Research Statement

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My research concerns the variety of pairs of almost commuting nilpotent matrices.

Introduction

There has been recent study in commuting varieties, see for instance [Lev02, Pan04, Knu05, PY07, Pop08, BI08]. I will review some of the classical results in this area.

Let G be a Lie group, and $\mathfrak{g}$ its Lie algebra. The commuting variety of $\mathfrak{g}$ is

$$\mathcal{C}(\mathfrak{g}) = \{(A,B) \in \mathfrak{g} \times \mathfrak{g} \mid [A,B] = 0\}$$

Richardson ([Ric79]) proves that if $\mathfrak{g}$ is reductive over a field of characteristic zero, then $\mathcal{C}(\mathfrak{g})$ is irreducible; in fact, $\mathcal{C}(\mathfrak{g}) = \overline{G \cdot (\mathfrak{h} \times \mathfrak{h})}$ where $\mathfrak{h}$ is a Cartan subalgebra.

For a complex semisimple Lie algebra $\mathfrak{g}$ we define

$$\mathcal{N}_2(\mathfrak{g}) := \{(A,B) \in \mathbf{n} \oplus \mathbf{n} \mid [A,B] = 0\}$$

where $\mathbf{n}$ is the nilcone of nilpotent elements in $\mathfrak{g}$. Baranovsky ([Bar01]) gives a proof of the following result (due to Grinberg).

Theorem 1. $\dim \mathcal{N}_2(\mathfrak{g}) = \dim \mathfrak{g}$ *and there exists a bijection between the set of irreducible components of* $\mathcal{N}_2(\mathfrak{g})$ *which have the maximal dimension* $\dim \mathfrak{g}$*, and the set of distinguished nilpotent adjoint classes in* $\mathfrak{g}$*, where an adjoint class is distinguished if one (and therefore any) of its elements* x *satisfies that its centralizer* $\mathfrak{z}_{\mathfrak{g}}(x) = \{y \in \mathfrak{g} \mid [y,x] = 0\}$ *is contained in* $\mathbf{n}$*.*

The Lie algebra $\mathfrak{g} = \mathfrak{sl}_n$ has a unique distinguished nilpotent class, and therefore a unique component of maximal dimension. Baranovsky proves the following:

Theorem 2. *The variety of commuting pairs of nilpotent operators of an* n*-dimensional vector space* $\mathcal{N}_2(\mathfrak{sl}_n)$ *is irreducible of dimension* $n^2 - 1$*.*

To prove this, he uses a connection between this variety and the Hilbert scheme of points. Recall (see [Nak99]) that one definition of the Hilbert scheme of points in the affine plane is

$$\mathcal{H}^n(\mathbb{A}^2) = \{(A,B,v) \in \mathfrak{gl}_n \times \mathfrak{gl}_n \times V \mid [A,B] = 0,\, \mathbb{C}[A,B]v = V\}/GL_n.$$

Since A and B commute we can simultaneously put them into upper triangular matrices. The Hilbert-Chow morphism $\mathcal{H}^n(\mathbb{A}^2) \to (\mathbb{A}^2)^n/S_n$ is defined by

$$\left(\begin{pmatrix} a_1 & \dots & * \\ \vdots & \ddots & \vdots \\ 0 & \dots & a_n \end{pmatrix}, \begin{pmatrix} b_1 & \dots & * \\ \vdots & \ddots & \vdots \\ 0 & \dots & b_n \end{pmatrix} \right) \mapsto ((a_1,b_1), \dots, (a_n,b_n)).$$

We have to mod out by the action of the symmetric group S_n since we can change the order of the eigenvalues. We define $\mathcal{H}^n_{[0]}$ to be the fiber of $n \cdot [0] = ((0,0), \dots, (0,0))$ under the Hilbert-Chow morphism.

In his proof, Baranovsky shows that $U = \{(A,B,v) \in \mathcal{N}_2(\mathfrak{g}) \times V \mid \mathbb{C}[A,B]v = V\}$ is irreducible, dense in $\mathcal{N}_2(\mathfrak{g}) \times V$ and has dimension n^2+n-1. $GL(V)$ acts faithfully on U, and the quotient $U/GL(V)$ is precisely $\mathcal{H}^n_{[0]}$, which is known to be irreducible of dimension $n-1$. Basili ([Bas03]) provides another proof of Theorem 2 that extends to the case when the characteristic of the field is either 0 or $\geq n/2$.

Baranovsky also conjectures that for any semisimple Lie algebra $\mathfrak{g}$ all irreducible components of the variety $\mathcal{N}_2(\mathfrak{g})$ of pairs of commuting nilpotent elements in $\mathfrak{g}$ have maximal dimension $\dim \mathfrak{g}$. The conjecture was proved by Premet ([Pre03]) when $\mathfrak{g}$ is the Lie algebra of a connected reductive algebraic group over an algebraic closed field of any characteristic.

