Homotopy Colimits and Universal Constructions

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In this talk I'll take of the gloves and say ‘∞-category’ for what I was calling ‘homotopy theory’ before. You may take that to mean any of your favorite models that we talked about last time (e.g. topological categories, complete Segal spaces, quasicategories, etc.)

(1) There aren’t enough colimits in the homotopy category of spaces.

Example 1.1 (Tyler Lawson). Suppose we had a pushout diagram in hSpaces*, the homotopy category of pointed spaces, which looked like this:

```
S^1 → → *
2 ↓ ↓
S^1 → → X
```

Then, by the universal property of pushouts, \([X,Y] = \{α ∈ π_1 Y : y^2 = 0\}\), i.e. \(X\) corepresents 2-torsion in the fundamental group. In particular, \([X,K(π,1)] = \text{Hom}(\mathbb{Z}/2, π)\) so that \(π_1 X = \mathbb{Z}/2\), and also \([X,S^0] = 0\) so \(X\) is connected. Thus, by choosing a CW-complex with the homotopy type of \(X\), we can build a map \(\mathbb{R}P^2 → X\) such that the composite

\[\mathbb{R}P^2 → X → \mathbb{R}P^∞\]

induces an isomorphism on fundamental groups, and hence on \(H_1\) as well. On the other hand, \([X, \mathbb{C}P^∞] = 0\) so the composite \(X → \mathbb{R}P^∞ → \mathbb{C}P^∞\) is nullhomotopic, and so is the composite after precomposing with the map from \(\mathbb{R}P^2\).

On the other hand, the maps \(\mathbb{R}P^2 → \mathbb{R}P^∞\) and \(\mathbb{R}P^∞ → \mathbb{C}P^∞\) both induce surjections on \(H^2\), a contradiction.

So that’s too bad. However, every homotopy theorist knows that, morally, the thing that deserves to be \(X\) in that diagram is \(\mathbb{R}P^2\), the cofiber of multiplication by 2 on \(S^1\). This leads to a natural question:

What is the universal property of the cofiber?

Well, we know we can compute it as the mapping cone which is obtained as the pushout of the diagram

```
S^1 → → \text{CS}^1 ≅ D^2
2 ↓ ↓
S^1 → → \mathbb{R}P^2
```

so this has a universal property as a pushout. But, in other cases, say when \(A ⊂ B\) is the inclusion of a subcomplex in a CW-complex, we know that \(B/A\) works perfectly well like the cofiber should (inducing long exact sequences in homology and so on.) We also know that, if we change the objects of our diagram by homotopy equivalent ones and then compute the mapping cone (say with the above formula) we will get a homotopy equivalent answer.

To give a satisfactory answer we will need some notation. Let \(K\) denote the diagram \(\bullet ← \bullet → \bullet\), and let \(K^\circ\) denote the square. Denote by Spaces the ∞-category of spaces and Fun(K, Spaces) the ∞-category of \(K\)-shaped diagrams in spaces. Then
Proposition 1.2. Let $\overline{p} : K^\triangleright \to \text{Spaces}$ be the diagram in the underlying $\infty$-category associated to a pushout square of spaces:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
C & \to & D
\end{array}
$$

where $f$ has the homotopy extension property (i.e. is a cofibration). Then let $p = \overline{p}|_K$ and, for any $Y \in \text{Spaces}$, let $\overline{Y}$ be the constant diagram on $Y$. The natural restriction map:

$$\text{Map}_{\text{Spaces}}(D,Y) \longrightarrow \text{Map}_{\text{Fun}(K,\text{Spaces})}(p,\overline{Y})$$

is a weak equivalence.

It is possible (though tedious) to prove this directly from the definition of mapping spaces via hammock localization. In a minute we will discuss some tools to prove this as a consequence of the general theory. Before we do, we may as well state the general definition of a homotopy colimit in an $\infty$-category:

Definition 1.3. If $K$ is any small $\infty$-category and $K^\triangleright$ is the cone on $K$ (i.e. the result of adding a single new object that receives one map from every object of $K$), then we say that $p : K^\triangleright \to \mathcal{C}$ exhibits the cone point $X$ as a homotopy colimit of $p = p|_K$ if, for every $Y \in \mathcal{C}$, the restriction map

$$\text{Map}_\mathcal{C}(X,Y) \longrightarrow \text{Map}_{\text{Fun}(K,\mathcal{C})}(p,\overline{Y})$$

is a weak equivalence.

Remark 1.4. Technically we have a zig-zag $\{(\text{cone pt.})\} \hookrightarrow K^\triangleright \hookleftarrow K$, but the map $\text{Fun}(K^\triangleright,\mathcal{C}) \to \mathcal{C}$ given by evaluation on the cone point is a trivial fibration, hence admits a section and any choice of section will do for the above definition.

Remark 1.5. It is not difficult to show directly that the collection of homotopy colimits for a diagram is ‘unique up to a contractible space of choices’ (if it exists!). Explicitly, if $\mathcal{E} \subset \text{Fun}(K^\triangleright,\mathcal{C})$ is the full subcategory spanned by those diagrams which are a homotopy colimit of their restriction to $K$, then the map $\mathcal{E} \to \text{Fun}(K,\mathcal{C})$ is a trivial fibration over its image.

Before we explore this definition further let me just give a bunch of examples of homotopy (co)limits. I won’t justify them, but most will be justified by the end of the talk.

Example 1.6. If $X$ is a (pointed) space then $\Sigma X$ and $\Omega X$ fit into homotopy colimit and homotopy limit diagrams, respectively:

$$
\begin{array}{ccc}
X & \longrightarrow & * \\
\downarrow & & \downarrow \\
\Sigma X & \to & \Omega X
\end{array}
$$

In fact, we can take these as definitions of loop and suspension in any pointed $\infty$-category with finite limits and colimits.

