $\pi_1 \times \dots \times \pi_r$  is irreducible and occurs as a subquotient with multiplicity one, the action of  $\mathfrak{Z}_\Omega$  on  $\pi_1 \times \dots \times \pi_r$  factors through a maximal ideal  $\chi : \mathfrak{Z}_\Omega \rightarrow \overline{\mathbb{Q}}_p$ . Since the cosocle is isomorphic to  $\pi$ , we deduce that  $\chi = \chi_\pi$ .

Theorem 3.7 implies that  $\pi$  is a quotient of  $\mathrm{c}\text{-Ind}_K^G \sigma(\tau)$ . Since  $\mathrm{c}\text{-Ind}_K^G \sigma(\tau)$  is projective there exists a  $G$ -equivariant map  $\varphi : \mathrm{c}\text{-Ind}_K^G \sigma(\tau) \rightarrow \pi_1 \times \dots \times \pi_r$  such that the composition with  $\pi_1 \times \dots \times \pi_r \twoheadrightarrow \pi$  is surjective. The  $G$ -cosocle of the cokernel of  $\varphi$  is zero. Since  $\pi_1 \times \dots \times \pi_r$  is of finite length, so is the cokernel of  $\varphi$ , and we deduce that  $\varphi$  is surjective. Since  $\mathfrak{Z}_\Omega$  acts naturally on everything,  $\varphi$  induces a surjection  $\bar{\varphi} : \mathrm{c}\text{-Ind}_K^G \sigma(\tau) \otimes_{\mathfrak{Z}_\Omega, \chi_\pi} \overline{\mathbb{Q}}_p \twoheadrightarrow \pi_1 \times \dots \times \pi_r$ .

Then [Dat99, Theorem 4.1] implies that the semisimplification of  $\mathrm{c}\text{-Ind}_K^G \sigma(\tau) \otimes_{\mathfrak{Z}_\Omega, \chi_\pi}$  is isomorphic to the semisimplification of  $\pi'_1 \times \dots \times \pi'_s$ , where  $\pi'_i$  are supercuspidal representations of  $\mathrm{GL}_{n'_i}(F)$  for some integers  $n'_i$ , such that  $n = n'_1 + \dots + n'_s$ . Since  $\pi$  is an irreducible subquotient of both  $\pi'_1 \times \dots \times \pi'_s$  and  $\pi_1 \times \dots \times \pi_r$ , [BZ77, Theorem 2.9] implies that  $\pi'_1 \times \dots \times \pi'_s$  and  $\pi_1 \times \dots \times \pi_r$  have the same semisimplification. Thus  $\mathrm{c}\text{-Ind}_K^G \sigma(\tau) \otimes_{\mathfrak{Z}_\Omega, \chi_\pi} \overline{\mathbb{Q}}_p$  and  $\pi_1 \times \dots \times \pi_r$  have the same semisimplification, which implies that  $\bar{\varphi}$  is an isomorphism, as required.  $\square$

**3.10. Corollary.** *Let  $\Omega$  be a Bernstein component corresponding to an inertial type  $\tau$  and let  $\mathfrak{Z}_\Omega$  be the centre of  $\Omega$ . Let  $\chi : \mathfrak{Z}_\Omega \rightarrow \overline{\mathbb{Q}}_p$  be a maximal ideal of  $\mathfrak{Z}_\Omega$ . Then the  $G$ -socle of  $\mathrm{c}\text{-Ind}_K^G \sigma(\tau) \otimes_{\mathfrak{Z}_\Omega, \chi} \overline{\mathbb{Q}}_p$  is irreducible and generic, and all the other irreducible subquotients are not generic.*

*Conversely, if an irreducible representation  $\pi$  in  $\Omega$  is generic then  $\pi$  is isomorphic to the  $G$ -socle of  $\mathrm{c}\text{-Ind}_K^G \sigma(\tau) \otimes_{\mathfrak{Z}_\Omega, \chi_\pi} \overline{\mathbb{Q}}_p$ .*

*Proof.* The first part follows from Propositions 3.8 and 3.9. The converse may be seen as follows. There exist supercuspidal representations  $\pi_i$  of  $\mathrm{GL}_{n_i}(F)$  such that  $n = n_1 + \dots + n_r$  and  $\pi$  is a subquotient of  $\pi_1 \times \dots \times \pi_r$ . If  $w$  is a permutation of  $\{1, \dots, r\}$  then  $\pi_1 \times \dots \times \pi_r$  and  $\pi_{w(1)} \times \dots \times \pi_{w(r)}$  have the same semisimplification by [BZ77, Theorem 2.9], so we may assume that the  $\pi_i$  satisfy the conditions of Proposition 3.8. Since the socle of  $\pi_1 \times \dots \times \pi_r$  is irreducible and occurs as a subquotient with multiplicity one, the action of  $\mathfrak{Z}_\Omega$  on  $\pi_1 \times \dots \times \pi_r$  factors through a maximal ideal, which is equal to  $\chi_\pi$ , as  $\pi$  occurs as a subquotient. If we let  $\pi'$  be the  $G$ -cosocle of  $\pi_1 \times \dots \times \pi_r$  then  $\pi'$  satisfies the conditions of Proposition 3.9, and we have  $\chi_{\pi'} = \chi_\pi$ . The assertion follows from Propositions 3.8 and 3.9.  $\square$

**3.11. Corollary.** *Let  $\pi$  be an irreducible smooth generic  $\overline{\mathbb{Q}}_p$ -representation of  $G$ , such that  $\mathrm{rec}_p(\pi)|_{I_F} \sim \tau$  and  $N = 0$  on  $\mathrm{rec}_p(\pi)$ . Then we have a natural isomorphism  $\mathrm{c}\text{-Ind}_K^G \sigma(\tau) \otimes_{\mathfrak{Z}_\Omega, \chi_\pi} \overline{\mathbb{Q}}_p \cong \pi$ .*

*Proof.* By Corollary 3.10, we see that the  $G$ -socle of  $\mathrm{c}\text{-Ind}_K^G \sigma(\tau) \otimes_{\mathfrak{Z}_{\Omega, \chi\pi}} \overline{\mathbb{Q}}_p$  is isomorphic to  $\pi$  and occurs with multiplicity one as a subquotient. Theorem 3.7 implies that  $\pi$  is a quotient of  $\mathrm{c}\text{-Ind}_K^G \sigma(\tau) \otimes_{\mathfrak{Z}_{\Omega, \chi\pi}} \overline{\mathbb{Q}}_p$ . This implies the assertion.  $\square$

**3.12. Rationality.** As a preparation for the next section we explain in this subsection how the results above remain true with a finite extension  $E$  of  $\mathbb{Q}_p$  as coefficient field, as long as  $E$  is sufficiently large (depending on the Bernstein component  $\Omega$ ). We do this by following various parts of the construction of the Bernstein centre in [Ren10], working over  $E$  rather than over  $\overline{\mathbb{Q}}_p$ , and we then deduce the results from those over  $\overline{\mathbb{Q}}_p$  by faithfully flat descent. Let  $(M, \omega)$  be the supercuspidal support of some irreducible representation in  $\Omega$ , let  $\mathcal{X}(M)$  be the group of unramified characters  $\chi : M \rightarrow \overline{\mathbb{Q}}_p^\times$ , and let

$$\mathcal{X}(M)(\omega) := \{\chi \in \mathcal{X}(M) : \omega \cong \omega \otimes \chi\}.$$

Let  $M^0$  be the intersection of the kernels of the characters  $\chi \in \mathcal{X}(M)$ , and let  $T$  be the intersection of the kernels of the  $\chi \in \mathcal{X}(M)(\omega)$ . The restriction of  $\omega$  to  $M^0$  is a finite direct sum of irreducible representations, see [Ren10, VI.3.2]. We fix one irreducible summand  $\rho$ . It follows from Lemme VI.4.4 of [Ren10] and its proof that  $\rho$  extends to a representation  $\rho_T$  of  $T$ , such that  $\mathrm{Ind}_T^M \rho_T$  is isomorphic to a finite direct sum of copies of  $\omega$ . Thus  $\chi \in \mathcal{X}(M)$  lies in  $\mathcal{X}(M)(\omega)$  if and only if the restriction of  $\chi$  to  $T$  is trivial. Thus the restriction to  $T$  induces a bijection

$$(3.13) \quad \mathcal{X}(M)/\mathcal{X}(M)(\omega) \xrightarrow{\cong} \mathcal{X}(T),$$

where  $\mathcal{X}(T)$  is the group of characters from  $T$  to  $\overline{\mathbb{Q}}_p^\times$ , which are trivial on  $M^0$ .

Let  $D$  be the Bernstein component (for  $M$ ) containing  $\omega$ , let  $\mathfrak{Z}_D$  be the centre of  $D$  and let  $\Pi(D) := \mathrm{c}\text{-Ind}_{M^0}^M \rho$ . It is shown in [Ren10, VI.4.1] that  $\Pi(D)$  is a projective generator for  $D$ . Thus we may identify  $\mathfrak{Z}_D$  with the centre of the ring  $\mathrm{End}_M(\Pi(D))$ . Since  $\rho$  is irreducible,  $\Pi(D)$  is a finitely generated  $\overline{\mathbb{Q}}_p[M]$ -module.

Let  $\mathfrak{Z}_\Omega$  be the Bernstein centre of  $\Omega$  and let  $\Pi(\Omega) := i_P^G \Pi(D)$ , where  $P$  is any parabolic subgroup with Levi subgroup  $M$ . It is shown in [Ren10, Thm.VI.10.1] that  $\Pi(\Omega)$  is a projective generator of  $\Omega$ , which is a finitely generated  $\overline{\mathbb{Q}}_p[G]$ -module. Thus we may identify  $\mathfrak{Z}_\Omega$  with the centre of the ring  $\mathrm{End}_G(\Pi(\Omega))$ .

It follows from Bushnell–Kutzko theory that  $\omega \cong \mathrm{c}\text{-Ind}_{\mathbf{J}}^M \Lambda$ , and  $\rho \cong \mathrm{c}\text{-Ind}_J^M \lambda$ , where  $\mathbf{J}$  is an open compact-mod-centre subgroup of  $M$ ,  $J$  is an open compact subgroup of  $M$ , and  $\Lambda, \lambda$  are (necessarily) finite-dimensional irreducible representations. We may realise both  $\Lambda$  and  $\lambda$  over a finite extension  $E$  of  $\mathbb{Q}_p$ . By compactly inducing these realisations, we deduce that both  $\omega$  and  $\rho$  can be realised over  $E$ . We denote these representations by  $\omega_E$  and  $\rho_E$ , respectively. It is shown in [Ren10, Lemme V.2.7] that  $\mathcal{X}(M)(\omega)$  is a finite group. Let  $W(D)$  be the subgroup of  $N_G(M)/M$  stabilising  $D$ . For each  $w \in W(D)$  there are precisely  $|\mathcal{X}(M)(\omega)|$  unramified characters  $\xi$  such that  $\omega^w \cong \omega \otimes \xi$ . Since the group  $M/M^0$  is finitely generated, by replacing  $E$  with a finite extension, we may assume that all the characters  $\xi$  are  $E$ -valued. By further enlarging  $E$  we may assume that  $\sqrt{q}$ , where  $q$  is the number of elements in the residue field of  $F$ , is contained in  $E$ . Then the modulus character of  $P$  is defined over  $E$ .

Let  $\Pi(D)_E := \mathrm{c}\text{-Ind}_{M^0}^M \rho_E$  and let  $\mathfrak{Z}_{D,E}$  be the centre of  $\mathrm{End}_M(\mathrm{c}\text{-Ind}_{M^0}^M \rho_E)$ . Since  $\rho_E \otimes_E \overline{\mathbb{Q}}_p \cong \rho$ , we have  $\Pi(D)_E \otimes_E \overline{\mathbb{Q}}_p \cong \Pi(D)$ . We may express the generators of  $\Pi(D)$  (as a  $\overline{\mathbb{Q}}_p[M]$ -module) as a finite  $\overline{\mathbb{Q}}_p$ -linear combination of elements of  $\Pi(D)_E$ .

The  $E[M]$ -submodule of  $\Pi(D)_E$  generated by these elements has to equal to  $\Pi(D)_E$ , as the quotient is zero once we extend the scalars to  $\overline{\mathbb{Q}}_p$ . In particular,  $\Pi(D)_E$  is a finitely generated  $E[M]$ -module.

Let  $\Pi(\Omega)_E := i_P^G \Pi(D)_E$  and let  $\mathfrak{Z}_{\Omega,E}$  be the centre of  $\text{End}_G(\Pi(\Omega)_E)$ . The smooth parabolic induction commutes with  $\otimes_E \overline{\mathbb{Q}}_p$ , as the set  $P \backslash G/H$  is finite for every open subgroup  $H$  of  $G$  and tensor products commute with inductive limits, so  $\Pi(\Omega)_E \otimes_E \overline{\mathbb{Q}}_p \cong \Pi(\Omega)$ . Since  $\Pi(\Omega)$  is a finitely generated  $\overline{\mathbb{Q}}_p[G]$ -module, arguing as in the previous paragraph we deduce that  $\Pi(\Omega)_E$  is a finitely generated  $E[G]$ -module.

The following observation (see for example Lemma 5.1 of [Paš13]) is very useful. If  $\pi$  and  $\pi'$  are representations of some group  $G$  on  $E$ -vector spaces, such that  $\pi$  is a finitely generated  $E[G]$ -module, then

$$(3.14) \quad \text{Hom}_{E[G]}(\pi, \pi') \otimes_E \overline{\mathbb{Q}}_p \cong \text{Hom}_{\overline{\mathbb{Q}}_p[G]}(\pi \otimes_E \overline{\mathbb{Q}}_p, \pi' \otimes_E \overline{\mathbb{Q}}_p).$$

It follows from (3.14) that

$$(3.15) \quad \text{End}_M(\Pi(D)_E) \otimes_E \overline{\mathbb{Q}}_p \cong \text{End}_M(\Pi(D)),$$

$$(3.16) \quad \text{End}_G(\Pi(\Omega)_E) \otimes_E \overline{\mathbb{Q}}_p \cong \text{End}_G(\Pi(\Omega)).$$

Let  $D_E$  be the full subcategory of smooth representation  $\omega'$  of  $M$  on  $E$ -vector spaces, such that  $\omega' \otimes_E \overline{\mathbb{Q}}_p$  is in  $D$ . It follows from (3.14) that  $\text{Hom}_M(\Pi(D)_E, \omega') \otimes_E \overline{\mathbb{Q}}_p \cong \text{Hom}_M(\Pi(D), \omega' \otimes_E \overline{\mathbb{Q}}_p)$ . Since  $\Pi(D)$  is a projective generator of  $D$ , we deduce that  $\Pi(D)_E$  is a projective generator of  $D_E$ . (This follows from the fact that  $\overline{\mathbb{Q}}_p$  is faithfully flat over  $E$ ; we will repeatedly use this fact below without further comment.) In particular,  $\mathfrak{Z}_{D,E}$  is naturally isomorphic to the centre of the category  $D_E$ .

