SOLVABLE FUSION CATEGORIES AND A CATEGORICAL BURNSIDE'S THEOREM

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The goal of this talk is to explain the classical representation-theoretic proof of Burnside's theorem in finite group theory, stating that a finite group of order $p^a q^b$ (where p, q are primes) is solvable, and then define the notion of a solvable fusion category and explain how to generalize Burnside's theorem to the categorical setting. This generalization is joint work with D. Nikshych and V. Ostrik, and is given in the paper math/0809.3031.

I am very grateful to V. Drinfeld for useful suggestions on how to improve the presentation.

1. The Burnside theorem

1.1. **The statement of Burnside's theorem.**

Theorem 1.1 (Burnside)**.** *Any group G of order $p^a q^b$, where p and q are primes and $a, b \in \mathbb{Z}_+$, is solvable.*

The first proof of this classical theorem was based on representation theory, and is reproduced below. Nowadays there is also a purely group-theoretical proof, but it is more complicated.

1.2. **Auxiliary results.**

In this subsection we prove two auxiliary theorems that are used in the proof of Burnside's theorem, but are interesting in their own right.

Theorem 1.2. *Let V be an irreducible complex representation of a finite group G and let C be a conjugacy class of G with $\gcd(|C|, \dim(V)) = 1$. Then for any $g \in C$, either $\chi_V(g) = 0$ or g acts as a scalar on V.*

Proof.

Lemma 1.3. *If $\varepsilon_1, \varepsilon_2 \dots \varepsilon_n \in \mathbb{C}$ are roots of unity such that $\dfrac{1}{n}(\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_n)$ is an algebraic integer, then either $\varepsilon_1 = \dots = \varepsilon_n$ or $\varepsilon_1 + \dots + \varepsilon_n = 0$.*

Proof. Let $a = \frac{1}{n}(\varepsilon_1 + \dots + \varepsilon_n)$. If not all ε_i are equal, then $|a| < 1$. Moreover, $|\sigma(a)| < 1$ for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. But the product of the numbers $\sigma(a)$ is an integer, so it equals to zero. □

Lemma 1.4. *Let V be an irreducible complex representation of a finite group G and let C be a conjugacy class of G. Then for $g \in C$, the number $\frac{|C|\chi_V(g)}{\dim V}$ is an algebraic integer.*

Date: November 11, 2008.

Proof. Let $P = \sum_{h\in C} h$. Then P is a central element of $\mathbb{Z}[G]$, so it acts on V by some scalar λ, which is an algebraic integer, since the center of $\mathbb{Z}[G]$ is integral over $\mathbb{Z}$. On the other hand, taking the trace of P in V, we get $|C|\chi_V(g) = \lambda \dim V$, $g \in C$, so $\lambda = \frac{|C|\chi_V(g)}{\dim V}$. □

Now we are ready to prove Theorem 1.2. Let $\dim V = n$. Let $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_n$ be the eigenvalues of $\rho_V(g)$. They are roots of unity, so $\chi_V(g)$ is an algebraic integer. Also, by Lemma 1.4, $\frac{1}{n}|C|\chi_V(g)$ is an algebraic integer. Since $\gcd(n, |C|) = 1$, this implies that

$$\frac{\chi_V(g)}{n} = \frac{1}{n}(\varepsilon_1 + \ldots + \varepsilon_n).$$

is an algebraic integer. Thus, by Lemma 1.3, we get that either $\varepsilon_1 = \ldots = \varepsilon_n$ or $\varepsilon_1 + \ldots + \varepsilon_n = \chi_V(g) = 0$. In the first case, since $\rho_V(g)$ is diagonalizable, it must be scalar. In the second case, $\chi_V(g) = 0$. The theorem is proved. □

Theorem 1.5. *Let G be a finite group, and let C be a conjugacy class in G of order p^k where p is prime and $k > 0$. Then G has a proper nontrivial normal subgroup.*

Proof. Choose an element $g \in C$. Since $g \neq e$, by orthogonality of columns of the character table,

$$\sum_{V\in \mathrm{Irr}G} \dim V \chi_V(g) = 0. \qquad (1)$$

We can divide Irr$G$ into three parts:

(1) the trivial representation,
(2) S, the set of irreducible representations whose dimension is divisible by p, and
(3) T, the set of non-trivial irreducible representations whose dimension is not divisible by p.

