ALGEBRAIC GEOMETRY — FOURTH HOMEWORK
(DUE FRIDAY FEB 24)

Please complete all the questions. For each question, please provide examples/graphs/pictures illustrating the ideas behind the question and your answer. Throughout you should presume given a field $k$ contained in an algebraically closed field $\Omega$, work with $\Omega$-valued points, and feel free to use the Nullstellensatz for $\Omega$ if necessary.

1. Consider the ideal $I = (XY - sZ^2, Y - tX, stXZ) \subseteq k[s, t, X, Y, Z]$. Think of $s, t$ as the coordinates on $\mathbb{A}^2$, and $X, Y, Z$ as the homogeneous coordinates on $\mathbb{P}^2$. Note that the generators of $I$ are homogeneous in $X, Y, Z$, and so $I$ cuts out a zero locus $Z_I(\Omega) \subseteq (\mathbb{A}^2 \times \mathbb{P}^2)(\Omega)$. What is the image of $Z_I(\Omega)$ in $\mathbb{A}^2(\Omega)$ under the projection?

2. (a) Prove that $\varphi : t \mapsto (t^2, t^3)$ defines a morphism from $\mathbb{A}^1(\Omega)$ to $Z_I(\Omega) \subseteq \mathbb{A}^2(\Omega)$, where $I \subseteq k[x, y]$ is the ideal $(y^2 - x^3)$.

(b) Show that $\varphi$ induces a bijection on points, but is not an isomorphism.

(c) Prove that there is a unique extension of $\varphi$ to a morphism $\tilde{\varphi} : \mathbb{P}^1(\Omega) \to Z_I(\Omega) \subseteq \mathbb{P}^2$, where $\tilde{I} \subseteq k[X, Y, Z]$ is the homogeneous ideal $(X^3 - Y^2Z)$.

3. Let $x, y$ be coordinates on $\mathbb{A}^2$, and let $X, Y$ be homogeneous coordinates on $\mathbb{P}^1$. Consider the ideal $I = (xY - yX) \subseteq k[x, y, X, Y]$, which cuts out an algebraic set $Z := Z_I(\Omega) \subseteq (\mathbb{A}^2 \times \mathbb{P}^1)(\Omega)$. Let $\pi : Z \to \mathbb{A}^2(\Omega)$ be the projection.

(a) Let $U = \mathbb{A}^2(\Omega) \setminus \{0\}$; note that $U$ is a Zariski open subset of $\mathbb{A}^2(\Omega)$. Show that the restriction of $\pi$ to $\pi^{-1}(U)$ induces an isomorphism $\pi^{-1}(U) \xrightarrow{\sim} U$.

(b) Prove that $\pi^{-1}(\{(0,0)\})$ is a Zariski closed subset of $Z$ that is isomorphic to $\mathbb{P}^1(\Omega)$.

[The algebraic set $Z$ is called the blow-up of $\mathbb{A}^2$ at the origin; the process of forming $Z$ from $\mathbb{A}^2$ is called blowing up. This question shows that blowing up at the origin leaves $\mathbb{A}^2$ unchanged away from the origin, but replaces the origin by a copy of $\mathbb{P}^1$ (which more canonically can be thought of the space of directions of lines passing through the origin).]
4. Maintain the notation $Z, \pi$ of the previous question. Let $C \subseteq \mathbb{A}^2(\Omega)$ be the curve cut out by $y^2 = x^3$ (i.e. $C = Z_j(\Omega)$, where $J = (y^2 - x^3) \subseteq k[x, y]$).

(a) Prove that $C \setminus \{(0, 0)\}$ is isomorphic to $\mathbb{A}^1(\Omega) \setminus \{0\}$. (This is a variation on Question 2 above.)

(b) Let $\tilde{C}$ denote the Zariski closure in $Z$ of $\pi^{-1}(C \setminus \{(0, 0)\})$. Prove that your isomorphism of (a) extends to an isomorphism between $\mathbb{A}^1(\Omega)$ and $\tilde{C}$.

[We call $\tilde{C}$ the proper transform of $C$ in the blow-up of $\mathbb{A}^2$ at the origin. This question shows that although $C$ is singular at the origin, its proper transform $\tilde{C}$ is non-singular (since it is isomorphic to the non-singular curve $\mathbb{A}^1$). A big part of the interest in blowing-up is that it provides a tool for resolving singularities.]

5. Let $C$ be a smooth curve in $\mathbb{P}^2(\Omega)$, thought of now as a Zariski closed subset. Let $\ell$ be a line in $\mathbb{P}^2(\Omega)$, and suppose that $\ell$ is not tangent to $C$ at any point.

If $P$ is a point of $C$, let $t_P$ denote the tangent line to $C$ at $P$. Define a map $f : C \to \ell$ via

\[ P \mapsto \text{the unique point of intersection of } t_P \text{ and } \ell. \]

Prove that $\varphi$ is a morphism.

6. Consider the projective plane curve $C$ cut out by the homogeneous equation

\[ Y^2Z = X^2(X + Z). \]

Write down a non-constant morphism from $\mathbb{P}^1$ to $C$.

7. Let $Q \subset \mathbb{P}^3(\Omega)$ be the quadric cut out by the homogeneous equation

\[ X^2 + Y^2 - Z^2 - W^2 = 0. \]

Find an injective morphism from $\mathbb{A}^2(\Omega)$ to $Q$. Can you extend this to a morphism from $\mathbb{P}^2(\Omega)$? If not, what is the largest open subset of $\mathbb{P}^2(\Omega)$ containing $\mathbb{A}^2(\Omega)$ to which you can extend it?

8. If $X$ is a topological space, $\{U_i\}_{i \in I}$ is an open cover of $X$, and $Z$ is a subset of $X$, prove that $Z$ is closed if and only if $Z \cap U_i$ is closed in $U_i$ (when $U_i$ is given the induced topology) for all $i$. [This verifies a fact used several times in class.]