Remark 1.7. It’s worth noting we can recover Puppe sequences in this setting:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
* & \xrightarrow{g} & C
\end{array}
$$

$$
\begin{array}{ccc}
C & \xrightarrow{h} & D \\
\downarrow & & \downarrow \\
\Sigma C & \to & \Sigma D
\end{array}
$$
If the first square is a homotopy pushout and either the right square or the big rectangle is a homotopy pushout, then every square/rectangle you see is a homotopy pushout (this is a formal consequence of the definition). In that situation we have \( D \cong \Sigma A \) (from the rectangle) and \( D \cong \text{cofib}(q) \) (from the square). We can keep going and we find the cofiber of \( h \) is \( \Sigma B \), and so on. There is a dual story using pullbacks and loops.

This is the genesis of every long exact sequence you’ve ever seen, and, in the land of ‘stable homotopy theory’ this tells you that ‘the homotopy category of a stable \( \infty \)-category is triangulated’, which is nice.

**Example 1.8.** If \( X \) is a space with an action of a group \( G \) then we can form the homotopy colimit over the elements of \( G \) and a model for this is \( EG \times_G X \) (the *homotopy orbits*). Similarly, the homotopy limit can be modeled by \( \text{Map}(EG, X) \) (the *homotopy fixed points*).

**Example 1.9.** If \( X_\bullet \) is a simplicial set viewed as a simplicial diagram of (discrete) spaces, then a model for the homotopy colimit is given by the geometric realization. Actually, if we modeled our spaces with simplicial sets, we could take \( X_\bullet \) itself (tautologically) as the homotopy colimit!

**Example 1.10.** More generally, if \( X_\bullet \) is any simplicial space, define a bi-simplicial set by \( X_{p,q} := \text{Sing}(X_p)_q \). Then \( \text{hocolim}_{\Delta^{op}} X_\bullet \) is modeled by \( |\text{diag}(X_\bullet, \cdot)| \). This is a formal consequence of the previous example and the fact that the diagonal map \( \Delta^{op} \to \Delta^{op} \times \Delta^{op} \) is cofinal, which means that homotopy colimits over the latter thing can be computed as homotopy colimits over the former thing. This fact is used over and over again in the game of homotopy theory. It is the generalization to \( \infty \)-categories of the nice behavior of reflexive coequalizers.

**Example 1.11.** If \( E \) is a spectrum (not that we know what that is yet) modeled by a sequence of spaces \((E_0, E_1, E_2, ...)\) then
\[
E = \text{hocolim} \left( \Sigma^\infty_+ E_0 \to \Sigma^\infty_+ E_1 \to \Sigma^\infty_+ E_2 \to \cdots \right).
\]
Together with the fact that smash products and taking homotopy groups commute with homotopy colimits, this recovers all your favorite formulae for the homology of a spectrum.

**Example 1.12.** The previous example also holds for genuine \( G \)-spectra, except the formula now looks like this:
\[
E = \text{hocolim} \left( \Sigma^\infty_+ E_0 \to \Sigma^\infty_+ V_1^\infty E_1 \to \cdots \right)
\]
where \( V_1 \subset V_2 \subset \cdots \) are real orthogonal representations of \( G \) eventually containing every finite dimensional real orthogonal representation of \( G \). Even if you know nothing about \( G \)-spectra, you can often use this to formally reduce questions to \( G \)-spaces (even finite \( G \)-CW-complexes) by manipulating homotopy colimits.

**Example 1.13.** If \( X \) is an \( E_\infty \)-space then a ‘delooping’ (quotes removed if \( X \) is group-like) can be defined as the homotopy pushout in \( E_\infty \)-spaces:

\[
\begin{array}{ccc}
X & \longrightarrow & 1 \\
\downarrow & & \downarrow \\
1 & \longrightarrow & BX
\end{array}
\]
(Note that if we took this homotopy pushout in *spaces* we would get \( \Sigma X \).) We can recover our favorite bar construction formula *formally* if we are willing to accept some hard but completely general facts: (1) The forgetful functor from \( E_\infty \)-spaces to spaces admits a left adjoint (the *free* \( E_\infty \)-space functor), (2) any \( E_\infty \)-space can be recovered as the homotopy colimit of the simplicial object \( \text{Free}_{E_\infty}(X) \), (3) the homotopy colimit of any simplicial object can be computed by taking the geometric realization of the underlying spaces.

**Example 1.14.** The previous example generalizes to any time we are in some situation where we have ‘stuff with algebraic structure’ and a forgetful functor, i.e. any (co)monadic situation. This gives a sophisticated reason for all the bar constructions that appear in the literature- they are a generalization of the familiar fact that we can compute colimits of algebraic gadgets by first resolving by free gadgets.
Example 1.15. So far we’ve just been taking homotopy colimits of diagrams indexed by *ordinary* categories. But we can do more. For example: we could take an ∞-category with one object- which could be modeled by a topological monoid. Then homotopy (co)limits again give homotopy orbits and homotopy fixed points, modeled by the usual things. More generally, we can associate to *any space* an ∞-category $\Pi_\infty X$ called its *fundamental ∞-groupoid*. The homotopy colimit of the constant diagram at a point in Spaces is just weakly equivalent to $X$ back again, and so we get a functor:

$$\hocolim : \text{Fun}(\Pi_\infty(X), \text{Spaces}) \longrightarrow \text{Spaces}/X$$

This turns out to be an equivalence of ∞-categories, generalizing the usual equivalence between covering spaces and representations of the fundamental groupoid (actually it’s a little weaker than this, since for that result we need to know our space $X$ was nice, but it’s morally what’s going on.)