Lemma 1.6. *There exists $V \in T$ such that $\chi_V(g) \neq 0$.*

Proof. If $V \in S$, the number $\frac{1}{p}\dim(V)\chi_V(g)$ is an algebraic integer, so

$$a = \sum_{V\in S} \frac{1}{p}\dim(V)\chi_V(g)$$

is an algebraic integer.

Now, by (1), we have

$$0 = \chi_{\mathbb{C}}(g) + \sum_{V\in S} \dim V \chi_V(g) + \sum_{V\in T} \dim V \chi_V(g) = 1 + pa + \sum_{V\in T} \dim V \chi_V(g).$$

This means that the last summand is nonzero. □

Now pick $V \in T$ such that $\chi_V(g) \neq 0$; it exists by Lemma 1.6. Theorem 1.2 implies that g (and hence any element of C) acts by a scalar in V. Now let H be the subgroup of G generated by elements ab^{-1}, $a, b \in C$. It is normal and acts trivially in V, so $H \neq G$, as V is nontrivial. Also $H \neq 1$, since $|C| > 1$. □

1.3. **Proof of Burnside's theorem.** Assume Burnside's theorem is false. Then there exists a nonabelian simple group G of order $p^a q^b$. Then by Theorem 1.5, this group cannot have a conjugacy class of order p^k or q^k, $k \geq 1$. So the order of any conjugacy class in G is either 1 or is divisible by pq. Adding the orders of conjugacy classes and equating the sum to $p^a q^b$, we see that there has to be more than one conjugacy class consisting just of one element. So G has a nontrivial center, which gives a contradiction.

2. Solvable fusion categories

2.1. **Integral and weakly integral categories.** Recall that a *fusion category* (over $\mathbb{C}$) is a semisimple rigid monoidal category with finitely many simple objects X_i and $\mathrm{End}X_i = \mathbb{C}$. A fusion category $\mathcal{C}$ is called *integral* if the Frobenius-Perron dimension of every (simple) object is an integer. This is equivalent to $\mathcal{C}$ being the category of representations of a finite dimensional semisimple quasi-Hopf algebra. $\mathcal{C}$ is said to be *weakly integral*, if its Frobenius-Perron dimension is an integer. It is known ([ENO]) that in such a category, the Frobenius-Perron dimension of any simple object is the square root of an integer.

There exist weakly integral categories which are not integral. An example is the Ising category, which has two invertible objects $\mathbf{1}_+$ and $\mathbf{1}_-$ and another simple object X such that $X \otimes X = \mathbf{1}_+ \oplus \mathbf{1}_-$. The Frobenius-Perron dimension of this category is 4, while the Frobenius-Perron dimension of X is $\sqrt{2}$. However, any weakly integral category of odd dimension is integral.

It is known that any weakly integral fusion category automatically has a canonical spherical structure with respect to which categorical dimensions coincide with the Frobenius-Perron dimensions [ENO, Propositions 8.23, 8.24].

In this talk, we will deal only with weakly integral categories. Thus, we will always use the canonical spherical structure, and use the word "dimension" for both the categorical and the Frobenius-Perron dimension.

