INTRODUCTION TO SYMMETRIC SPECTRA. LECTURE 1

In this lecture we will concentrate on the algebraic (rather than homotopy-theoretic) aspects of the theory of symmetric spectra. In particular, the discussion of model categories will be postponed until the next lecture, and the highlight of this talk will be the construction of a closed symmetric monoidal structure on the category of symmetric spectra. (The existence of such a structure is one of the main technical advantages of the theory of symmetric spectra).

The main reference we have used in preparing these notes is

[HSS] M. Hovey, B. Shipley, and J. Smith, "Symmetric spectra", Journal of the AMS **13** (1999), no. 1, 149 – 208.

A very good background reference for this lecture and the next one is

[GM] S. Gelfand and Yu. Manin, "Methods of Homological algebra"

In particular, [GM] contains a detailed discussion of simplicial sets and model categories. Two of the more classical references on simplicial sets and homotopy theory are

[GZ] P. Gabriel and M. Zisman, "Calculus of fractions and homotopy theory"

[M] P. May, "Simplicial objects in algebraic topology"

1. Simplicial sets.

The goal of this section is mostly to fix the notation. Let $\triangle_{\text{big}}$ denote the category of finite linearly ordered sets and nondecreasing maps, and let $\triangle \subset \triangle_{\text{big}}$ be the full subcategory formed by the objects

$$[n] = \{0, 1, 2, \dots, n\}, \qquad n \in \mathbb{Z},\ n \geq 0.$$

Definition: If $\mathcal{C}$ is a category, a **simplicial object** in $\mathcal{C}$ is a functor $X : \triangle^{\text{op}} \to \mathcal{C}$. Thus, for any $n \in \mathbb{Z}_{\geq 0}$, we have an object

$$X_{(n)} := X([n]) \in \mathcal{O}b(\mathcal{C}),$$

and for every nondecreasing map $f : [m] \to [n]$, we have a morphism

$$X(f) : X_{(n)} \to X_{(m)}$$

in $\mathcal{C}$, such that $X(g \circ f) = X(f) \circ X(g)$ whenever $g \circ f$ is defined. Of course, simplicial objects in $\mathcal{C}$ form a category, $\text{Funct}(\triangle^{\text{op}}, \mathcal{C})$. As we have seen in the previous lecture (on the K-theory of Waldhausen categories), sometimes it is also convenient to consider Q-simplicial objects (which generalize simplicial objects, bisimplicial objects, etc.).

Definition: Let Q be a finite set. We write $\triangle^Q \subset \triangle^Q_{\text{big}}$ for the products of copies of the categories $\triangle$ and $\triangle_{\text{big}}$ indexed by Q. If the reader prefers, she/he may think of Q as a discrete category (whose set of objects is Q and which has no non-identity morphisms) and define

$$\triangle^Q = \text{Funct}(Q, \triangle), \qquad \triangle^Q_{\text{big}} = \text{Funct}\,(Q, \triangle_{\text{big}})\,.$$

If $\mathcal{C}$ is any category, we define the **category of Q-simplicial objects** of $\mathcal{C}$ as

$$\mathcal{S}^Q\mathcal{C} := \text{Funct}\left(\left(\triangle^Q\right)^{\text{op}}, \mathcal{C}\right).$$

The **diagonal** of a Q-simplicial object $X : \left(\triangle^Q\right)^{\text{op}} \to \mathcal{C}$ is the simplicial object $\text{diag}(X) : \triangle^{\text{op}} \to \mathcal{C}$ obtained by composing X with the diagonal inclusion $\triangle^{\text{op}} \hookrightarrow \left(\triangle^Q\right)^{\text{op}}$.

Remark: The following comment was used implicitly in the previous lecture. Let $\Sigma_Q = \text{Aut}(Q)$ denote the group of permutations of the set Q (if $|Q| = m$, then $\Sigma_Q \cong \Sigma_m$, the symmetric group on m letters). Then Σ_Q acts on $\triangle^Q$ by permuting the factors, and hence there is an obvious notion of a **Σ_Q-equivariant** Q-simplicial object of $\mathcal{C}$, namely, it is a functor $X : \left(\triangle^Q\right)^{\text{op}} \to \mathcal{C}$ equipped with a compatible collection of isomorphisms of functors $\sigma^* X \xrightarrow{\simeq} X$ for all $\sigma \in \Sigma_Q$. In this situation, the diagonal $\text{diag}(X)$ inherits an *action* of Σ_Q. (This is what was used in the construction of the K-theory of a Waldhausen category as a *symmetric* spectrum that we have presented.) We leave the precise formulation of this fact to the reader.

The main example of simplicial objects for us is the category of **simplicial sets**, i.e., simplicial objects in the category of sets. We denote the category of simplicial sets simply by $\mathcal{S}$. In what follows it will be convenient to use the language introduced by Grothendieck; thus a simplicial set is the same thing as a presheaf of sets on the category $\triangle$. When no confusion is possible, we will omit the word "sets" and simply speak of a "presheaf on $\triangle$". By abstract nonsense, the category $\mathcal{S}$ is complete and cocomplete. For each $n \in \mathbb{Z}_{\geq 0}$, we have the representable presheaf $\triangle[n] \in \mathcal{S}$, defined by

$$\triangle[n]_{(m)} = \triangle([m], [n]).$$

(Here we use the convention that if $\mathcal{C}$ is a category and X, Y are objects of $\mathcal{C}$, then $\mathcal{C}(X, Y)$ denotes the set of morphisms $X \to Y$ in $\mathcal{C}$.) By Yoneda's lemma, we then have, for any $X \in \mathcal{S}$,

$$\mathcal{S}(\triangle[n], X) \cong X_{(n)} \quad \text{canonically.}$$

In particular, let $* = \triangle[0]$; this is a simplicial set with $*_{(m)}$ consisting of one point for each $m \in \mathbb{Z}_{\geq 0}$, and it is clear that $*$ is a terminal object in $\mathcal{S}$.