Similarly we let  $\Omega_E$  be the full subcategory of smooth representations  $\pi'$  of  $G$  on  $E$ -vector spaces, such that  $\pi' \otimes_E \overline{\mathbb{Q}}_p$  is in  $\Omega$ . The same argument as above gives that  $\Pi(\Omega)_E$  is a projective generator of  $\Omega_E$  and  $\mathfrak{Z}_{\Omega,E}$  is naturally isomorphic to the centre of  $\Omega_E$ .

**3.17. Lemma.** *Let  $\mathcal{A}$  be an  $E$ -algebra and let  $\mathcal{Z}$  be an  $E$ -subalgebra of  $\mathcal{A}$ . If  $\mathcal{Z} \otimes_E \overline{\mathbb{Q}}_p$  is the centre of  $\mathcal{A} \otimes_E \overline{\mathbb{Q}}_p$  then  $\mathcal{Z}$  is the centre of  $\mathcal{A}$ .*

*Proof.* For each  $z \in \mathcal{Z}$ , we define an  $E$ -linear map  $\phi_z : \mathcal{A} \rightarrow \mathcal{A}$ ,  $a \mapsto az - za$ . Since  $z \otimes 1$  is central in  $\mathcal{A} \otimes_E \overline{\mathbb{Q}}_p$  by assumption, we deduce that  $(\text{Im } \phi_z) \otimes_E \overline{\mathbb{Q}}_p = 0$ , which implies that  $\text{Im } \phi_z = 0$ . We deduce that  $\mathcal{Z}$  is contained in  $\mathcal{Z}(\mathcal{A})$ , the centre of  $\mathcal{A}$ . If  $z \in \mathcal{Z}(\mathcal{A})$  then  $z \otimes 1$  is contained in the centre of  $\mathcal{A} \otimes_E \overline{\mathbb{Q}}_p$  and thus in  $\mathcal{Z} \otimes_E \overline{\mathbb{Q}}_p$  by assumption. Hence  $(\mathcal{Z}(\mathcal{A})/\mathcal{Z}) \otimes_E \overline{\mathbb{Q}}_p = 0$ , which implies that  $\mathcal{Z} = \mathcal{Z}(\mathcal{A})$ .  $\square$

**3.18. Lemma.** *The isomorphism (3.15) induces an isomorphism  $\mathfrak{Z}_{D,E} \otimes_E \overline{\mathbb{Q}}_p \cong \mathfrak{Z}_D$ .*

*Proof.* Since  $\Pi(D) \cong \text{Ind}_T^M(\text{c-Ind}_{M^0}^T \rho)$  and induction is a functor, we have an inclusion  $\text{End}_T(\text{c-Ind}_{M^0}^T \rho) \subset \text{End}_M(\Pi(D))$ . It follows from [Ren10, Thm.VI.4.4] that this inclusion identifies  $\text{End}_T(\text{c-Ind}_{M^0}^T \rho)$  with the centre of  $\text{End}_M(\Pi(D))$ . The assertion follows from Lemma 3.17 applied to  $\mathcal{Z} = \text{End}_T(\text{c-Ind}_{M^0}^T \rho_E)$  and  $\mathcal{A} = \text{End}_M(\Pi(D)_E)$ .  $\square$

Let  $\text{Irr}(D)$  be the set of irreducible representations in  $D$ . Every such irreducible representation is of the form  $\omega \otimes \chi$  for some  $\chi \in \mathcal{X}(M)$ . We thus have a bijection

$\mathcal{X}(M)/\mathcal{X}(M)(\omega) \xrightarrow{\cong} \text{Irr}(D)$ ,  $\chi \mapsto \omega \otimes \chi$ . Composing this bijection with (3.13) we obtain a bijection

$$(3.19) \quad \text{Irr}(D) \xrightarrow{\cong} \mathcal{X}(T).$$

Now  $\mathcal{X}(T)$  is naturally isomorphic to the set of homomorphisms of  $\overline{\mathbb{Q}}_p$ -algebras from  $\overline{\mathbb{Q}}_p[T/M^0]$  to  $\overline{\mathbb{Q}}_p$ . It is explained in [Ren10, VI.4.4] that we have identifications

$$\mathfrak{Z}_D \cong \text{End}_T(\text{Ind}_{M^0}^T \rho) \cong \overline{\mathbb{Q}}_p[T/M^0];$$

see Théorème VI.4.4 for the first isomorphism and Proposition VI.4.4 for the second, so that (3.19) induces a natural bijection between  $\text{Irr}(D)$  and  $\text{MaxSpec } \mathfrak{Z}_D$ .

The group  $W(D)$  acts on  $\text{Irr}(D)$  by conjugation. For each  $w \in W(D)$  let  $\xi \in \mathcal{X}(M)$  be any character such that  $\omega^w \cong \omega \otimes \xi$ , and let  $\xi_w$  be the restriction of  $\xi$  to  $T$ . Then  $\xi_w$  depends only on  $w$  and not on the choice of  $\xi$ . If  $\chi \in \mathcal{X}(M)$  then  $(\omega \otimes \chi)^w \cong \omega \otimes \chi^w \xi$ . Thus the action of  $W(D)$  on  $\mathcal{X}(T)$  via (3.19) is given by  $w \cdot \chi = \chi^w \xi_w$ . It is immediate that this action is induced by the action of  $W(D)$  on  $\overline{\mathbb{Q}}_p[T/M^0]$  given by  $w \cdot (tM^0) = \xi_w^{-1}(t)t^w M^0$ .

**3.20. Lemma.** *The action of  $W(D)$  on  $\mathfrak{Z}_D$  preserves  $\mathfrak{Z}_{D,E}$ .*

*Proof.* Since  $\omega$  and  $\rho$  can both be defined over  $E$ , so can the representation  $\rho_H$  defined at the beginning of [Ren10, VI.4.4], and in particular so can  $\rho_T$ , its restriction to  $T$ . Hence, if we identify  $\mathfrak{Z}_D$  with  $\overline{\mathbb{Q}}_p[T/M^0]$  as in [Ren10, Prop. VI.4.4] then  $\mathfrak{Z}_{D,E}$  is identified with  $E[T/M^0]$ . Since the characters  $\xi_w$  are  $E$ -valued by the choice of  $E$ , we get the assertion.  $\square$

**3.21. Lemma.**  $\mathfrak{Z}_{\Omega,E} = \mathfrak{Z}_{D,E}^{W(D)}$ .

*Proof.* Since  $\Pi(\Omega) = i_P^G \Pi(D)$  and parabolic induction is a functor, we have an inclusion  $\mathfrak{Z}_D \subset \text{End}_G(\Pi(\Omega))$ . It follows from the discussion immediately preceding the proof of Theorem VI.10.4 of [Ren10] that this inclusion identifies  $\mathfrak{Z}_D^{W(D)}$  with the centre of  $\text{End}_G(\Pi(\Omega))$ . The assertion follows from Lemma 3.17 applied to  $\mathcal{Z} = \mathfrak{Z}_{D,E}^{W(D)}$  and  $\mathcal{A} = \text{End}_G(\Pi(\Omega)_E)$ .  $\square$

**3.22. Proposition.** *The isomorphism (3.16) induces an isomorphism  $\mathfrak{Z}_{\Omega,E} \otimes_E \overline{\mathbb{Q}}_p \cong \mathfrak{Z}_\Omega$ .*

*Proof.* Using Lemmas 3.18 and 3.21 we obtain

$$\mathfrak{Z}_{\Omega,E} \otimes_E \overline{\mathbb{Q}}_p \cong \mathfrak{Z}_{D,E}^{W(D)} \otimes_E \overline{\mathbb{Q}}_p \cong (\mathfrak{Z}_{D,E} \otimes_E \overline{\mathbb{Q}}_p)^{W(D)} \cong \mathfrak{Z}_D^{W(D)} \cong \mathfrak{Z}_\Omega,$$

the last isomorphism following from [Ren10, VI.10.4] as in the proof of Lemma 3.21.  $\square$

**3.23. Lemma.**  $\mathfrak{Z}_{\Omega,E}$  coincides with the ring  $E[\mathcal{B}]$  constructed in [Che09, Prop. 3.11].

*Proof.* Let  $\Delta$  be the subgroup of  $\mathcal{X}(M) \rtimes W(D)$  consisting of pairs  $(\xi, w)$ , such that  $\omega^w \cong \omega \otimes \xi$ . This subgroup acts naturally on  $E[M/M^0]$ . The map  $\xi \mapsto (\xi, 1)$  identifies  $\mathcal{X}(M)(\omega)$  with a normal subgroup of  $\Delta$  and the quotient is isomorphic to  $W(D)$ . We have

$$\overline{\mathbb{Q}}_p[M/M^0]^\Delta \cong (\overline{\mathbb{Q}}_p[M/M^0]^{\mathcal{X}(M)(\omega)})^{W(D)} \cong \overline{\mathbb{Q}}_p[T/M^0]^{W(D)} \cong \mathfrak{Z}_D^{W(D)},$$

see [Ren10, Rem. VI.4.4] for the second isomorphism. Chenevier defines  $E[\mathcal{B}]$  to be  $E[M/M^0]^\Delta$ . This subring gets identified with  $E[T/M^0]^{W(D)}$  inside  $\overline{\mathbb{Q}}_p[T/M^0]^{W(D)}$ ,

and with  $\mathfrak{Z}_{D,E}^{W(D)}$  inside  $\mathfrak{Z}_D^{W(D)}$ , see the proof of Lemma 3.20. The assertion follows from Lemma 3.21.  $\square$

Let  $\sigma(\tau)$  be the representation of  $K$  given by Theorem 3.7. After replacing  $E$  by a finite extension we may assume that there exists a representation  $\sigma(\tau)_E$  of  $K$  on an  $E$ -vector space, such that  $\sigma(\tau)_E \otimes_E \overline{\mathbb{Q}}_p \cong \sigma(\tau)$ . Then  $\text{c-Ind}_K^G \sigma(\tau)_E$  is an object in  $\Omega_E$ . Since  $\mathfrak{Z}_{\Omega,E}$  is the centre of  $\Omega_E$  it acts on  $\text{c-Ind}_K^G \sigma(\tau)_E$ , thus inducing a homomorphism  $\mathfrak{Z}_{\Omega,E} \rightarrow \text{End}_G(\text{c-Ind}_K^G \sigma(\tau)_E)$ .

**3.24. Lemma.** *The map  $\mathfrak{Z}_{\Omega,E} \rightarrow \text{End}_G(\text{c-Ind}_K^G \sigma(\tau)_E)$  is an isomorphism.*

*Proof.* It follows from Theorem 4.1 of [Dat99] and Proposition 3.22 above that the map is an isomorphism once we extend scalars to  $\overline{\mathbb{Q}}_p$ . This implies the assertion.  $\square$

Let  $\mathcal{R} := \text{End}_G(\Pi(\Omega))$ . Since  $\Pi(\Omega)$  is a projective generator the functors  $M \mapsto M \otimes_{\mathcal{R}} \Pi(\Omega)$  and  $\pi \mapsto \text{Hom}_G(\Pi(\Omega), \pi)$  induce an equivalence of categories between the category of right  $\mathcal{R}$ -modules and  $\Omega$ . If  $\pi$  is irreducible, then the action of  $\mathfrak{Z}_{\Omega}$  on  $\pi$  factors through  $\chi_{\pi} : \mathfrak{Z}_{\Omega} \rightarrow \overline{\mathbb{Q}}_p$ . It follows from [Ren10, Lem.VI.10.4] that  $\mathcal{R}$  is a finitely generated  $\mathfrak{Z}_{\Omega}$ -module, which implies that the module corresponding to  $\pi$  is a finite dimensional  $\overline{\mathbb{Q}}_p$ -vector space. Since  $\mathfrak{Z}_{\Omega,E}$  is a finitely generated algebra over  $E$ ,  $E(\chi_{\pi}) := \chi_{\pi}(\mathfrak{Z}_{\Omega,E})$  is a finite extension of  $E$ .

In the above  $E$  was only required to be sufficiently large. Thus if  $E'$  is a subfield of  $\overline{\mathbb{Q}}_p$  containing  $E$ , then we let  $\Omega_{E'}$ ,  $\Pi(\Omega)_{E'}$  be the corresponding objects defined over  $E'$  instead of  $E$ . Then  $\Pi(\Omega)_{E'}$  is a projective generator of  $\Omega_{E'}$  and the functors  $M \mapsto M \otimes_{\mathcal{R}_{E'}} \Pi(\Omega)_{E'}$  and  $\pi \mapsto \text{Hom}_G(\Pi(\Omega)_{E'}, \pi)$  induce an equivalence of categories between the category of right  $\mathcal{R}_{E'}$ -modules and  $\Omega_{E'}$ , where  $\mathcal{R}_{E'} := \text{End}_G(\Pi(\Omega)_{E'})$ .

**3.25. Lemma.** *Every irreducible generic  $\pi \in \Omega$  can be realised over  $E(\chi_{\pi})$ .*

*Proof.* To ease the notation we let  $E' := E(\chi_{\pi})$  and let  $\pi' := \text{c-Ind}_K^G \sigma(\tau)_E \otimes_{\mathfrak{Z}_{\Omega,E}} E'$ . Then

$$\pi' \otimes_{E'} \overline{\mathbb{Q}}_p \cong \text{c-Ind}_K^G \sigma(\tau) \otimes_{\mathfrak{Z}_{\Omega, \chi_{\pi}}} \overline{\mathbb{Q}}_p \cong \pi_1 \times \dots \times \pi_r,$$

where the last isomorphism is given by Proposition 3.9. Hence,  $\pi' \otimes_{E'} \overline{\mathbb{Q}}_p$  is of finite length, which implies that  $\text{Hom}_G(\Pi(\Omega), \pi' \otimes_{E'} \overline{\mathbb{Q}}_p)$  is a finite dimensional  $\overline{\mathbb{Q}}_p$ -vector space, which implies that  $M' := \text{Hom}_G(\Pi(\Omega)_E, \pi')$  is a finite dimensional  $E'$ -vector space.

If  $\pi''$  is an irreducible  $E'$ -subrepresentation of  $\pi'$ , and if we define  $M'' := \text{Hom}_G(\Pi(\Omega)_{E'}, \pi'')$ , then  $M''$  is an irreducible  $\mathcal{R}_{E'}$ -module which is finite dimensional over  $E'$ . It follows from [Bou12, Cor.12.7.1a)] that  $M'' \otimes_{E'} \overline{\mathbb{Q}}_p$  is a semi-simple  $\mathcal{R}$ -module. Hence,  $\pi'' \otimes_{E'} \overline{\mathbb{Q}}_p$  is a semi-simple  $G$ -representation. Proposition 3.8 implies that the  $G$ -socle of  $\pi'' \otimes_{E'} \overline{\mathbb{Q}}_p$  is irreducible and is isomorphic to  $\pi$ . Thus  $\pi'' \otimes_{E'} \overline{\mathbb{Q}}_p \cong \pi$ .  $\square$

Henceforth for each Bernstein component  $\Omega$  we will fix a sufficiently large  $E$  as above and work with it. Agreeing on this, we will *omit  $E$  from the notation* when there is no danger of confusion. For instance we will write  $\mathfrak{Z}_{\Omega}$ ,  $\sigma(\tau)$ , and so on, in place of  $\mathfrak{Z}_{\Omega,E}$ ,  $\sigma(\tau)_E$  and so on. Note that we fixed a choice of  $E$  in Section 2; however, it is harmless to replace our patched module  $M_{\infty}$  with its base extension to the ring of integers in any larger choice of  $E$ , and we will do so without further comment.

