

ALGEBRAIC GEOMETRY — FIRST HOMEWORK
(DUE FRIDAY JAN 20)

Please complete all the questions. For each question (even question 2, if you can see how) please provide examples/graphs/pictures illustrating the ideas behind the question and your answer.

1. Suppose that the field k is algebraically closed. Prove that an affine conic (i.e. a degree 2 curve in the affine plane) is smooth at every point if and only if the conic is either irreducible, or the union of two distinct parallel lines.

2. Prove that if $f \in k[x_1, \dots, x_n]$ is homogeneous of degree n , then
$$\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} = nf.$$

3. Suppose that f and g are non-zero homogeneous polynomials in two variables, and that f and g are of different degrees. Show that if the curve $f + g = 0$ has a singular point other than the point $(0, 0)$, then it is reducible (i.e. $f + g$ is a reducible polynomial). [Hint: Remember the formula from the previous exercise.]

4. Suppose that f is an irreducible cubic polynomial in two variables. Prove that the affine curve $f = 0$ has at most one singular point. [Hint: Use the previous exercise.]

5. Let $f \in \mathbb{R}[x, y]$ be a non-constant polynomial, and suppose that $(x_0, y_0) \in \mathbb{R}^2$ lies on the curve C cut out by $f = 0$. Show that (x_0, y_0) is an *isolated point* in the set of real points of C if the tangent cone to f at (x_0, y_0) has no real linear factors.

6. Let $f \in \mathbb{C}[x, y]$ be a non-constant polynomial, cutting out the plane curve C . Prove that the set of complex points of C is *not* a compact topological space.

7. Let $f \in \mathbb{R}[x, y]$ be a non-constant polynomial, cutting out the plane curve C . Prove that if C has no real points at infinity, then the set of real points of C is a compact topological space. Is the converse true?

8. Let $f \in \mathbb{R}[x, y]$ be a non-constant polynomial, cutting out the plane curve C , and suppose that P is a real point at infinity of C , at which the intersection multiplicity of C and the line at infinity equals 1.

(a) Explain why C is smooth at P .

(b) Let ℓ be the tangent line to P at C . Explain why ℓ is *not* the line at infinity.

(c) Prove that the set of real points of C has as an asymptote along the line ℓ .

9. (a) Let C be a projective plane curve over \mathbb{C} . Show that $C(\mathbb{C})$, endowed with its topology as a closed subset of $\mathbb{P}^2(\mathbb{C})$, has no isolated points. (If you need to, feel free to assume that C is a smooth curve, so that you can apply the holomorphic form of the implicit function theorem.)

(b) Let $f \in \mathbb{C}[x, y]$ be non-constant, cutting out the affine plane curve C_f . Let F be the homogenization of f , and let C_F be the projective plane curve cut out by F . Show that $C_F(\mathbb{C})$ is the closure in $\mathbb{P}^2(\mathbb{C})$ of $C_f(\mathbb{C})$. (Hint: use part (a).)

(c) If we replace \mathbb{C} by \mathbb{R} , are the analogues of (a) and (b) true or false? Give arguments or counterexamples as appropriate. (Hint: look back at Questions 5 and 7.)

10. Let k be an algebraically closed field, and let C be a projective plane conic (i.e. a projective plane curve of degree 2).

(a) Prove that there is a choice of coordinates for \mathbb{P}^2 so that the equation of C has one of the following forms:

$$XY - Z^2, \quad XY, \quad \text{or} \quad X^2.$$

(b) Prove that C is smooth at every one of its points if and only if the equation for C is irreducible if and only if its equation can be put in the first form of (a).

(c) Prove that C is singular at every one of its points if and only if the equation for C can be put in the third form of (a).

(d) Prove that C has exactly one singular point if and only if the equation for C can be put in the second form of (a). What is the geometric description of this singular point?

Remark. If the equation for C can be put in the first form of (a), we say that C is *smooth*, or *irreducible*. (For conics, and only for conics, smooth and irreducible are equivalent, if we are over an algebraically closed field.) If the equation for C can be put in the second form of (a), we say that C is *two lines crossing*. If the equation for C can be put in the third form of (a), we say that C is a *double line*.

11. We have defined the intersection multiplicity of a point lying on the intersection of a line and a plane curve, and we have defined projective plane curves. Now you can prove a first case of Bézout's Theorem:

Suppose that k is algebraically closed. Let C be a projective plane curve of degree d , and let ℓ be a line. Show that either $\ell \subset C$ (i.e. the equation for ℓ divides the equation for C), or else that there are finitely many points $P \in \ell(k) \cap C(k)$, and

$$d = \sum_{P \in \ell(k) \cap C(k)} \text{multiplicity of intersection of } \ell \text{ and } C \text{ at } P.$$

- 12.** (a) If k is infinite and C is an affine plane curve, prove that $C(k)$ is a proper subset of $\mathbb{A}^2(k)$.
(b) If k is infinite and C is a projective plane curve, prove that $C(k)$ is a proper subset of $\mathbb{P}^2(k)$.
(c) If k is finite, give counterexamples to each of (a) and (b).