An initial object, $\varnothing$, also exists in $\mathcal{S}$ (though it is not representable); it is defined by $\varnothing_{(m)} = \varnothing$ for all $m \in \mathbb{Z}_{\geq 0}$.

The category of **pointed simplicial sets** (i.e., the category of arrows $* \to X$ in $\mathcal{S}$) is denoted by $\mathcal{S}_*$; note that it is naturally equivalent to the category of simplicial objects in the category of pointed sets.

2. Geometric realization.

Arguably, the notion of geometric realization could be "discovered" by a beginner in the following way. It is known that for a general category $\mathcal{D}$, every functor $\mathcal{D}^{\text{op}} \to \mathcal{S}ets$ is a colimit of representable ones. Thus if we require the geometric realization functor

$$|\cdot| : \mathcal{S} \longrightarrow \mathcal{T}op = \text{category of topological spaces}$$

to commute with colimits and satisfy the (obviously very natural) property

$$(*) \qquad |\triangle[n]| = \left\{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \;\middle|\; x_i \geq 0, \sum x_i = 1 \right\},$$

then the geometric realization of any simplicial set would be uniquely determined.

Let us give some details. For the abstract categorical aspect, we refer the reader to [GZ], § II.1.2. Fix a simplicial set $X : \triangle^{\mathrm{op}} \to \mathcal{S}ets$. For each $n \geq 0$, consider the simplicial set $X_{(n)} \times \triangle[n]$ (i.e., the disjoint union = coproduct of copies of $\triangle[n]$ indexed by $X_{(n)}$), and form the disjoint union

$$P^0 := \coprod_{n \geq 0} X_{(n)} \times \triangle[n] \in \mathcal{S}.$$

Next, for each arrow $f : [m] \to [n]$ in $\triangle$, let us write $d(f) = m$ and $c(f) = n$ (here "d" stands for "domain" and "c" for "codomain") and form

$$P^1 := \coprod_{f \in \mathrm{Ar}\, \triangle} X_{(c(f))} \times \triangle[d(f)] \in \mathcal{S}.$$

There are two natural morphisms

$$P^1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} P^0 \,,$$

namely, α is induced by the morphisms

$$X_{(c(f))} \times \triangle[d(f)] \xrightarrow{X(f) \times \mathrm{id}} X_{(d(f))} \times \triangle[d(f)],$$

and β is induced by the morphisms

$$X_{(c(f))} \times \triangle[d(f)] \xrightarrow{\mathrm{id} \times \triangle[f]} X_{(c(f))} \times \triangle[c(f)]$$

for all $f \in \mathrm{Ar}\, \triangle$, where

$$\triangle[f] : \triangle[m] \to \triangle[n]$$

is the morphism induced by $f : [m] \to [n]$.

We also have a natural morphism

$$P^0 \xrightarrow{\gamma} X$$

induced by the compositions

$$X_{(n)} \times \triangle[n] \cong \mathcal{S}(\triangle[n], X) \times \triangle[n] \xrightarrow{\mathrm{ev}} X$$

(the last arrow is the obvious "evaluation map"). The following result is very important; its proof is left as an easy but instructive exercise.

Theorem: The diagram $P^1 \overset{\alpha}{\underset{\beta}{\rightrightarrows}} P^0 \xrightarrow{\gamma} X$ is a coequalizer in $\mathcal{S}$.

Remark: The reason for discussing this basic material in such detail is that its understanding is necessary for the later understanding of why the symmetric monoidal structures on pointed simplicial sets, symmetric spectra, etc., are *closed.*

The discussion above motivates the following

Definition:*) For each $n \in \mathbb{Z}_{n\geq 0}$, define the **standard (topological) n-simplex** $|\triangle[n]|$ by the formula $(*)$. If X is a simplicial set, its **geometric realization** $|X|$ is defined as the coequalizer of the pair

$$(**) \qquad \coprod_{f\in \mathrm{Ar}\,\triangle} X_{(c(f))} \times |\triangle[d(f)]| \overset{|\alpha|}{\underset{|\beta|}{\rightrightarrows}} \coprod_{n\geq 0} X_{(n)} \times |\triangle[n]| \ ,$$

where $|\alpha|$ and $|\beta|$ are defined in a way completely analogous to the one above, and we only point out that for an arrow $f : [m] \to [n]$ in $\triangle$, we let

$$|\triangle[f]| : |\triangle[m]| \to |\triangle[n]|$$

be the unique affine map of simplices which on the vertices coincides with f.

Remarks: (1) The definition of $|X|$ is obviously functorial. For any morphism $\varphi : X \to Y$ of simplicial sets, the corresponding continuous map of the geometric realizations is denoted by $|\varphi|$.

