ÉTALE COHOMOLOGY SEMINAR LECTURE 5

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1. Sheaves on the Étale Site (Continued)

The following definition is an obvious generalization of our notion of a local ring for the étale topology.

Definition 1.1. Let X be a scheme, $\overline{x}$ a geometric point of X, and $\mathcal{F}$ a presheaf on X_{et}. The **stalk** $\mathcal{F}_{\overline{x}}$ of $\mathcal{F}$ at $\overline{x}$ is defined as $\mathcal{F}_{\overline{x}} = \varinjlim \mathcal{F}(U)$, where the limit runs over all étale neighborhoods $(U, \overline{u})$ of $\overline{x}$.

Example 1.2. Let k be a field. Recall that an étale k-algebra is a finite direct product of finite, separable extensions of k. Suppose $\mathcal{F}$ is a sheaf of abelian groups on $\operatorname{Spec} k$. The sheaf condition implies that if k''/k' is a finite Galois extension, where k' is finite and separable over k, then

$$\mathcal{F}(k') \to \mathcal{F}(k'') \rightrightarrows \mathcal{F}(k'' \otimes_{k'} k'')$$

is exact. It is a straightforward exercise to check that $k'' \otimes_{k'} k'' \cong (k'')^{[k'':k']}$ in such a way that the first inclusion of k'' is the identity on each factor, while the second inclusion twists by the action of a unique element of the Galois group on each factor. The sheaf condition also implies that products are sent to direct sums under $\mathcal{F}$, so it follows that $\mathcal{F}(k') \cong \mathcal{F}(k'')^{\operatorname{Gal}(k''/k')}$.

Since finite Galois extensions are cofinal among all extensions, it follows that the sheaf $\mathcal{F}$ is uniquely determined by its stalk over the geometric point k^{sep}. This identification determines an equivalence of categories between sheaves of abelian groups on $(\operatorname{Spec} k)_{\text{et}}$ and discrete $\operatorname{Gal}(k^{\text{sep}}/k)$-modules, where the forward direction sends a sheaf to its stalk over k^{sep}, and the reverse direction sends a discrete $\operatorname{Gal}(k^{\text{sep}}/k)$-module M to the sheaf $\mathcal{F}_M(A) = \operatorname{Hom}_{\operatorname{Gal}(k^{\text{sep}}/k)}(\operatorname{Hom}_{k-\text{alg}}(A, k^{\text{sep}}), M)$.

Definition 1.3. We call a sheaf $\mathcal{F}$ a **skyscraper sheaf** if $\mathcal{F}_{\overline{x}} = 0$ for any geometric point $\overline{x} \to X$ with image lying outside a finite set of points in X.

For Λ an abelian group, $\overline{x} \to X$ a geometric point, we define, for $U \to X$ étale,

$$\Lambda^{\overline{x}}(U) = \bigoplus_{\operatorname{Hom}_X(\overline{x}, U)} \Lambda.$$

If the image x of $\overline{x}$ is closed, then this is a skyscraper sheaf with support at x. To give a morphism $\mathcal{F} \to \Lambda^{\overline{x}}$ is to give compatible systems of morphisms $\mathcal{F}(U) \to \Lambda$ for every étale neighborhood $(U, \overline{u})$ of $\overline{x}$. This is equivalent to giving a morphism $\mathcal{F}_{\overline{x}} \to \Lambda$. Thus,

$$\operatorname{Hom}(\mathcal{F}, \Lambda^{\overline{x}}) \cong \operatorname{Hom}(\mathcal{F}_{\overline{x}}, \Lambda).$$

2. The Category of Sheaves

Let X be a scheme. We denote by $\mathrm{Sh}(X_{\mathrm{et}})$ the category of sheaves of abelian groups on X_{et}. A morphism of sheaves is simply a natural transformation of functors. This category is clearly additive with the obvious biproduct, $(\mathcal{F} \oplus \mathcal{G})(U) = \mathcal{F}(U) \oplus \mathcal{G}(U)$. We will see that, as in the case of Zariski sheaves, this category is abelian.