## 4. LOCAL-GLOBAL COMPATIBILITY

The goal of this section is to prove that the patched module  $M_\infty$  satisfies local-global compatibility, in the following sense: the  $G$ -action on  $M_\infty$  (obtained by patching global objects) will induce a tautological Hecke action on certain patched modules for particular  $K$ -types. On the other hand, we will define a second Hecke action via an interpolation of the classical local Langlands correspondence. We will then prove that these two Hecke actions coincide. The details are made explicit below.

Note that it is plausible that  $M_\infty$  should satisfy local-global compatibility, since it is patched together from spaces of algebraic modular forms; the difficulty in proving this is that the modules at finite level are all  $p$ -power torsion, while local-global compatibility is usually defined after inverting  $p$ , so that we need to establish some integral control on the compatibility. Some of our arguments were inspired by the treatment of the two-dimensional crystalline case in [Kis07, §3.6], and somewhat related considerations in the arguments of [Kis09a].

Let  $\sigma$  be a locally algebraic type for  $G = \mathrm{GL}_n(F)$  defined over  $E$ . Then by definition  $\sigma$  is an absolutely irreducible representation of  $K = \mathrm{GL}_n(\mathcal{O}_F)$  over  $E$  of the form  $\sigma_{\mathrm{sm}} \otimes \sigma_{\mathrm{alg}}$ , where  $\sigma_{\mathrm{sm}}$  is a smooth type for  $K$  (i.e.  $\sigma_{\mathrm{sm}} = \sigma(\tau)$  for some inertial type  $\tau$ ) and  $\sigma_{\mathrm{alg}}$  is the restriction to  $K$  of an irreducible algebraic representation of  $\mathrm{Res}_{F/\mathbb{Q}_p} G$ ; we will sometimes also write  $\sigma_{\mathrm{alg}}$  for the corresponding  $G$ -representation. (So, all of our locally algebraic types are “potentially crystalline”, in the sense that they detect representations for which  $N = 0$ .) Set  $\mathcal{H}(\sigma) := \mathrm{End}_G(\mathrm{c}\text{-Ind}_K^G \sigma)$ .

We say that a continuous representation  $r : G_F \rightarrow \mathrm{GL}_n(E)$  is *potentially crystalline of type  $\sigma$*  if it is potentially crystalline with inertial type  $\tau$  and Hodge–Tate weights prescribed by  $\sigma_{\mathrm{alg}}$ . We will also say a global representation has type  $\sigma$  if it restricts to such an  $r$ . Let  $R_{\mathfrak{p}}^\square(\sigma)$  be the local framed universal deformation ring of type  $\sigma$  at  $\mathfrak{p}$  (i.e. the unique reduced and  $p$ -torsion free quotient of  $R_{\mathfrak{p}}^\square$  corresponding to potentially crystalline lifts of type  $\sigma$ ).

Let  $\mathcal{X} = \mathrm{Spf} R_{\mathfrak{p}}^\square(\sigma)$ , with ideal of definition taken to be the maximal ideal, and let  $\mathcal{X}^{\mathrm{rig}}$  denote its rigid generic fiber (as constructed in [dJ95, §7]). Note that  $\mathcal{X}^{\mathrm{rig}} = \cup_j U_j$  is an increasing union of affinoids, and in fact is a quasi-Stein rigid space, since it is a closed subspace of an open polydisc, which is an increasing union of closed polydiscs. By a standard abuse of notation, we will write  $\mathcal{O}_{\mathcal{X}^{\mathrm{rig}}}$  for the ring of rigid-analytic functions on  $\mathcal{X}^{\mathrm{rig}}$ . Then  $\mathcal{O}_{\mathcal{X}^{\mathrm{rig}}} = \varprojlim_j \Gamma(U_j, \mathcal{O}_{U_j})$  and we equip it with the inverse limit topology. We note that by [dJ95, Lemma 7.1.9], there is a bijection between the points of  $\mathcal{X}^{\mathrm{rig}}$  and the closed points of  $\mathrm{Spec} R_{\mathfrak{p}}^\square(\sigma)[1/p]$ .

The universal lift over  $R_{\mathfrak{p}}^\square(\sigma)$  gives rise to a continuous family of representations  $\rho^{\mathrm{rig}} : G_F \rightarrow \mathrm{GL}_n(\mathcal{O}_{\mathcal{X}^{\mathrm{rig}}})$ . (The continuity of  $\rho^{\mathrm{rig}}$  is equivalent to that of each of the representations  $G_F \rightarrow \mathrm{GL}_n(\Gamma(U_j, \mathcal{O}_{U_j}))$  obtained by restricting elements of  $\mathrm{GL}_n(\mathcal{O}_{\mathcal{X}^{\mathrm{rig}}})$  to  $U_j$ .) If  $x$  is a point of  $\mathcal{X}^{\mathrm{rig}}$  with residue field  $E_x$ , we denote by  $\rho_x : G_F \rightarrow \mathrm{GL}_n(E_x)$  the specialisation of  $\rho^{\mathrm{rig}}$  at  $x$ . We define the locally algebraic  $G$ -representation  $\pi_{1.\mathrm{alg},x} := \pi_{\mathrm{sm}}(\rho_x) \otimes_E \sigma_{\mathrm{alg}} = \pi_{\mathrm{sm}}(\rho_x) \otimes_{E_x} \pi_{\mathrm{alg}}(\rho_x)$ . (Recall the notation  $\pi_{\mathrm{sm}}(\rho_x)$  and  $\pi_{\mathrm{alg}}(\rho_x)$  from §1.5; in particular,  $\pi_{\mathrm{sm}}(\rho_x) = r_p^{-1}(\mathrm{WD}(\rho_x)^{F\text{-ss}})$ .) Note that  $\mathcal{H}(\sigma)$  acts via a character on the space  $\mathrm{Hom}_K(\sigma, \pi_{1.\mathrm{alg},x})$ , the latter being

one-dimensional (by Theorem 3.7 together with the argument of [ST06, Lemma 1.4]).

The following theorem, which may be of independent interest, gives our interpolation of the local Langlands correspondence. Its proof will occupy much of this section.

**4.1. Theorem.** *There is an  $E$ -algebra homomorphism  $\eta : \mathcal{H}(\sigma) \rightarrow R_{\mathfrak{p}}^{\square}(\sigma)[1/p]$  which interpolates the local Langlands correspondence  $r_{\mathfrak{p}}$ . More precisely, for any closed point  $x$  of  $\mathrm{Spec} R_{\mathfrak{p}}^{\square}(\sigma)[1/p]$ , the action of  $\mathcal{H}(\sigma)$  on  $\mathrm{Hom}_K(\sigma, \pi_{1,\mathrm{alg},x})$  factors as  $\eta$  composed with the evaluation map  $R_{\mathfrak{p}}^{\square}(\sigma)[1/p] \rightarrow E_x$ .*

We begin by proving the following weaker result, showing the existence of a rigid analytic local Langlands map.

**4.2. Proposition.** *There is an  $E$ -algebra homomorphism  $\eta : \mathcal{H}(\sigma) \rightarrow \mathcal{O}_{\mathcal{X}^{\mathrm{rig}}}$  which interpolates the local Langlands correspondence  $r_{\mathfrak{p}}$ . More precisely, for any point  $x \in \mathcal{X}^{\mathrm{rig}}$ , the action of  $\mathcal{H}(\sigma)$  on  $\mathrm{Hom}_K(\sigma, \pi_{1,\mathrm{alg},x})$  factors as  $\eta$  composed with the evaluation map  $\mathcal{O}_{\mathcal{X}^{\mathrm{rig}}} \rightarrow E_x$ .*

Let  $\rho_{\mathrm{WD}}^{\mathrm{rig}} : W_F \rightarrow \mathrm{GL}_n(\mathcal{O}_{\mathcal{X}^{\mathrm{rig}}})$  be the family of Weil–Deligne representations associated to the family of representations  $\rho^{\mathrm{rig}}$  by Corollaire 3.19 of [Che09]. More precisely, this result of Chenevier generalises Théorème C of [BC08] to the case of an arbitrary reduced quasi-compact and quasi-separated rigid space. Applying it to each  $U_j$ , we obtain a compatible family of Weil–Deligne representations  $\rho_{\mathrm{WD}}^{\mathrm{rig}} : W_F \rightarrow \mathrm{GL}_n(\Gamma(U_j, \mathcal{O}_{U_j}))$  and thus a Weil–Deligne representation  $\rho_{\mathrm{WD}}^{\mathrm{rig}} : W_F \rightarrow \mathrm{GL}_n(\mathcal{O}_{\mathcal{X}^{\mathrm{rig}}})$ . Note that  $\rho_{\mathrm{WD}}^{\mathrm{rig}}$  has  $N = 0$ .

For a point  $x$  of  $\mathcal{X}^{\mathrm{rig}}$ , we denote by  $\rho_{\mathrm{WD},x}$  the specialisation of  $\rho_{\mathrm{WD}}^{\mathrm{rig}}$  at  $x$ . Then  $\rho_{\mathrm{WD},x}|_{I_F} \simeq \tau$  for all points  $x$  of  $\mathcal{X}^{\mathrm{rig}}$ . Recall that  $\mathfrak{Z}_{\Omega}$  is the Bernstein centre for the Bernstein component  $\Omega$  corresponding to  $\sigma(\tau)$ .

**4.3. Proposition.** *There exists a unique  $E$ -algebra map  $I : \mathfrak{Z}_{\Omega} \rightarrow \mathcal{O}_{\mathcal{X}^{\mathrm{rig}}}$  such that for any point  $x$  of  $\mathcal{X}^{\mathrm{rig}}$  with residue field  $E_x$ , the smooth  $G$ -representation  $\pi_x$  corresponding to  $\rho_{\mathrm{WD},x}$  via the local Langlands correspondence  $\mathrm{rec}_{\mathfrak{p}}$  determines via specialisation the map  $x \circ I : \mathfrak{Z}_{\Omega} \rightarrow E_x$ .*

*Proof.* Consider the following map, obtained by specialisation:

$$\gamma_G : \mathfrak{Z}_{\Omega} \rightarrow \prod_{x \in \mathcal{X}^{\mathrm{rig}}} E'_x,$$

where  $\gamma_G$  is defined on the factor corresponding to  $x$  by evaluating  $\mathfrak{Z}_{\Omega}$  at the closed point in the Bernstein component  $\Omega$  determined via local Langlands by  $x$ , and  $E'_x/E_x$  is a sufficiently large finite extension.

Consider as well the following map, also obtained by specialisation:

$$\gamma_{\mathrm{WD}} : \mathcal{O}_{\mathcal{X}^{\mathrm{rig}}} \rightarrow \prod_{x \in \mathcal{X}^{\mathrm{rig}}} E'_x.$$

This is an injection since  $\mathcal{X}^{\mathrm{rig}}$  is reduced and since each  $\Gamma(U_j, \mathcal{O}_{U_j})$  is Jacobson. (The Jacobson property is true of any affinoid algebra. To see that  $\mathcal{X}^{\mathrm{rig}}$  is reduced, it is enough to check it on completed local rings at closed points, but these are the same as the completed local rings of  $R_{\mathfrak{p}}^{\square}(\sigma) \left[ \frac{1}{p} \right]$  by [dJ95, Lemma 7.1.9]. The latter is reduced (being a localization of  $R_{\mathfrak{p}}^{\square}(\sigma)$ , which is reduced by definition)

and excellent, since  $R_{\mathfrak{p}}^{\square}(\sigma)$  is a complete, local, noetherian ring (and thus excellent by [Gro65, Scholie 7.8.3(iii)]). The reducedness of the completed local rings now follows from [Gro65, Scholie 7.8.3(v)].

In order to define our map  $I$ , it suffices to show that the image of  $\mathfrak{Z}_{\Omega}$  under  $\gamma_G$  is contained in the image of  $\gamma_{\text{WD}}$ . Let  $T : W_F \rightarrow \mathfrak{Z}_{\Omega}$  be the pseudo-representation constructed in Proposition 3.11 of [Che09]. (Note that Chenevier's  $E[\mathcal{B}]$  is our  $\mathfrak{Z}_{\Omega}$  by Lemma 3.23.) By the construction of  $T$ , we have  $\gamma_G \circ T = \gamma_{\text{WD}} \circ \text{tr}(\rho_{\text{WD}}^{\text{rig}})$ . Therefore the proof of the proposition is reduced to Lemma 4.5 below.  $\square$

Write  $v : W_F \rightarrow \mathbb{Z}$  for the valuation map assigning 1 to any lift of the geometric Frobenius. Let  $\phi \in W_F$  be an element of valuation 1. For  $w \in W_F$  and any  $I_F$ -representation  $r_0$ , let  $r_0^w$  be the twist  $r_0^w(\gamma) := r_0(w^{-1}\gamma w)$ .

**4.4. Lemma.** *Let  $r$  be an irreducible continuous representation of  $W_F$  over  $\overline{\mathbb{Q}}_p$ .*

- (1) *The restriction  $r|_{I_F}$  decomposes as a direct sum of non-isomorphic irreducible  $I_F$ -representations  $\bigoplus_{i=1}^f r_1^{\phi^i}$  for some integer  $f \geq 1$ . If  $t \in \mathbb{Z}$  then  $r(\phi^t)$  respects the decomposition (i.e.  $r(\phi^t)$  sends  $r_1^{\phi^i}$  into itself for  $1 \leq i \leq f$ ) exactly when  $f \mid t$ .*
- (2) *We have  $\text{tr}(r(w)) \neq 0$  for some  $w \in W_F$  of valuation  $t$  if and only if  $f \mid t$ .*
- (3) *The unramified characters  $\chi$  of  $W_F$  satisfying  $r \otimes \chi \simeq r$  are exactly the characters of order dividing  $f$ .*

*Proof.* (1) The representation  $r|_{I_F}$  factors through a finite quotient  $I_F/H$ , so it decomposes as a direct sum of irreducible  $I_F$ -representations  $\bigoplus_{i=1}^f r_i$ , for some integer  $f \geq 1$ . The fact that  $r$  is irreducible as a  $W_F$ -representation implies that  $r(\phi)$  acts transitively on (the representation spaces of) the  $r_i$ . Up to reordering the  $r_i$ , we may assume that it sends  $r_i$  to  $r_{i+1}$ , where  $r_{f+1} := r_1$ . Moreover, we also deduce that  $r_{i+1} \simeq r_i^{\phi}$  and that  $r_1 \simeq r_1^{\phi^f}$ . Finally, all the representations  $r_i$  are non-isomorphic, since if there was an isomorphism between them, we could define a proper  $W_F$ -subrepresentation of  $r$  and thus contradict the irreducibility of  $r$ . (More precisely, if we had an isomorphism  $r_1 \simeq r_{1+s}$  for some  $1 \leq s < f$ , then we could assume that  $f = sf'$  for some integer  $f'$  and get  $I_F$ -isomorphisms

$$\alpha_{sk} : r_1 \oplus \cdots \oplus r_s \xrightarrow{\sim} r_{1+sk} \oplus \cdots \oplus r_{s(1+k)}$$

for each  $1 \leq k < f'$ . In that case, we could take the  $I_F$ -subrepresentation of  $r$  generated by  $v + \alpha_s(v) + \cdots + \alpha_{s(f'-1)}(v)$  with  $v \in r_1 \oplus \cdots \oplus r_s$ ; it is easy to check that this space is also stable under  $\phi$  if we choose the  $\alpha_{sk}$  appropriately.) The fact that  $r(\phi)$  induces a cyclic permutation of the  $f$  irreducible constituents implies the statement about  $r(\phi^t)$ .

