Algebra 1 : Fourth homework — due Monday, November 3

Do the following exercises from Fulton and Harris:

4.4, 4.6, 4.17

Also do the following exercises:

1. Let G be a finite group, and let $\sum_{g\in G} a_g g$ be an element of $\mathbf{C}[G]$ with the properties that (a) each a_g is a non-negative real number; (b) the coefficient sum $\sum_{g\in G} a_g = 1$; (c) the coefficient a_1 is strictly positive. Let H denote the subgroup of G generated by the elements g for which $a_g > 0$. Then prove that $\lim_{n\to\infty} (\sum_g a_g)^n$ exists and is equal to $\frac{1}{|H|} \sum_{h\in H} a_h h$.

2. Give an example to show that the assumption in ex. 1 that $a_1 > 0$ is necessary.

3. Recall that for $n \geq 1$, and any field k, we let $\mathbf{P}^{n-1}(k)$ denote the set of lines in k^n.

(a) Show that the natural action of $\mathrm{GL}_n(k)$ on k^n induces a transitive action of $\mathrm{GL}_n(k)$ on $\mathbf{P}^{n-1}(k)$, and compute the stabilizer of the line $k \times 0 \times \cdots \times 0$ under this action.

(b) Taking k to be a finite field $\mathbf{F}_q$, use the result of part (a) to inductively compute the order of $\mathrm{GL}_n(\mathbf{F}_q)$.

4. Describe all the conjugacy classes in (a) $\mathrm{GL}_2(\mathbf{R})$; (b) $\mathrm{GL}_2(\mathbf{Q})$.