There has been some work in varieties related to the commuting variety. Let F be an algebraically closed field and

$$M_n^{(k)} = \{(A,B) \in \mathfrak{gl}_\mathfrak{n} \times \mathfrak{gl}_\mathfrak{n} \mid \mathrm{rk}([A,B]) \leq k\}$$

where $\mathfrak{gl}_n := \mathfrak{gl}_n(F)$ is the algebra of endomorphisms of an n-dimensional F-vector space. It is known that $M_n^{(k)}$ is irreducible for $k \neq 1$ (see [Ger61, MT55] for $k=0$, [Hul81] for $k \geq 2$), but $M_n^{(1)}$ has $n-1$ irreducible components of dimension n^2+2n-1 ([Gur80, Neu89]).

In [GG06], Gan and Ginzburg study the scheme

$$\mathcal{M} = \{(X,Y,i,j) \in \mathfrak{g} \times \mathfrak{g} \times V \times V^* \mid [X,Y] + ij = 0\},$$

where $V = \mathbb{C}^n$, $\mathfrak{g} = \mathrm{End}(V) = \mathfrak{gl}_n(\mathbb{C})$. We regard the elements of V and V^* as column and row vectors, respectively. The scheme $\mathcal{M}$ has $n+1$ components, the ideal is a complete intersection, and the scheme is reduced of equidimension n^2+n. The components of $\mathcal{M}$ are the closures of the sets $\mathcal{M}_0, \mathcal{M}_1, \ldots, \mathcal{M}_n$, defined as

$$\mathcal{M}_t = \{(X,Y,i,j) \in \mathcal{M} \mid Y \text{ has pairwise distinct eigenvalues and } \dim \mathbb{C}\langle X,Y\rangle i = t,\ \dim j\mathbb{C}\langle X,Y\rangle = n-t\}$$

where $\mathbb{C}\langle X,Y\rangle i$ (resp. $j\mathbb{C}\langle X,Y\rangle$) is the smallest subspace of V (resp. V^*) containing i (resp. j) and invariant under X and Y.

$\mathfrak{g} \times \mathfrak{g} \times V \times V^*$ can be identified with $T^*(\mathfrak{g} \times V)$, which has a symplectic structure. GL_n acts diagonally on $\mathfrak{g} \times V$ by $G \cdot (X,v) = (GXG^{-1}, Gv)$, and therefore it acts on $T^*(\mathfrak{g} \times V) \simeq \mathfrak{g} \times \mathfrak{g} \times V \times V^*$ by $G \cdot (X,Y,i,j) = (GXG^{-1}, GYG^{-1}, Gi, jG^{-1})$. This action admits a moment map $\mu : \mathfrak{g} \times \mathfrak{g} \times V \times V^* \to \mathfrak{g}$, $(X,Y,i,j) = [X,Y] + ij$. Let $\mathcal{M}_{\mathrm{nil}} = \{(X,Y,i,j) \in \mathcal{M} \mid Y \text{ is nilpotent}\}$. This is a closed subset of $\mathcal{M}$ which, viewed as inside $T^*(\mathfrak{g} \times V)$, is a (not necessarily reduced) complete intersection that is Lagrangian.

Their results have applications to the study of the commuting variety, and Cherednik algebras: Consider the (not necessarily reduced) commuting scheme $\mathcal{Z} = \mathrm{Spec}\ \mathbb{C}[X,Y]/I$ where $\mathbb{C}[X,Y] = C[\mathfrak{g} \times \mathfrak{g}]$ stands for the polynomial algebra, and $I \subset \mathbb{C}[X,Y]$ stands for the ideal generated by the n^2 entries of the matrix $[X,Y]$. It is a standing open question whether or not the scheme Z is reduced, i.e., whether or not $\sqrt{I} = I$. They prove that $(\sqrt{I})^G = I^G$. They also give a construction of the spherical subalgebra of a Cherednik algebra in terms of quantum hamiltonian reduction.

Results

I have studied the the variety

$$\mathcal{N} := \{(X,Y,i,j) \in \mathbf{n} \times \mathbf{n} \times V \times V^* \mid [X,Y] + ij = 0\}$$

and proved that it has n irreducible components: 2 of dimension n^2+n-1 corresponding to the case where the matrices commute, and $n-2$ of dimension n^2+n-2, corresponding to the noncommutative pairs.

