ALGEBRAIC GEOMETRY — SIXTH HOMEWORK (DUE FRIDAY FEBRUARY 28)

Please complete all the questions.

1. Let $X \to Y$ be a morphism of affine varieties over a field k. Show that the induced morphism $k[Y] \to k[X]$ on rings of regular functions is injective if and only if the original morphism $X \to Y$ has dense image.

2. Prove that the Segre embedding from $\mathbb{P}^m(\Omega) \to \mathbb{P}^n(\Omega)$ to $\mathbb{P}^N(\Omega)$ (where $N = mn + m + n$) is injective, with image equal to an algebraic set in $\mathbb{P}^N(\Omega)$. [This was asserted in class; your job is to check it carefully.]

3. Consider the projective plane curve C cut out by the homogeneous equation

$$Y^2 Z = X^2(X + Z).$$

Write down a non-constant morphism from $\mathbb{P}^1$ to C; carefully check that what you've written down *is* a morphism.

4. If X is a topological space, $\{U_i\}_{i \in I}$ is an open cover of X, and Z is a subset of X, prove that Z is closed if and only if $Z \cap U_i$ is closed in U_i (when U_i is given the induced topology) for all i. [This verifies a claim made in class.]

5. Let C be a *smooth* projective plane curve. Let ℓ be a fixed line in the projective plane, and assume that ℓ is *not* tangent to C at any of the points where it intersects C. If $P \in C$, let t_P denote the tangent line. Define a map $C \to \ell$ via

$$P \mapsto \text{ the intersection point of } \ell \text{ and } t_P.$$

Prove that this is a *morphism* from C to ℓ.

6. Let $Q \subset \mathbb{P}^3(\Omega)$ be the quadric cut out by the homogeneous equation

$$X^2 + Y^2 - Z^2 - W^2 = 0.$$

Find an injective morphism from $\mathbb{A}^2(\Omega)$ to Q. Can you extend this to a morphism from $\mathbb{P}^2(\Omega)$? If not, what is the largest open subset of $\mathbb{P}^2(\Omega)$ containing $\mathbb{A}^2(\Omega)$ to which you *can* extend it?