(2) Since  $r(w)$  is not supported on the diagonal unless  $f \mid t$  we get the only if part. For the if part, assume that  $f \mid t$ . By part 1, the matrix  $r(\phi^t)$  has the same block decomposition as  $r|_{I_F}$ . Note that the group algebra of  $I_F/H$  surjects onto  $\bigoplus_{i=1}^f \text{End}_{\overline{\mathbb{Q}}_p}(r_i)$ , since the  $r_i$  are non-isomorphic irreducible representations of the finite group  $I_F/H$ . Therefore, there is some linear combination of matrices  $\sum_{h \in I_F} c_h \cdot r(h)$  which has non-zero trace against the non-zero matrix  $r(\phi^t)$ . This implies that  $\text{tr}(r(h \cdot \phi^t)) \neq 0$  for some  $h \in I_F$ .

(3) Observe that  $r \otimes \chi \simeq r$  if and only if  $\chi(w)\text{tr}r(w) = \text{tr}r(w)$ ,  $\forall w \in W$ . The latter condition is equivalent via part 2 to the condition that  $\chi(w) = 1$  for all  $w \in W_F$  such that  $f|v(w)$ , or equivalently that  $\chi^f = 1$ . Hence part 3 is verified.  $\square$

**4.5. Lemma.** *The image of  $T$  generates  $\mathfrak{Z}_\Omega$  as an  $E$ -algebra.*

*Proof.* It suffices (by the faithful flatness of the extension  $\overline{\mathbb{Q}_p}/E$ ) to prove the result after replacing  $E$  with  $\overline{\mathbb{Q}_p}$ . Since the inertial type  $\tau$  factors through a finite quotient  $I_F/H$ , it decomposes as a direct sum  $\oplus_{i=1}^r (\tau_i)_i^d$ , where the  $\tau_i$  are non-isomorphic inertial types such that  $\sigma(\tau_i)$  is cuspidal. As in the proof of Proposition 3.11 of [Che09], the Bernstein component  $\Omega$  decomposes as  $\Omega_1 \times \cdots \times \Omega_r$  and  $\mathfrak{Z}_\Omega = \otimes_{i=1}^r \mathfrak{Z}_{\Omega_i}$ , where each  $\Omega_i$  corresponds to the simple type  $\sigma((\tau_i)_i^d)$ . If we let  $T_i : W_F \rightarrow \mathfrak{Z}_{\Omega_i}$  be the pseudo-representation associated to  $\Omega_i$  by Proposition 3.11 of [Che09], then by definition  $T(g) := \sum_{i=1}^r T_i(g)$ . It suffices to show that the image of  $T$  generates each  $\mathfrak{Z}_{\Omega_i}$  for  $i = 1, \dots, r$ , where  $\mathfrak{Z}_{\Omega_i}$  is regarded as a subalgebra of  $\mathfrak{Z}_\Omega$ .

Let  $r_i$  be an irreducible  $W_F$ -representation such that  $r_i|_{I_F} \simeq \tau_i$ , and let  $f_i$  be the integer associated to  $r_i$  by Lemma 4.4. By choosing  $\text{rec}_p^{-1}(r_i)^{\otimes d_i}$  as a base point, each closed point of  $\text{Spec } \mathfrak{Z}_{\Omega_i}$  may be represented by an unramified character  $\chi_i = (\chi_{i,1}, \dots, \chi_{i,d_i})$  (or more precisely by  $\otimes_{j=1}^{d_i} (\text{rec}_p^{-1}(r_i) \otimes \chi_{i,j})$  up to a permutation of factors), where the  $\chi_{i,j}$  are unramified characters of  $F^\times$ . Then each  $T_i(g)$  is defined by

$$T_i(g)(\chi_i) := \text{tr}(r_i)(g) \sum_{j=1}^{d_i} \chi_{i,j}(\text{Art}_F(g)).$$

Consider elements  $g$  of the form  $h \cdot \phi^{t_i}$ , with  $h \in I_F$  and  $t_i$  a multiple of  $f_i$ . By Lemma 4.4(1), the matrix  $r_i(\phi^{t_i})$  is non-zero and consists of  $f_i$  blocks which match the block decomposition of  $\tau_i$ . Because the constituents of  $\tau_i$  are non-isomorphic for different  $i$ 's, we may choose the  $c_h$  such that

$$\sum_{h \in I_F/H} c_h \cdot \text{tr}(r_i(h) \cdot r_i(\phi^{t_i})) \neq 0$$

and  $\sum_{h \in I_F/H} c_h \cdot r_{i'}(h) = 0$  for  $i' \neq i$ . In particular, this means that  $\sum_{h \in I_F/H} c_h \cdot T(h\phi^{t_i}) \in \mathfrak{Z}_{\Omega_i}$ .

Now notice that  $\text{Art}_F(\phi^{t_i}) \in F^\times$  has valuation  $t_i$ , which by Lemma 4.4(3) and Remark 3.5 coincides with the valuation of  $\det(\pi_{E_i})^{t_i/f_i}$ , where  $E_i/F$  is the extension in Lemma 3.4 for the cuspidal type  $\sigma(\tau_i)$ . Therefore,  $\sum_{j=1}^{d_i} \chi_{i,j}(\text{Art}_F(g))$  as a regular function on the Bernstein component  $\mathfrak{Z}_{\Omega_i}$  is exactly  $\sum_{j=1}^{d_i} X_{ij}^{t_i/f_i}$ , where  $X_{i1}$  is a generator of the Bernstein centre for  $\sigma(\tau_i)$  and the  $X_{ij}$  are obtained from  $X_{i1}$  from the Weyl group action. This implies that for our choice of coefficients  $c_h$ ,  $\sum_{h \in I_F/H} c_h \cdot T(h\phi^{t_i})$  is equal to  $\sum_{j=1}^{d_i} X_{ij}^{t_i/f_i} \in \mathfrak{Z}_{\Omega_i}$ , up to a non-zero scalar.

Note that we can ensure that  $t_i/f_i$  is any integer. Therefore, we can generate all elements in  $\mathfrak{Z}_{\Omega_i}$  which are sums of  $k$ -th powers of the  $d_i$  generators of the Bernstein centre for a single supercuspidal, for any integer  $k$ . However, since  $\mathfrak{Z}_{\Omega_i}$  is obtained by taking Weyl-group invariants, it is generated as a  $\overline{\mathbb{Q}_p}$ -algebra by the elementary symmetric polynomials in those  $d_i$  variables together with the product of the inverses of the variables. Over  $\overline{\mathbb{Q}_p}$ , which is a field of characteristic 0, we may take the sums of powers of  $d_i$  variables as generators for the elementary symmetric polynomials in those variables. We may also generate the product of the inverses of the variables from sums of powers with negative exponents.  $\square$

**4.6. Remark.** While the proof of Lemma 4.5 is slightly technical, the lemma itself is rather natural; it expresses the idea that local Langlands should make sense

in families, and hence that the family of  $G$ -representations parameterised by  $\mathfrak{Z}_\Omega$  — and thus the parameter ring  $\mathfrak{Z}_\Omega$  itself — should be completely determined by the corresponding family of Weil group representations, which are encoded by the  $\mathfrak{Z}_\Omega$ -valued pseudo-representation  $T$ .

If we let  $\mathfrak{A}_\Omega$  denote the  $E$ -subalgebra of  $\mathfrak{Z}_\Omega$  generated by the image of  $T$ , then this is a finite type  $E$ -algebra, and we have a morphism  $\mathrm{Spec} \mathfrak{Z}_\Omega \rightarrow \mathrm{Spec} \mathfrak{A}_\Omega$ . It is not hard to see (e.g. by applying local Langlands over the fraction field of  $\mathfrak{A}_\Omega$ ) that this is a birational map, which is in fact a bijection on points (as one sees by applying local Langlands at the closed points). Unfortunately, we were unable to find a completely conceptual proof in general that this morphism is an isomorphism of varieties over  $E$ .

In the case when  $\mathfrak{Z}_\Omega$  parameterises supercuspidal representations, one can see this as follows: it suffices to check that one obtains an isomorphism after passing to the formal completion at each closed point  $x \in \mathrm{Spec} \mathfrak{Z}_\Omega$ . Let  $\pi_x$  be the supercuspidal  $G$ -representation corresponding to  $x$ , and let  $T_x : W_F \rightarrow E_x$  the specialisation of  $T$  to the image of  $x$  in  $\mathrm{Spec} \mathfrak{A}_\Omega$ . Let  $R_x$  be the universal formal deformation ring of  $T_x$ , so that we have morphisms

$$\mathrm{Spf} \widehat{\mathfrak{Z}_{\Omega_x}} \rightarrow \mathrm{Spf} \widehat{\mathfrak{A}_{\Omega_x}} \rightarrow \mathrm{Spf} R_x,$$

the second being induced by  $T$ . Let  $r_x : W_F \rightarrow \mathrm{GL}_n(E'_x)$  denote the (absolutely) irreducible representation attached to  $\pi_x$  via local Langlands, where  $E'_x/E_x$  is a finite extension, and let  $T'_x$  denote the composite  $T_x : W_F \rightarrow E_x \rightarrow E'_x$ . Then  $T'_x$  is the pseudo-representation attached to  $r_x$ . Since  $r_x$  is irreducible, the universal formal deformation rings of  $r_x$  and  $T'_x$  coincide ([Nys96, Théorème 3], [Rou96, Corollaire 6.2]), and are thus both given by  $R_x \otimes_{E_x} E'_x$ . A direct analysis, using that the source and target are both obtained simply by forming unramified twists, and that local Langlands gives a bijection on isomorphism classes that is compatible with twisting, shows that the composite of the base change to  $E'_x$  of the above morphisms is an isomorphism. Since the first of them is dominant, it is also an isomorphism. Thus the morphism  $\mathrm{Spec} \mathfrak{Z}_\Omega \rightarrow \mathrm{Spec} \mathfrak{A}_\Omega$  is a bijection on closed points and induces isomorphisms after completing at each closed point. From the latter, we see that it is étale and radiciel, hence an open immersion by [Gro67, Théorème 17.9.1] and, since it is also surjective, we see that it is in fact an isomorphism.

One could use a variant of the argument in first paragraph of the proof of Lemma 4.5 to reduce the general case of the lemma to the case when  $\mathfrak{Z}_\Omega$  parameterises a family of supercuspidal representations, where the preceding argument then applies. In this way, one could give a slightly more conceptual proof of the Lemma.

*Proof of Proposition 4.2.* We adopt the notation of §3.12. In particular  $M$  is the Levi subgroup in the supercuspidal support of some (thus any) irreducible representation in  $\Omega$ , and  $\mathcal{X}(M)$  is the group of unramified characters of  $M(F)$ . The group automorphism  $\mathcal{X}(M) \xrightarrow{\sim} \mathcal{X}(M)$  given by  $\chi_M \mapsto \chi_M |\det|^{\frac{1-n}{2}}$  gives rise to an  $E$ -isomorphism  $\mathrm{Spec} \mathfrak{Z}_D \xrightarrow{\sim} \mathrm{Spec} \mathfrak{Z}_D$ . The latter map is invariant under the  $W(D)$ -action (the point is that  $|\det|$  is invariant under  $G$ -conjugation) so it descends to an  $E$ -isomorphism  $\mathrm{Spec} \mathfrak{Z}_\Omega \xrightarrow{\sim} \mathrm{Spec} \mathfrak{Z}_\Omega$  in view of Lemma 3.21. Let  $\mathrm{tw} : \mathfrak{Z}_\Omega \rightarrow \mathfrak{Z}_\Omega$  denote the induced isomorphism.

We have a natural isomorphism  $\iota_\sigma : \mathcal{H}(\sigma_{\mathrm{sm}}) \xrightarrow{\sim} \mathcal{H}(\sigma)$ ; viewing Hecke algebras as endomorphism-valued functions on  $G$ , this is given by  $\psi \mapsto \psi \cdot \sigma_{\mathrm{alg}}$ . (This is a

*a priori* only an injection, but in fact is an isomorphism by the proof of Lemma 1.4 of [ST06].) Now we construct  $\eta$  as the following composite map

$$\mathcal{H}(\sigma) \xrightarrow{\iota_\sigma^{-1}} \mathcal{H}(\sigma_{\text{sm}}) \simeq \mathfrak{Z}_\Omega \xrightarrow{\text{tw}} \mathfrak{Z}_\Omega \xrightarrow{I} \mathcal{O}_{\mathcal{X}^{\text{rig}}},$$

where the second map comes from Lemma 3.24. Note that  $\eta$  is already defined over  $E$ . To verify the desired interpolation property of  $\eta$ , we let  $x : \mathcal{O}_{\mathcal{X}^{\text{rig}}} \rightarrow E_x$  be an  $E$ -algebra map. Then  $x \circ I \circ \text{tw} : \mathfrak{Z}_\Omega \rightarrow E_x$  gives the supercuspidal support of  $\pi_{\text{sm}}(\rho_x) = r_p^{-1}(\text{WD}(\rho_x)^{F\text{-ss}})$  by Proposition 4.3; indeed, since  $I$  interpolates  $\text{rec}_p$  by that proposition,  $I \circ \text{tw}$  interpolates  $r_p$ . In order to complete the proof, we can and do base change to  $\overline{\mathbb{Q}}_p$ . Then Proposition 3.9 shows us that  $\mathfrak{Z}_\Omega$  acts on  $\text{Hom}_K(\sigma_{\text{sm}}, \pi_{\text{sm}}(\rho_x))$  through  $x \circ I \circ \text{tw}$ .

To conclude, it is enough to observe that the action of  $\mathcal{H}(\sigma_{\text{sm}})$  on the space  $\text{Hom}_K(\sigma_{\text{sm}}, \pi_{\text{sm}}(\rho_x))$  is compatible with the  $\mathfrak{Z}_\Omega$ -action on the same space via the isomorphism  $\mathcal{H}(\sigma_{\text{sm}}) \simeq \mathfrak{Z}_\Omega$ , and also with the  $\mathcal{H}(\sigma)$ -action on  $\text{Hom}_K(\sigma, \pi_{1.\text{alg},x})$  via the canonical isomorphisms between the algebras (via  $\iota_\sigma$ ) and the modules. These are readily checked.  $\square$

In order to deduce Theorem 4.1 from Proposition 4.2, we will now use the results of [Hu09] to show that the image of  $\eta$  is bounded, in the sense that for any  $h \in \mathcal{H}(\sigma)$ , the valuation of  $\eta(h)$  at each point of  $\mathcal{X}^{\text{rig}}$  is uniformly bounded.