(2) The definition of $|X|$ given above works equally well for simplicial topological spaces, i.e., simplicial objects in $\mathcal{T}op$. Namely, in $(**)$, one needs to view $X_{(c(f))}$ and $X_{(n)}$ as topological spaces, and the product has to be taken in $\mathcal{T}op$; everything else goes through in the same way.

(3) There is an alternate description of $|X|$, which is easily seen to be equivalent to the one given above. Namely, $|X|$ can be defined as the quotient $\left(\coprod_{n\geq 0} X_{(n)} \times |\triangle[n]|\right) \Big/ \sim$, where $\sim$ is the equivalence relation generated by the relation

$$(f(x), p) \approx (x, |\triangle[f]|(p))$$

for all $[m] \xrightarrow{f} [n]$, $x \in X_{(n)}$, $p \in |\triangle[m]|$. (The quotient is equipped with the quotient topology.) This description is probably the most classical one.

(4) For some purposes the definition of geometric realization we gave is not the best one. A more "modern" definition is obtained by extending a given $X \in \mathcal{S}$ to a functor $\triangle^{\mathrm{op}}_{\mathrm{big}} \to \mathcal{S}ets$ (such an extension is unique, since objects of $\triangle_{\mathrm{big}}$ have no nontrivial automorphisms), and defining the geometric realization of such a functor in a "coordinate-free" way. See: [V. Drinfeld, "On the notion of geometric realization"].

3. Smash products in $\mathcal{S}_*$.

Since the categories $\mathcal{S}ets$ and $\mathcal{S}ets_*$ (the latter is that of pointed sets) are complete and cocomplete, so are the categories $\mathcal{S}$ and $\mathcal{S}_*$ (by abstract nonsense). Two special cases of this statement deserve explicit mention.

First, we have quotients. If $j : Y \to X$ is a morphism in $\mathcal{S}$ (which the reader may assume to be injective), the **quotient** X/Y is defined as the pushout

$$\begin{array}{ccc} Y & \xrightarrow{j} & X \\ \downarrow & & \downarrow \\ * & \longrightarrow & X/Y \end{array}$$

Note that by construction, X/Y is naturally defined as an object of $\mathcal{S}_*$. This makes sense, in particular, when $Y = \varnothing$; then we write $X_+ := X/\varnothing \cong X \sqcup *$. This defines a functor

$$\mathcal{S} \to \mathcal{S}_*, \qquad X \mapsto X_+,$$

which is left adjoint to the forgetful functor $\mathcal{S}_* \to \mathcal{S}$.

Next we have products. The categorical product of two objects $X, Y \in \mathcal{S}$ is defined by $X \times Y$; explicitly,

$$(X \times Y)_{(n)} = X_{(n)} \times Y_{(n)} \qquad \forall n \geq 0.$$

We also introduce the **smash product** in $\mathcal{S}_*$, which is different from the categorical product in $\mathcal{S}_*$; if $X, Y \in \mathcal{S}_*$, we define

$$X \wedge Y = (X \times Y)/((X \times *) \cup (* \times Y)).$$

Theorem: The smash product defines a closed symmetric monoidal structure on $\mathcal{S}_*$.

The fact that it is a symmetric monoidal structure is a straightforward exercise. For the fact that it is closed, we recall first that it means the following: for any $X, Y \in \mathcal{S}_*$ there exists $\mathrm{Hom}(X, Y) \in \mathcal{S}_*$ and a functorial isomorphism

$$\mathcal{S}_*(Z \wedge X, Y) \cong \mathcal{S}_*(Z, \mathrm{Hom}(X, Y)) \tag{$\dagger$}$$

for all $Z \in \mathcal{S}_*$. To prove the existence of this inner hom, we observe first that if we take $Z = \triangle[n]_+$ for some $n \geq 0$, then

$$\begin{aligned} \mathrm{Hom}(X, Y)_{(n)} &\cong \mathcal{S}(\triangle[n], \mathrm{Hom}(X, Y)) && \text{(Yoneda)} \\ &\cong \mathcal{S}_*(\triangle[n]_+, \mathrm{Hom}(X, Y)) && \text{(adjunction)} \\ &\cong \mathcal{S}_*(\triangle[n]_+ \wedge X, Y), && \text{(by } (\dagger)) \end{aligned}$$

which already determines $\mathrm{Hom}(X, Y)$. Conversely, if we define $\mathrm{Hom}(X, Y)$ by

$$\mathrm{Hom}(X, Y)_{(n)} := \mathcal{S}_*(\triangle[n]_+ \wedge X, Y)$$

for all $n \geq 0$, then one can check that ($\dagger$) will be satisfied for all $z \in \mathcal{S}_*$, not just for the representable ones.

Remark: This construction of inner hom is really a special case of the following important general fact. A functor

$$\mathcal{PS}h(\mathcal{D})^{\mathrm{op}} \to \mathcal{S}ets,$$

where $\mathcal{D}$ is an arbitrary category and $\mathcal{PS}h(\mathcal{D}) = \mathrm{Funct}(\mathcal{D}^{\mathrm{op}}, \mathcal{S}ets)$, is representable if and only if it transforms colimits into limits. The "only if" direction is true (more or less by definition) when $\mathcal{PS}h(\mathcal{D})$is replaced by any category, but the "if" direction is a special property of the category of presheaves.

