Roman Bezrukavnikov's synopsis of his April 2003 talks on the Thomason-Trobaugh theorem from [TT].

Some results and proofs from [TT] were later improved and clarified in [T] and [N], and a nice exposition can be found in [BvdB]. I was mostly following [BvdB]. Anyway, each of the results formulated below was taken from one of the papers [TT, T, N, BvdB], but in some of the cases I do not remember from which one.

Let X be a quasi-compact, quasi-separated scheme, Sh_X be the category of sheaves of O-modules on X, and $QCoh_X \subset Sh_X$ be the full subcategory of quasi-coherent sheaves. Let $D(X)$ be the full subcategory in the derived category of Sh_X consisting of complexes with quasi-coherent cohomology sheaves. My understanding is that the functor $D(QCoh_X) \to D(X)$ is an equivalence if X either Noetherian or semi-separated (the latter means that the diagonal morphism is affine); SGA6, and a paper by Boksted and Neeman contain this statement with slightly stronger assumptions.

For a closed $Z \subset X$ let $D_Z(X) \subset D(X)$ be the subcategory of complexes whose cohomology sheaves are supported on Z.

For an additive category C and a set S of objects in C we set $S^\perp = \{X \in C \mid Hom(s, X) = 0 \forall s \in S\}$; we say S generates C if $S^\perp = 0$ (it's natural to say S is dense, but I follow the terminology of [BvdB]).

Recall that if C has arbitrary (small) direct sums, then $X \in C$ is *compact* if $Hom(C, \ )$ commutes with direct sums.

Lemma. Compact objects in $D(X)$ are exactly the perfect complexes.

Let $D_{perf}(X) \subset D(X)$ be the category of perfect complexes, and $D_Z^{perf}(X) = D_{perf}(X) \cap D_Z(X)$.

Theorem 1. If X (as above) is quasi-compact, quasi-separated then $D(X)$ is generated by a single perfect complex. Moreover, if an open $U \subset X$ is quasi-compact, and $Z = X - U$, then $D_Z(X)$ is generated by a perfect complex.

Corollary. Each of the categories $D_{perf}(X)$, $D_Z(X)$ contains an object F such that the thick subcategory generated by F is the whole category.

Theorem 2. [TT] Let $X \supset U$ be as above.

a) The functor $D_{perf}(X)/D_Z^{perf}(X) \to D_{perf}(U)$ is a full imbedding.

b) Any object in $D_{perf}(U)$ is a direct summand of $j^*(F)$ for some $F \in D_{perf}(X)$.

c) An object $F \in D_{perf}(U)$ is isomorphic to $j^*(G)$ for some $G \in D_{perf}(X)$

iff its class in the Grothendieck group lies in the image of the restriction map $K_0(X) \to K_0(U)$. (In particular, for any perfect complex F the object $F \oplus F[1]$ can be extended to a perfect complex on X).

The formal structure of the proof is as follows.

(0) First, Theorem 2(c) follows from 2(b) by the classification of dense subcategories in a triangulated category [T].

(1) Thm 1 (for given X, Z) implies Thm 2(a,b) for the same X, Z by some results about general triangulated categories (see below).

(2) Thm 1 is obvious when X is affine.

(3) Thm 2 is excision-friendly: if we know it for X, Z and have an open imbedding $X \to \overline{X}$, then we easily deduce it for $\overline{X}, Z$. (Here we use 2(c) also: to represent a perfect complex F on $\overline{X}$ as a direct summand of an extendable complex we use that $(F \oplus F[1])|_X$ is extendable, and hence $F \oplus F[1]$ is extendable).

Thus (from (1), (2)) we get Thm 2 for all X, Z such that Z is contained in an affine open $U \subset X$.

(4) Thm 2 allows to do gluing in proving Thm 1: if we know Thm 1(a) for $X - Z$, Thm 1(b) for (X, Z), and Thm 2 for X, Z, then we know Thm 1(a) for X. (The key point here is to be able to extend a perfect generator from an open subset; Thm 2(b) allows to do this after innocuously adding something.)

¿From (2), (3), (4) we get Thm 1(a) for all X by induction in the affine covering. Similarly we have

(4′) If $N \subset X$ is closed, and we know Thm 1(b) for $(X - N, Z - Z \cap N)$, and 1(b) for $(X, N \cap Z)$, along with Thm 2 for $X, N \cap Z$ then we can deduce Thm 1(b) for X, Z.

Thus we get Thm 1(b) by induction in affine covering as well.

The "mass center" of this argument is in (part (b) of) the following Lemma.

For a set of objects S in a triangulated category let $\langle S \rangle$ be the thick subcategory generated by S; and let $\overline{\langle S \rangle}$ be the smallest strict full triangulated subcategory closed under arbitrary direct sums. (Notice $\overline{\langle S \rangle}$ is thick by the "telescopic" construction ([N], 1.6, p553)).

Lemma. [N] Let D be a triangulated category with arbitrary (small) direct sums, and S be a set of compact objects in D. Let $D_c \subset D$ be the full subcategory of compact objects. Then

a) $\overline{\langle S\rangle} \cap D_c = \langle S\rangle$.

b) The subcategory $\overline{\langle S\rangle}$ is left admissible, i.e. the inclusion functor has a left adjoint (equivalently, the projection functor $D \to D/\overline{\langle S\rangle}$ has a right adjoint).

c) The functor $D_c/\langle S\rangle \to D/\overline{\langle S\rangle}$ is a full imbedding.

Here the key part is (b), see [N], Lemma 1.7, pp 553-554.

The rest follows formally using the following

Sublemma. A morphism from a compact object to an object of $\overline{\langle S\rangle}$ factors through an object of $\langle S\rangle$.

Also the following statement helps to derive Corollary of Theorem 1.

Corollary. (to Lemma) If $S^\perp = 0$ then $D = \overline{\langle S\rangle}$, and $D_c = \langle S\rangle$.

There is an alternative proof of Theorem 2, at least in the quasi-projective case. First, Thm 1 is almost obvious for X quasi-projective (if X is locally closed in P^n, then the sum of $O(i)$, $i = 0, .., n$ is a generator for $D(X)$; and if Z is the zero locus of a section of a vector bundle, then twisting a generator of $D(X)$ with the Koszul complex one gets a generator for $D_Z(X)$). So we also get a proof of Theorem 2.

But, there is yet another way to get Theorem 2(b) in the quasi-projective case; see the original proof, [TT], Lemma 5.5.1, p 343.

A more explicit (and attractive) version of this argument was rediscovered by Kontsevich (unpublished): given a perfect complex F it is easy to extend it to a bounded above (but maybe not below) complex of vector bundles. Then the claim is that one can take G to be the stupid truncation $\tau^{b\hat{e}te}_{\geq n}$ $n \ll 0$ of that complex. The fact that F is indeed a direct summand of $j^*(G)$ follows from vanishing of the corresponding Ext group, which is a consequence of F having finite homological dimension.

References

[BvdB] Bondal, van der Bergh, "Generators and representability of functors in commutative and noncommutative geometry.", preprint, math.AG/0204218

[N] Neeman, "The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of

Bousfield and Ravenel." Ann. Sci. École Norm. Sup. (4) 25 (1992), no. 5, 547–566.

[T] Thomason, "The classification of triangulated subcategories." Compositio Math. 105 (1997), no. 1, 1–27.

[TT] Thomason, Trobaugh "Higher algebraic K-theory of schemes and of derived categories." The Grothendieck Festschrift, Vol. III, 247–435, Progr. Math., 88, Birkhäuser Boston, Boston, MA, 1990.