Example 1.16. There is an ∞-category of ∞-categories, $\text{Cat}_\infty$ and it is (co)complete. If $C$ is a pointed ∞-category with (co)limits, we can formally define

$$\text{Sp}(C) := \text{holim} \left( \cdots \Omega \xrightarrow{} C \xrightarrow{\Omega} \cdots \right)$$

and this is a model for the *stabilization of* $C$, or *spectrum objects* in $C$ which is supposed to capture the idea that we’ve inverted the loops functor. This is a fancy realization of the original Boardmann-Vogt definition of spectra, and again allows us to formally conclude lots of things about spectra.

(2) Ok, so hopefully you’re starting to get the idea that homotopy (co)limits are pretty ubiquitous. I’d like to begin to justify some of the examples I gave. The problem for us is that, if we want to verify that some construction we built in some relative category $(\mathcal{C}, W)$ is a homotopy colimit, we need to have an understanding of *mapping spaces* in the underlying homotopy theory of $\mathcal{C}$ and in $\text{Fun}(K, \mathcal{C})$.

Let me actually write down Dwyer and Kan’s definition of this mapping space, just to give you an idea of what a mess we’re in:

**Definition 2.17 (Hammock localization).** Let $(\mathcal{C}, W)$ be a relative category and $X, Y \in \mathcal{C}$ objects. We define a simplicial set $\text{Map}_{L_H}(X, Y)$ with $k$-simplices of the form:

Here $n$ is arbitrary- each column of arrows alternates in direction, and left-facing arrows are weak equivalences. The columns could start with either a left or right-facing arrow. We don’t allow any column to be only identity maps, and faces and degeneracies are given by deleting or inserting rows and then making the diagram ‘reduced’ to get rid of identity columns and such.

Now let me try to explain where ‘fibrations’ and ‘cofibrations’ come from. The idea is that this simplicial set is huge. It would be better if we could find a much smaller, weakly equivalent simplicial set. In particular,
we would expect something like a ‘deformation’ retract from this big simplicial set to a smaller one. Here is a candidate for such a thing: Just look at zig-zags like this

\[
\begin{array}{ccc}
X & \xleftarrow{\sim} & X_0 \\
& \searrow & \downarrow \\
& X_0 & \leftrightsquigarrow Y_0 & \xrightarrow{\sim} & Y
\end{array}
\]

Indeed, given some longer zig-zag, say

\[
\begin{array}{ccc}
C_0 & \xrightarrow{\sim} & C_2 \\
& \searrow & \downarrow \\
X & \xrightarrow{\sim} & C_1 & \xrightarrow{\sim} & Y
\end{array}
\]

we might try to form the pullback of the middle square (if it exists):

\[
\begin{array}{ccc}
X_0 & \xleftarrow{\sim} & ? \\
& \searrow & \downarrow \\
& X_0 & \leftrightsquigarrow C_0 & \xrightarrow{\sim} C_1 & \xrightarrow{\sim} Y
\end{array}
\]

and this would seem to reduce the length of our zig-zag, as long as we could ensure that the ? is still a weak equivalence. So the general idea is to search for some collection of weak equivalences which are stable under pullback, and so that every other weak equivalence is ‘close’ to being of that form. Dually, we could do the same trick with pushouts. If we write down exactly what we need, it turns out that we’d come up with the definition of what Barwick and Kan call a partial model category.

Since I’ve alluded to it before, let me now give some different conditions you can put on a relative category that makes working with it a little easier:

1. (Calculus of fractions) This is basically like assuming that those pullbacks exist, but it’s weaker: we just assume we can complete such squares in some way with a weak equivalence in the right place. We also assume something called ‘right cancellability’ which I won’t spell out.

   Bonus: Together these conditions imply that the homotopy category exists and the hom-sets can be computed in a relatively simple way using just one zig and one zag.

2. (Homotopical categories) A homotopical category is a relative category whose weak equivalences contain all isomorphisms and which satisfy the 2 out of 6 property: if we have three composable morphisms \( h, g, f \) and \( hg \) and \( gf \) are weak equivalences, so are each of \( f \), \( g \), and \( h \).

   Bonus: When \( \mathcal{C} \) admits a calculus of fractions, the 2 out of 6 property is equivalent to the statement that weak equivalences are exactly the maps that become isomorphisms in the localization \( \mathcal{C}[W^{-1}] \).

3. (Fibration categories) A fibration category is a homotopical category which has a final object and some collection of morphisms called fibrations, denoted by \( \rightarrow \). Objects \( x \) such that \( x \rightarrow * \) is a fibration are called fibrant. Fibrations which are also weak equivalences are called trivial fibrations. We ask that if \( f : A \rightarrow C \) is a morphism between fibrant objects and \( p : B \rightarrow C \) is a (trivial) fibration, then \( B \times_C A \rightarrow A \) is a (trivial) fibration. We also require that any map \( f : A \rightarrow C \) with \( C \) fibrant factors as a weak equivalence followed by a fibration. Dualizing gives the notion of a cofibration category.

   Bonus: Everything from before and the hammock localization has Kan-complexes as morphism spaces with a simpler expression for the mapping spaces using a smaller number of zig-zags. Also get the existence of some finite homotopy limits in the underlying \( \infty \)-category for free.
4. (Partial model categories) A partial model category is a homotopical category with subcategories $U, V \subset W$ of the weak equivalences such that

(i) Pushouts along maps in $U$ exist and are in $U$,
(ii) Pullbacks along maps in $V$ exist and are in $V$,
(iii) the maps $w \in W$ may be functorially factored as $w = vu$ where $v \in V$ and $u \in U$.