4. Examples of simplicial sets.

1) We define the **simplicial spheres** as follows. First of all, by convention,

$$S^0 := \triangle[0]_+ = * \sqcup * \in \mathcal{S}_*.$$

The **simplicial circle** is defined by

$$\mathcal{S}_* \ni S^1 := \triangle[1]/\partial\triangle[1] = \mathrm{coequalizer}(\triangle[0] \rightrightarrows \triangle[1]),$$

where the two maps $\triangle[0] \rightrightarrows \triangle[1]$ are induced by the only two maps $[0] \rightrightarrows [1]$. More generally, for each $n \geq 2$, we define

$$\mathcal{S}_* \ni S^n := S^1 \wedge S^1 \wedge \ldots \wedge S^1 \qquad \text{(n copies)}.$$

Note that this is NOT isomorphic to the more classical version of S^n defined as $\partial\triangle[n+1]$, i.e., $\triangle[n+1]$ with the interior removed. The advantages of the definition above are that S^n has a distinguished base point and comes with a natural action of the symmetric group Σ_n. This will be important when we define symmetric spectra.

2) If X is a *topological space*, we define the **total singular complex** $\mathrm{Sing}(X)$ by

$$\mathrm{Sing}(X)_{(n)} = \mathcal{T}op(|\triangle[n]|, X) \qquad \forall n \geq 0,$$
$$\mathrm{Sing}(X)_f = |\triangle[f]|^* \qquad \forall [m] \xrightarrow{f} [n].$$

The functor $\mathrm{Sing} : \mathcal{T}op \to \mathcal{S}$ is right adjoint to the geometric realization functor. We refer the reader to [M], Ch. III, for details.

3) If $\mathcal{C}$ is a small category, its **nerve** $N_\bullet\mathcal{C}$ is the simplicial set defined by

$$N_{(n)}\mathcal{C} = \mathrm{Funct}([n], \mathcal{C}) \qquad \forall n \geq 0$$

(recall that $[n]$ can be viewed as a category). In fact, if $\mathcal{C}at$ is the category of small categories, then

$$N_\bullet : \mathcal{C}at \to \mathcal{S}$$

is a fully faithful functor ([GZ], § II.4.3).

More generally, if $\mathcal{C}$ is a Q-simplicial category, i.e., a Q-simplicial object in $\mathcal{C}at$, **it is clear that $N_\bullet\mathcal{C}$ is a (Q+1)-simplicial set**. We write $B\mathcal{C} = |N_\bullet\mathcal{C}|$ and call it the **classifying space** of $\mathcal{C}$.

5. Spectra and symmetric spectra.

Def 1: A **spectrum** E is given by the data of:
(i) a sequence $E_0, E_1, \ldots, E_n, \ldots$ of pointed simplicial sets, and
(ii) a sequence of (pointed) morphisms

$$\sigma : S^1 \wedge E_n \to E_{n+1} \qquad \forall n \geq 0,$$

called the **structure maps** of the spectrum. The category of spectra is denoted $\mathcal{S}\mathrm{p}^{\mathbb{N}}$.

Remarks. (1) We could take the E_n's to be pointed topological spaces, or pointed CW complexes, etc. This gives the original definition of a spectrum due to Lima and Whitehead. Note also that some people call this a prespectrum, and reserve the name "spectrum" for the situation where the structure maps are monomorphisms (i.e., cofibrations with respect to an appropriate model category structure on pointed simplicial sets). There are also other versions.

(2) If E is a spectrum, then for each $k \in \mathbb{Z}$, we can define the **k-th homotopy group** of E as

$$\pi_k(E) := \varinjlim_{n \geq \max\{0,k\}} \pi_{k+n}(E_n),$$

where the colimit is taken with respect to the compositions

$$\pi_{k+n}(E_n) \xrightarrow[\nearrow]{\text{susp.}} \pi_{k+n+1}(S^1 \wedge E_n) \xrightarrow[\nearrow]{} \pi_{k+n+1}(E_{n+1}).$$
$$\text{induced by } S^{k+n+1} \cong S^1 \wedge S^{k+n} \qquad\qquad \text{induced by } \sigma$$

We say that a morphism $f : E \to E'$ of spectra is a **stable homotopy equivalence** if the homomorphism $\pi_k(f) : \pi_k(E) \to \pi_k(E')$ is an isomorphism for all $k \in \mathbb{Z}$. Inverting all stable homotopy equivalences in $\mathcal{S}\mathrm{p}^{\mathbb{N}}$, we obtain a certain category called **the stable homotopy category**. It is equivalent to the s.h. category originally introduced by Boardman; his construction was written up by Vogt [Vo70].

This category has been extensively studied by algebraic topologists, and as far as I understand, there is no doubt that it is the right category to look at after inverting stable homotopy equivalence. In particular, each of the many different known constructions of categories of spectra all have the property that the associated stable homotopy category is equivalent to the one introduced by Boardman.

(3) However, there is a problem with the definition of the category of spectra given above. An important property of Boardman's stable homotopy category is that it has a symmetric monoidal structure; however, it is not induced from any such structure on the category of spectra. All other categories of spectra defined in a similar manner share this disadvantage. This has caused algebraic topologists a lot of frustration: for example, the absence of a monoidal structure means that one does not have a good notion of a ring spectrum. (This problem was solved in some sense by Peter May and collaborators, who introduced the notion of an E_∞ ring spectrum.)

