CHAPTER 2

Geometry of $\mathcal{D}$-schemes

... иногда же, удалясь в близь лежащую рощу, играл на флейтравере, оставя друга молодаго между гробов одного, яко бы для того, что б издали ему приятнее было слушать музыку.

М. И. Ковалинский,
"Жизнь Григория Сковороды.
Писана 1794 года в древнем вкусе." †

2.1. $\mathcal{D}$-modules: Recollections and notation

The language of $\mathcal{D}$-modules plays for us about the same role as the language of linear algebra for the usual commutative algebra and algebraic geometry. Good references are [**Ber**], [**Ba**], [**Kas2**] (we will not use the theory of holonomic $\mathcal{D}$-modules though). This section collects some information that will be of use; we recommend the reader to skip it, returning when necessary.

We discuss left and right $\mathcal{D}$-modules in 2.1.1, basic $\mathcal{D}$-module functoriality in 2.1.2, Kashiwara's lemma in 2.1.3, the exactness of the pull-back functor for locally complete intersection maps in 2.1.4, locally projective $\mathcal{D}$-modules in 2.1.5, the sheafified middle de Rham cohomology functor h in 2.1.6, the induced $\mathcal{D}$-modules in 2.1.8–2.1.10 (we refer to [**S1**] for all details) and the quasi-induced $\mathcal{D}$-modules in 2.1.11, maximal constant quotients and de Rham homology in 2.1.12, modifications of a $\mathcal{D}$-module at a point and completions of the local de Rham cohomology in 2.1.13–2.1.15, and a variant of the same subject when the point moves in 2.1.16. Finally, in 2.1.17 we prove an auxiliary result from commutative algebra which explains how the formalism of I-adic topology can be rendered to the setting of arbitrary (possibly infinitely generated) modules.

All schemes and algebras are defined over a fixed base field k of characteristic zero.

The reader may prefer to work all the way with super objects following conventions of 1.1.16.

2.1.1. Left and right $\mathcal{D}$-modules; homotopical flatness. For a scheme X we denote by $\mathcal{M}_{\mathcal{O}}(X)$ the category of $\mathcal{O}$-modules on X ($:=$ quasi-coherent sheaves of $\mathcal{O}_X$-modules). If X is smooth, then $\mathcal{M}^r(X)$ (resp. $\mathcal{M}^\ell(X)$) denotes the category

† "... sometimes he played flute in a nearby grove, abandoning the young friend betwixt the tombs, as if the music were to be better appreciated from afar."
M. I. Kovalinski, "The life of Gregori Skovoroda, written in 1794 in ancient gusto."

of right (resp. left) $\mathcal{D}$-modules on X ($:=$ sheaves of $\mathcal{D}_X$-modules quasi-coherent as $\mathcal{O}_X$-modules). Notice that left $\mathcal{D}$-modules are the same as $\mathcal{O}$-modules equipped with an integrable connection. We mostly deal with right $\mathcal{D}$-modules and often call them simply $\mathcal{D}$-modules writing $\mathcal{M}(X) := \mathcal{M}^r(X)$. One has the obvious forgetting of $\mathcal{D}$-action functors $o^r\colon \mathcal{M}(X) \to \mathcal{M}_{\mathcal{O}}(X)$, $o^\ell\colon \mathcal{M}^\ell(X) \to \mathcal{M}_{\mathcal{O}}(X)$.

If G, L are left $\mathcal{D}$-modules, then $G \otimes L = G \underset{\mathcal{O}_X}{\otimes} L$ is a left $\mathcal{D}$-module in a natural way. Therefore $\mathcal{M}^\ell(X)$ is a tensor category with a unit object $\mathcal{O}_X$, and o^ℓ is a tensor functor.

If M is a right $\mathcal{D}$-module and L a left $\mathcal{D}$-module, then $M \underset{\mathcal{O}_X}{\otimes} L$ is naturally a right $\mathcal{D}$-module (a field $\tau \in \Theta_X \subset \mathcal{D}_X$ acts as $(m \otimes \ell)\tau = m\tau \otimes \ell - m \otimes \tau\ell$) denoted by $M \otimes L$. The sheaf $\omega_X = \Omega_X^{\dim X}$ (we consider it as an *even* super line) has a canonical right $\mathcal{D}$-module structure, namely, $\nu\tau = -\mathcal{L}ie_\tau(\nu)$ for $\nu \in \omega_X$. The functor $\mathcal{M}^\ell(X) \to \mathcal{M}(X)$, $L \mapsto L^r := L\omega_X = \omega_X \otimes L$, is an equivalence of categories. The inverse equivalence is $M \mapsto M^\ell := M \otimes \omega_X^{-1}$.

We refer to complexes of left (right) $\mathcal{D}$-modules as left (right) $\mathcal{D}$-*complexes* on X; sometimes right $\mathcal{D}$-complexes are called simply $\mathcal{D}$-complexes. We call the identification of derived categories

$$D\mathcal{M}^\ell(X) \xrightarrow{\sim} D\mathcal{M}^r(X), \qquad L \mapsto L\omega_X[\dim X], \tag{2.1.1.1}$$

the canonical equivalence. So $D\mathcal{M}(X)$ carries two standard $\mathcal{M}^r$ and $\mathcal{M}^\ell$ t-structures, differing by shift by $\dim X$. The composition $D\mathcal{M}(X) = D\mathcal{M}^\ell(X) \xrightarrow{o^\ell} D\mathcal{M}_{\mathcal{O}}(X)$ is denoted simply by o^ℓ.

