

Algebra 1 : Second homework — due Monday, October 10

Please read Sections 3.1 and 3.2 of the text carefully. Some of the exercises from these sections are included in the list below, but you should also work through those that I haven't assigned in the privacy of your own homes.

Do the following exercises from Fulton and Harris:

2.2, 2.3, 2.4, 2.5, 2.22, 2.25, 2.27

Also do the following exercises:

1. If G is a group, and $k[G]$ denotes the group ring of G over a field k , we may regard G as a representation of $G \times G$ via $(g_1, g_2) \cdot x = g_1 x g_2^{-1}$ for any $(g_1, g_2) \in G \times G$ and any $x \in k[G]$.

(a) Prove that the preceding definition *does* make $k[G]$ a representation of $G \times G$.

(b) For any G -representation V over k , show that $\text{Hom}_G(V, k[G])$, where the G -equivariance is understood with respect to the G -action on V and the action of the first factor of $G \times G$ on $k[G]$, becomes a representation of G via the action of the second factor of $G \times G$ on $k[G]$.

(c) If G is finite and V is any G -representation over k , show that the map

$$\text{Hom}_G(V, k[G]) \rightarrow \text{Hom}(V, k),$$

given by evaluation at the identity of G , is an isomorphism of G -representations (equipping its target with the contragredient G -action). (If G is infinite, the same result would hold true if we replaced $k[G]$ by the space of *all* k -valued functions on G , rather than just those with finite support.)

(d) If G is finite and k is algebraically closed of characteristic 0, use part (c) to prove that there is an isomorphism of $G \times G$ -representations

$$k[G] \cong \bigoplus_V V \otimes V^*,$$

where V ranges over a set of isomorphism class representatives for the irreducible representations of G , and $G \times G$ acts on $V \otimes V^*$ with the first factor of G acting on V , and the second factor of G acting via the contragredient action on V^* .

(e) Suppose that H is a subgroup of the finite group G , and that k is algebraically closed of characteristic 0. As usual, equip the quotient G/H with the action of G via left multiplication, and consider the associated permutation representation $k[G/H]$. Show that $k[G/H] = k[G]^H$, where the fixed points are taken with respect to the *right* action of H , and combine this with the result of (d) to conclude that

$$k[G/H] \cong \bigoplus_V V \otimes (V^*)^H,$$

where G acts on each summand of the right hand side through its action on the left-hand factor.

2. Prove that *any* representation (not necessarily finite-dimensional) of a finite group G over a field of characteristic zero (or more generally, a field in which $|G|$ is invertible) is a direct sum of irreducible representations.

3. Note that the bijection $g \mapsto g^{-1}$ gives an isomorphism between any group G and its opposite group G^{op} . This in turn induces an isomorphism of k -algebras $k[G] \cong k[G]^{op}$. Give an example of a ring which is not isomorphic to its opposite. Can you find an example that is a finite-dimensional algebra over a field? That is actually finite?