In the late 1990s / early 2000s at least three solutions of this problem were proposed. Two of them were developed by Peter May and collaborators, resulting in the theories of "S-modules" and "orthogonal spectra". The third one was suggested in a work of Hovey, Shipley, and Smith; it results in the category of symmetric spectra and is one of the main subjects of the present series of talks.

One of the main advantages of the category of symmetric spectra is that it has a "smash product" which is associative and symmetric "on the nose". On the other hand, the theory of symmetric spectra is technically much simpler than the theory of S-modules, yet it suffices for all our applications, which is why we have chosen to work with it.**)

Def 2: A **symmetric spectrum** is a spectrum E in the sense of Definition 1, equipped with the additional data of the action of the symmetric group Σ_n on E_n by automorphisms for each $n \geq 0$, such that $\forall p \geq 1$, the composition

$$\begin{array}{ccccccc} S^p \wedge E_n & \longrightarrow & & & & & E_{p+n} \\ \| & & & & & & \uparrow \sigma \\ (S^1 \wedge \ldots \wedge S^1) \wedge E_n & \longrightarrow & (S^1 \wedge \ldots \wedge S^1) \wedge E_{n+1} & \longrightarrow & \cdots & \longrightarrow & S^1 \wedge E_{n+p-1} \end{array}$$

induced by the structure maps of the spectrum is $\Sigma_p \times \Sigma_n$-equivariant, where the action of $\Sigma_p \times \Sigma_n$ on E_{n+p} is induced by the obvious injection $\Sigma_p \times \Sigma_n \hookrightarrow \Sigma_{p+n}$.

A **morphism** $f : E \to F$ of symmetric spectra is a collection of Σ_n-equivariant morphisms $f_n : E_n \to F_n$ in $\mathcal{S}_* \; \forall n \geq 0$, which commute with the structure maps in the obvious sense. The category of symmetric spectra is denoted by $\mathcal{S}\mathrm{p}^{\Sigma}$, and the forgetful functor is denoted by

$$\mathcal{U} : \mathcal{S}\mathrm{p}^{\Sigma} \to \mathcal{S}\mathrm{p}^{\mathbb{N}}$$

(here $\mathcal{U}$ stands for "underlying spectrum").

Remark: With the correct notion of stable homotopy equivalence of symmetric spectra, a map $f : E \to F$ such that $\mathcal{U}(f) : \mathcal{U}(E) \to \mathcal{U}(F)$ is an s.h.e. is itself an s.h.e., but not vice versa!

Example: If K is a pointed simplicial set, its **symmetric suspension spectrum** $\Sigma^\infty K$ is the symmetric spectrum defined by

$$(\Sigma^\infty K)_n = S^n \wedge K,$$

where the Σ_n-action comes from its natural action on S^n, and the structure maps

$$\sigma : S^1 \wedge S^n \wedge K \to S^{n+1} \wedge K$$

are induced by the obvious isomorphisms

$$S^1 \wedge S^n \cong S^{n+1}.$$

In particular, the **symmetric sphere spectrum** is defined by $S = \Sigma^\infty S^0 = (S^0, S^1, S^2, \ldots)$.

6. Simplicial structure on $\mathcal{S}\mathrm{p}^\Sigma$.

If E is a symmetric spectrum and K is a pointed simplicial set, then $E \wedge K$ is defined in the obvious way: $(E \wedge K)_n = E_n \wedge K$, where Σ_n acts trivially on K, and the structure maps are $\sigma_{\mathrm{new}} = \sigma_{\mathrm{old}} \wedge \mathrm{id}_K$.

On the other hand, if E and F are symmetric spectra, we define the simplicial set $\mathrm{Map}_{\mathcal{S}\mathrm{p}^\Sigma}(E, F)$ of maps from E to F by

$$\mathrm{Map}_{\mathcal{S}\mathrm{p}^\Sigma}(E, F)_{(n)} = \mathcal{S}\mathrm{p}^\Sigma(E \wedge \triangle[n]_+, F).$$

Of course, the definition models the construction of inner hom on the category of simplicial sets. With this notation, we have the following

Theorem: (1) The assignment

$$K \rightsquigarrow (E \mapsto E \wedge K)$$

defines a right action of $\mathcal{S}_*$ (as a monoidal category) on $\mathcal{S}\mathrm{p}^\Sigma$.

(2) We have bifunctorial isomorphisms

$$\mathcal{S}\mathrm{p}^\Sigma(E \wedge K, F) \cong \mathcal{S}_*(K, \mathrm{Map}_{\mathcal{S}\mathrm{p}^\Sigma}(E, F))$$

for all $E, F \in \mathcal{S}\mathrm{p}^\Sigma$ and all $K \in \mathcal{S}_*$. In particular, we have the evaluation map

$$\mathrm{ev}_{E,F} : E \wedge \mathrm{Map}_{\mathcal{S}\mathrm{p}^\Sigma}(E, F) \to F,$$

defined as the adjoint of

$$\mathrm{id} : \mathrm{Map}_{\mathcal{S}\mathrm{p}^\Sigma}(E, F) \to \mathrm{Map}_{\mathcal{S}\mathrm{p}^\Sigma}(E, F).$$