For details about the next definitions see [**Sp**].[1] We say that a complex M of $\mathcal{O}$-modules is *homotopically* $\mathcal{O}_X$*-flat* if for every acyclic complex F of $\mathcal{O}$-modules the complex $M \otimes F$ is acyclic. A $\mathcal{D}$-complex $M \in C\mathcal{M}(X)$ is homotopically $\mathcal{O}_X$-flat if it is homotopically flat as a complex of $\mathcal{O}_X$-modules. Equivalently, this means that for every acyclic $L \in C\mathcal{M}^\ell(X)$ the complex $M \otimes L$ is acyclic. Similarly, $M \in C\mathcal{M}(X)$ is *homotopically* $\mathcal{D}_X$*-flat* if for every L as above the complex of sheaves of k-vector spaces $M \underset{\mathcal{D}_X}{\otimes} L$ is acyclic. A $\mathcal{D}$-complex M is homotopically $\mathcal{D}_X$-flat if and only if it is homotopically $\mathcal{O}_X$-flat and for every $P \in C\mathcal{M}^\ell(X)$ the canonical projection $DR(M \otimes P) \to h(M \otimes P)$ (see 2.1.7 below) is a quasi-isomorphism.

2.1.2. Functoriality. A morphism $f\colon X \to Y$ of smooth varieties yields canonical functors

$$D\mathcal{M}(X) \underset{f^!}{\overset{f_*}{\rightleftarrows}} D\mathcal{M}(Y) \tag{2.1.2.1}$$

that satisfy standard compatibilities (the base change property, adjunction property for proper f, etc.).

If f is quasi-finite, then f_* is left exact, and if f is affine, then f_* is right exact with respect to the $\mathcal{M}^r$ t-structure.

[1] Our "homotopically flat" is Spaltenstein's "K-flat".

The diagram of functors

$$\begin{array}{ccc} D\mathcal{M}_{\mathcal{O}}(X) & \xleftarrow{Rf^!} & D\mathcal{M}_{\mathcal{O}}(Y) \\ \uparrow o^r & & \uparrow o^r \\ D\mathcal{M}(X) & \xleftarrow{f^!} & D\mathcal{M}(Y) \\ \downarrow o^\ell & & \downarrow o^\ell \\ D\mathcal{M}_{\mathcal{O}}(X) & \xleftarrow{Lf^*} & D\mathcal{M}_{\mathcal{O}}(Y) \end{array} \tag{2.1.2.2}$$

commutes; here $Rf^!$, Lf^* are standard pull-back functors for $\mathcal{O}$-modules (recall that for any $F \in D\mathcal{M}_{\mathcal{O}}(Y)$ one has $(Lf^*F) \otimes \omega_X[\dim X] = Rf^!(F \otimes \omega_Y[\dim Y])$). In particular, $f^!$ is right exact with respect to the $\mathcal{M}^\ell$ t-structure. We often denote the corresponding functor $\mathcal{M}^\ell(Y) \to \mathcal{M}^\ell(X)$ by f^*; this is the usual pull-back of an $\mathcal{O}$-module equipped with a connection.

If X_i, $i \in I$, is a finite collection of smooth schemes, then we have the exterior tensor product functors for both left and right $\mathcal{D}$-modules $\prod \mathcal{M}^r(X_i) \to \mathcal{M}^r(\prod X_i)$, $\prod \mathcal{M}^\ell(X_i) \to \mathcal{M}^\ell(\prod X_i)$, $(M_i) \mapsto \boxtimes M_i$. The canonical equivalence (2.1.1.1) identifies the corresponding derived functors.[2] The functors f_*, $f^!$ are compatible with $\boxtimes$: if $f_i\colon X_i \to Y_i$ are morphisms of smooth schemes, then $(\prod f_i)_*(\boxtimes M_i) = \boxtimes(f_{i*}M_i)$, $(\prod f_i)^!(\boxtimes N_i) = \boxtimes(f_i^! N_i)$.

Assume that all X_i coincide with X; let $\Delta^{(I)}\colon X \to X^I$ be the diagonal embedding. For $L_i \in D\mathcal{M}^\ell(X)$ one has a canonical isomorphism

$$\overset{\mathbb{L}}{\otimes} L_i = \Delta^{(I)!}(\boxtimes L_i). \tag{2.1.2.3}$$

In particular, for $L_i \in \mathcal{M}^\ell(X)$ one has

$$\otimes L_i = \Delta^{(I)*}(\boxtimes L_i). \tag{2.1.2.4}$$

2.1.3. If $i\colon X \hookrightarrow Y$ is a closed embedding, then i_* is exact with respect to the $\mathcal{M}^r$ t-structure, and its right adjoint $i^!$ is left exact. They define mutually inverse equivalences between $\mathcal{M}(X)$ and the full subcategory $\mathcal{M}(Y)_X \subset \mathcal{M}(Y)$ that consists of $\mathcal{D}_Y$-modules that vanish on $Y \setminus X$ (Kashiwara's lemma).

