Algebra 1 : Third homework — due Monday, October 17

Do the following exercises from Fulton and Harris:

4.17, 4.50

Also do the following exercises:

1. If V is a three dimensional representation of a group G over a field k, and $\wedge^2 V$ is reducible, prove that V itself is reducible.

2. Let H be a subgroup of a group G, and let U_1 and U_2 be two H-representations over a field k. If $f_1 \in \operatorname{Ind}_H^G U_1$ and $f_2 \in \operatorname{Ind}_H^G U_2$ (so f_i is a function $G \to U_i$ such that $f_i(hg) = hf_i(g)$ for all $g \in G$, $h \in H$), define $f_1 \cdot f_2 : G \to U_1 \otimes U_2$ via

$$(f_1 \cdot f_2)(g) := f_1(g) \otimes f_2(g).$$

Show that $f_1 \otimes f_2 \mapsto f_1 \cdot f_2$ defines a G-equivariant morphism

$$(\operatorname{Ind}_H^G U_1) \otimes (\operatorname{Ind}_H^G U_2) \to \operatorname{Ind}_H^G (U_1 \otimes U_2).$$

3. Let G be a finite group, let H be a subgroup, and let $\underline{1}_H$ (resp. $\underline{1}_G$) denote the trivial representation of H (respectively G). Prove that there is a G-equivariant surjection $\operatorname{Ind}_H^G \underline{1}_H \to \underline{1}_G$, which is unique up to scaling.

4. Let G be a finite group and H a subgroup, and let U be a finite-dimensional representation of H over $\mathbf{C}$. If U^* denotes the contragredient to U, then note that there is a natural H-equivariant map

$$U \otimes U^* \to \underline{1}_H. \qquad (*)$$

Combining this with (2) and (3), one obtains G-equivariant maps

$$(\operatorname{Ind}_H^G U) \otimes (\operatorname{Ind}_H^G U^*) \to \operatorname{Ind}_H^G (U \otimes U^*) \to \operatorname{Ind}_H^G \underline{1}_H \to \underline{1}_G.$$

(The first map arises from (2), the second from functoriality of induction applied to (*), and the third from (3).) Show that this map gives a non-degenerate pairing between $\operatorname{Ind}_H^G U$ and $\operatorname{Ind}_H^G U^*$, and hence realizes the latter as the contragredient of the former.

5. Use (4) to prove that any permutation representation of a finite group over $\mathbf{C}$ is isomorphic its own contragredient.