These axioms are supposed to model how trivial cofibrations and trivial fibrations behave in spaces, and they allow us to carry out the program from before of reducing zig-zags.

Bonus: It is easy to write down the hammock localization, and the homotopy category doesn’t invert any more arrows than you told it to. This is as far as you’ll ever need to go because every $\infty$-category underlies a partial model category.

5. (Model categories) A model category is simultaneously a cofibration and fibration category with the added requirements that we every diagram

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow & & \downarrow \\
B & \longrightarrow & Y
\end{array}
\]

of solid arrows where the left vertical map is a cofibration and the right vertical map is a fibration admits a dotted lift if one of the vertical arrows is also a weak equivalence. We also add a bunch more conditions for the heck of it:

(a) Every map can be functorially factored as a cofibration (resp. trivial cofibration) followed by a trivial fibration (resp. fibration),
(b) Cofibrations are exactly those that satisfy the left lifting property with respect to trivial fibrations, and dually for fibrations,
(c) All our special arrows are closed under retracts.

Bonus: Very explicit computation of mapping spaces here: if $X$ is cofibrant and $Y$ is fibrant, just take the nerve of the category of spans and weak equivalences of spans.

Modulo set theory, every (co)complete $\infty$-category can be explicitly modeled by a simplicial model category- we’ll even write down which one in a minute. Since model categories are the oldest method for abstract homotopy theory, there are the most references- starting with Quillen’s excellent book where he invented them!

6. (Simplicial model categories) These are model categories enriched in simplicial sets that have the following interplay between their simplicial and model structures: for every cofibration $i : A \rightarrow B$ and fibration $X \rightarrow Y$ the morphism of simplicial sets

\[
\text{Hom}(B, X) \longrightarrow \text{Hom}(A, X) \times_{\text{Hom}(A, Y)} \text{Hom}(B, Y)
\]

is a fibration and is a weak equivalence if either $i$ or $p$ is a weak equivalence. This says that the space of factorizations we are guaranteed is contractible. Bonus: If $X$ is cofibrant and $Y$ is fibrant, then the
given simplicial set $\text{Hom}_C(X,Y)$ is a Kan complex which is weakly equivalent to the mapping space in the underlying $\infty$-category (e.g. in the hammock localization).

**Remark 2.18.** Any category with (co)fibrations can be embedded as a (homotopically full) subcategory of a simplicial model category (a result of Cisinski) so, in some sense, there’s not too much danger in always working with these more restrictive objects.

**Remark 2.19.** Most things you like will be model categories, but the important part is that (1) they will often be model categories in more than one way, and (2) functors between them may not behave well with respect to your favorite model category structure in sight. Both of these things will happen for $G$-spectra, for example.

Don’t get so upset by those definitions- there are plenty of references for model categories out there that are very friendly, like Hovey’s book, with lots of examples. It’s good to know that you can model your favorite homotopy theories using a model category, just like it’s useful to know that you can choose local coordinates on a manifold. However, as you know from working with manifolds, many constructions are easier if you give a universal description and then construct and compute locally so that you don’t need to worry about checking the ‘transition functions’.

(3) Okay, at this point we’ve defined colimits using a universal property and discussed how one might try to compute mapping spaces between objects in some homotopy theory coming from a relative category. That will help us to identify homotopy colimits after we’ve built them, but we still need to build them!

Here’s a first guess for what to do: If our relative category $(\mathcal{C}, W)$ just use those! Of course, we already saw that this doesn’t work for spaces. The problem is that ordinary colimits don’t respect weak equivalences of diagrams. This happens a lot, where we have some functor

$$F : \mathcal{C} \longrightarrow \mathcal{D}$$

which doesn’t respect weak equivalences. We’d be okay, though, if we could find a (not necessarily full) subcategory $\mathcal{C}' \subset \mathcal{C}$ such that (1) $F$ preserves weak equivalences on $\mathcal{C}'$, and (2) there is a replacement functor $r : \mathcal{C} \rightarrow \mathcal{C}'$ and a natural weak equivalence $r \rightarrow \text{id}_C$ or $\text{id}_C \rightarrow r$. Such functors are called **left deformable** and **right deformable**, respectively. In this case we call the resulting functor $F \circ r$ the **left (resp. right)** derived functor of $F$. One can show that there is a natural moduli space of such derived functors and it is contractible if it is nonempty.

**Example 3.20.** The functor $\mathbb{Z}/2 \otimes (-) : \text{Ch}(\mathbb{Z}) \longrightarrow \text{Ch}(\mathbb{Z})$ does not respect weak equivalences, but it preserves weak equivalences between complexes of *free* modules. Every complex can be functorially replaced by a complex of free modules, and so we get a left derived functor $\mathbb{Z}/2 \otimes^L (-)$ whose homology groups compute $\text{Tor}$. This can be generalized to any category of modules. It should be noted that for categories of sheaves we have to do something different because there aren’t enough projective- however, there are often enough *flat* objects.

**Example 3.21.** The functor which takes a simplicial commutative $k$-algebra $R$ and computes the Kähler differentials levelwise does not preserve weak equivalences, but it does so between *free* $k$-algebras. It turns out that there are enough of these to deform the functor and we get a left derived function $R \mapsto L_{R/k}$ called the *cotangent complex* whose homotopy groups are the *André-Quillen cohomology*. This example was one of Quillen’s main motivations for inventing model categories in the first place.