(3) The *composition pairing*

$$\mathrm{Map}_{\mathcal{S}\mathrm{p}^\Sigma}(X, Y) \wedge \mathrm{Map}_{\mathcal{S}\mathrm{p}^\Sigma}(Y, Z) \to \mathrm{Map}_{\mathcal{S}\mathrm{p}^\Sigma}(X, Z),$$

defined as the adjoint of the composition

$$X \wedge \mathrm{Map}(X, Y) \wedge \mathrm{Map}(Y, Z) \xrightarrow{\mathrm{ev}_{X,Y} \wedge \mathrm{id}} Y \wedge \mathrm{Map}(Y, Z) \xrightarrow{\mathrm{ev}_{Y,Z}} Z,$$

is associative, and the adjoint $S^0 \to \mathrm{Map}_{\mathcal{S}\mathrm{p}^\Sigma}(E,E)$ of the isomorphism $E \wedge S^0 \to E$ is a unit for the composition. Hence $\mathcal{S}\mathrm{p}^\Sigma$ is naturally enriched over $\mathcal{S}_*$.

(For more information, see [HSS], § 1.3.)

In other words, $\mathcal{S}\mathrm{p}^\Sigma$ *is a simplicial category with a closed right action of* $\mathcal{S}_*$. The rest of the lecture is devoted to a construction of a closed symmetric monoidal structure on $\mathcal{S}\mathrm{p}^\Sigma$.

7. Symmetric sequences. Before defining smash product on $\mathcal{S}\mathrm{p}^\Sigma$, it is convenient to define it on a larger category (here, "larger" is understood in the sense that there are more objects as well as more morphisms).

Definition: Consider the groupoid Σ whose objects are the finite sets $\overline{n} = \{1, 2, \ldots, n\}$ for all $n \geq 0$ (with the convention that $\overline{0} = \varnothing$) with morphisms

$$\Sigma(\overline{m}, \overline{n}) = \varnothing \quad \text{for} \quad m \neq n,$$
$$\text{and} \quad \Sigma(\overline{n}, \overline{n}) = \Sigma_n \quad \text{for all} \quad n \geq 0.$$

It is (non-uniquely) equivalent to the groupoid of finite sets and bijections.

If $\mathcal{C}$ is a category, the category of **symmetric sequences** in $\mathcal{C}$ is defined as

$$\mathcal{C}^\Sigma = \mathrm{Funct}(\Sigma, \mathcal{C}).$$

In more down-to-earth terms, a symmetric sequence in $\mathcal{C}$ is a sequence $\{X_n\}_{n\geq 0}$ of objects of $\mathcal{C}$, together with an action of Σ_n on each X_n (with no compatibility conditions).

In particular, we have the category $\mathcal{S}_*^\Sigma$ of symmetric sequences of pointed simplicial sets. By definition, there is a faithful (but certainly not full!) forgetful functor

$$\mathcal{S}\mathrm{p}^\Sigma \to \mathcal{S}_*^\Sigma.$$

Note, on the other hand, that the category of symmetric sequences is technically much simpler than the category of symmetric spectra, due to the formal computation

$$\begin{aligned} \mathcal{S}_*^\Sigma &= \mathrm{Funct}(\Sigma, \mathcal{S}_*) \\ &= \mathrm{Funct}(\Sigma, \mathrm{Funct}(\triangle^{\mathrm{op}}, \mathcal{S}ets_*)) \\ &\cong \mathrm{Funct}(\Sigma \times \triangle^{\mathrm{op}}, \mathcal{S}ets_*) \\ &\cong \mathrm{Funct}((\Sigma^{\mathrm{op}} \times \triangle)^{\mathrm{op}}, \mathcal{S}ets_*). \end{aligned}$$

Thus, the category of symmetric sequences (hereafter, we omit the words "of pointed simplicial sets" when no confusion can arise) is equivalent to the category of pointed presheaves of sets on $\Sigma^{\mathrm{op}} \times \triangle$. (Footnote: in fact, $\Sigma^{\mathrm{op}} \simeq \Sigma$ by elementary group theory.) In particular, the formalism of presheaves applies equally well to this category.

Remark: This is not stated explicitly in [HSS], and for this reason § 2.1 in that paper is somewhat hard to read.

Definition: The **tensor product** of two symmetric sequences $X, Y \in \mathcal{S}_*^\Sigma$ is the symmetric sequence $X \otimes Y$ defined by

$$(X \otimes Y)_n = \bigvee_{p+q=n} (\Sigma_n)_+ \bigwedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q).$$

Let us explain the notation.

1) $\bigvee$ denotes the **wedge** operation (= coproduct) in the category $\mathcal{S}_*$. If $\{X_i\}_{i\in I}$ is any collection of pointed simplicial sets, their wedge $\bigvee_{i\in I} X_i$ is obtained from the disjoint union $\coprod_{i\in I} X_i$ by identifying all the base points to a single base point.

2) Σ_n can be viewed as a constant simplicial set (= constant presheaf on $\triangle$) defined by

$$\begin{aligned} (\Sigma_n)_{(m)} &= \Sigma_n && \text{for all } m \geq 0, \\ (\Sigma_n)_f &= \mathrm{id}_{\Sigma_n} && \text{for all } f \in \operatorname{Ar} \triangle. \end{aligned}$$

Then $(\Sigma_n)_+$ is the pointed simplicial set associated to Σ_n.