Recall that  $\sigma = \sigma_{\text{alg}} \otimes \sigma_{\text{sm}}$ , where  $\sigma_{\text{sm}} = \sigma(\tau)$  for some inertial type  $\tau : I_F \rightarrow \text{GL}_n(E)$ . Let  $x : R_{\mathfrak{p}}^\square(\sigma)[1/p] \rightarrow E_x$  be a closed point, so that  $E_x$  is a finite extension field of  $E$ . Then  $x$  defines a local Galois representation  $\rho_x : G_F \rightarrow \text{GL}_n(E_x)$  which is potentially crystalline, and has Hodge–Tate weights determined by the highest weight  $\xi$  of  $\sigma_{\text{alg}}$ . Set  $\pi_x := \pi_{\text{sm}}(\rho_x)$ . Recall that  $\pi_{1.\text{alg},x}$  is the locally algebraic representation defined over  $E_x$  corresponding to  $\rho_x$  (see §1.5), whose smooth part is  $\pi_x$  and which determines the character  $\chi_{\pi_x} \circ \iota_\sigma^{-1} : \mathcal{H}(\sigma) \rightarrow E_x$  (via the action of  $\mathcal{H}(\sigma)$  on  $\text{Hom}_K(\sigma, \pi_{1.\text{alg},x})$ ).

Let  $P = MN$  be a parabolic subgroup of  $G$ , with Levi  $M$  and unipotent radical  $N$ . Let  $Z(M)$  be the centre of  $M$ , let  $N_0 \subset N$  be a compact open subgroup and define  $Z(M)^+ := \{t \in Z(M) \mid tN_0t^{-1} \subset N_0\}$ . When  $P$  is a standard (upper) parabolic, the subgroup  $Z(M)^+$  of  $Z(M)$  consists of elements with non-decreasing  $p$ -adic valuations on the diagonal. Then [Eme06a] defines a Jacquet module functor  $J_P$  on locally analytic representations of  $G$ .

We will consider the following condition on a locally analytic representation  $V$  of  $G$ .

**4.7. Condition.** For every parabolic subgroup  $P = MN$  as above, with modulus character  $\delta_P$ , every  $\chi : Z(M) \rightarrow E^\times$  such that  $\text{Hom}_{Z(M)}(\chi, J_P(V)) \neq 0$ , and every  $t \in Z(M)^+$ , we have  $|\chi(t)\delta_P(t)^{-1}|_p \leq 1$ .

As above, we write  $\tau = \bigoplus_{i=1}^r (\tau_i)^{d_i}$ , where the  $\tau_i$  are pairwise non-isomorphic  $I_F$ -representations corresponding via the inertial local Langlands correspondence to cuspidal types  $\sigma(\tau_i)$  of  $\text{GL}_{e_i}(\mathcal{O}_F)$ . Let  $M = \prod_{i=1}^r \text{GL}_{e_i}(F)^{d_i}$  be a standard Levi of  $G$ , with corresponding standard parabolic  $P = MN$ .

From now on until the end of this section, we will replace  $\pi_x$  (as well as  $\chi_{\pi_x}$ ,  $\sigma_{\text{sm}}$ , and so on) by its base extension to  $\overline{\mathbb{Q}}_p$ . Then as recalled in Section 3.1,  $\pi_x$  is the unique irreducible quotient of a normalised parabolic induction  $\tilde{\pi}_x := i_P^G \pi_{x,M}$ , where  $\pi_{x,M} = \bigotimes_{i=1}^r (\bigotimes_{j=1}^{d_i} \pi_{x,i,j})$  such that each  $\pi_{x,i,j}$  is a supercuspidal

representation of  $\mathrm{GL}_{e_i}(F)$  containing the type  $\sigma(\tau_i)$  and where for each  $i$ , we have  $\pi_{x,i,j} \not\cong \pi_{x,i,j'}(1)$  for  $1 \leq j' < j \leq d_i$ .

By Proposition 3.9 we have a  $G$ -equivariant homomorphism  $\varphi : \mathrm{c}\text{-Ind}_K^G \sigma_{\mathrm{sm}} \rightarrow \tilde{\pi}_x$ , which identifies  $\tilde{\pi}_x$  with  $\mathrm{c}\text{-Ind}_K^G \sigma_{\mathrm{sm}} \otimes_{\mathcal{H}(\sigma_{\mathrm{sm}}), \chi_{\pi_x}} \overline{\mathbb{Q}}_p$ . We identify  $W_{[M, \pi_x, M]}$  with  $\prod_{i=1}^r S_{d_i}$  in the obvious way, where  $S_{d_i}$  is the symmetric group on  $\{1, \dots, d_i\}$ . Note that  $W_{[M, \pi_x, M]}$  and the identification are independent of  $x$ .

**4.8. Lemma.** *Let  $\chi(\pi_{x,i,j})$  denote the central character of  $\pi_{x,i,j}$ . For an element  $w = \{w_i\}_{i=1}^r \in W_{[M, \pi_x, M]}$ , define characters  $\chi_{x,w} : Z(M) \rightarrow \overline{\mathbb{Q}}_p^\times$  by  $\chi_{x,w} = \bigotimes_{i=1}^r \bigotimes_{j=1}^{d_i} \chi(\pi_{x,i,w_i(j)})$ .*

*For every  $t \in Z(M)^+$ , there exists a constant  $C_t$  such that  $|\chi_{x,w}(t)|_p \leq C_t$  for all points  $x$  of  $\mathcal{X}^{\mathrm{rig}}$ .*

*Proof.* We know that  $\sigma_{\mathrm{alg}} \otimes \tilde{\pi}_x$ , after twisting by a unitary character  $(\det | \det |)^{1-n}$  (this twist is discussed at the beginning of §5 below), corresponds to the potentially crystalline Galois representation  $\rho_x$  with Hodge–Tate weights determined by  $\sigma_{\mathrm{alg}}$ , in the sense that  $\tilde{\pi}_x | \det |^{1-n} \leftrightarrow \mathrm{WD}(\rho_x)^{F\text{-ss}}$  via the modified local Langlands correspondence as in Section 4 of [BS07]. Moreover, note that by Lemma 4.2 of Section 4 of [BS07],  $\sigma_{\mathrm{alg}} \otimes \tilde{\pi}_x$  actually has a model over a sufficiently large finite extension of  $\mathbb{Q}_p$ , so the characters  $\chi_{x,w}$  then take values in some sufficiently large finite extension  $E'_x/\mathbb{Q}_p$ .

By the equivalence between parts (ii) and (iv) of Theorem 1.2 of [Hu09] (where our coefficient field is taken to be  $E'_x$ ), we know that  $\sigma_{\mathrm{alg}}(\det)^{1-n} \otimes \tilde{\pi}_x | \det |^{1-n}$  has a unitary central character and satisfies Condition 4.7 [Eme06a]. Therefore so does  $\sigma_{\mathrm{alg}} \otimes \tilde{\pi}_x$ .

Note that by Proposition 4.3.6 of [Eme06a] we have  $J_P(\sigma_{\mathrm{alg}} \otimes \tilde{\pi}_x) \xrightarrow{\sim} \sigma_{\mathrm{alg}}^N \otimes r_P^G(\tilde{\pi}_x) \delta_P^{1/2}$ , where  $r_P^G$  is the normalised Jacquet functor for smooth representations. Putting this formula together with Condition 4.7, we see that then we have  $|\chi(t)|_p \leq |\sigma_{\mathrm{alg}}^N(t) \cdot \delta_P(t)^{-1/2}|^{-1}$  for every  $\chi$  occurring in  $r_P^G(\tilde{\pi}_x)$ .

Now, Proposition 3.2(2) of [Hu09] computes  $r_P^G(\tilde{\pi}_x)$  (observe that in the notation of *loc. cit.*, all  $b_i$  are 1 in our case) and shows that the characters  $\chi_{x,w} = \bigotimes_{i=1}^r \bigotimes_{j=1}^{d_i} \chi(\pi_{x,i,w_i(j)})$  of  $Z(M)$  for all sets  $w$  of permutations  $w_i$  of  $\{1, \dots, d_i\}$  occur in  $r_P^G(\tilde{\pi}_x)$ . The result follows.  $\square$

Let  $Z(M)^{++} \subset Z(M)$  be the subgroup generated by elements with the property that the  $p$ -adic valuations are non-decreasing on the diagonal of each block  $\mathrm{GL}_{e_i}^{d_i}$ . Clearly,  $Z(M)^+ \subset Z(M)^{++}$ .

**4.9. Corollary.** *The conclusion of Lemma 4.8 holds for all  $t \in Z(M)^{++}$ .*

*Proof.* There is a permutation of  $\{1, \dots, r\}$  which induces a permutation on the factors  $\mathrm{GL}_{e_i}^{d_i}$  of  $M$  such that the image of  $t$  under that permutation has non-decreasing  $p$ -adic valuations. Let  $M'$  be the Levi subgroup of  $G$  with the permuted blocks as factors. Abstractly,  $M' \simeq M$  and by Proposition 6.4 of [Zel80], we know that the induction  $i_P^G \pi_M$  is independent of the ordering of the  $\tau_i$ . We conclude by applying Lemma 4.8 to the Levi  $M'$  instead of  $M$ .  $\square$

As discussed in Sections 3.3 and 3.6, we have a semisimple Bushnell–Kutzko type  $(J, \lambda)$  such that  $\sigma_{\mathrm{sm}}$  is a direct summand of  $\mathrm{Ind}_J^K(\lambda)$ , and the natural map  $s_P : \mathcal{H}(G, \lambda) \rightarrow \mathcal{H}(\sigma_{\mathrm{sm}})$  induces an isomorphism  $Z(\mathcal{H}(G, \lambda)) \xrightarrow{\sim} Z(\mathcal{H}(\sigma_{\mathrm{sm}}))$ . In particular,

this means that  $\tilde{\pi}_x|_J$  contains  $\lambda$ . Then in the notation of Section 3.3,  $\pi_{x,M}$  contains the type  $(J \cap M, \lambda_M)$ . Let  $\chi_{\pi_{x,M}}$  be the character by which  $\mathcal{H}(M, \lambda_M)$  acts on  $\text{Hom}_{J \cap M}(\lambda_M, \pi_M)$ .

**4.10. Corollary.** *Let  $t \in Z(M)$  and let  $\nu_t \in \mathcal{H}(M, \lambda_M)$  be an intertwiner supported on  $t(J \cap M)$ . Then there exists a constant  $C_t$  such that for all points  $x$  of  $\mathcal{X}^{\text{rig}}$  we have  $|\chi_{\pi_{x,M}}(\nu_t)|_p \leq C_t$ .*

*Proof.* Assume  $\nu_t \neq 0$ . Note that since  $t \in Z(M)$ ,  $\nu_t(t)$  commutes with the action of  $J \cap M$  on  $\lambda_M$  and since  $\lambda_M$  is irreducible we deduce that  $\nu_t(t)$  is a nonzero scalar. Rescaling, we may assume that  $\nu_t(t) := \text{id}_{\lambda_M}$ . Let  $s \in Z(M)^{++}$  be such that  $s = {}^w t$  for some  $w \in W_{[M, \pi_{x,M}]}$ . It follows from the definitions that  $\chi_{\pi_{x,M}}(\nu_t) = \chi_{x,w}(s)$ , and the corollary then follows from Corollary 4.9.  $\square$

**4.11. Corollary.** *Let  $\nu \in \mathcal{H}(M, \lambda_M)$ . Then there exists a constant  $C_\nu$  such that  $|\chi_{\pi_{x,M}}(\nu)|_p \leq C_\nu$  for all points  $x$  of  $\mathcal{X}^{\text{rig}}$ .*

*Proof.* Since  $\nu$  has compact support and we only need some bounded constant  $C_\nu$ , it suffices to prove the claim in the case that  $\nu$  is an element of the basis of  $\mathcal{H}(M, \lambda_M)$  given by Lemma 3.4. Let  $\nu'$  be the  $e$ -fold convolution of  $\nu$  with itself, where  $e$  is as in the statement of Lemma 3.4; then from Corollary 4.10 applied to  $\nu'$ , we see that there is some constant  $C_\nu$  such that  $|\chi_{\pi_{x,M}}(\nu')|_p \leq C_\nu^e$  for all  $x$ . This implies that  $|\chi_{\pi_{x,M}}(\nu)|_p \leq C_\nu$ , as required.  $\square$

**4.12. Proposition.** *For any  $\nu \in \mathcal{H}(\sigma)$ , there is a constant  $C_\nu$  such that*

$$|\chi_{\pi_x}(\iota_\sigma^{-1}(\nu))|_p \leq C_\nu$$

for all points  $x$  of  $\mathcal{X}^{\text{rig}}$ .

*Proof.* Recall from Section 3.6 that we have an isomorphism

$$(s_P \circ t_P) : \mathcal{H}(M, \lambda_M)^{W_{[M, \pi_{x,M}]}} \rightarrow \mathcal{H}(\sigma_{sm}).$$

Set  $\nu_t := (s_P \circ t_P)^{-1}(\iota_\sigma^{-1}(\nu))$ . Corollary 4.11 implies that there is a constant  $C_\nu$  such that  $|\chi_{\pi_{x,M}}(\nu_t)|_p \leq C_\nu$  for all  $x$ .

Now, in order to conclude, we just need to relate  $\chi_{\pi_{x,M}}(\nu_M)$  to  $\chi_{\pi_x}((s_P \circ t_P)(\nu_M))$  for any  $\nu_M \in \mathcal{H}(M, \lambda_M)^{W_{[M, \pi_{x,M}]}}$ . As recalled in Section 3.3,  $i_P^G$  corresponds on the level of Hecke modules to pushforward along the map  $t_P$ . More precisely, if we let  $\mathcal{M} := \text{Hom}_{J \cap M}(\lambda_M, \pi_{x,M})$  and  $\mathcal{N} := \text{Hom}_J(\lambda, \pi_x)$ , then  $\mathcal{N} \simeq \text{Hom}_{\mathcal{H}(M, \lambda_M)}(\mathcal{H}(G, \lambda), \mathcal{M})$ . Here we view  $\mathcal{H}(G, \lambda)$  as a left  $\mathcal{H}(M, \lambda_M)$ -module via  $t_P$  and the action of  $\mathcal{H}(G, \lambda)$  on the space of homomorphisms is via right translation. For  $z \in Z(\mathcal{H}(G, \lambda))$ , we note that the right action is also a left action, so the eigenvalue of  $z$  on  $\mathcal{N}$  is the same as the eigenvalue of  $(t_P)^{-1}(z)$  on  $\mathcal{M}$ . On the other hand,  $\text{Hom}_K(\sigma, \pi_x) \simeq e_K \mathcal{N}$  for the idempotent  $e_K$  in  $\mathcal{H}(G, \lambda)$  which defines  $\sigma$ . Therefore, any eigenvalue of  $e_K * z$  on  $e_K \mathcal{N}$  is an eigenvalue of  $z$  on  $\mathcal{N}$ . We deduce that  $\chi_{\pi_{x,M}}(\nu_M) = \chi_{\pi_x}((s_P \circ t_P)(\nu_M))$ .  $\square$

*Proof of Theorem 4.1.* By [Kis08, Thm 3.3.8] we know that  $R_{\mathfrak{p}}^\square(\sigma)[1/p]$  is a regular ring. Proposition 7.3.6 of [dJ95] (which is applicable because  $R_{\mathfrak{p}}^\square(\sigma)[1/p]$  is in particular normal) then implies that the ring of rigid analytic functions on  $\mathcal{X}^{\text{rig}}$  whose absolute value is bounded by 1 coincides precisely with the normalisation of

$R_{\mathfrak{p}}^{\square}(\sigma)$  in  $R_{\mathfrak{p}}^{\square}(\sigma)[1/p]$ , and so the ring of bounded rigid analytic functions on  $\mathcal{X}^{\text{rig}}$  is equal to  $R_{\mathfrak{p}}^{\square}(\sigma)[1/p]$ .