2.2. **Graded categories and extensions.** ([ENO, GNk])

Let $\mathcal{C}$ be a fusion category and let G be a finite group. We say that $\mathcal{C}$ is *graded* by G if $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$, and for any $g, h \in G$, one has $\otimes : \mathcal{C}_g \times \mathcal{C}_h \to \mathcal{C}_{gh}$. The fusion category $\mathcal{C}_e$ corresponding to the neutral element $e \in G$ is called *the trivial component* of the G-graded category $\mathcal{C}$. If $\mathcal{C}_g \neq 0$ for all $g \in G$, one says that $\mathcal{C}$ is a *G-extension* of $\mathcal{C}_e$. It it easy to show that in this case $\dim(\mathcal{C}) = |G| \cdot \dim(\mathcal{C}_e)$

The simplest example of a graded category is a *pointed* category, i.e. a category where all simple objects are invertible, or, equivalently, a category of the form $\mathrm{Vec}_{G,\omega}$ of vector spaces graded by a finite group G with associativity defined by a cohomology class $\omega \in H^3(G, \mathbb{C}^*)$. Another example is $\mathrm{Rep}(G)$, which is graded by the character group $Z_G^\vee$ of the center Z_G of G.

2.3. **Equivariantization.** Let $\mathcal{C}$ be a category with an action of a group G. In this case one can define the category $\mathcal{C}^G$ of G-equivariant objects in $\mathcal{C}$. Objects of this category are objects X of $\mathcal{C}$ equipped with an isomorphism $u_g : g(X) \to X$ for all $g \in G$, such that

$$u_{gh} \circ \gamma_{g,h} = u_g \circ g(u_h),$$

where $\gamma_{g,h} : g(h(X)) \to gh(X)$ is the natural isomorphism associated to the action. This category is called the *G-equivariantization* of $\mathcal{C}$.

It is easy to show that if $\mathcal{C}$ is a fusion category, then so is $\mathcal{C}^G$, and $\dim(\mathcal{C}^G) = |G|\dim(\mathcal{C})$. For example, $\mathrm{Vec}^G = \mathrm{Rep}(G)$ (for the trivial action of G on Vec). A more interesting example is the following. Let K be a normal subgroup of G. Then we have a natural action of G/K on $\mathrm{Rep}(K)$, and $\mathrm{Rep}(K)^{G/K} = \mathrm{Rep}(G)$.

2.4. **Definition and properties of solvable fusion categories.**

Definition 2.1. A fusion category $\mathcal{C}$ is **solvable** if it can be included in a sequence of fusion categories $\mathrm{Vec} = \mathcal{C}_0, \mathcal{C}_1, ..., \mathcal{C}_n = \mathcal{C}$ such that for each $i \in [1, n]$, $\mathcal{C}_i$ is a G_i-extension of $\mathcal{C}_{i-1}$ or a G_i-equivariantization of $\mathcal{C}_{i-1}$ for some group G_i of prime order.

In particular, it is easy to show that the category $\mathrm{Rep}(G)$ of representations of a finite group G is solvable if and only if G is a solvable group, and the same holds for the category $\mathrm{Vec}_{G,\omega}$, which motivates the definition.

Proposition 2.2. *Any solvable fusion category $\mathcal{C} \neq Vec$ contains a nontrivial invertible object.*

Proof. The proof is by induction in the dimension of $\mathcal{C}$. The base of induction is clear, and only the induction step needs to be justified. If $\mathcal{C}$ is an extension of a smaller solvable category $\mathcal{D}$, then either $\mathcal{D} \neq \mathrm{Vec}$ and the statement follows from the induction assumption, or $\mathcal{D} = \mathrm{Vec}$ and $\mathcal{C}$ is pointed, so the statement is obvious. On the other hand, if $\mathcal{C}$ is a $\mathbb{Z}/p$-equivariantization of a smaller solvable category $\mathcal{D}$, then $\mathrm{Rep}(\mathbb{Z}/p)$ sits inside $\mathcal{C}$, so we are done. □

Remark 2.3. If a fusion category $\mathcal{C}$ is "well understood", then its extensions and equivariantizations by groups G of prime order can be classified in fairly explicit terms. This means that solvable fusion categories can in principle by described in terms of group theory.