**Example 3.22.** On $G$-spectra there is a construction which takes a sort of symmetric power indexed by a finite $G$-set instead of being indexed by just a set. It turns out that in the usual model structures on $G$-spectra this does not preserve weak equivalences between cofibrant objects. Luckily, this functor is still deformable (in fact, there is another model structure called the “positive model structure” which fixes this.) It would be nice to find a model-independent interpretation of this thing so that we knew what we are doing...
We are primarily interested in the example \( \text{colim} : \text{Fun}(D, \mathcal{C}) \to \mathcal{C} \). This doesn’t preserve weak equivalences. However, if \( d \in D \) is some object and \( A \in \mathcal{C} \), then we can form the silly functor \( x \mapsto \coprod_{\text{Hom}_D(d,x)} A \) and it is easy to check that the colimit of this functor is just \( A \). (Indeed, the functor was left Kan extended from the constant functor.) Maps between such silly functors which are weak equivalences levelwise then evidently induce weak equivalences on colimits (as long as we have 2 out of 3). In good cases, for essentially every homotopical category we come across in life, there will be enough functors built from pushouts and filtered colimits and retracts of functors like these to get a left derived functor of the colimit, called the \textbf{homotopy colimit}.

Luckily, we are not tied to any particular model because in one of our favorite examples we can do better.

**Example 3.23.** If \( X_* \) is a simplicial set viewed as a discrete simplicial space then \( \text{hocolim} \ X_* \) is modeled by the geometric realization. (More generally this calculation holds for what are called \textit{Reedy fibrant} simplicial spaces, which are simplicial spaces such that the \( k \)th level sits in a fibration over a space built out of gluing together information from levels \( k-1 \) and below. For example, the natural map \( X_2 \to X_1 \times X_0 \times X_1 \) is a fibration, and so one.) The proof of this is very cute and uses the fact that we can view the colimit as evaluating a certain coend (in two variables) and we can fibrantly replace the second variable by \( \Delta^* \). This fact also remains true for nice simplicial model categories (and uses that funky axiom in an essential way).

We should point out that, at this point, there is potential for confusion: we have two different things called homotopy colimits. Combining our description of mapping spaces in relative categories that have enough fibrant or cofibrant objects of some sort one can check that these two notions agree, i.e. one can check that any choice of left derived functor of a colimit functor automatically satisfies the universal property of a homotopy colimit.

**Theorem 3.24.** If \( \mathcal{C} \) is a relative category with colimits and the left derived functor \( \text{Lcolim} \) exists for some diagram shape \( K \), then it gives a homotopy colimit of this diagram in the underlying \( \infty \)-category.

**Remark 3.25.** As I’ve said before, deriving functors is a bit of an ad hoc way to build functors between \( \infty \)-categories. You lose track of what your functor is actually telling you. It is better if we can give some intrinsically \( \infty \)-categorical universal property of the functor we want. Often we can do this: for example, \( \otimes^L \) can be obtained as the adjoint to \( \text{RHom} \) which itself is just the actual internal hom of the \( \infty \)-category underlying chain complexes (albeit enhanced canonically to a spectrum): its homotopy groups compute \( \text{Ext} \), for example. The cotangent complex also admits an invariant description but that would take use too far afield.

**Remark 3.26.** So far we’ve only talked about homotopy colimits indexed on diagrams which are actually \textit{categories} as opposed to, say, simplicial categories. The latter is possible- though it turns out that any one of these can be replaced by a homotopy colimit over an ordinary category, so life isn’t so bad.

(4) When \( \mathcal{C} \) is a category and \( f : K \to K' \) is a map of small categories we get a pullback functor

\[
f^* : \text{Fun}(K', \mathcal{C}) \to \text{Fun}(K, \mathcal{C})
\]

We like it when this has a adjoints, which it will if \( \mathcal{C} \) is (co)complete: we call a left adjoint \( f_! \) \emph{left Kan extension} and a right adjoint \( f_* \) \emph{right Kan extension}. When \( K' \) is just a \emph{point} this recovers the colimit and limit, resp., of a diagram. It is extremely useful to have an analog of this story for \( \infty \)-categories, and to do so we will need the notion of an adjoint. This isn’t so bad:

**Definition 4.27.** Suppose we have a pair of functors

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{f} & \mathcal{D} \\
\downarrow{g} & & \\
\mathcal{D} & & 
\end{array}
\]
between ∞-categories. We say that \( f \) is a left adjoint to \( g \) if there is a morphism \( u : \text{id}_\mathcal{C} \to g \circ f \) in \( \text{Fun}(\mathcal{C}, \mathcal{C}) \) with the property that, for every \( C \in \mathcal{C} \) and \( D \in \mathcal{D} \), the composition

\[
\text{Map}_\mathcal{D}(f(C), D) \to \text{Map}_\mathcal{C}(gf(C), g(D)) \xrightarrow{u} \text{Map}_\mathcal{C}(C, g(D))
\]

is an equivalence.

If \( \mathcal{C} \) and \( \mathcal{D} \) are homotopical categories and we have an adjunction

\[
\mathcal{C} \xleftarrow{F} \xrightarrow{G} \mathcal{D}
\]

we may wonder when this induces an adjunction on the underlying ∞-categories. Usually it is too much to ask that each functor preserve weak equivalences (as we discussed before), but maybe each admits a left or right derived functor.

I guess there isn’t a proof in the literature at this level of generality that you then get an adjunction of the underlying ∞-categories, but I believe it should be true. A related result for model categories (where we don’t even need to assume functorial factorization exists) was proven by Mazel-Gee (generalizing a result of Lurie). Here we need to know the definition of

**Definition 4.28.** A **Quillen pair** is an adjunction

\[
\mathcal{C} \xleftarrow{F} \xrightarrow{G} \mathcal{D}
\]

between model categories so that any of the following hold:

1. \( F \) preserves cofibrations and trivial cofibrations,
2. \( G \) preserves fibrations and trivial fibrations,
3. \( F \) preserves cofibrations and \( G \) preserves fibrations,
4. \( F \) preserves trivial cofibrations and \( G \) preserves trivial fibrations.