Lemma 2.1. *Let $\alpha : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on X_{et}. Then the following are equivalent.*

(a) *α is surjective, i.e., $\mathcal{F} \xrightarrow{\alpha} \mathcal{G} \to 0$ is exact.*
(b) *α is locally surjective, i.e., for every $U \in X_{\mathrm{et}}$, $s \in \mathcal{G}(U)$, there exists a covering $(U_i \to U)_{i \in I}$ such that for all $i \in I$, there exists $t_i \in \mathcal{F}(U_i)$ with $\alpha(t_i) = s|_{U_i}$.*
(c) *α is surjective on stalks, i.e., for every geometric point $\overline{x} \to X$, $\mathcal{F}_{\overline{x}} \xrightarrow{\alpha_{\overline{x}}} \mathcal{G}_{\overline{x}} \to 0$ is exact.*

Proof. (b) $\Rightarrow$ (a). Suppose $\beta : \mathcal{G} \to \mathcal{H}$ is a morphism of sheaves such that $\beta \circ \alpha = 0$. Let $s \in \mathcal{G}(U)$ for some étale $U \to X$. Then there is some étale cover $(U_i \to U)_{i \in I}$, $s_i \in \mathcal{F}(U_i)$, such that $\alpha(s_i) = s|_{U_i}$ for all $i \in I$. Thus, $\beta(s)|_{U_i} = (\beta \circ \alpha)(s_i) = 0$, and so $\beta(s) = 0$ by the sheaf condition on $\mathcal{H}$. Hence α is surjective.

(a) $\Rightarrow$ (c). Fix a geometric point $\overline{x} \to X$. Let $\Lambda = \mathrm{coker}(\mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}})$. We wish to show $\Lambda = 0$. The surjective cokernel map $\mathcal{G}_{\overline{x}} \to \Lambda$ corresponds to a map $\mathcal{G} \to \Lambda^{\overline{x}}$. But the composition $\mathcal{F} \to \mathcal{G} \to \Lambda^{\overline{x}}$ corresponds to the composition $\mathcal{F}_{\overline{x}} \to \mathcal{G}_{\overline{x}} \to \Lambda$, which is the zero morphism. Since $\mathcal{F} \to \mathcal{G} \to 0$ is exact, this implies that our map $\mathcal{G} \to \Lambda^{\overline{x}}$ is the zero morphism, and hence the image of $\mathcal{G}_{\overline{x}}$ in Λ is zero. Since this map was surjective, we must have $\Lambda = 0$ as desired.

(c) $\Rightarrow$ (b). It is clear that if $U \to X$ is étale, $\overline{u} \to U$ a geometric point of U, then $\mathcal{F}_{\overline{u}} \cong \mathcal{F}_{\overline{x}}$, where $\overline{x}$ is the geometric point of $\overline{x}$ given by composition of $\overline{u}$ with $U \to X$. Thus, for every $\overline{u} \to U$, $\mathcal{F}_{\overline{u}} \to \mathcal{G}_{\overline{u}}$ is surjective. This means that for each $\overline{u} \to U$, there is some étale neighborhood $(V, \overline{v})$ of $\overline{u}$ with $s|_V$ in the image of $\mathcal{F}(V) \xrightarrow{\alpha|_V} \mathcal{G}(V)$. Taking the union of these V as a cover of U, we have that α is locally surjective. ∎

There is an obvious forgetful functor from $\mathrm{Sh}(X_{\mathrm{et}})$ to the category $\mathrm{PreSh}(X_{\mathrm{et}})$ of presheaves of abelian groups on X_{et}. Just as in the case of Zariski sheaves, this functor has a left adjoint, called *sheafification*. A simple construction of the sheafification of a presheaf $\mathcal{F}$ is given by $\mathcal{F}^{\mathrm{sh}} = \prod_{x \in X} (\mathcal{F}_{\overline{x}})^{\overline{x}}$, where $\overline{x}$ is any geometric point lying over x. It is straightforward to check that $\mathcal{F}^{\mathrm{sh}}$ satisfies the appropriate universal property.

Proposition 2.2. *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ be a sequence of sheaves of abelian groups on X_{et}. Then the following are equivalent.*

(a) *The sequence is exact.*
(b) *For every étale $U \to X$, the sequence $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$ is exact.*
(c) *For every geometric point $\overline{x} \to X$, the sequence $0 \to \mathcal{F}'_{\overline{x}} \to \mathcal{F}_{\overline{x}} \to \mathcal{F}''_{\overline{x}}$ is exact.*

Proof. Since the forgetful functor $\mathrm{Sh}(X_{\mathrm{et}}) \to \mathrm{PreSh}(X_{\mathrm{et}})$ admits a left adjoint, it is left exact. Condition (b) is precisely the condition of left exactness in the category of presheaves, so (a) and (b) are equivalent. Since taking direct limits preserves exactness, (b) implies (c).