2.5. **The main theorem.**

Theorem 2.4. *(The categorical Burnside theorem) Any fusion category of Frobenius-Perron dimension p^aq^b, where p, q are primes and a, b are nonnegative integers, is solvable.*

Corollary 2.5. *Any fusion category of Frobenius-Perron dimension p^aq^b, where p, q are primes and a, b are nonnegative integers contains a nontrivial invertible object. In particular, every semisimple quasi-Hopf algebra of dimension p^aq^b has a nontrivial 1-dimensional representation.*

3. Proof of the categorical Burnside theorem

In this section we will explain the proof of the categorical Burnside theorem. For brevity, we will assume that p and q are odd; in the even case, the proof is slightly longer, but ideologically the same.

The proof is based on the theory of modular tensor categories, so before starting the proof, we have to review relevant parts of this theory.

3.1. **Müger centralizer and Müger center.**

Let $\mathcal{C}$ be a braided fusion category, and $\mathcal{D} \subset \mathcal{C}$ a full subcategory. The *Müger centralizer* $\mathcal{D}'$ of $\mathcal{D}$ in $\mathcal{C}$ is the category of all objects $Y \in \mathcal{C}$ such that for any $X \in \mathcal{D}$ the squared braiding on $X \otimes Y$ is the identity. The *Müger center* of $\mathcal{C}$ is the Müger centralizer $\mathcal{C}'$ of the entire category $\mathcal{C}$.

3.2. **Ribbon categories and the s-matrix.** A fusion category $\mathcal{C}$ is said to be *ribbon* if it is equipped with a braiding and a spherical structure[1]. An important invariant of such a category is its *s-matrix*, which is a matrix of complex numbers whose rows and columns are labeled by simple objects of $\mathcal{C}$. By definition, the entry s_{XY} of this matrix corresponding to simple objects X and Y is the quantum trace of the squared braiding $\beta_{YX}\beta_{XY}$ on $X \otimes Y$. Note that $s_{XY} = s_{YX}$ and $s_{\mathbf{1}Y} = s_{Y\mathbf{1}} = \dim Y$.

3.3. **Modular categories.**

Theorem 3.1. ([M4]) *The following two conditions on a ribbon category $\mathcal{C}$ are equivalent:*

(i) The Müger center of $\mathcal{C}$ is trivial (i.e., equivalent to the category of vector spaces);

(ii) The s-matrix of $\mathcal{C}$ is nondegenerate.

Definition 3.2. A ribbon category $\mathcal{C}$ is said to be *modular* if any of the two conditions of Theorem 3.1 is satisfied[2].

Thus, a ribbon category is modular if it is "as far as possible" from being symmetric.

3.4. **The Drinfeld center.** An important example of a modular category is *the Drinfeld center* $Z(\mathcal{C})$ of a spherical fusion category $\mathcal{C}$. By definition, the center of a fusion category $\mathcal{C}$ is the category of pairs (X, ϕ), where $X \in \mathcal{C}$ and ϕ is an isomorphism of the functor of right multiplication by X with the functor of left multiplication by X, satisfying a natural hexagon axiom. It is easy to see that this is naturally a braided category.

Theorem 3.3. ([BK],[ENO]) *The category $Z(\mathcal{C})$ is semisimple, i.e. it is a fusion category. One has $\dim Z(\mathcal{C}) = (\dim \mathcal{C})^2$. In addition, if $\mathcal{C}$ is spherical, then $Z(\mathcal{C})$ is canonically a modular category.*

As an example consider the case when $\mathcal{C} = \mathrm{Vec}_G$. the category of sheaves on G, i.e. vector spaces graded by G, with the usual associativity isomorphism. In this case, it is easy to show that $Z(\mathcal{C})$ is the category of $(\mathrm{Vec}_G)^G$ of G-equivariant sheaves on G acting on itself by conjugation (which is natural to expect by analogy with the center of $\mathbb{Z}[G]$). Indeed, an object of $Z(\mathcal{C})$ is a G-graded space $V = \oplus_{g\in G} V_g$ together with an isomorphism $\phi : h \otimes V \to V \otimes h$, $h \in G$ satisfying the hexagon relation, which is the same as a system of isomorphisms $u_h : V_{hgh^{-1}} \to V_g$ satisfying the axiom of an equivariant sheaf.