In this case there are always left and right derived functors by just restricting to \( F \) to cofibrant objects to define \( LF \) and restricting \( G \) to fibrant objects to define \( RG \). We say it is a **Quillen equivalence** if, moreover, \( LF \) induces an equivalence on homotopy categories (or dually for \( RG \)).

Quillen pairs give adjunctions of ∞-categories, and Quillen equivalences give equivalences of ∞-categories, so that’s nice. Using this it is usually possible to construct Kan extensions for your favorite ∞-categories by deriving the usual Kan extension functors.

An especially nice example of an adjoint pair is a **localization**.

**Definition 4.29.** We say that a functor \( L : \mathcal{E} \to \mathcal{D} \) is a **localization** if it admits a fully faithful right adjoint.

These put the ‘relations’ in ‘generators and relations’ for building homotopy theories, and come up all over the place. They are often built by taking a relative category and then putting in more weak equivalences. In model category land this is the theory of **Bousfield localizations**.

This theory says: if we have a nice simplicial model category, and some set of morphisms, then we can put a new model structure which adds those morphisms to the weak equivalences and keeps the cofibrations the same. This receives a left Quillen functor from the original guy, and is still nice and simplicial. See [Dugger] and [Hirschorn] (though cellular model categories aren’t so in vogue nowadays.)

**Example 4.30.** Given any homology theory \( E \) there is a localization of spaces and spectra, \( \text{Spaces}_E \) and \( \text{Spectra}_E \), which is the full subcategory spanned by objects \( X \) such that whenever \( A \to B \) is an \( E_* \)-equivalence, \( \text{map}(B, X) \to \text{map}(A, X) \) is a weak equivalence. These localizations were first constructed by Bousfield (both as a bare homotopy category, and again with model categories).
Example 4.31. More generally, whenever $F : \mathcal{C} \to \mathcal{D}$ is a functor then we’d like to be able to formally invert those morphisms in $\mathcal{C}$ which become equivalences under $F$. We can always do this modulo set theory and a mild assumption on $F$. The lingo is: if $\mathcal{C}$ is presentable and $F$ is accessible then we’re good. This recovers the previous example taking $\mathcal{C}$ to be spaces or spectra, $\mathcal{D}$ to be spectra, and the functor to be $X \mapsto X \wedge E$.

(5) We are long past due for some more examples. Let’s consider the homotopy theory of spaces with an action of $G$ where equivalences are underlying weak equivalences. I’ll write this as $\text{Fun}(BG, \text{Spaces})$ where $BG$ is the group $G$ considered as a category with one object. Let’s justify our earlier formula for the homotopy orbits, which is supposed to be a functor $(-)_{hG} : \text{Fun}(BG, \text{Spaces}) \to \text{Spaces}$

that just takes the homotopy colimit of our diagram. Well, we have a silly inclusion $i : \bullet \to BG$ where $\bullet$ is the trivial category with one object and only an identity morphism. One nice thing is that we know how to define homotopy colimits over the trivial category. Denote by $\epsilon$ the other obvious functor $BG \to \bullet$ so that $\epsilon! = \text{hocolim}$. Then here’s the observation:

Spaces $\xrightarrow{i!} \text{Fun}(BG, \text{Spaces}) \xrightarrow{\epsilon!} \text{Spaces}$

commutes- we can compute the homotopy orbits of diagrams which are left Kan extended.

But what is $i_!(X)$ anyway? We can write down a formula for it:

$$i_!(X) = \text{hocolim}_D X$$

where $D$ is the category whose objects are maps $\bullet \to \bullet$ in $BG$ and there are no nontrivial morphisms. This is a discrete category and we know how to compute that homotopy colimit! It’s just $\coprod_{g \in G} X$ where $G$ acts by translating. In other words, it’s $G \times X$.

Now here’s the other piece of good news: we can resolve any $G$-space by free ones like this. That is, if $X$ is a $G$-space, build a simplicial object $B\bullet X$ by

$$B_nX = (i_!i^*)^{n+1}X \cong G^{\times n+1} \times X$$

where the simplicial maps come from the unit of the adjunction.

There is an evident map $B\bullet X \to X$ which induces a map

$$\text{hocolim} B\bullet X \to X$$

I claim this is a weak equivalence. Indeed, we need only check this after applying $i^*$, which detects weak equivalences and preserves homotopy colimits. But then it’s easy: we have an explicit map going the other direction which ‘contracts’ our simplicial object, and it is standard that this gives the equivalence we want.

Now we just compute:

$$X_{hG} = \text{hocolim}_{BG} \text{hocolim}_{\Delta^op} B_nX$$

$$= \text{hocolim}_{\Delta^op} \text{hocolim}_{BG} B_nX$$

(hocolims commute with hocolims)

$$= \text{hocolim}_{\Delta^op} ((i_!i^*)^{n+1}X)_{hG}$$

$$= \text{hocolim}_{\Delta^op} (i_!i^*)^{n}X$$

(hocolim of left Kan extension is easy)

$$= \text{hocolim}_{\Delta^op} G^{\times n} \times X$$

$$= EG \times_G X$$

The “=” signs should be taken lightly, and the last one comes from our earlier observation that homotopy colimits over $\Delta^op$ are the same as geometric realization in good cases, together with the definition of $EG$.