Our functors can be described explicitly as follows. Let $\mathcal{I} \subset \mathcal{O}_Y$ be the ideal of X. If $N \in \mathcal{M}(Y)$, then $i^!N$ coincides, as an $\mathcal{O}$-module, with the subsheaf of N that consists of sections killed by $\mathcal{I}$. For a $L \in \mathcal{M}^\ell(Y)$ one has $i^*L = L/\mathcal{I}L$. For $M \in \mathcal{M}(X)$ one has

$$i_*M = i_\cdot(M \underset{\mathcal{D}_X}{\otimes} i^*\mathcal{D}_Y). \tag{2.1.3.1}$$

Here we consider $\mathcal{D}_Y$ as a left $\mathcal{D}_Y$-module, and the right $\mathcal{D}_Y$-module structure on i_*M comes from the right action of $\mathcal{D}_Y$ on $\mathcal{D}_Y$ (as on a left $\mathcal{D}$-module). The identification $M \xrightarrow{\sim} i^!i_*M$ is $m \mapsto m \otimes 1$.

For $L \in \mathcal{M}^\ell(Y)$ and $M \in \mathcal{M}(X)$ there is a canonical isomorphism

$$i_*(M \otimes i^*L) \xrightarrow{\sim} (i_*M) \otimes L. \tag{2.1.3.2}$$

[2] There is a canonical isomorphism of super lines $\omega_{\Pi X_i}[\dim \Pi X_i] \xrightarrow{\sim} \boxtimes(\omega_{X_i}[\dim X_i])$; in particular, it does not depend on the ordering of I.

The corresponding isomorphism $M \otimes i^* L \xrightarrow{\sim} i^!((i_* M) \otimes L)$ sends $m \otimes \ell$ to $(m \otimes 1) \otimes \ell$.

The reader can skip the next remark.

Remark. We will use only $\mathcal{D}$-modules on *smooth* algebraic varieties. However at some places (such as 2.1.4) the smoothness restriction on X is unnatural. Here is a short comment about the notion of $\mathcal{D}$-module on an arbitraly singular scheme. One can assign to any k-scheme X of locally finite type an abelian k-category $\mathcal{M}(X)$ whose objects are called "$\mathcal{D}$-modules on X" together with a conservative[3] left exact functor $o : \mathcal{M}(X) \to \mathcal{M}_{\mathcal{O}}(X)$. If X is smooth, then $\mathcal{M}(X)$ is the category of *right* $\mathcal{D}$-modules on X and o is the functor o^r from 2.1.1. The important point is that the notion of *left* $\mathcal{D}$-module on an arbitrary singular X makes no sense. The standard functors from 2.1.2 exist in this setting[4] and satisfy the usual properties. According to Kashiwara's lemma, if one has a closed embedding $i : X \hookrightarrow Y$ where Y is smooth, then i_* identifies $\mathcal{M}(X)$ with the full subcategory of usual right $\mathcal{D}$-modules on Y which vanish on $Y \smallsetminus X$; for $M \in \mathcal{M}(X)$ one has $o(M) = i^!(o^r i_* M)$. There exist two (different yet equivalent) ways to define $\mathcal{M}(X)$ (see [**S2**][5] and [**BD**] 7.10). The method of [**S2**] is to consider all possible closed embeddings $i_Y : U \hookrightarrow Y$ where $U \subset X$ is an open subset and Y is a smooth scheme; a $\mathcal{D}$-module on X is a rule M which assigns to every such datum a $\mathcal{D}$-module $i_{Y*}M$ on Y in a compatible manner. [**BD**] presents a crystalline approach: a $\mathcal{D}$-module M on X is a rule that assigns to any U as above and an infinitesimal thickening[6] $i_Z : U \hookrightarrow Z$ an $\mathcal{O}$-module $M_Z = o(i_{Z*}M)$ on Z in a compatible (with respect to morphisms of Z's) manner.

2.1.4. The pull-back functor. If f is a smooth morphism then f^* is exact with respect to the $\mathcal{M}^\ell$ t-structure. Left $\mathcal{D}$-modules satisfy the smooth descent property, so the categories $\mathcal{M}^\ell(U)$, U is a smooth X-scheme, form a sheaf of abelian categories on the smooth topology X_{sm}, which we denote by $\mathcal{M}^\ell(X_{sm})$. One defines then the category $\mathcal{M}^\ell(Z)$ for any smooth algebraic stack Z in the usual way.

Remark. If our schemes are not necessarily smooth (so we are in the setting of Remark in 2.1.3) but f is a smooth morphism, then the functor $f^\dagger := f^![-\dim X/Y] : D\mathcal{M}(Y) \to D\mathcal{M}(X)$ is t-exact (i.e., $f^\dagger(\mathcal{M}(Y)) \subset \mathcal{M}(X)$), so we have a canonical exact functor $\mathcal{M}(Y) \to \mathcal{M}(X)$ which we denote also by $f^\dagger$. The smooth descent property holds. Of course, if our schemes are smooth, then $f^\dagger$ coincides with the pull-back functor $f^* : \mathcal{M}^\ell(Y) \to \mathcal{M}^\ell(X)$.[7]

The above picture generalizes to the case of locally complete intersection morphisms. Namely, assume that $f : X \to Y$ makes X a locally complete intersection over Y of pure relative dimension $d \geq 0$ (i.e., X can be locally represented as a closed subscheme of $Y \times \mathbb{A}^n$ defined by $n - d$ equations and the fibers of f have pure dimension d). Set $f^\dagger := f^![-d] : D\mathcal{M}(Y) \to D\mathcal{M}(X)$.

Proposition. *The functor $f^\dagger$ is t-exact, i.e., $f^\dagger M$ is a $\mathcal{D}$-module for every $\mathcal{D}$-module M on Y.*

We give two proofs. The first one makes sense even if $\mathcal{D}$-modules are replaced, say, by ℓ-adic perverse sheaves. The second one is based on the relation between $\mathcal{D}$-modules and $\mathcal{O}$-modules.

