ALGEBRAIC GEOMETRY — FIFTH HOMEWORK (DUE MONDAY MARCH 16)

Please complete all the questions. As usual, we consider $k \subset K$ with K assumed to satisfy the Nullstellensatz (equivalently, as we now know, K is algebraically closed).

1. Let $X \to Y$ be a morphism of affine varieties over a field k. Show that the induced morphism $k[Y] \to k[X]$ on rings of regular functions is injective if and only if the original morphism $X \to Y$ has dense image.

2. Let I be an ideal in $k[x_1, \dots, x_n]$, let $Z \hookrightarrow \mathbb{A}^n(K)$ be the Zariksi closed subset cut out by I, let $J \subseteq k[X_0, \dots, X_n]$ denote the corresponding homogeneous ideal — i.e. the ideal generated by the homogenizations of all the elements of I — and let $W \hookrightarrow \mathbb{P}^n(K)$ denote the Zariski closed subset cut out by the homogeneous ideal J. Show that W coincides with the Zariski closure of Z in $\mathbb{P}^n(K)$. [Slogan: projective closure equals Zariksi closure.]

3. (a) Prove that the Segre embedding $\mathbb{P}^m(K) \times \mathbb{P}^n(K) \hookrightarrow \mathbb{P}^N(K)$ (where $N = mn+m+n$) is injective, with image equal to an algebraic set in $\mathbb{P}^N(\Omega)$. [This was sketched in class; your job is to check it carefully.]

(b) Recall that there are two ways to define a Zariksi topology on $\mathbb{P}^m(K) \to \mathbb{P}^n(K)$: (i) Using bihomogeneous polynomials in $X_0, \dots, X_m$ and $Y_0, \dots, Y_n$; (ii) by using the Segre embedding of (a) to regard $\mathbb{P}^m(K) \times \mathbb{P}^n(K)$ as a closed subset of $\mathbb{P}^N(K)$. Verify that these two topologies coincide.

4. (a) If $d \geq 1$, show that the map $\mathbb{P}^1(K) \to \mathbb{P}^d(K)$ defined via $[X_0 : X_1] \to [X_0^d : X_0^{d-1}X_1 : \dots : X_0X_1^{d-1} : X_1^d]$ is (i) well-defined, and (ii) is a morphism.

(b) By elimination theory, the image of the morphism of (a) is closed. In the cases when $d = 2$ and 3, find explicit equations that cut out its image. (The image in the case $d = 3$ is called a *twisted cubic curve* in $\mathbb{P}^3$. What is the traditional name for the image in the case $d = 2$?)

(c) [This is really a remark, rather than a question.] Generalizing (a) gives a map $\mathbb{P}^n(K) \to \mathbb{P}^N(K)$, where $N := \binom{n+d}{n} - 1$, which is called the *degree d Veronese embedding.* In the case when $n = 2$, we get the morphisms $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ (for $d = 2$) and $\mathbb{P}^2 \hookrightarrow \mathbb{P}^9$ (for $d = 3$) which came up in our study of linear systems of conics and cubics.