Thus, irreducible objects of $Z(\mathcal{C})$ correspond to pairs (C, V), where C is a conjugacy class in G and V an irreducible representation of the centralizer Z_g of an element g of C: namely, $X_{C,V}$ is the equivariant sheaf supported on C, whose fiber at g is V as a representation of Z_g.

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[1]The motivation for this terminology is the fact that a ribbon category gives rise to invariants of ribbon tangles, constructed by Reshetikhin and Turaev.

[2]This terminology comes from the fact that the Grothendieck ring of a modular category is a representation of the modular group $SL_2(\mathbb{Z})$.

We may think about equivariant sheaves as representations of the semidirect product $\mathbb{C}[G] \ltimes \mathrm{Fun}(G, \mathbb{C})$, called the quantum double of G. In terms of this presentation, the braiding is given by the formula

$$\beta(v \otimes w) = \sum_{h \in G} \delta_h w \otimes hv.$$

Example 3.4. If C is a conjugacy class in G and V an irreducible representation of G then it is easy to compute using the above formula for the braiding (and the fact that the quantum trace in this case coincides with the usual trace) that $s_{(C,\mathbf{1}),(1,V)} = |C|\chi_V(g)$, $g \in C$. Thus, we see that the s-matrix of $Z(\mathrm{Rep}G)$ is essentially an extension of the character table of G.

Remark 3.5. It is not hard to show that the Drinfeld center of the category Rep$G$ is the same as that of Vec_G.

3.5. **The Verlinde formula.** One of the main results on modular categories is the Verlinde formula, relating the fusion coefficients N_{ij}^k of a modular category $\mathcal{C}$ with its s-matrix:

Proposition 3.6. *One has*

$$\sum_k N_{ij}^k s_{kr} = \frac{s_{ir}s_{jr}}{s_{0r}},$$

where 0 *is the label for the unit object.*

(As the s-matrix is nondegenerate, this formula shows that the fusion coefficients N_{ij}^k can be expressed via the s-matrix).

Proof. The formula follows easily from the hexagon relation for the braiding. □

Corollary 3.7. *The eigenvalues of the matrix N_i with entries N_{ij}^k are s_{ir}/s_{0r}, so s_{ir}/s_{0r} is an algebraic integer.*

Note that in Example 3.4 this yields that $\chi_V(g)$ is an algebraic integer, and also that $|C|\chi_V(g)/\dim V$ is an algebraic integer (Lemma 1.4).

Corollary 3.8. *For every $r \neq 0$,*

$$\sum_k d_k s_{kr} = 0,$$

where $d_k = s_{0k}$ are the dimensions of the simple objects.

Proof. We have $\sum_j d_j N_{ij}^k = d_i d_k$, so multiplying the Verlinde formula (Proposition 3.6) by d_j and summing over j, we get

$$d_i \sum_k d_k s_{kr} = (s_{ir}/s_{0r}) \sum_j d_j s_{jr},$$

and the result follows as $d_i \neq s_{ir}/s_{0r}$ for some i by nondegeneracy of s. □

3.6. **Müger's theorem.**

Theorem 3.9. *Let $\mathcal{C}$ be a ribbon category, and $\mathcal{D}$ a modular subcategory in $\mathcal{C}$. Then $\mathcal{C}$ is naturally equivalent, as a ribbon category, to $\mathcal{D} \boxtimes \mathcal{D}'$.*

3.7. **The Anderson-Moore-Vafa theorem.**

Theorem 3.10. *(see e.g.* [BK]*) The squared braiding $\beta_{YX}\beta_{XY}$ in a ribbon category acts on multiplicity spaces in $X \otimes Y$ by multiplication by roots of unity.*

3.8. **The divisibility theorems.** (see [ENO])

Theorem 3.11. *Let $\mathcal{C}$ be a modular category. Then for any simple object $X \in \mathcal{C}$, the ratio* $\dim(\mathcal{C})/\dim(X)^2$ *is an algebraic integer.*

Theorem 3.12. *The Frobenius-Perron dimension of a fusion subcategory of a fusion category $\mathcal{C}$ divides the Frobenius-Perron dimension of $\mathcal{C}$.*

3.9. **Modular categories with a simple object of prime power dimension.** In this subsection we will generalize to modular categories Theorems 1.2 and 1.5, which played a central role in the proof of Burnside's theorem.