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[3]Which means that $o(M) \neq 0$ for $M \neq 0$.

[4]We assume that our schemes are quasi-compact and separated.

[5]M. Saito considers $\mathcal{D}$-modules in the analytic setting; the algebraic picture is similar.

[6]I.e., i_Z is a closed embedding $i_Z : U \hookrightarrow Z$ defined by a nilpotent ideal.

[7]We identify $\mathcal{M}$ with $\mathcal{M}^\ell$ as in 2.1.1.

First proof. For any $f : X \to Y$ with fibers of dimension $\leq d$ one has $H^i f^! M = 0$ for $i < -d$.[8] So it remains to show that $H^i f^! M = 0$ for $i > -d$. We can assume that $f = \pi i$ where $\pi : \tilde{Y} \to Y$ is smooth of pure relative dimension n and $i : X \hookrightarrow \tilde{Y}$ is a closed embedding such that $\tilde{Y} \setminus i(X)$ is covered by $n - d$ open subsets U_j affine over $\tilde{Y}$. Now compute $i_* f^! M = i_* i^! \pi^* M[n]$ using the U_j's. The direct image with respect to an affine open embedding is an exact functor, so $H^i f^! M = 0$ for $i > -d$, and we are done.

Second proof. We can assume that there is a closed embedding $X \hookrightarrow Y \times \mathbb{A}^n$ such that $i(X) \subset Y \times \mathbb{A}^n$ is defined by $n - d$ equations. One can find subschemes $X_k \subset Y \times \mathbb{A}^n$, $k \in \mathbb{N}$, such that $X_k \subset X_{k+1}$, $(X_k)_{red} = i(X_{red})$, each X_k is a locally complete intersection over Y, and every subscheme $Z \subset Y \times \mathbb{A}^n$ with $Z_{red} = i(X_{red})$ is contained in some X_k. The $\mathcal{O}_{Y \times \mathbb{A}^n}$-module corresponding to $H^p f^! M$ is the direct limit of the $\mathcal{O}_{X_k}$-modules $H^p f_k^! M_{\mathcal{O}}$, where f_k is the morphism $X_k \to Y$ and $M_{\mathcal{O}}$ is the $\mathcal{O}_Y$-module corresponding to M. But X_k is Cohen-Macaulay over Y, so $H^p f_k^! M_{\mathcal{O}} = 0$ for $p \neq -d$. □

REMARK. In the above proposition it is not enough to require X to be Cohen-Macaulay over Y (a counterexample: Y is a point and X is the scheme of 2×3 matrices of rank ≤ 1).

2.1.5. Vector $\mathcal{D}$-bundles. If X is affine, then the projectivity of a $\mathcal{D}_X$-module is a local property for the Zariski or étale topology of X (see 2.3.6 below where this fact is proven in a more general situation). For arbitrary X we call a locally projective right $\mathcal{D}_X$-module of finite rank a *vector $\mathcal{D}_X$-bundle* or a *vector $\mathcal{D}$-bundle on X*.

For a vector $\mathcal{D}_X$-bundle V its *dual* V° is $\mathcal{H}om_{\mathcal{D}_X}(V^\ell, \mathcal{D}_X)$. So a section of V° is a morphism of left $\mathcal{D}_X$-modules $V^\ell \to \mathcal{D}_X$, and the right $\mathcal{D}_X$-module structure on V° comes from the right $\mathcal{D}_X$-action on $\mathcal{D}_X$. It is clear that V° is again a vector $\mathcal{D}_X$-bundle, and $(V^\circ)^\circ = V$. For further discussion see 2.2.16.

A vector $\mathcal{D}_X$-bundle need not be a locally free $\mathcal{D}_X$-module as the following example from [**CH**] (see also [**BGK**]) shows.

EXAMPLE. Assume that X is an affine curve, and let $V \subset \mathcal{D}_X$ be a $\mathcal{D}$-submodule such that V equals $\mathcal{D}_X$ outside of a point $x \in X$. Then V is a projective $\mathcal{D}_X$-module (since the category of $\mathcal{D}$-modules has homological dimension 1).[9] By 2.1.13 below, such V's are the same as open *k-vector subspaces* of the formal completion O_x of the local ring $\mathcal{O}_x$.

If V is locally free, then (shrinking X if necessary) we can assume that it is generated by a single section ϕ; i.e., $V = \phi\mathcal{D}_X \subset \mathcal{D}_X$. Since $\phi\mathcal{D}_X = \mathcal{D}_X$ on $X \smallsetminus \{x\}$, the differential operator ϕ has order 0; i.e., $V = t^n \mathcal{D}_X$ for some $n \geq 0$ (where t is a parameter at x). Therefore locally free V's correspond to open *ideals* in O_x.

2.1.6. The middle de Rham cohomology sheaf. For $M \in \mathcal{M}(X)$ set $h(M) := M \underset{\mathcal{D}_X}{\otimes} \mathcal{O}_X = M/M\Theta_X$. For a local section $m \in M$ we denote by $\overline{m}$ its class

[8]To prove this, choose a stratification of Y such that for each stratum Y_α the morphism $X_\alpha := (f^{-1}(Y_\alpha))_{red} \to Y_\alpha$ is smooth and look at the sheaves $H^i \nu_\alpha^! f^! M$, $\nu_\alpha : X_\alpha \hookrightarrow X$.

[9]In fact, it is enough to know that $\mathcal{D}_X/V$ has a projective resolution of length 1, and this follows from Kashiwara's lemma.

in $h(M)$. We consider $h(M)$ as a sheaf on the étale topology of X. This will be relevant in Chapter 4; for the present chapter Zariski localization works as well.