Let $\mathcal{N}_{r,s} = \{(X,Y,i,j) \in \mathcal{N} \mid \dim \mathbb{C}\langle X,Y\rangle i = r, \dim j\mathbb{C}\langle X,Y\rangle = s\}$, $\mathcal{N}'_{r,s} = \{(X,Y,i,j) \in \mathcal{N} \mid \dim \mathbb{C}\langle X,Y\rangle i \leq r, \dim j\mathbb{C}\langle X,Y\rangle \leq s\}$. Clearly $\mathcal{N}_{r,s}$ is a closed set containing $\mathcal{N}_{r,s}$. Since X and Y can be put in upper triangular form simultaneously ([EG02], Lemma 12.7), we have that $[X,Y]$ is not only strictly upper triangular, but the entries that are located two positions over the diagonal are zero too. Therefore,

$$\mathcal{N} = \mathcal{N}_{0,n} \cup \mathcal{N}_{n,0} \cup \bigcup_{0<r+s<n} \mathcal{N}_{r,s}.$$

Following Baranovsky ([Bar01]), we have that the closures of $\mathcal{N}_{0,n}$ and $\mathcal{N}_{n,0}$ are $\mathcal{N}'_{0,n}$ and $\mathcal{N}'_{n,0}$ which can be identified (using his notation) with $\mathcal{N}_2(\mathfrak{sl}_n) \times V$ and $\mathcal{N}_2(\mathfrak{sl}_n) \times V^*$, respectively.

Since

$$\mathcal{N} = \mathcal{N}'_{0,n} \cup \mathcal{N}'_{n,0} \cup \bigcup_{0<r+s<n} \overline{\mathcal{N}}_{r,s},$$

the irreducible components of $\mathcal{N}$ are the non-redundant closed sets on the right hand side. Our goal is to identify those irreducible components. The following Lemma proves the non-redundancy of $\mathcal{N}_{t,n-1-t}$, $1 \leq t \leq n-2$.

Lemma 3. *$\mathcal{N}_{t,n-1-t} \cap \overline{\mathcal{N}}_{r,s} = \emptyset$ unless $r=t$ and $s=n-1-t$.*

The main result is thus the following.

Theorem 4. *The irreducible components of $\mathcal{N}$ are precisely $\overline{\mathcal{N}}_{t,n-1-t}$, $1 \leq t \leq n-2$, $\mathcal{N}'_{0,n}$ and $\mathcal{N}'_{n,0}$.*

In order to do this, I prove the following

Theorem 5.
For $1 \leq t \leq n-1$, $\mathcal{N}_{t,n-1-t}$ (and therefore $\overline{\mathcal{N}}_{t,n-1-t}$) is irreducible and has dimension n^2-n+2.

As in Baranovsky's proof, the idea is to describe the GL_n orbits using the Hilbert scheme.

Lemma 6. *Let $(X,Y,i,j) \in \mathcal{N}$. Both $j\mathbb{C}\langle X,Y\rangle$ (resp. $\mathbb{C}\langle X,Y\rangle i$) and its perpendicular complement $(j\mathbb{C}\langle X,Y\rangle)^\perp = \{v \in V \mid uv = 0\ \forall u \in j\mathbb{C}\langle X,Y\rangle\}$ (resp. $(\mathbb{C}\langle X,Y\rangle i)^\perp = \{z \in V^* \mid zw = 0\ \forall w \in \mathbb{C}\langle X,Y\rangle i\}$) are $\mathbb{C}[x,y]$-modules, where x and y act as X and Y respectively.*

Remark 7. *In our case, Lemma 6 is less clear than in Baranovsky's proof, since X and Y do not commute.*

From now on we write $j\mathbb{C}[X,Y]$ (resp. $\mathbb{C}[X,Y]i$) instead of $j\mathbb{C}\langle X,Y\rangle$ (resp. $\mathbb{C}\langle X,Y\rangle i$). Lemma 6 implies that there is a well-defined ideal $\{p(x,y) \in \mathbb{C}[x,y] \mid p(X,Y)v = 0, \forall v \in \mathbb{C}[X,Y]i\}$; and since i is a cyclic vector for X and Y in $\mathbb{C}[X,Y]i$ that ideal is equal to $\{p(x,y) \in \mathbb{C}[x,y] \mid p(X,Y)i = 0\}$ and its length is equal to $\dim \mathbb{C}[X,Y]i$. Therefore there is a well-defined map $\mathcal{N}_{r,s} \to H^r_{[0]}$. Since $\mathbb{C}[x,y]$ acts on the $(n-r)$-dimensional space $(j\mathbb{C}[X,Y])^\perp$, there is also a well-defined ideal $\{p(x,y) \in \mathbb{C}[x,y] \mid p(X,Y)v = 0, \forall v \in (j\mathbb{C}[X,Y])^\perp\}$, but the length of this ideal may not be $n-r$ since there may not be a cyclic vector for X and Y in $(j\mathbb{C}[X,Y])^\perp$.