Theorem 3.13. *Let X and Y be two simple objects of an integral braided category $\mathcal{C}$ with coprime dimensions*[3] d_X, d_Y*. Then one of two possibilities hold:*

(i) *X and Y projectively centralize each other (i.e. the square of the braiding on $X \otimes Y$ is a scalar);*
(ii) $s_{X,Y} = 0$.

Note that in Example 3.4, this reduces exactly to Theorem 1.2, and the proof is an almost literal repetition of the proof of Theorem 1.2.

Proof. Since $\mathcal{C}$ is integral, it is ribbon. So it suffices to consider the case when $\mathcal{C}$ is modular, since any ribbon category can be embedded into a modular one (its Drinfeld center). In this case, by Corollary 3.7, $\frac{s_{X,Y}}{d_X}$ and $\frac{s_{X,Y}}{d_Y}$ are algebraic integers. Since d_X and d_Y are coprime, $\frac{s_{X,Y}}{d_X d_Y}$ is also an algebraic integer.

By Theorem 3.10, the eigenvalues of the squared braiding $\beta_{YX}\beta_{XY}$ are roots of unity. Therefore, $s_{X,Y}$ is a sum of $d_X d_Y$ roots of unity, and thus by Lemma 1.3 $\frac{s_{X,Y}}{d_X d_Y}$ is either a root of unity (in which case the square of the braiding must be a scalar, option (i)), or 0 (option (ii)). □

Theorem 3.14. *Let $\mathcal{E}$ be an integral modular category which contains a simple object X with dimension $d_X = p^r$, $r > 0$, where p is a prime. Then $\mathcal{E}$ contains a nontrivial symmetric subcategory.*

Proof. We first show that $\mathcal{E}$ contains a nontrivial proper subcategory. Assume not. Take any simple $Y \neq \mathbf{1}$ with d_Y coprime to d_X. We claim that $s_{X,Y} = 0$. Indeed, otherwise by Theorem 3.13 X and Y projectively centralize each other, hence Y centralizes $X \otimes X^*$, so the Müger centralizer of the category generated by Y is a nontrivial proper subcategory, a contradiction.

Now, by Corollary 3.8,

$$\sum_{Y \in \mathrm{Irr}\mathcal{E}} \frac{s_{X,Y}}{d_X} d_Y = 0.$$

As we have shown, all the nonzero summands in this sum, except the one for $Y = \mathbf{1}$, come from objects Y of dimension divisible by p. Therefore, all the summands in this sum except for the one for $Y = \mathbf{1}$ (which equals 1) are divisible by p. This is a contradiction.

[3]Here and below, we use shortened notation d_X for the dimension $\dim(X)$.

Now we prove the corollary by induction in FPdim($\mathcal{E}$). Let $\mathcal{D}$ be a nontrivial proper subcategory of $\mathcal{E}$. If $\mathcal{D}$ is not modular, its Müger center is a nontrivial proper symmetric subcategory in $\mathcal{E}$, and we are done. Otherwise, $\mathcal{D}$ is modular, and by Theorem 3.9, $\mathcal{E} = \mathcal{D} \boxtimes \mathcal{D}'$. Thus $X = X_1 \otimes X_2$, where $X_1 \in \mathcal{D}$, $X_2 \in \mathcal{D}'$ are simple. Since the dimension of X_1 or X_2 is a positive power of p, we get the desired statement from the induction assumption applied to $\mathcal{D}$ or $\mathcal{D}'$ (which are modular categories of smaller dimension). □

Note that although Theorem 3.14 is not quite a generalization of Theorem 1.5, its proof is rather parallel to the proof of Theorem 1.5.

3.10. **Proof of the categorical Burnside theorem.** We are going to prove the following theorem, which will imply the categorical Burnside Theorem.