3) The group $\Sigma_p \times \Sigma_q$ acts on Σ_{p+q} by right translations, and hence it also acts on the pointed simplicial set $(\Sigma_{p+q})_+$.

4) If K and L are pointed simplicial sets and G is a discrete group acting on K on the right and on L on the left, we define $K \underset{G}{\wedge} L$ as the quotient of $K \wedge L$ by the diagonal action of G given by

$$g : (k, l) \mapsto (kg^{-1}, gl).$$

Thus $(\Sigma_n)_+ \bigwedge_{\Sigma_p\times\Sigma_q} (X_p \wedge Y_q)$ is now defined, and moreover inherits a left action of Σ_n from its action on $(\Sigma_n)_+$. Hence $X \otimes Y$ is indeed a symmetric sequence.

Remark: The construction of

$$(\Sigma_n)_+ \bigwedge_{\Sigma_p\times\Sigma_q} (X_p \wedge Y_q)$$

is reminiscent of the construction of an induced representation.

Theorem: The tensor product defines a closed symmetric monoidal structure on $\mathcal{S}_*^\Sigma$.

For the proof of this fact we refer the reader to [HSS], § 2.1. As remarked above, the existence of inner homs is essentially a formal statement. Note also that inner homs can be constructed in two steps. First, we have a right action of $\mathcal{S}_*$ on $\mathcal{S}_*^\Sigma$ defined by

$$X \in \mathcal{S}_*^\Sigma,\ K \in \mathcal{S}_* \rightsquigarrow X \wedge K \in \mathcal{S}_*^\Sigma,$$

where $(X \wedge K)_n = X_n \wedge K$ with the diagonal action of Σ_n, induced by the given action on X_n and the trivial action on K.

This action is closed, i.e., for $X, Y \in \mathcal{S}_*^\Sigma$ we have a "simplicial set of maps" $\mathrm{Map}_{\mathcal{S}_*^\Sigma}(X, Y)$ such that there are functorial isomorphisms

$$\mathcal{S}_*^\Sigma(X \wedge K, Y) \cong \mathcal{S}_*\left(K, \mathrm{Map}_{\mathcal{S}_*^\Sigma}(X, Y)\right)$$
$$\text{for all } \ X, Y \in \mathcal{S}_*^\Sigma \ \text{ and } \ K \in \mathcal{S}_*.$$

Thus $\mathcal{S}_*^\Sigma$ also becomes a simplicial category. The inner hom can then be constructed using this simplicial structure; we note that if $X, Y \in \mathcal{S}_*^\Sigma$, then $\mathrm{Hom}_{\mathcal{S}_*^\Sigma}(X, Y)$ is a symmetric sequence whose 0-th term is $\mathrm{Map}_{\mathcal{S}_*^\Sigma}(X, Y)$. We will also describe the braiding on $\mathcal{S}_*^\Sigma$ (and omit all further details). The symmetry isomorphism $X \otimes Y \xrightarrow{\simeq} Y \otimes X$ is induced from isomorphisms

$$(\Sigma_{p+q})_+ \wedge X_p \wedge Y_q \xrightarrow{\simeq} (\Sigma_{p+q})_+ \wedge Y_q \wedge X_p \tag{†}$$

that commute with the diagonal action of $\Sigma_p \times \Sigma_q \cong \Sigma_q \times \Sigma_p$ and thereby descend to

$$(\Sigma_{p+q})_+ \bigwedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q) \xrightarrow{\cong} (\Sigma_{p+q})_+ \bigwedge_{\Sigma_q \times \Sigma_p} (Y_q \wedge X_p).$$

A moment's thought will convince the reader that the only natural way of defining an isomorphism (††) with this property is

$$(\alpha, x, y) \mapsto (\alpha \cdot \rho_{q,p}, y, x),$$

where $\rho_{q,p} \in \Sigma_{p+q}$ is the (q,p)-shuffle, defined by

$$\rho_{q,p}(i) = \begin{cases} i+p & \text{if } \ 1 \leq i \leq q, \\ i-q & \text{if } \ q+1 \leq i \leq q+p. \end{cases}$$

The proof that the isomorphism $X \otimes Y \cong Y \otimes X$ induced by (††) is a braiding on $\mathcal{S}_*^\Sigma$ whose square is 1 is completely straightforward.

Remark: It is easy to verify that the unit object for the monoidal structure on $\mathcal{S}_*^\Sigma$ is the symmetric sequence $(S^0, *, *, *, \ldots)$.

8. Smash products in $\mathcal{S}p^\Sigma$. The construction is based on the following fundamental observations. 1) Let $S^0 \in \mathcal{S}_*^\Sigma$ denote the **sphere sequence**, defined by

$$(S^0)_n = S^n = S^1 \wedge \ldots \wedge S^1 \qquad (n \text{ terms})$$

with the natural action of Σ_n. Then S^0 is a *commutative monoid* in $\mathcal{S}_*^\Sigma$. The multiplication map

$$S^0 \otimes S^0 \xrightarrow{m} S^0$$

comes from the natural maps

$$(\Sigma_{p+q})_+ \bigwedge_{\Sigma_p \times \Sigma_q} S^p \wedge S^q \longrightarrow S^{p+q}$$

induced by the action of Σ_{p+q} on $S^p \wedge S^q \cong S^{p+q}$. The unit morphism

$$\mathbb{1}_{\mathcal{S}_*^\Sigma} = (S^0, *, *, *, \ldots) \to (S^0, S^1, S^2, \ldots) = S^0$$

is the obvious one. The fact that S^0 is a commutative monoid is expressed by the commutativity of the diagram

$$\begin{array}{ccc} S^0 \otimes S^0 & \xrightarrow{\text{braiding}} & S^0 \otimes S^0 \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle m} & \\ & S^0 & \end{array}$$

which can be easily checked by hand.