This example is not so special. The same argument proves the following:
**Theorem 5.32.** If $K$ is a category and $f : K \to \mathcal{C}$, then a model for the homotopy colimit (if it exists) is given by:

\[
\text{hocolim}_{\Delta^{op}} \left( \prod_{x \in K} F(x) \right) \cong \prod_{x \to y} F(x) \cong \prod_{x_0 \to x_1 \to x_2} F(x_0)
\]

In particular, when we can compute homotopy colimits over $\Delta^{op}$ by geometric realizations, this gives an explicit choice for a model of the homotopy colimit of a functor.

You should think of this as a generalization of the familiar fact from ordinary category theory that any colimit is given as the reflexive coequalizer of the first two steps in that simplicial object above.

**Remark 5.33.** In fact, the whole genesis of bar constructions in homotopy theory comes from the following set-up: you have some $\infty$-category $\mathcal{C}$ and some type of $\infty$-category of objects in $\mathcal{C}$ with extra algebraic structure, called $\text{Alg}_\mathcal{C}(\mathcal{C})$. Homotopy colimits inside of this category of ‘algebras’ are easy to compute on free things, so you take a presentation (= simplicial resolution) compute there, and then take the geometric realization of your answer. Key words to look up are ‘(co)monads’ and the ‘Barr-Beck theorem’. This yoga recovers our formula for delooping an $E_\infty$-space.

(6) Now that we sort of have a handle on homotopy colimits, we’ll introduce a construction where we take some $\infty$-category of objects in $\mathcal{C}$ and some type of $\infty$-category freely from this by taking homotopy colimits formally.

In the land of ordinary category theory, this is done by taking presheaves of sets. In homotopy theory we always replace sets with spaces, so we will look at presheaves of spaces (or simplicial presheaves.)

**Definition 6.34.** A presheaf on an $\infty$-category $\mathcal{C}$ is just a functor $\mathcal{C}^{op} \to \text{Spaces}$. We denote the $\infty$-category of presheaves by $\text{Psh}(\mathcal{C}) := \text{Fun}(\mathcal{C}^{op}, \text{Spaces})$.

**Theorem 6.35.** Suppose $\mathcal{C}$ is small. There is a fully faithful functor $j : \mathcal{C} \to \text{Psh}(\mathcal{C})$ with the following property: For any cocomplete $\infty$-category $\mathcal{D}$, the restriction map

\[
j^* : \text{Fun}'(\text{Psh}(\mathcal{C}), \mathcal{D}) \to \text{Fun}(\mathcal{C}, \mathcal{D})
\]

is a trivial fibration, where $\text{Fun}'$ denotes the full subcategory of $\text{Fun}$ spanned by homotopy colimit preserving functors. Moreover, $j$ preserves all homotopy limits which exist in $\mathcal{C}$.

**Remark 6.36.** It is possible to characterize $f$ up to a contractible space of choices using something called the Grothendieck construction together with a universal property of that ‘twisted arrow category’ I introduced last time for computing mapping spaces.

**Corollary 6.37.** Modulo set-theory, every complete and cocomplete $\infty$-category underlies a simplicial model category (and vice versa).

**Proof sketch:** The set-theory I’m glossing over ensures that we can obtain $\mathcal{C}$ as a nice localization of $\text{Psh}(\mathcal{C}_0)$ where $\mathcal{C}_0$ is some small $\infty$-category (this is called being presentable). Lurie’s straightening construction allows us to realize $\mathcal{C}_0$ as coming from a small simplicial category $\mathcal{C}_0$, and then $\mathcal{C}$ underlies a Bousfield localization of the injective model structure on simplicial presheaves on $\mathcal{C}_0$.

**Corollary 6.38.** Modulo set-theory, every $\infty$-category underlies a partial model category.

**Example 6.39.** So Spaces is presheaves on a point, which means that it is the free cocomplete $\infty$-category on one object. So if $\mathcal{C}$ is a cocomplete $\infty$-category and $A \in \mathcal{C}$ I get a functor $\text{Spaces} \to \mathcal{C}$, which we denote by $X \mapsto X \otimes A$. So every cocomplete homotopy theory is canonically tensored over spaces. Explicitly, when $X$ is discrete this is just $\prod_{x \in X} A$, and when $X$ is the homotopy colimit of some simplicial set $X_\bullet$, then $X \otimes A$ is the homotopy colimit of $X_\bullet \otimes A$. When $\mathcal{C}$ is some category of commutative-like algebras and $X = S^1$ this thing is called “Hochschild homology”.
Let \( \mathcal{O}_G \) denote the collection of finite, transitive \( G \)-sets for a finite group \( G \), and \( \text{Spaces}^G \) the \( \infty \)-category associated to, say, \( G \)-CW complexes and equivariant homotopy equivalences (or \( G \)-spaces and maps inducing weak equivalences on fixed points for all \( H \subset G \).)

**Proposition 6.40** (Guillou, Elmendorf, ...). There is an equivalence of \( \infty \)-categories

\[
A : \text{Psh}(\mathcal{O}_G) \to \text{Spaces}^G
\]

**Proof sketch.** We get a (homotopy colimit preserving) functor for free because \( \text{Spaces}^G \) is cocomplete and contains \( \mathcal{O}_G \). This has an evident right adjoint \( \text{Fix}(\cdot) \) which takes a space and sends it to its ‘fixed point diagram’ (that comes from a functor on relative categories which preserves weak equivalences by our definition of weak equivalences.)