Notice that for any $M \in \mathcal{M}(X)$, $L \in \mathcal{M}^\ell(X)$ one has $M \underset{\mathcal{D}_X}{\otimes} L = h(M \otimes L)$.

The reader can skip the next technical lemma returning to it when necessary.

LEMMA. (i) *If M is a non-zero coherent $\mathcal{D}_X$-module then $h(M) \neq 0$.*

(ii) *If M is a coherent $\mathcal{D}_X$-module with no O_X-torsion then for every $\mathcal{D}_X$-module L the map* $\mathrm{Hom}(L, M) \to \mathrm{Hom}(h(L), h(M))$ *is injective.*

Proof. (i) (a) Using induction by the dimension of the support of M, we can assume, by Kashiwara's lemma (see 2.1.3), that M is non-zero at the generic point of X. Replacing X by its open subset, we can assume that M is a free $\mathcal{O}_X$-module.[10]

(b) Let us show that for any (not necessarily coherent) non-zero $\mathcal{D}_X$-module M which is free as an $\mathcal{O}_X$-module one has $h(M) \neq 0$. We use induction by $\dim X$. Replacing X by its open subset, we can find a smooth hypersurface $i: Y \hookrightarrow X$ such that the determinant of the normal bundle to Y is trivial. Let $j: U \hookrightarrow X$ be its complement. Set $M_Y := i^!(j_*j^*M/M) = (i * M^\ell)^r$; this is a non-zero $\mathcal{D}$-module on Y which is free as an $\mathcal{O}_Y$-module by the condition on Y. Hence $h(M_Y) \neq 0$ by the induction hypothesis. Notice that $j_\cdot j^\cdot h(M) = h(j_*j^*M)$ and $h(i_*M_Y) = i_\cdot h(M_Y)$. Since h is right exact and i_*M_Y is a quotient of j_*j^*M, we see that $h(M) \neq 0$.

(ii) (a) Since h is right exact, it suffices to show that for every non-zero $L \subset M$ the map $h(L) \to h(M)$ is non-zero. The condition on M assures that we can replace X by any open subset, so we can assume that L, M, and M/L are free $\mathcal{O}_X$-modules.

(b) Let us show that for every (not necessarily coherent) non-zero $\mathcal{D}_X$-modules $L \subset M$ such that L, M, M/L are free $\mathcal{O}_X$-modules, the map $h(L) \to h(M)$ is non-zero. We use induction by $\dim X$ and follow notation from (i)(b) above. It suffices to show that the map $h(j_*j^*L) \to h(j_*j^*M)$ is non-zero. It admits as a quotient the map $h(L_Y) \to h(M_Y)$ which is non-zero by induction by $\dim X$. □

2.1.7. The de Rham complex. Denote by $DR(M)$ the de Rham complex of M, so $DR(M)^i := M \underset{\mathcal{O}_X}{\otimes} \Lambda^{-i}\Theta_X$. One has $h(M) = H^0 DR(M)$; in fact, as an object of the derived category of sheaves of k-vector spaces on X_{Zar} or $X_{\text{ét}}$, $DR(M)$ equals to $M \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{O}_X$. In particular, if M is a locally projective $\mathcal{D}_X$-module (say, a vector $\mathcal{D}_X$-bundle), then the projection $DR(M) \to h(M)$ is a quasi-isomorphism.

Set $\Gamma_{DR}(X, M) := \Gamma(X, DR(M))$, $R\Gamma_{DR}(X, M) := R\Gamma(X, DR(M))$.

For a morphism $f\colon X \to Y$ of smooth schemes there is a canonical isomorphism $Rf_\cdot(DR(M)) \xrightarrow{\sim} DR(f_*M)$ in the derived category of sheaves on Y; here $f_\cdot$ is the sheaf-theoretical direct image, and $Rf_\cdot$ is its derived functor. For a more precise setting for this quasi-isomorphism; see 2.1.11. We mention two particular cases:

(a) The global de Rham cohomology coincides with the push-forward by projection $\pi\colon X \to$ (point): one has $H^\cdot_{DR}(X, M) := H^\cdot(X, DR(M)) = H^\cdot \pi_* M$.

(b) If $i\colon X \hookrightarrow Y$ is a closed embedding, then the above quasi-isomorphism arises as a canonical embedding of complexes $i_\cdot DR(M) \hookrightarrow DRi_*(M)$ defined by the obvious embedding of $\mathcal{O}_Y$-modules $i_\cdot M \hookrightarrow i_*M$. Thus $i_\cdot h(M) = h(i_*M)$.

[10]Indeed, choose a good filtration on M; then, according to [EGA IV] 6.9.2, $\mathrm{gr} M$ is $\mathcal{O}_X$-free over a non-empty open $V \subset X$.