The result then follows immediately from Proposition 4.12 and the defining property of  $\eta$ .  $\square$

We now return to the global setting. Fix a  $K$ -stable  $\mathcal{O}$ -lattice  $\sigma^{\circ}$  in  $\sigma$ . Set

$$M_{\infty}(\sigma^{\circ}) := \left( \text{Hom}_{\mathcal{O}[[K]]}^{\text{cont}}(M_{\infty}, (\sigma^{\circ})^d) \right)^d.$$

Here we are considering continuous homomorphisms for the profinite topology on  $M_{\infty}$  and the  $p$ -adic topology on  $(\sigma^{\circ})^d$ . We equip  $\text{Hom}_{\mathcal{O}[[K]]}^{\text{cont}}(M_{\infty}, (\sigma^{\circ})^d)$  with the  $p$ -adic topology. Note that  $M_{\infty}(\sigma^{\circ})$  is an  $\mathcal{O}$ -torsion free, profinite, linear-topological  $\mathcal{O}$ -module.

**4.13. Lemma.** *There is a natural isomorphism of topological  $\mathcal{O}$ -modules*

$$M_{\infty}(\sigma^{\circ}) \cong \varprojlim_n \left( \text{Hom}_{\mathcal{O}[[K]]}^{\text{cont}}(M_{\infty}, (\sigma^{\circ}/\varpi^n)^{\vee}) \right)^{\vee}.$$

*Proof.* Let  $H := \text{Hom}_{\mathcal{O}[[K]]}^{\text{cont}}(M_{\infty}, (\sigma^{\circ})^d)$ , so that  $M_{\infty}(\sigma^{\circ}) = H^d$ . Then  $H^d \cong \varprojlim_n \text{Hom}_{\mathcal{O}}(H, \mathcal{O}/\varpi^n) \cong \varprojlim_n \text{Hom}_{\mathcal{O}}(H/\varpi^n, \mathcal{O}/\varpi^n) \cong \varprojlim_n (H/\varpi^n)^{\vee}$ . Since  $M_{\infty}$  is a projective  $\mathcal{O}[[K]]$ -module, the short exact sequence

$$0 \rightarrow (\sigma^{\circ})^d \xrightarrow{\varpi^n} (\sigma^{\circ})^d \rightarrow (\sigma^{\circ})^d/\varpi^n \rightarrow 0$$

yields an isomorphism  $H/\varpi^n \cong \text{Hom}_{\mathcal{O}[[K]]}^{\text{cont}}(M_{\infty}, (\sigma^{\circ})^d/\varpi^n)$ . Finally,  $(\sigma^{\circ})^d/\varpi^n \cong \text{Hom}_{\mathcal{O}}(\sigma^{\circ}/\varpi^n, \mathcal{O}/\varpi^n) \cong (\sigma/\varpi^n)^{\vee}$ .  $\square$

**4.14. Remark.** One may modify the proof of Lemma 4.13 to show that  $M_{\infty}(\sigma^{\circ})$  is naturally isomorphic to  $\left( \text{Hom}_{\mathcal{O}[[K]]}^{\text{cont}}(M_{\infty}, (\sigma^{\circ})^{\vee}) \right)^{\vee}$ .

**4.15. Remark.**  $M_{\infty}(\sigma^{\circ})$  is essentially the patched module constructed in Section 5.5 of [EG13], although as the conventions and constructions of the current paper differ slightly from those of [EG13] (see e.g. the difference in the choices of  $v_1$ , noted in the discussion of Subsection 2.3, as well as Remark 2.7) we will not make this precise.

Let  $\mathcal{H}(\sigma^{\circ}) := \text{End}_G(\text{c-Ind}_K^G \sigma^{\circ})$ ; this is an  $\mathcal{O}_E$ -subalgebra of  $\mathcal{H}(\sigma)$ . Note that since  $\sigma^{\circ}$  is a free  $\mathcal{O}$ -module of finite rank, it follows from the proof of Theorem 1.2 of [ST02] that Schikhof duality induces an isomorphism

$$\text{Hom}_{\mathcal{O}[[K]]}^{\text{cont}}(M_{\infty}, (\sigma^{\circ})^d) \cong \text{Hom}_K(\sigma^{\circ}, (M_{\infty})^d).$$

Frobenius reciprocity gives  $\text{Hom}_K(\sigma^{\circ}, (M_{\infty})^d) \cong \text{Hom}_G(\text{c-Ind}_K^G \sigma^{\circ}, (M_{\infty})^d)$ . Thus  $M_{\infty}(\sigma^{\circ})$  is equipped with a tautological Hecke action of  $\mathcal{H}(\sigma^{\circ})$ , which commutes with the action of  $R_{\infty}$ .

Let  $R_{\infty}(\sigma)$  be the quotient of  $R_{\infty}$  which acts faithfully on  $M_{\infty}(\sigma^{\circ})$ . (It follows from Lemma 2.16 of [Paš12] that this is independent of the choice of lattice  $\sigma^{\circ} \subset \sigma$ .)

Set  $R_{\infty}(\sigma)' := R_{\infty} \otimes_{R_{\mathfrak{p}}^{\square}} R_{\mathfrak{p}}^{\square}(\sigma)$ .

- 4.16. Lemma.** (1)  $R_{\infty}(\sigma)$  is a reduced  $\mathcal{O}$ -torsion free quotient of  $R_{\infty}(\sigma)'$ .  
 (2) If  $h \in \mathcal{H}(\sigma^{\circ})$  is such that  $\eta(h) \in R_{\mathfrak{p}}^{\square}(\sigma)$ , then the action of  $h$  on  $M_{\infty}(\sigma^{\circ})$  agrees with the action of  $\eta(h)$  via the natural map  $R_{\mathfrak{p}}^{\square}(\sigma) \rightarrow R_{\infty}(\sigma)'$ .

*Proof.* (1) That  $R_\infty(\sigma)$  is  $\mathcal{O}$ -torsion free follows immediately from the fact that by definition it acts faithfully on the  $\mathcal{O}$ -torsion free module  $M_\infty(\sigma^\circ)$ . The fact that it is actually a quotient of  $R_\infty(\sigma)'$  is then essentially an immediate consequence of classical local-global compatibility at  $\tilde{\mathfrak{p}}$ , but to see this will require a little unraveling of the definitions. Note that if  $N$  is sufficiently large, then  $K_N$  acts trivially on  $(\sigma^\circ)^d/\varpi^N$ . Using Lemma 4.13, we see that

$$\begin{aligned} M_\infty(\sigma^\circ)^\vee &= \varprojlim_N \mathrm{Hom}_{\mathcal{O}[\Gamma_{2N}]} \left( (M_\infty/\mathfrak{b}_N)_{K_{2N}}, (\sigma^\circ)^d/\varpi^N \right)^\vee \\ &= \varprojlim_N \mathrm{Hom}_{\mathcal{O}[\Gamma_{2N}]} \left( (M_{1, Q_{N'(N)}}^\square/\mathfrak{b}_N)_{K_{2N}}, (\sigma^\circ)^d/\varpi^N \right)^\vee, \end{aligned}$$

so it suffices to show that if  $N \gg 0$  then the action of  $R_\infty$  on

$$\mathrm{Hom}_{\mathcal{O}[[K]]}^{\mathrm{cont}} \left( M_{1, Q_{N'(N)}}^\square, (\sigma^\circ)^d/\varpi^N \right)$$

factors through  $R_\infty(\sigma)'$ . Now, by definition we have

$$\begin{aligned} M_{1, Q_{N'(N)}}^\square &= \\ & \mathrm{pr}^\vee \left( S_{\xi, \tau} \left( U_1(Q_{N'(N)})_{2N'(N)}, \mathcal{O}/\varpi^{N'(N)} \right)_{\mathfrak{m}_{Q_{N'(N)}}}^\vee \right) \otimes_{R_{S_{Q_{N'(N)}}}^{\mathrm{univ}}} R_{S_{Q_{N'(N)}}}^{\square T}, \end{aligned}$$

so it suffices to prove the same result for

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}[[K]]}^{\mathrm{cont}} \left( S_{\xi, \tau} \left( U_1(Q_{N'(N)})_{2N'(N)}, \mathcal{O}/\varpi^{N'(N)} \right)_{\mathfrak{m}_{Q_{N'(N)}}}^\vee \right) \otimes_{R_{S_{Q_{N'(N)}}}^{\mathrm{univ}}} R_{S_{Q_{N'(N)}}}^{\square T}, \\ (\sigma^\circ)^d/\varpi^N, \end{aligned}$$

which is equal to

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}[[K]]}^{\mathrm{cont}} \left( S_{\xi, \tau} \left( U_1(Q_{N'(N)})_{2N'(N)}, \mathcal{O}/\varpi^{N'(N)} \right)_{\mathfrak{m}_{Q_{N'(N)}}}^\vee, \right. \\ \left. (\sigma^\circ)^d/\varpi^N \right) \otimes_{R_{S_{Q_{N'(N)}}}^{\mathrm{univ}}} \left( R_{S_{Q_{N'(N)}}}^{\square T} \right)^\vee, \end{aligned}$$

which in turn equals

$$S_{\xi, \tau} \left( U_1(Q_{N'(N)})_0, (\sigma^\circ)^d/\varpi^N \right)_{\mathfrak{m}_{Q_{N'(N)}}} \otimes_{R_{S_{Q_{N'(N)}}}^{\mathrm{univ}}} \left( R_{S_{Q_{N'(N)}}}^{\square T} \right)^\vee.$$

Therefore it would suffice to prove the same result for

$$S_{\xi, \tau} \left( U_1(Q_{N'(N)})_0, (\sigma^\circ)^d \right)_{\mathfrak{m}_{Q_{N'(N)}}} \otimes_{R_{S_{Q_{N'(N)}}}^{\mathrm{univ}}} \left( R_{S_{Q_{N'(N)}}}^{\square T} \right)^\vee.$$

If  $\mathbb{T}$  denotes the image of  $\mathbb{T}^{S_p \cup Q_{N'(N)}, \mathrm{univ}}$  in the endomorphism ring

$$\mathrm{End}_{\mathcal{O}} \left( S_{\xi, \tau} \left( U_1(Q_{N'(N)})_0, (\sigma^\circ)^d \right)_{\mathfrak{m}_{Q_{N'(N)}}} \right),$$

then the action of  $R_{S_{Q_{N'(N)}}}^{\mathrm{univ}}$  on  $S_{\xi, \tau} \left( U_1(Q_{N'(N)})_0, (\sigma^\circ)^d \right)_{\mathfrak{m}_{Q_{N'(N)}}}$  is given by an  $\mathcal{O}$ -algebra homomorphism  $R_{S_{Q_{N'(N)}}}^{\mathrm{univ}} \rightarrow \mathbb{T}$ . Since the space of automorphic forms  $S_{\xi, \tau} \left( U_1(Q_{N'(N)})_0, (\sigma^\circ)^d \right)_{\mathfrak{m}_{Q_{N'(N)}}}$  is  $\mathcal{O}$ -torsion free (by the choice of  $U_{m, v_1}$  in Section 2.3) and the algebra  $\mathbb{T}$  is reduced (by the usual comparison between algebraic modular forms and classical automorphic forms, and the semisimplicity of the space of cuspidal automorphic forms; cf. [CHT08, Corollary 3.3.3 and §3.4]), then by the definition of  $R_{\mathfrak{p}}^\square(\sigma)$ , we need to show that if  $\mathbb{T} \rightarrow \overline{\mathbb{Q}}_p$  is a closed point, then

the restriction to  $G_{F_{\tilde{\mathfrak{p}}}}$  of the corresponding Galois representation  $G_{\tilde{F}^+, T \cup Q_{N'(N)}} \rightarrow \mathcal{G}_n(\overline{\mathbb{Q}}_p)$  is potentially crystalline of type  $\sigma$ ; but this is immediate from classical local-global compatibility (see e.g. Theorem 1.1 of [Car12]).

Finally, to see that  $R_\infty(\sigma)$  is reduced, we note that by part (2) of Lemma 4.17 below, the ring  $R_\infty(\sigma)[1/p]$  is a direct factor of the regular (by [Kis08, Thm 3.3.8]) ring  $R_\infty(\sigma')[1/p]$ . Thus  $R_\infty(\sigma)[1/p]$  is regular, and in particular reduced, and hence so is its subring  $R_\infty(\sigma)$ . (The reader can easily check that this reducedness is not used in the proof of Lemma 4.17, and hence no circularity is involved in this argument.)

(2) Again, this is essentially an immediate consequence of classical local-global compatibility at  $\tilde{\mathfrak{p}}$ , but a little explanation is needed in order to make this plain.

Note first that the natural action of  $\mathcal{H}(\sigma^\circ)$  on  $M_\infty(\sigma^\circ)$  is induced via Frobenius reciprocity from the  $G$ -action on  $M_\infty$ , which is patched from the partial  $G$ -actions defined by

$$\bar{\alpha}_{N'(N)} : (M_\infty/\mathfrak{b}_N)_{K_{2N}} \rightarrow \text{c-Ind}_{K^{\mathbb{Z}}}^{G_N}((M_\infty/\mathfrak{b}_N)_{K_N})$$

and these in turn, after taking homomorphisms into  $(\sigma^\circ)^d/\varpi^N$ , induce partial  $\mathcal{H}(\sigma^\circ)$ -actions on spaces of algebraic modular forms of weight  $(\sigma^\circ)^d$ . (More precisely, the  $G$ -action on  $\varpi$ -adically completed cohomology

$$\tilde{S}_{\xi, \tau}(U_1^{\mathfrak{p}}(Q_{N'(N)}), \mathcal{O}) := \varprojlim_s \left( \varinjlim_m S_{\xi, \tau}(U_1(Q_{N'(N)})_m, \mathcal{O}/\varpi^s) \right)$$

gives rise, via Frobenius reciprocity and the identification

$$S_{\xi, \tau}(U_1^{\mathfrak{p}}(Q_{N'(N)})_0, (\sigma^\circ)^d) \simeq \text{Hom}_K \left( \sigma^\circ, \tilde{S}_{\xi, \tau}(U_1^{\mathfrak{p}}(Q_{N'(N)}), \mathcal{O}) \right),$$

to a natural action of  $\mathcal{H}(\sigma^\circ)$  on  $S_{\xi, \tau}(U_1(Q_{N'(N)})_0, (\sigma^\circ)^d)$ . We see therefore, as in part (1), that it is enough to consider the natural action of each  $h \in \mathcal{H}(\sigma^\circ)$  on the spaces

$$S_{\xi, \tau}(U_1(Q_{N'(N)})_0, (\sigma^\circ)^d)_{\mathfrak{m}_{Q_{N'(N)}}} \otimes_{R_{S_{Q_{N'(N)}}}^{\text{univ}}} \left( R_{S_{Q_{N'(N)}}}^{\square_T} \right)^\vee$$

for  $N \gg 0$ . In addition to the natural action of  $\mathcal{H}(\sigma^\circ)$ , this is equipped with an action of  $R_{\tilde{\mathfrak{p}}}^{\square}(\sigma)$  via the composite  $R_{\tilde{\mathfrak{p}}}^{\square} \rightarrow R^{\text{loc}} \rightarrow R_{S_{Q_{N'(N)}}}^{\square_T}$ , which factors through  $R_{\tilde{\mathfrak{p}}}^{\square}(\sigma)$  by part (1). By classical local-global compatibility and the defining property of the morphism  $\eta$  of Theorem 4.1, we see that, after inverting  $p$ , the action of  $h$  on this space agrees with the action of  $\eta(h)$ . The desired result now follows from the fact that  $S_{\xi, \tau}(U_1(Q_{N'(N)})_0, (\sigma^\circ)^d)_{\mathfrak{m}_{Q_{N'(N)}}}$  is  $\mathcal{O}$ -torsion free.  $\square$

We now use the usual commutative algebra arguments underlying the Taylor–Wiles–Kisin method to study the support of  $M_\infty(\sigma^\circ)$ .