Lemma 8. *Let $(X,Y,i,j) \in \mathcal{N}_{t,n-1-t}$, $1 \leq t \leq n-2$. There exists a cyclic vector for X and Y in $(j\mathbb{C}[X,Y])^\perp$. Therefore there is a well-defined map $\mathcal{N}_{t,n-1-t} \to \mathcal{H}^{t+1}_{[0]}(\mathbb{A}^2)$, $(X,Y,i,j) \mapsto \{p(x,y) \in \mathbb{C}[x,y] \mid p(X,Y)v = 0, \forall v \in (j\mathbb{C}[X,Y])^\perp\}$.*

We can explicitly describe the image of this map.

Theorem 9. *The map* $\mathcal{N}_{t,n-1-t} \to \mathcal{H}^{t+1}_{[0]}(\mathbb{A}^2)$ *induced by the actions of* X *and* Y *on* $(j\mathbb{C}[X,Y])^{\perp}$ *is dominant. The image of any element of* $\mathcal{N}_{t,n-1-t}$ *has the form* $\langle y^{t+1}, x - a_1 y - \cdots - a_t y^t\rangle$ *or* $\langle x^{t+1}, y - a_1 x - \cdots - a_t x^t\rangle$ *for some* $a_1, \ldots, a_{t-1} \in \mathbb{C}$.

Clearly elements of a GL_n-orbit have the same image under the map $\mathcal{N}_{t,n-1-t} \to \mathcal{H}^{t+1}_{[0]}(\mathbb{A}^2)$; we prove the following stronger result.

Theorem 10. *The fibers of the map* $\Psi : \mathcal{N}_{t,n-1-t} \to \mathcal{H}^{t+1}_{[0]} \times \mathcal{H}^{n-t}_{[0]}$ *are the* $GL_n(\mathbb{C})$*-orbits, and the isotropy of each element of* $\mathcal{N}_{t,n-1-t}$ *is one-dimensional.*

The map $\Psi : \mathcal{N}_{t,n-1-t} \to \mathcal{H}^{t+1}_{[0]} \times \mathcal{H}^{n-t}_{[0]}$ is dominant, and the fibers have the same dimension and are irreducible. With this information we can prove the irreducibility of $\mathcal{N}_{t,n-1-t}$ and calculate its dimension. This proves Theorem 5

To complete the proof of Theorem 4 I proved the following:

Theorem 11. *If* $0 < r + s < n - 1$ *then* $\mathcal{N}_{r,s} \subseteq \bigcup_{0<t<n-1} \overline{\mathcal{N}}_{t,n-1-t}$. *Therefore*

$$\mathcal{N} = \mathcal{N}'_{0,n} \cup \mathcal{N}'_{n,0} \cup \bigcup_{t=1}^{n-2} \overline{\mathcal{N}}_{t,n-1-t}$$

The main idea of the proof is to consider the fibers of the map $(X, Y, i, j) \mapsto (X|_{\mathbb{C}[X,Y]i}, Y|_{\mathbb{C}[X,Y]i}, i)$ and prove that they are birrational to $\mathcal{K}_{n-r} \times (V'^{*})^{r+1}$ where $\mathcal{K}_{n-r}$ is the variety of commuting nilpotent endomorphisms of $\mathbb{C}^{n-r}$.

Research plans

- Find an analogous result in characteristic $p > 0$. Most of the results mentioned here extend to the case where $p \geq n/2$ since the Hilbert scheme is irreducible.
- In order to extend these and another analogous results to other Lie algebras, one must generalize find a way to generalize the property of almost commutativity. One possibility is to define that two elements of a simple algebra almost commute if their Lie bracket is in the unique nonzero nilpotent orbit of minimal dimension ([CM93]). Note that even in the case when two not necessarily nilpotent matrices almost commute, it is still true that its Lie bracket is nilpotent. This idea is due to George McNinch.
- I would like to study the connections between the commuting or almost commuting variety and vector bundles over projective spaces ([OSS80]). For instance, the image of $\mathcal{N}_{t,n-1-t} \to \mathcal{H}^{t+1}_{[0]}$ is a vector bundle over $\mathbb{P}^2$.
- $\mathcal{N}$ projects into the variety $\{(a, b) \in \mathbf{n} \times \mathbf{n} \,|\, \mathrm{rk}(a + b) \leq 1\}$, $(X, Y, i, j) \mapsto (XY, -YX)$. There is a numerical evidence that suggests that irreducible components are mapped into irreducible components.

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