2) If $X \in \mathcal{S}_*^\Sigma$, the additional structure of a symmetric *spectrum* on X is equivalent to the additional structure of a left S^0-module on X. We sketch an explanation and leave the details to the reader as a very instructive exercise. First of all, a left S^0-module on X is a morphism $S^0 \otimes X \to X$ satisfying a certain condition. This is the same as giving a

morphism $S^0 \to \mathrm{Hom}_{\mathcal{S}_*^\Sigma}(X,X)$ of monoids in $\mathcal{S}_*^\Sigma$ (the inner hom on the right can be easily made into a commutative monoid). Since S^0 is clearly "generated by S^1 as a monoid" in a suitable sense, such a morphism of monoids is determined by "where S^1 goes". Now it is not hard to check that to specify where S^1 goes in a way that extends to all of S^0 is equivalent to equipping X with a symmetric spectrum structure.

Once again, for all the details we refer the reader to [HSS], § 2.2.

We finally come to the key construction of the section. Note that if $X \in \mathcal{S}_*^\Sigma$ is a left S^0-module, then, because S^0 is commutative, we can turn X into a right S^0-module using the composition

$$X \otimes S^0 \xrightarrow{\text{braiding}} S^0 \otimes X \xrightarrow{\text{action}} X.$$

Now if X and Y are symmetric spectra, we look at the corresponding right S^0-module structure $\rho_X : X \otimes S^0 \to X$ and left S^0-module structure $\lambda_Y : S^0 \otimes Y \to Y$.

Definition: The **smash product** or **tensor product** of X and Y is defined by

$$X \wedge Y = X \underset{S^0}{\otimes} Y = \mathrm{Coequalizer}\left(X \otimes S^0 \otimes Y \underset{\mathrm{id}_X \otimes \lambda_Y}{\overset{\rho_X \otimes \mathrm{id}_Y}{\rightrightarrows}} X \otimes Y \right).$$

The category $\mathcal{S}_*^\Sigma$ is obviously complete and cocomplete, so $X \underset{S^0}{\otimes} Y$ is well defined, and, moreover, inherits a left action of S^0, and hence becomes a symmetric spectrum.

Analogy: If $\Bbbk$ is a field, S is a commutative $\Bbbk$-algebra, and M, N are S-modules, one can construct their tensor product over S as

$$M \underset{S}{\otimes} N = \mathrm{Coeq}\left(M \underset{\Bbbk}{\otimes} S \underset{\Bbbk}{\otimes} N \rightrightarrows M \underset{\Bbbk}{\otimes} N \right).$$

Theorem ([HSS], Theorem 2.2.10): The smash product is a closed symmetric monoidal structure on $\mathcal{S}p^\Sigma$.

We omit the proof and end with the following important

Definition: A **symmetric ring spectrum** is a monoid in $\mathcal{S}p^\Sigma$.

As far as I understand, the possibility of introducing this definition was one of the original motivation behind the construction of the category $\mathcal{S}p^\Sigma$.

Post-lecture footnotes

*) As Drinfeld explained during the talk, there is a somewhat different point of view on the notion of geometric realization. Namely, one can think of a simplicial set X as a **right $\mathbb{\Delta}$-module**. On the other hand, we have a natural left $\mathbb{\Delta}$-module, which is the functor $\triangle : \mathbb{\Delta} \to \mathcal{T}op$ obtained by sending $[n]$ to the standard n-simplex $|\triangle[n]|$. Then, with the appropriate understanding of the tensor product, we have

$$|X| = X \underset{\mathbb{\Delta}}{\otimes} \triangle \in \mathcal{T}op.$$

**) As remarked by Peter May during the lecture, the theory of orthogonal spectra is no more complicated than the theory of symmetric spectra, and, in fact, it is more convenient for certain purposes (for example, it is the case that a morphism of orthogonal spectra is a stable homotopy equivalence if and only if it induces an isomorphism on all the stable homotopy groups). However, we have made a choice to work with symmetric spectra throughout the minicourse; moreover, as far as I understand, orthogonal spectra are more of a topological theory and it would be much harder to phrase it in the language of simplicial sets.

Final comment. All of the second half of the lecture (i.e., starting with § 5) is very formal in the sense that the category of simplicial sets could be replaced with any closed symmetric monoidal category (e.g., that of vector spaces over a field), and all the constructions and results would go through just as well.

Another remark is that to get a topological theory one needs to take this closed symmetric monoidal category to be some convenient category of "spaces" , but here the word "space" could also have several different meanings. For instance, we can look at the categories of

- simplicial sets;
- weakly Hausdorff compactly generated topological spaces;
- topological spaces having the homotopy type of a CW complex.

It is possible to develop a theory of spectra and symmetric spectra in each of these contexts. The main reason we choose to work with simplicial sets is that these are the objects that arise most naturally in the applications to Beilinson's work. However, it is not necessary to work in $\mathcal{S}_*$ to understand this lecture.