- 4.17. Lemma.** (1) *The module  $M_\infty(\sigma^\circ)$  is finitely generated over  $R_\infty(\sigma)$ , and moreover  $M_\infty(\sigma^\circ)[1/p]$  is locally free of rank one over  $R_\infty(\sigma)[1/p]$ . The topology on  $M_\infty(\sigma^\circ)$  coincides with its  $\mathfrak{m}$ -adic topology, where  $\mathfrak{m}$  denotes the maximal ideal of  $R_\infty(\sigma)$ .*
- (2) *The support of  $M_\infty(\sigma^\circ)$  in  $\text{Spec } R_\infty(\sigma)'$  is a union of irreducible components of  $R_\infty(\sigma)'$ .*
- (3) *Let  $\overline{R}_\infty(\sigma)$  be the normalisation of  $R_\infty(\sigma)$  inside  $R_\infty(\sigma)[1/p]$ . Then the action of  $\mathcal{H}(\sigma^\circ)$  on  $M_\infty(\sigma^\circ)$  induces an  $\mathcal{O}$ -algebra map  $\alpha : \mathcal{H}(\sigma^\circ) \rightarrow \overline{R}_\infty(\sigma)$ .*

*Proof.* Since  $M_\infty$  is a finite projective  $S_\infty[[K]]$ -module, the module  $M_\infty(\sigma^\circ)$  is finite and projective (equivalently, free) over  $S_\infty$ . Indeed, we may write  $M_\infty$  as a direct summand of  $S_\infty[[K]]^r$  for some  $r \geq 0$ , and so  $\mathrm{Hom}_{\mathcal{O}[[K]]}^{\mathrm{cont}}(M_\infty, (\sigma^\circ)^d)^d$  is a direct summand of  $\mathrm{Hom}_{\mathcal{O}[[K]]}^{\mathrm{cont}}(S_\infty[[K]]^r, (\sigma^\circ)^d)^d \cong \left(\mathrm{Hom}_{\mathcal{O}[[K]]}^{\mathrm{cont}}(S_\infty[[K]], (\sigma^\circ)^d)^d\right)^r$ . Thus it suffices to note that since  $S_\infty[[K]] \cong S_\infty \widehat{\otimes}_{\mathcal{O}} \mathcal{O}[[K]]$  as an  $\mathcal{O}[[K]]$ -module, there is a natural isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}[[K]]}^{\mathrm{cont}}(S_\infty[[K]], (\sigma^\circ)^d)^d &\cong \mathrm{Hom}_{\mathcal{O}[[K]]}^{\mathrm{cont}}(\mathcal{O}[[K]], \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(S_\infty, (\sigma^\circ)^d))^d \\ &\cong \mathrm{Hom}_{\mathcal{O}}^{\mathrm{cont}}(S_\infty \otimes_{\mathcal{O}} \sigma^\circ, \mathcal{O})^d \cong S_\infty \otimes_{\mathcal{O}} \sigma^\circ. \end{aligned}$$

Since the  $S_\infty$ -action on  $M_\infty(\sigma^\circ)$  factors through the action of  $R_\infty$ , which in turn factors through  $R_\infty(\sigma)$  by definition, we find that in particular  $M_\infty(\sigma^\circ)$  is finitely generated over  $R_\infty(\sigma)$ .

Since the identification of  $M_\infty$  as a direct summand of  $S_\infty[[K]]$  is compatible with the natural topologies on each of  $M_\infty$  and  $S_\infty[[K]]$ , one easily verifies that the topology on  $M_\infty(\sigma^\circ)$  coincides with its  $\mathfrak{n}$ -adic topology, where  $\mathfrak{n}$  denotes the maximal ideal of  $S_\infty$ . Furthermore, since by definition  $R_\infty(\sigma)$  embeds into  $\mathrm{End}_{S_\infty}(M_\infty(\sigma^\circ))$ , we find that  $R_\infty(\sigma)$  is finite as an  $S_\infty$ -algebra, and so in particular the  $\mathfrak{n}$ -adic topology and  $\mathfrak{m}$ -adic topology on  $M_\infty(\sigma^\circ)$  coincide (where, as in the statement of the lemma,  $\mathfrak{m}$  denotes the maximal ideal of  $R_\infty(\sigma)$ ). Thus the topology on  $M_\infty(\sigma^\circ)$  coincides with its  $\mathfrak{m}$ -adic topology.

By Lemma 3.3 of [BLGHT11], Lemma 2.4.19 of [CHT08], and Theorem 3.3.8 of [Kis08], we see that the ring  $R_\infty(\sigma)'$  is equidimensional of the same Krull dimension as  $S_\infty$ . Since  $M_\infty(\sigma^\circ)$  is free of finite rank over  $S_\infty$ , and the image of  $R_\infty(\sigma)'$  in  $\mathrm{End}(M_\infty(\sigma^\circ))$  is an  $S_\infty$ -algebra, we see that the depth of  $M_\infty(\sigma^\circ)$  as an  $R_\infty(\sigma)'$ -module is at least the Krull dimension of  $S_\infty$ . Since this is equal to the Krull dimension of  $R_\infty(\sigma)'$ , it follows immediately from Lemma 2.3 of [Tay08] that the support of  $M_\infty(\sigma^\circ)$  is a union of irreducible components of  $R_\infty(\sigma)'$ . (Of course, conjecturally  $R_\infty(\sigma)$  is actually equal to  $R_\infty(\sigma)'$ .)

That  $M_\infty(\sigma^\circ)[1/p]$  is locally free over  $R_\infty(\sigma)[1/p]$  follows by an argument of Diamond (*cf.* [Dia97]). More precisely,  $R_{\mathfrak{p}}^\square(\sigma)[1/p]$  and each of the rings  $R_{\mathfrak{v}}^{\sigma, \xi, \tau}[1/p]$  for places  $v \mid p$ ,  $v \neq \mathfrak{p}$  are regular by Theorem 3.3.8 of [Kis08], and  $R_{\mathfrak{v}_1}^\square$  is smooth by the choice of  $v_1$ , so  $R_\infty(\sigma)'$  is regular. Therefore  $R_\infty(\sigma)[1/p]$  is also regular, so  $M_\infty(\sigma^\circ)[1/p]$  is locally free over  $R_\infty(\sigma)[1/p]$  by Lemma 3.3.4 of [Kis09b] (or rather by its proof, which goes over unchanged to our setting, where we do not assume that  $R_\infty(\sigma)[1/p]$  is a domain).

That it is actually locally free of rank one can be checked at finite level, where it follows from the multiplicity one assertion in Theorem 3.7, and the choice of  $v_1$  (and the fact that we have fixed the action mod  $p$  of the Hecke operators at  $\tilde{v}_1$ ).

This completes the proof of parts (1) and (2), and so we turn to proving (3). To this end, let  $\mathcal{A}$  be the  $R_\infty$ -subalgebra of the endomorphism algebra of  $M_\infty(\sigma^\circ)$  generated by  $\mathcal{H}(\sigma^\circ)$ . Since  $M_\infty(\sigma^\circ)$  is a finite type  $R_\infty(\sigma)$ -module, we see that  $\mathcal{A}$  is a finite  $R_\infty(\sigma)$ -algebra. Since  $M_\infty(\sigma^\circ)[1/p]$  is in fact locally free of rank one over  $R_\infty(\sigma)[1/p]$  we have  $\mathrm{End}_{R_\infty(\sigma)[1/p]}(M_\infty(\sigma^\circ)[1/p]) = R_\infty(\sigma)[1/p]$ , so that  $\mathcal{A}[1/p] = R_\infty(\sigma)[1/p]$ . So the natural map  $\mathcal{H}(\sigma^\circ) \rightarrow \mathcal{A}$  lands inside  $\overline{R}_\infty(\sigma)$ .  $\square$

The morphism  $\alpha$  of Lemma 4.17 induces an  $E$ -algebra morphism  $\alpha : \mathcal{H}(\sigma) \rightarrow R_\infty(\sigma)[1/p]$ .

4.18. **Theorem.** *The map  $\alpha$  coincides with the composition*

$$\mathcal{H}(\sigma) \xrightarrow{\eta} R_{\mathfrak{p}}^{\square}(\sigma)[1/p] \rightarrow R_{\infty}(\sigma)[1/p].$$

*Proof.* Note firstly that if  $h \in \mathcal{H}(\sigma^{\circ})$  is such that  $\eta(h) \in R_{\mathfrak{p}}^{\square}(\sigma)$ , then the two maps agree on  $h$  by Lemma 4.16 (2). Since  $R_{\infty}(\sigma)$  is  $p$ -torsion free by Lemma 4.16 (1), it is therefore enough to show that  $\mathcal{H}(\sigma)$  is spanned over  $E$  by such elements.

Now,  $\mathcal{H}(\sigma^{\circ})$  certainly spans  $\mathcal{H}(\sigma)$  over  $E$ , so it is enough to show that for any element  $h' \in \mathcal{H}(\sigma^{\circ})$ , we have  $\eta(p^C h') \in R_{\mathfrak{p}}^{\square}(\sigma)$  for some  $C \geq 0$ ; but this is obvious.  $\square$

4.19. *Remark.* It follows from Lemma 4.17 (2) that the locus of closed points of  $\text{Spec } R_{\mathfrak{p}}^{\square}(\sigma)[1/p]$  which come from closed points of  $\text{Spec } R_{\infty}(\sigma)[1/p]$  is a union of irreducible components, which we call the set of *automorphic components* of  $\text{Spec } R_{\mathfrak{p}}^{\square}(\sigma)[1/p]$ . (Note that we do not know *a priori* that this notion is independent of the choice of global setting, although of course we expect that in fact every component of  $\text{Spec } R_{\mathfrak{p}}^{\square}(\sigma)[1/p]$  is an automorphic component.)

4.20. *Remark.* We expect that  $\eta(\mathcal{H}(\sigma^{\circ}))$  is contained in the normalisation of  $R_{\mathfrak{p}}^{\square}(\sigma)$  in  $R_{\mathfrak{p}}^{\square}(\sigma)[1/p]$ ; it may well be possible to prove this via our methods, but as we do not need this result, we have not pursued it. It is easy to see that the analogous result holds for the quotient of  $R_{\mathfrak{p}}^{\square}(\sigma)$  corresponding to the automorphic components in the sense of Remark 4.19.

## 5. THE BREUIL–SCHNEIDER CONJECTURE

Continue to assume that  $p \nmid 2n$ , and that  $F$  is a finite extension of  $\mathbb{Q}_p$ . If  $r : G_F \rightarrow \text{GL}_n(E)$  is a de Rham representation of regular weight then we say that  $r$  is generic if  $\pi_{\text{sm}}(r)$  is generic. In this case, we set

$$\text{BS}(r) := \pi_{\text{alg}}(r) \otimes \pi_{\text{sm}}(r).$$

(In fact, our  $\text{BS}(r)$  differs from the definition made in [BS07] in that  $\pi_{\text{alg}}(r)$  and  $\pi_{\text{sm}}(r)$  are their analogues in [BS07] times the characters  $\det^{n-1}$  and  $|\det|^{n-1}$ , respectively. Since  $(\det|\det|)^{n-1}$  is a unitary character, this makes no difference to the following conjecture. See also Section 2.4 of [Sor12] for a discussion of the difference between these conventions.) The following is [BS07, Conjecture 4.3] (in the open direction, in the generic case).

5.1. **Conjecture.** *If  $r : G_F \rightarrow \text{GL}_n(E)$  is de Rham and has regular weight, then  $\text{BS}(r)$  admits a nonzero unitary Banach completion.*

5.2. *Remark.* In fact, it seems reasonable (particularly in the light of Corollary 5.4 below) to conjecture that there is even a nonzero *admissible* completion.

Fix a representation  $r : G_F \rightarrow \text{GL}_n(E)$ , and assume from now on that  $r$  is potentially crystalline of regular weight, and that  $r$  is generic. By Remark 2.13 (and possibly replacing  $E$  with a finite extension if necessary), we may replace  $r$  with a conjugate representation so that  $r : G_F \rightarrow \text{GL}_n(\mathcal{O})$ , and  $\bar{r}$  satisfies the hypotheses of Section 2. We can therefore carry out the construction of Section 2, obtaining the patched module  $M_{\infty}$ . Recall that  $r$  is induced from an  $\mathcal{O}$ -algebra homomorphism  $x : R_{\mathfrak{p}}^{\square} \rightarrow \mathcal{O}$ , which we extended to an  $\mathcal{O}$ -algebra homomorphism

$y : R_\infty \rightarrow \mathcal{O}$ . Then  $V(r)$  is obtained from the fiber of  $(M_\infty)^d[1/p]$  above the closed point of  $R_\infty[1/p]$  determined by  $y$ .

Write  $\sigma_{\text{sm}}(r)$  for  $\sigma(\tau)$ ,  $\sigma_{\text{alg}}(r)$  for  $\pi_{\text{alg}}(r)|_K$ , and let  $\sigma := \sigma_{\text{alg}}(r) \otimes \sigma_{\text{sm}}(r)$ , keeping in mind the convention at the end of Section 3.12. (Also enlarge  $E$  to another finite extension if necessary as explained in that section.) As above, we write  $\mathcal{H}(\sigma)$  for  $\text{End}_G(\text{c-Ind}_K^G \sigma)$ , which is isomorphic to  $\text{End}_G(\text{c-Ind}_K^G \sigma(\tau))$  via  $\iota_\sigma$ , so that  $\pi_{\text{sm}}(r)$  determines a character  $\chi_{\pi_{\text{sm}}(r)} : \mathcal{H}(\sigma) \rightarrow E$ . Since  $r$  is generic, we see from Corollary 3.11 that  $\pi_{\text{sm}}(r) \cong (\text{c-Ind}_K^G \sigma_{\text{sm}}(r)) \otimes_{\mathcal{H}(\sigma), \chi_{\pi_{\text{sm}}(r)}} E$ . Tensoring with  $\pi_{\text{alg}}(r)$ , we have

$$\text{BS}(r) \cong (\text{c-Ind}_K^G \sigma) \otimes_{\mathcal{H}(\sigma), \chi_{\pi_{\text{sm}}(r)}} E.$$

Since our representations  $V(r)$  are unitary Banach representations, and since (because we have assumed that  $r$  is generic)  $\text{BS}(r)$  is irreducible, in order to prove Conjecture 5.1 it would be enough to check that  $\text{Hom}_G(\text{BS}(r), V(r)) \neq 0$ . While we cannot at present do this in general, we are able to reinterpret the problem in terms of automorphy lifting theorems, and deduce new cases of Conjecture 5.1. In particular, Corollary 5.5 below gives the first general results in the principal series case.