Notice also that in situation (b) for $N \in D\mathcal{M}(Y)$ the obvious morphism $DR(Ri^! N) \to Ri^! DR(N)$ is a quasi-isomorphism.[11]

LEMMA. *If X is an affine curve, then the map $H^0_{DR}(X, M) \to \Gamma(X, h(M))$ is surjective.*

Proof. Consider the de Rham complex $M^\ell \xrightarrow{d} M$; set $K := \mathrm{Ker}(d)$, $I := \mathrm{Im}(d)$. Notice that $H^2(X, K) = 0$ (since $\dim X = 1$) and $H^1(X, M^\ell) = 0$ (since X is affine). Thus $H^1(X, I) = 0$, which implies that $\Gamma(X, M) \twoheadrightarrow \Gamma(X, h(M))$. □

2.1.8. Induced $\mathcal{D}$-modules. See [**S1**] for all the details.

Let X be a smooth scheme. For an $\mathcal{O}$-module B on X we have the induced right $\mathcal{D}$-module $B_{\mathcal{D}} := B \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \in \mathcal{M}(X)$. The de Rham complex $DR(B_{\mathcal{D}})$ is acyclic in non-zero degrees, and the projection $DR(B_{\mathcal{D}})^0 = B \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \to B$, $b \otimes \partial \mapsto b\partial(1)$, yields an isomorphism $h(B_{\mathcal{D}}) \xrightarrow{\sim} B$. In other words, there is a canonical quasi-isomorphism

$$\nu_B \colon DR(B_{\mathcal{D}}) \to B. \tag{2.1.8.1}$$

If C is another $\mathcal{O}$-module, then for any $\varphi \in \mathrm{Hom}_{\mathcal{M}(X)}(B_{\mathcal{D}}, C_{\mathcal{D}})$ the corresponding map $h(\varphi)\colon h(B_{\mathcal{D}}) \to h(C_{\mathcal{D}})$ is a differential operator $B \to C$. In fact, the map

$$h\colon \mathrm{Hom}(B_{\mathcal{D}}, C_{\mathcal{D}}) \to \mathrm{Diff}(B, C), \tag{2.1.8.2}$$

where Diff is the vector space of all differential operators, is an isomorphism. The inverse map assigns to a differential operator $\psi\colon B \to C$ the morphism $\psi_{\mathcal{D}}\colon B_{\mathcal{D}} \to C_{\mathcal{D}}$, $b \otimes \partial \mapsto \psi_b \partial$, where $\psi_b \in C \otimes \mathcal{D}_X = \mathrm{Diff}(\mathcal{O}_X, C)$ is the differential operator $\psi_b(f) = \psi(bf)$.

Let $\mathcal{D}iff(X)$ be the category whose objects are $\mathcal{O}$-modules on X and where morphisms are differential operators. We see that the induction $B \mapsto B_{\mathcal{D}}$ defines a fully faithful functor $\mathcal{D}iff(X) \hookrightarrow \mathcal{M}(X)$.

For $N \in \mathcal{M}^\ell(X)$, $B \in \mathcal{M}_{\mathcal{O}}(X)$ there is a canonical isomorphism of $\mathcal{D}$-modules

$$(B \underset{\mathcal{O}_X}{\otimes} N)_{\mathcal{D}} \xrightarrow{\sim} B_{\mathcal{D}} \otimes N, \quad (b \otimes n) \otimes \partial \mapsto (b \otimes n)\partial. \tag{2.1.8.3}$$

Therefore $N, B \mapsto B \otimes N := B \underset{\mathcal{O}_X}{\otimes} N$ is an action of the tensor category $\mathcal{M}^\ell(X)$ on $\mathcal{D}iff(X)$; the induction functor $\cdot_{\mathcal{D}}$ commutes with $\mathcal{M}^\ell(X)$-actions.[12]

The "identity" functor from $\mathcal{D}iff(X)$ to the category $Sh(X)$ of sheaves of k-vector spaces on X is faithful, so we know what a $\mathcal{D}iff(X)$*-structure* on $P \in Sh(X)$ is. Explicitly, this is an equivalence class of quasi-coherent $\mathcal{O}_X$-module structures on P, where two $\mathcal{O}_X$-module structures P_1, P_2 are said to be equivalent if $id_P : P_1 \to P_2$ is a differential operator.

REMARK. For an $\mathcal{O}$-module B the structure action of $\mathcal{O}_X$ on B yields an $\mathcal{O}_X$-action on the $\mathcal{D}$-module $B_{\mathcal{D}}$. Therefore $B_{\mathcal{D}}$ is an $(\mathcal{O}_X, \mathcal{D}_X)$-bimodule. This bimodule satisfies the following property: the ideal of the diagonal $\mathrm{Ker}(\mathcal{O}_X \underset{k}{\otimes} \mathcal{O}_X \xrightarrow{\cdot} \mathcal{O}_X)$ acts on $B_{\mathcal{D}}$ in a locally nilpotent way. As follows from Kashiwara's lemma, the

[11] Use the standard exact triangle $i_* Ri^! N \to N \to Rj_* j^* N \to i_* Ri^! N[1]$ for $j : Y \smallsetminus i(X) \hookrightarrow Y$.

[12] Here $\mathcal{M}^\ell(X)$ acts on $\mathcal{M}(X)$ by the usual tensor product.

functor $B \mapsto B_{\mathcal{D}}$ from the category of $\mathcal{O}$-modules (and $\mathcal{O}$-linear maps) to that of $(\mathcal{O}_X, \mathcal{D}_X)$-bimodules satisfying the above property is an equivalence of categories.

EXAMPLES. (i) Let $X \hookrightarrow Y$ be a closed embedding. Suppose that the formal neighborhood $\hat{Y}$ of X in Y admits a retraction $\hat{Y} \to X$. Let E be an $\mathcal{O}_Y$-modules E supported (set-theoretically) on X. Then E, considered as a sheaf of k-vector spaces on X, admits a *canonical* $\mathcal{D}iff(X)$-structure. Indeed, every retraction $\pi : \hat{Y} \to X$ yields an $\mathcal{O}_X$-module structure on E, and its equivalence class does not depend on the choice of π.