What we need to check is that the counit map

\[
id \to \text{Fix} \circ A
\]

is an equivalence (this means that \( A \) is fully faithful) and that \( A \) is essentially surjective. The second claim is easier, it follows from the observation that every \( G \)-space is weakly equivalent to a \( G \)-CW complex and these are built from orbits by using homotopy colimits. The second claim is equivalent to the assertion that \( \text{Fix} \circ A \) preserves homotopy colimits, which in turn is equivalent to the assertion that the functor \( (\cdot)^H : \text{Spaces}^G \to \text{Spaces} \) preserves homotopy colimits.

This is not obvious because the statement doesn’t hold on the nose for colimits in any of your favorite relative models modeling the situation. However, the result is true if I restrict to \emph{discrete} objects on both sides, and you are formally reduced, by the results we’ve mentioned above, to checking that this fixed point functor preserves geometric realizations of simplicial \emph{discrete} \( G \)-spaces, and this you can do by hand!

**Remark 6.41.** From the point of view of presheaves, the fixed point functor \( X \mapsto X^H \) is just evaluation on \( G/H \). As such, it factors through the restriction to the full subcategory of \( \mathcal{O}_G \) which just has \{\( G/H \)\} and its automorphisms. Its automorphisms are sometimes denoted \( W(H) \) and called the Weyl group of \( H \). You can check that \( W(H) = N_G(H)/H \).

**Remark 6.42.** From the proposition and the above remark we can think of \( G \)-spaces as the data of: (1) a ‘Borel’ \( W(H) \)-space for every subgroup \( H \subset G \), and (2) a bit of glue. Soon we will meet \( G \)-spectra and it’s important to remember that the above proposition does not generalize so easily- \( G \)-spectra are not spectra indexed over the orbit category. However, this latter description does generalize thanks to work of Saul Glasman: a genuine \( G \)-spectrum is just (1) a collection of plain old spectra with an action \( W(H) \) for each \( H \subset G \), and (2) some glue.

**Example 6.43.** Let’s say we didn’t know what topological spaces were (who does?), but we knew wanted to do homotopy theory and we knew we liked manifolds. Then here’s what you could do. Start with the category \( \text{Mfld} \) of manifolds, and build the \emph{free} homotopy theory on these, that’s \( \text{Psh}^{\text{Mfld}} \). Boom- you’ve got homotopy theory.

Except that you want to remember that if you have an open cover of a manifold then you can glue together the pieces to get back your manifold. So you want to establish the relation that, if \( U_\bullet \) is something like a Cech cover of a manifold \( M \), then the map \( \text{hocolim} U_\bullet \to M \) is an equivalence. If you formally impose that condition (so you localize), then you’ve built a new thing which we’ll call \( \text{Shv}^{\text{Mfld}} \).

Finally, you want to do homotopy theory like you know it, so you’d better make \( \times \mathbb{R} \to X \) an equivalence as well. Great. If you do that you get a new \( \infty \)-category and it’s called \( \text{Shv}_\mathbb{R}^{\text{Mfld}} \). Pretty straightforward. Turns out: you’ve just built \emph{ordinary} homotopy theory!

That is: the sheaf represented by a point (viewed as a 0-dimensional manifold) induces a map

\[
\text{Spaces} \to \text{Shv}_\mathbb{R}^{\text{Mfld}}
\]

and this is an equivalence of \( \infty \)-categories.
Example 6.44. The above example works word for word if you work with $G$-manifolds for $G$ a finite group. (For a compact Lie group I think you may need to add the extra requirement that the map $V \to X$ is a weak equivalence for every equivariant vector bundle over $X$, but maybe you don’t.)

Example 6.45. If you play the same game with smooth schemes and replace $\mathbb{R}$ with $\mathbb{A}^1$ you’ve invented unstable motivic homotopy theory.

There are variants on freely adjoining colimits- you could freely adjoin only filtered colimits and geometric realizations, especially if you already know what you want your finite coproducts to look like. This thing is sometimes called the nonabelian derived category and it’s modeled by the full subcategory of $\text{Psh}(\mathcal{C})$ of product preserving functors, which can be denoted $D(\mathcal{C})$ or $\text{Psh}_\Sigma(\mathcal{C})$.

Example 6.46. Let $\text{Free}_R$ denote the category of finitely generated free $R$-modules where $R$ is some ring. Then $D(\text{Free}_R)$ is equivalent to the $\infty$-category underlying the relative category of $\text{Ch}(R)_{\geq 0}$ and quasi-isomorphisms, i.e. the usual derived category.

Example 6.47. Let $\text{Poly}_k$ denote the category of finitely generated free $k$-algebras, i.e. polynomial algebras. Then $D(\text{Poly}_k)$ is equivalent to the $\infty$-category underlying the relative category of simplicial commutative $k$-algebras and weak equivalences of simplicial sets.

Example 6.48. Take your favorite $\infty$-category $\mathcal{C}$ of complicated algebraic objects. If every object can be written as (filtered colimits of) geometric realizations of diagrams involving objects of a smaller category $\mathcal{C}_0$ that you totally understand, then you’re $\infty$-category $\mathcal{C}$ is a localization of $D(\mathcal{C}_0)$. This is how Goerss-Hopkins obstruction theory works, for example, where $\mathcal{C}$ is the $\infty$-category of $E$-local $E_\infty$-rings in spectra and $\mathcal{C}_0$ is the collection of free algebras on certain finite cellular spectra that are algebraic in the sense that their $E$-homology is projective as a module over $E_+$ and their universal coefficient spectral sequence collapses.

Similarly, if $\mathcal{C}$ is the opposite category of 2-complete spaces then you can take $\mathcal{C}_0$ to be the subcategory of finite products of Eilenberg-MacLane objects $K(\mathbb{F}_2, n)$, and we end up with an approximation of 2-complete spaces by simplicial unstable modules over the Steenrod algebra.

At this point I should probably stop.