**5.3. Theorem.** *Suppose that  $p \nmid 2n$ , and that  $r : G_F \rightarrow \text{GL}_n(E)$  is a generic potentially crystalline representation of regular weight. If  $r$  corresponds to a closed point on an automorphic component of  $R_{\mathfrak{p}}^\square(\sigma)[1/p]$  (in the sense of Remark 4.19), then  $\text{BS}(r)$  admits a non-zero unitary admissible Banach completion.*

*Proof.* As remarked above, it suffices to show that  $\text{Hom}_G(\text{BS}(r), V(r)) \neq 0$ . Proposition 2.20 of [Paš12] implies that

$$\dim_E \text{Hom}_K(\sigma, V(r)) = \dim_E M_\infty(\sigma^\circ) \otimes_{R_\infty, y} E.$$

(More specifically, in the notation of that paper we take  $R = R_\infty$ ,  $\Theta = \sigma^\circ$ ,  $V = \sigma$ ,  $N = M_\infty$ , and  $\mathfrak{m}^\circ = \mathcal{O}$ , regarded as an  $R_\infty$ -module via  $y$ . Note that [Paš12] assumes that  $N$  is finitely generated as an  $R[[K]]$ -module, which is satisfied in our case:  $M_\infty$  is a finitely generated  $R_\infty[[K]]$ -module, since it is finite over  $S_\infty[[K]]$  and the  $S_\infty$ -action on  $M_\infty$  factors through a map  $S_\infty \rightarrow R_\infty$ .)

Together with Frobenius reciprocity, this shows that  $\text{Hom}_G(\text{c-Ind}_K^G \sigma, V(r)) = \text{Hom}_K(\sigma, V(r)) \neq 0$ , as  $y$  is in the support of  $M_\infty(\sigma^\circ)[1/p]$  by assumption. Since  $\text{BS}(r) \cong (\text{c-Ind}_K^G \sigma) \otimes_{\mathcal{H}(\sigma), \chi_{\pi_{\text{sm}}(r)}} E$ , we need only show that the action of  $\mathcal{H}(\sigma^\circ)$  on  $M_\infty(\sigma^\circ) \otimes_{R_\infty, y} \mathcal{O}$  factors through the character  $\chi_{\pi_{\text{sm}}(r)}$ ; but this is immediate from Theorem 4.18.  $\square$

**5.4. Corollary.** *Suppose that  $p \nmid 2n$  and that  $r : G_F \rightarrow \text{GL}_n(E)$  is de Rham of regular weight, and is potentially diagonalizable in the sense of [BLGGT13]. Suppose also that  $r$  is generic. Then  $\text{BS}(r)$  admits a nonzero unitary admissible Banach completion.*

*Proof.* By Theorem 5.3, we need only prove that  $r$  corresponds to a point on an automorphic component of  $R_{\mathfrak{p}}^\square(\sigma)[1/p]$ . Recalling that  $y$  was chosen to correspond to the potentially diagonalizable representation  $r_{\text{pot.diag}}$  at the places  $v \mid p$ ,  $v \neq \mathfrak{p}$ , this follows from Theorem A.4.1 of [BLGG13], which constructs a global automorphic Galois representation corresponding to a point on the same component of  $R_{\mathfrak{p}}^\square(\sigma)[1/p]$  as  $r$  (cf. the proof of Corollary 4.4.3 of [GK12]).

Indeed, the existence of a global automorphic Galois representation corresponding to a point on  $R_{\mathfrak{p}}^{\square}(\sigma)[1/p]$  shows that  $S_{\xi,\tau}(U_0, (\sigma^\circ)^d)_{\mathfrak{m}}[1/p]$  is supported at this point. We have  $R_{\mathfrak{p}}^{\square}(\sigma)[1/p]$ -equivariant isomorphisms

$$\begin{aligned} S_{\xi,\tau}(U_0, (\sigma^\circ)^d)_{\mathfrak{m}}[1/p] &\simeq \mathrm{Hom}_K(\sigma^\circ, \tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}})[1/p] \\ &\simeq \mathrm{Hom}_{\mathcal{O}[[K]]}^{\mathrm{cont}}(M_\infty/\mathfrak{a}M_\infty, (\sigma^\circ)^d)[1/p], \end{aligned}$$

where the former comes by identifying locally algebraic vectors in completed cohomology and the latter follows from Schikhof duality and Corollary 2.9. These isomorphisms imply that  $(M_\infty(\sigma^\circ)/\mathfrak{a}M_\infty(\sigma^\circ))^d[1/p]$  is supported at the same point of  $R_{\mathfrak{p}}^{\square}(\sigma)[1/p]$  coming from a global automorphic Galois representation. Therefore,  $M_\infty(\sigma^\circ)^d[1/p]$  is supported at a point of  $R_\infty(\sigma)[1/p]$  on the same component as  $r$ . Finally, we conclude that the component of  $R_{\mathfrak{p}}^{\square}(\sigma)[1/p]$  corresponding to  $r$  is automorphic in the sense of Remark 4.19.  $\square$

**5.5. Corollary.** *Suppose that  $p > 2$ , that  $r : G_F \rightarrow \mathrm{GL}_n(E)$  is de Rham of regular weight, and that  $r$  is generic. Suppose further that either*

- (1)  $n = 2$ , and  $r$  is potentially Barsotti–Tate, or
- (2)  $F/\mathbb{Q}_p$  is unramified,  $r$  is crystalline,  $n \neq p$  and  $r$  has Hodge–Tate weights in the extended Fontaine–Laffaille range; that is, for each  $\kappa : F \hookrightarrow E$ , any two elements of  $\mathrm{HT}_\kappa(r)$  differ by at most  $p - 1$ .

*Then  $\mathrm{BS}(r)$  admits a nonzero unitary admissible Banach completion.*

*Proof.* By Corollary 5.4, it is enough to check that our hypotheses imply that  $r$  is potentially diagonalizable; in case (1), this is Lemma 4.4.1 of [GK12], and in case (2), it is the main result of [GL12].  $\square$

**5.6. Remark.** The attentive reader will have noticed that since throughout the paper we assumed that  $E$  is sufficiently large (and allowed it to be enlarged in the course of making our argument), we have not proved cases of Conjecture 5.1 as it is written, but rather an apparently weaker version, which allows a finite extension of scalars. However, Conjecture 5.1 is an immediate consequence of this version, in the following way: given an (admissible) unitary Banach completion of  $\mathrm{BS}(r) \otimes_E E'$ , where  $E'/E$  is a finite extension, we may regard this completion as being a representation over  $E$ , and then the closure of  $\mathrm{BS}(r)$  inside it gives the required representation.

**5.7. Remark.** It is natural to wonder whether our methods also prove the conjecture when  $r$  is non-generic, in which case the representation  $\mathrm{BS}(r)$  defined in [BS07] is reducible. Our methods should still give a nonzero element of  $\mathrm{Hom}_G(\mathrm{BS}(r), V(r))$  in these cases, and it will then suffice to prove that this homomorphism is injective. We anticipate that in most cases none of the constituents of  $\mathrm{BS}(r)$  other than the socle will admit unitary completions, from which the injectivity would follow. For example, when  $n = 2$ , the representation  $\mathrm{BS}(r)$  has length at most 2, and the other constituent is a finite-dimensional algebraic representation, which can only admit a unitary completion if it is one-dimensional.

Similar issues arise if we study potentially semistable representations  $r$  which are not potentially crystalline. The definition of  $\sigma(\tau)$  can be modified to extend to this case, but even if  $r$  is generic, we no longer expect to have an isomorphism  $(\mathrm{c}\text{-Ind}_K^G \sigma) \otimes_{\mathcal{H}(\sigma), \chi_{\pi_{\mathrm{sm}}(r)}} E \xrightarrow{\sim} \mathrm{BS}(r)$ ; rather, there should be a surjection from the

left hand side (which will no longer be irreducible) to the right-hand side, with a non-trivial kernel. Again, we expect that typically none of the constituents of this kernel will admit non-trivial unitary completions.

5.8. *Remark.* Our techniques also apply to the converse question of describing the representations  $r$  such that  $V(r)$  has non-zero locally algebraic vectors. Although we have not discussed the general potentially semistable case in this paper, it is not hard to modify the proofs of Lemma 4.16(1) and Theorem 5.3 to show that if  $r : G_F \rightarrow \mathrm{GL}_n(E)$  is a representation with the property that  $V(r)$  has non-zero locally algebraic vectors, then  $r$  is a regular de Rham representation. It is presumably also possible to develop the theory of Section 3 to encompass representations with non-zero monodromy, and then deduce further that the locally algebraic vectors in  $V(r)$  are exactly  $\mathrm{BS}(r)$  (at least if  $r$  is generic).

## 6. RELATIONSHIP WITH A HYPOTHETICAL $p$ -ADIC LOCAL LANGLANDS CORRESPONDENCE

In this section we describe the relationship between our construction and a hypothetical  $p$ -adic local Langlands correspondence.

6.1. **A hypothetical formulation of the  $p$ -adic local Langlands correspondence.** Perhaps the strongest hypothesis one might make regarding a  $p$ -adic local Langlands correspondence is the following: that given  $\bar{r} : G_F \rightarrow \mathrm{GL}_n(\mathbb{F})$  with trivial endomorphisms<sup>1</sup> and with associated deformation ring  $R_{\bar{r}}$ , there exists a finitely generated  $R_{\bar{r}}[[K]]$ -module  $L_\infty$  which is  $\mathcal{O}$ -torsion free, and is equipped with an  $R_{\bar{r}}$ -linear  $G$ -action which extends the  $K$ -action arising from its  $R_{\bar{r}}[[K]]$ -module structure.

Given such an object  $L_\infty$ , then for any  $r : G_F \rightarrow \mathrm{GL}_n(E)$  arising from an  $\mathcal{O}_E$ -valued point  $x$  of  $\mathrm{Spec} R_{\bar{r}}$ , we may associate a unitary Banach space representation  $B(r) := (L_\infty \otimes_{R_{\bar{r},x}} \mathcal{O}_E)^d[1/p]$  of  $G$ , which should be “the” representation of  $G$  associated to  $r$  via the  $p$ -adic local Langlands correspondence.

One might conjecture that such a structure should exist and satisfy the following properties:

6.1.1. *Relationship with classical local Langlands.* For any potentially semistable lift  $r : G_F \rightarrow \mathrm{GL}_n(E)$  of  $\bar{r}$  with regular Hodge–Tate weights, the locally algebraic vectors of  $B(r)$  are isomorphic to  $\mathrm{BS}(r)$ .

6.1.2. *Local-global compatibility.* Using the notation for completed cohomology and the Hecke algebra that acts on it introduced in the discussion preceding Corollary 2.9, there is an isomorphism of  $\mathbb{T}_{\xi,\tau}^{S_p}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}}[G]$ -modules

$$\tilde{S}_{\xi,\tau}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}}^d \cong \mathbb{T}_{\xi,\tau}^{S_p}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}} \otimes_{R_{\bar{r}}} L_\infty$$

(here  $\mathbb{T}_{\xi,\tau}^{S_p}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}}$  is regarded as an  $R_{\bar{r}}$ -algebra via the natural maps  $R_{\bar{r}} \rightarrow R_S^{\mathrm{univ}} \rightarrow \mathbb{T}_{\xi,\tau}^{S_p}(U^{\mathfrak{p}}, \mathcal{O})_{\mathfrak{m}}$ , where the first morphism corresponds to restricting global Galois representations to the decomposition group at  $\mathfrak{p}$ , and the second morphism

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<sup>1</sup>We make this assumption in what follows for simplicity, since the discussion is purely hypothetical in any case. If  $\bar{r}$  admits non-trivial endomorphisms, then we would instead work with the framed deformation ring  $R_{\bar{r}}^\square$  in everything that follows, and the representation  $L_\infty$  would be endowed with a further equivariant structure for the group  $\mathrm{GL}_n$  acting by “change of framing”.

is induced by the universal automorphic deformation of  $\bar{\rho}$ ), as well as analogous isomorphisms when we add auxiliary  $Q_N$ -level structure, as in the patching process.

**6.2. The relationship between  $L_\infty$  and  $M_\infty$ .** If Condition 6.1.2 held, then we would find that our patched  $R_\infty[G]$ -module  $M_\infty$  is obtained by restricting  $R_\infty \otimes_{R_{\bar{r}}} L_\infty$  to the smallest closed subscheme of  $\text{Spec } R_\infty$  which contains the support of all the modules  $(M_1, Q_{N'(N)}^\square \otimes_{S_\infty} S_\infty/\mathfrak{b}_N)_{K_{2N}}$  that arise in the patching process. Optimistic conjectures (of “big  $R$  equals big  $\mathbb{T}$ ”-type) might suggest that this support is all of  $R_\infty$ , and thus that  $M_\infty$  is obtained from  $L_\infty$  simply by pulling-back it back along the natural map  $\text{Spec } R_\infty \rightarrow \text{Spec } R_{\bar{r}}$ . For this reason, we are hopeful that our patched representation  $M_\infty$  is a good candidate for (the pull-back to  $R_\infty$  of)  $p$ -adic local Langlands.

**6.3. The relationship with the Fontaine–Mazur conjecture.** Note that if both Conditions 6.1.1 and 6.1.2 held, together with an appropriate “big  $R$  equals big  $\mathbb{T}$ ” result, then we would find that if  $\rho$  is de Rham deformation of  $\bar{\rho}$  corresponding to a point of  $R_S^{\text{univ}}$ , then it would contribute to the locally algebraic vectors of completed cohomology. However, locally algebraic vectors in the completed cohomology arise precisely from algebraic automorphic forms (see [Eme06b]), and hence we would conclude that  $\rho$  would be an automorphic Galois representation. (This is an abstraction of the strategy used to deduce the Fontaine–Mazur conjecture for odd two-dimensional representations of  $G_{\mathbb{Q}}$  in [Eme].)

**6.4. Concluding remarks.** The preceding discussion shows that the question of whether one can in fact relate  $M_\infty$  to a purely local correspondence which satisfies the above two conditions is closely related to the Fontaine–Mazur conjecture for deformations of  $\bar{\rho}$ . Given this, we expect it to be a difficult question to precisely determine the support of  $M_\infty$  and the related modules  $M_\infty(\sigma^\circ)$  in general, and we likewise expect it to be difficult to analyze the extent to which  $M_\infty$  arises from a purely local construction over  $\text{Spec } R_{\bar{r}}$ .

Furthermore, it seems quite possible that our hypothetical formulation of the  $p$ -adic local Langlands correspondence is too naive; even if a local correspondence of some kind exists, it may be of a more subtle nature. In this case, we would still expect it to have a strong relationship to our patched modules  $M_\infty$ , but perhaps not as direct as the one considered in the above discussion.

In any case, whatever the eventual truth might be, the preceding discussion suggests that the further investigation of the patched representation  $M_\infty$  is a problem of substantial interest, which we hope to return to in future work.

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