A particular situation we will use is the diagonal embedding $X \hookrightarrow X^n$. Notice that for E as above the corresponding $\mathcal{D}_X$-module coincides with $\Delta^!(E \underset{\mathcal{O}_{X^n}}{\otimes} \mathcal{D}_{X^n})$.

(ii) For a right $\mathcal{D}$-module M its de Rham complex $DR(M)$ is a complex in $\mathcal{D}iff(X)$. If L is a left $\mathcal{D}$-module, then $DR(L \otimes M) = L \otimes DR(M)$.

(iii) Let $i: X \hookrightarrow Y$ be a closed embedding. For B as above consider the canonical embeddings of $\mathcal{O}_Y$-modules $i.B \hookrightarrow i.(B \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X) \hookrightarrow i_*(B_{\mathcal{D}})$. The composition induces a canonical morphism of $\mathcal{D}_Y$-modules γ: $(i.B)_{\mathcal{D}} \to i_*(B_{\mathcal{D}})$ which is an isomorphism.

2.1.9. For any $M \in \mathcal{M}(X)$ the de Rham complex $DR(M)$ is a complex in $\mathcal{D}iff(X)$ (the de Rham differential is a first order differential operator), so we have the corresponding complex $DR(M)_{\mathcal{D}}$ of $\mathcal{D}$-modules. It is acyclic in degrees $\neq 0$, and the projection $DR(M)^0_{\mathcal{D}} = M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \to M$, $m \otimes \partial \mapsto m\partial$, defines an isomorphism $H^0 DR(M)_{\mathcal{D}} \xrightarrow{\sim} M$. In other words we have a canonical quasi-isomorphism

$$\mu_M: DR(M)_{\mathcal{D}} \to M. \tag{2.1.9.1}$$

2.1.10. One may replace modules by complexes of modules to get the functors between the DG categories of complexes

$$DR: C\mathcal{M}(X) \to C\mathcal{D}iff(X), \qquad \cdot_{\mathcal{D}}: C\mathcal{D}iff(X) \to C\mathcal{M}(X) \tag{2.1.10.1}$$

together with natural morphisms ν, μ as in (2.1.8.1) and (2.1.9.1).

The category $\mathcal{D}iff(X)$ is not abelian. Following Saito, a morphism ψ: $B \to C$ of complexes in $\mathcal{D}iff(X)$ is called a quasi-isomorphism if $\psi_{\mathcal{D}}$ is. One defines the derived category $D\mathcal{D}iff(X)$ by inverting quasi-isomorphisms.

REMARK. A morphism $\psi : B \to C$ in $C\mathcal{D}iff(X)$ is called a *naive quasi-isomorphism* if it is a quasi-isomorphism of complexes of plain sheaves of vector spaces. It is clear that ψ is a quasi-isomorphism if and only if for any $\mathcal{O}$-flat left $\mathcal{D}$-module N the morphism $id_N \otimes \psi : N \otimes B \to N \otimes C$ is a naive quasi-isomorphism. If both B, C are bounded above complexes of *coherent* $\mathcal{O}$-modules, then every naive quasi-isomorphism ψ is necessarily a quasi-isomorphism.[13] For general quasi-coherent B, C this may be false (for example, take $B = DR(M)$ where M is a constant sheaf equal to the field of fractions of $\mathcal{D}_X$, $C = 0$).

Both μ and ν are quasi-isomorphisms. They are compatible in the following sense: for every $B^{\cdot} \in C\mathcal{D}iff(X)$, $M^{\cdot} \in C\mathcal{M}(X)$ the morphisms $DR(\mu_M), \nu_{DR(M)}$: $DR(DR(M)_{\mathcal{D}}) \to DR(M)$ are homotopic; the same is true for the morphisms

[13]Let M be the top non-zero cohomology of the cone of $\psi_{\mathcal{D}}$. Then M is a coherent $\mathcal{D}$-module such that $h(M) = 0$. We are done by the lemma from 2.1.6.

$(\nu_B)_{\mathcal{D}}, \mu_{B_{\mathcal{D}}}: DR(B_{\mathcal{D}})_{\mathcal{D}} \to B_{\mathcal{D}}$. Therefore DR and $\cdot_{\mathcal{D}}$ define mutually inverse equivalences between the derived categories:

$$D\mathcal{M}(X) \overset{\sim}{\leftrightarrows} D\mathcal{D}iff(X). \tag{2.1.10.2}$$

For a morphism of schemes $f\colon X \to Y$ there is an obvious sheaf-theoretic push-forward functor $Rf.\colon D\mathcal{D}iff(X) \to D\mathcal{D}iff(Y)$. Equivalence (2.1.10.2) identifies it with f_*: there are canonical isomorphisms

$$DR(f_* M) = Rf.DR(M), \qquad f_*(B_{\mathcal{D}}) = (Rf.(B))_{\mathcal{D}}. \tag{2.1.10.3}$$

For particular cases see 2.1.7 and Example (iii) in 2.1.8.

2.1.11. Quasi-induced $\mathcal{D}$-modules and $\mathcal{D}$-complexes. The full subcategory of induced $\mathcal{D}$-modules, i.e., $\mathcal{D}$-modules isomorphic to $N_{\mathcal{D}}$ for some $\mathcal{O}$-module N, does not look reasonable. For example, a direct summand of an induced $\mathcal{D}$-module need not be induced (as the example from 2.1.5 shows); induced $\mathcal{D}$-modules do not have a local nature. A more friendly category of quasi-induced $\mathcal{D}$-modules is defined as follows.