Finally, suppose (c) holds, and $s' \in \mathcal{F}'(U)$ is mapped to zero in $\mathcal{F}(U)$. Then its image is zero on every stalk, so s' is zero at every stalk of $\mathcal{F}'$. Thus $s = 0$. Similarly, if $s \in \mathcal{F}(U)$ maps to zero in $\mathcal{F}''(U)$, then s must restrict to an element of $\mathcal{F}'_x$ on every stalk, from which it follows that $s \in \mathcal{F}'(U)$. Thus (c) implies (b). □

Corollary 2.3. *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a sequence of sheaves of abelian groups on X_{et}. Then the following are equivalent.*

(a) *The sequence is exact.*
(b) *For every étale $U \to X$, the sequence $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$ is exact, and $\mathcal{F} \to \mathcal{F}''$ is locally surjective.*
(c) *For every geometric point $\overline{x} \to X$, the sequence $0 \to \mathcal{F}'_{\overline{x}} \to \mathcal{F}_{\overline{x}} \to \mathcal{F}''_{\overline{x}} \to 0$ is exact.*

Corollary 2.4. *The category* $\mathrm{Sh}(X_{\text{et}})$ *is abelian.*

Proof. The kernel of a morphism is the usual presheaf kernel, which is obviously a sheaf. The cokernel of a morphism is the sheafification of the presheaf cokernel. Images and coimages are isomorphic because they are isomorphic on stalks. □

We now give examples of two important exact sequences on the étale site. Note that these sequences are clearly *not* exact in the Zariski topology. This is our first real sign that étale cohomology may be more powerful than Zariski cohomology.

Example 2.5 (The Kummer sequence)**.** Let $n \in \mathbb{Z}$ be relatively prime to the characteristics of all residue fields of X. We define the **Kummer sequence** of degree n to be the sequence

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{t \mapsto t^n} \mathbb{G}_m \to 0.$$

We wish to show this sequence is exact. By Corollary 2.3, this is equivalent showing that the stalks are exact at every geometric point $\overline{x} \to X$. If $A = \mathcal{O}_{X,\overline{x}}$, then the corresponding sequence of stalks is

$$0 \to \mu_n(A) \to A^\times \xrightarrow{t \mapsto t^n} A^\times \to 0.$$

This sequence is clearly left-exact. To show it is exact, we must show that every element $a \in A^\times$ has an nth root. The derivative of the polynomial $p(t) = t^n - a \in A[t]$ is nt^{n-1}, which has nonzero image $\overline{p} \in (A/\mathfrak{m})[t]$ because $n \neq 0$ in $A/\mathfrak{m}$. Thus $\overline{p}$ is separable. Since A is strictly Henselian, p splits into linear factors. In particular, this implies that a possesses an nth root.

Example 2.6 (The Artin-Schreier sequence)**.** Let X be a scheme such that every residue field of X has characteristic $p \neq 0$. (For example, X could be an algebraic variety over a field k of characteristic p.) We define the **Artin-Schreier sequence** to be the sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \xrightarrow{t \mapsto t^p - t} \mathbb{G}_a \to 0.$$

Again, to prove exactness, we must check exactness of

$$0 \to \mathbb{Z}/p\mathbb{Z} \to A \xrightarrow{t \mapsto t^p - t} A \to 0$$

for every $A = \mathcal{O}_{X,\overline{x}}$. Again, this is clearly left-exact. We note that , for any $a \in A$, the polynomial $p(t) = t^p - t - a$ has derivative $pt^p - 1$, which maps to $-1 \neq 0$ in

$(A/\mathfrak{m})[t]$. Thus, by the reasoning above, a is in the image of $t \mapsto t^p - t$, and so the sequence is exact.

Note the geometric significance of these results. For example, the exactness of the Kummer sequence says that étale-locally, every invertible function a on some affine open set $U = \operatorname{Spec} B$ in X has an nth root. But this follows immediately from the fact that the map $B \to C = B[t]/(t^n - a)$ defines an étale cover; the function t on $\operatorname{Spec} C$ is an nth root of A.

References

1. James S. Milne, *Lectures on Etale Cohomology*, `http://www.jmilne.org/math/`, 2008.