Theorem 3.15. *Let $\mathcal{E}$ be an integral modular category of dimension $p^r q^s$, where p, q are primes, and r, s nonnegative integers. If $\mathcal{E}$ is not pointed, then it contains a Tannakian subcategory of the form Rep(G), where G is a group of prime order (p or q).*

Let us explain how Theorem 3.15 implies categorical Burnside Theorem. Let $\mathcal{C}$ be a fusion category of dimension $p^a q^b$, where p, q are odd primes, and $a, b \geq 0$ are nonnegative integers. We prove that $\mathcal{C}$ is solvable by induction in $a + b$.

The category $\mathcal{C}$ is integral. Also, we can clearly assume that $\mathcal{C}$ is not pointed. Then the center $\mathcal{E} := Z(\mathcal{C})$ is not pointed. So by Theorem 3.15, $\mathcal{E}$ contains a Tannakian subcategory of the form Rep(G), where G is a group of prime order. Consider the image of this subcategory in $\mathcal{C}$ under the forgetful functor $Z(\mathcal{C}) \to \mathcal{C}$. If the image is the category of vector spaces, then it is easy to show that $\mathcal{C}$ is graded by a group of prime order, so it is an extension by a group of prime order of another fusion category $\mathcal{C}_1$, which is solvable by the induction assumption, so $\mathcal{C}$ is solvable.

Otherwise, the forgetful functor embeds Rep(G) into $\mathcal{C}$, and it can be shown that $\mathcal{C}$ is the equivariantization of another fusion category $\mathcal{C}_G$, called the de-equivariantization of $\mathcal{C}$. Namely, let A be the algebra of functions on G, regarded as an algebra in $Z(\mathcal{C})$; then $\mathcal{C}_G$ is the category of A-modules in $\mathcal{C}$ (since A is commutative, this is a fusion category).

The category $\mathcal{C}_G$ is solvable by the induction assumption. So $\mathcal{C}$ is solvable. This implies the categorical Burnside theorem.

The rest of the subsection is devoted to the proof of Theorem 3.15.

Proposition 3.16. *Let $\mathcal{E}$ be an integral modular of dimension $p^a q^b$, $a + b > 0$, where p, q are odd primes. Then $\mathcal{E}$ contains a nontrivial invertible object.*

Proof. Assume the contrary. We may assume that $a > 0$. By Theorem 3.11, the squared dimensions of simple objects of $\mathcal{E}$ divide $p^a q^b$. Therefore, by the "sum of squares" formula, $\mathcal{E}$ must contain a simple object of dimension p^r or q^r, $r > 0$. Hence, by Theorem 3.14, it contains a nontrivial symmetric subcategory $\mathcal{D}$. But by Deligne's theorem [De], any symmetric odd-dimensional fusion category is Tannakian, so $\mathcal{D}$ is of the form Rep(G) for some finite group G. By Theorem 3.12, the order of G is of the form $p^m q^n$. Therefore, by the usual Burnside theorem for finite groups (saying that a group of order $p^m q^n$ is solvable), $\mathcal{E}$ contains nontrivial invertible objects. Contradiction. □

Consider now the subcategory $\mathcal{B}$ spanned by all invertible objects of $\mathcal{E}$. Proposition 3.16 implies that this subcategory is nontrivial. If $\mathcal{B}$ is modular, then by Theorem 3.9, $\mathcal{E} = \mathcal{B} \boxtimes \mathcal{B}'$, where $\mathcal{B}'$ is another modular category, which is nontrivial (as $\mathcal{E}$ is not pointed), but has no nontrivial invertible objects. Thus, by Proposition 3.16, this case is impossible.

Therefore, $\mathcal{B}$ is not modular. Consider the Müger center $\mathcal{Z}$ of $\mathcal{B}$. It is a nontrivial pointed symmetric subcategory in $\mathcal{E}$, which by Theorem 3.12 has odd dimension. Thus it clearly contains a subcategory $\mathrm{Rep}(G)$, where G is a group of prime order. Theorem 3.15 is proved.

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