ALGEBRAIC GEOMETRY — SECOND HOMEWORK (DUE FRIDAY JAN 24)

There are 7 questions, plus a bonus question. Questions 1, 2, 4, 5, 6, and 7 are straightforward and precise; please do them, illustrating them with examples as appropriate. Question 3 is precise, but may be tricky, because our foundations of the notion of "dual curve" are slightly imprecise; do what you can. There is also a bonus question; this is open-ended, and is intended just to get you thinking about an interesting and non-obvious geometric phenomenon: given 8 (general) points in the plane, they determine a 9th point.

1. Let k be an algebraically closed field, and let C be a projective plane conic (i.e. a projective plane curve of degree 2).

(a) Prove that there is a choice of coordinates for $\mathbb{P}^2$ so that the equation of C has one of the following forms:

$$XY - Z^2, \quad XY, \quad \text{or} \quad X^2.$$

(b) Prove that C is smooth at every one of its points if and only if the equation for C is irreducible if and only if its equation can be put in the first form of (a).

(c) Prove that C is singular at every one of its points if and only if the equation for C can be put in the third form of (a).

(d) Prove that C has exactly one singular point if and only if the equation for C can be put in the second form of (a). What is the geometric description of this singular point?

Remark. If the equation for C can be put in the first form of (a), we say that C is *smooth*, or *irreducible*. (For conics, and only for conics, smooth and irreducible are equivalent, if we are over an algebraically closed field.) If the equation for C can be put in the second form of (a), we say that C is *two lines crossing*. If the equation for C can be put in the third form of (a), we say that C is a *double line*.

2. Let C be a smooth projective plane conic (over an algebraically closed field k). Prove (e.g. by direct calculation) that the dual curve to C is again a smooth conic.

3. Let C be a smooth projective plane curve, let $P \in C(k)$, and let ℓ denote the tangent line to C at P. Let C^* denote the dual curve to C, in the dual plane $(\mathbb{P}^2)^*$ (the plane that parameterizes lines in $\mathbb{P}^2$). Let

P^* denote the line in $(\mathbb{P}^2)^*$ that parameterizes the lines passing through the point P. Note that (by definition of C^*), the line ℓ corresponds to a point of C^*. Prove that P^* is the tangent line to C^* at its point ℓ. [If you can't give a rigorous proof with the foundations you have available, that's okay: try to understand why the statement is true, and explain what you can. E.g. you could check it in the case of a conic, using your explicit computations from the previous question. Also, feel free to assume that $k = \mathbb{R}$ of $\mathbb{C}$ if you find that it helps; then you could try to argue more topologically.]

4. We have defined the intersection multiplicity of a point lying on the intersection of a line and a plane curve, and we have defined projective plane curves. Now you can prove a first case of Bézout's Theorem:

Suppose that k is algebraically closed. Let C be a projective plane curve of degree d, and let ℓ be a line. Show that either $\ell \subset C$ (i.e. the equation for ℓ divides the equation for C), or else that there are finitely many points $P \in \ell(k) \cap C(k)$, and

$$d = \sum_{P \in \ell(k) \cap C(k)} \text{multiplicity of intersection of } \ell \text{ and } C \text{ at } P.$$

5. (a) If k is infinite and C is an affine plane curve, prove that $C(k)$ is a proper subset of $\mathbb{A}^2(k)$.

(b) If k is infinite and C is a projective plane curve, prove that $C(k)$ is a proper subset of $\mathbb{P}^2(k)$.

(c) If k is finite, give counterexamples to each of (a) and (b).

6. Show that the space of degree d projective plane curves is a copy of $\mathbb{P}^{d(d+3)/2}$.