We say that $M \in \mathcal{M}(X)$ is *quasi-induced* if for every $\mathcal{O}_X$-flat $L \in \mathcal{M}^\ell(X)$ one has $Tor^{\mathcal{D}_X}_{>0}(M, L) = 0$; i.e., the canonical projection $DR(M \otimes L) \to h(M \otimes L)$ is a quasi-isomorphism.[14] Similarly, a $\mathcal{D}_X$-complex $M \in C\mathcal{M}(X)$ is *homotopically quasi-induced* if for every homotopically $\mathcal{O}_X$-flat $L \in C\mathcal{M}^\ell(X)$ one has $M \overset{L}{\underset{\mathcal{D}_X}{\otimes}} L \xrightarrow{\sim} M \underset{\mathcal{D}_X}{\otimes} L$; i.e., the canonical projection $DR(M \otimes L) \to h(M \otimes L)$ is a quasi-isomorphism.

Here is an intrinsic characterization of quasi-induced $\mathcal{D}$-modules. Let η be a (not necessarily closed) point of X, $i_\eta : \eta \hookrightarrow X$ the embedding. For $C \in D\mathcal{M}(X)$ denote by $i^!_\eta C$ the fiber of $i^!_Y C$ at η, where Y is a non-empty open smooth subvariety of the closure of η and i_Y is the embedding $Y \hookrightarrow X$. So $i^!_\eta C$ belongs to the derived category of modules over the ring of differential operators on η. Let us say that C is *strongly non-negative* if for every $\eta \in X$ the Tor-dimension of $i^!_\eta C$ is non-negative. Equivalently, C is strongly non-negative if and only if for every $\eta \in X$ the Tor-dimension of the stalk C_η of C at η is $\leq \operatorname{codim} \eta$. It is easy to see that strong non-negativity implies the usual one (i.e., $H^{<0}C = 0$) and that strong non-negativity is preserved under direct images with respect to arbitrary morphisms.

LEMMA. *$M \in \mathcal{M}(X)$ is quasi-induced if and only if it is strongly non-negative.*

Proof. Suppose M is quasi-induced. We can assume that X is affine. Take any $\eta \in X$ and $L \in \mathcal{M}^\ell(X)$. Let P be a left resolution of L such that $P_a = 0$ for $a > \operatorname{codim} \eta$ and $P_0, \dots, P_{\operatorname{codim}\eta - 1}$ are projective $\mathcal{D}$-modules. Localizing at η, we see that every $P_{a\eta}$ is a flat $\mathcal{O}_\eta$-module. Thus $(M \overset{L}{\underset{\mathcal{D}_X}{\otimes}} L)_\eta = h(P_\eta)$; hence $Tor^{\mathcal{D}_X}_a(M, L)_\eta = 0$ for $a > \operatorname{codim} \eta$; q.e.d.

Suppose M is strongly non-negative. Then $M \otimes L$ is strongly non-negative for any $\mathcal{O}_X$-flat $L \in \mathcal{M}^\ell(X)$. So it suffices to check that $H^a DR(M) = 0$ for every $a < 0$. Suppose this is not true. Let η be a point of smallest codimension such that $DR(M)_\eta$ has non-zero negative cohomology. Since $DR(i^!_\eta M) = i^!_\eta DR(M)$, one has

[14] This local property amounts to the corresponding global property which says that for every $\mathcal{O}_X$-flat $L \in \mathcal{M}^\ell(X)$ one has $R\Gamma_{DR}(X, M \otimes L) \xrightarrow{\sim} R\Gamma(X, h(M \otimes L))$.

$H^a DR(i^!_\eta M) = H^a DR(M)_\eta$ for $a < 0$. Therefore $i^!_\eta M$ has negative Tor-dimension, and we are done. □

REMARKS. (i) Assume that $k = \mathbb{C}$. A holonomic complex with regular singularities is strongly non-negative if and only if the complex of (usual, not perverse) sheaves corresponding to its Verdier dual is non-positive.

(ii) An induced $\mathcal{D}$-module is quasi-induced. The direct image of a quasi-induced $\mathcal{D}$-module under an affine morphism is quasi-induced. The exterior tensor product of quasi-induced $\mathcal{D}$-modules is quasi-induced. A holonomic $\mathcal{D}$-module is quasi-induced if and only if its support has dimension 0. A tensor product of an $\mathcal{O}_X$-flat $\mathcal{D}$-module and a quasi-induced $\mathcal{D}$-module is quasi-induced.

(iii) If X is a curve, then a $\mathcal{D}$-module is quasi-induced if and only if its restriction to any open subset does not contain a non-trivial lisse submodule.

2.1.12. Maximal constant quotients and de Rham homology. Suppose that X is connected and $\dim X = n$.

For $M \in \mathcal{M}(X)$ set $H_a^{DR}(X, M) := H_{DR}^{-a}(X, M[n])$; this is the *de Rham homology* of X with coefficients in M. Notice that $H_a^{DR}(X, M)$ vanish unless $a \in [0, 2n]$; if X is affine, the vanishing holds unless $a \in [n, 2n]$. The vector space $H_0^{DR}(X, M)$ vanishes if X is non-proper.

We say that $M \in \mathcal{M}(X)$ is *constant* if M^ℓ is generated by global horizontal sections. The category of constant $\mathcal{D}$-modules is closed under subquotients. It identifies canonically with the category of vector spaces by means of the functors $M \mapsto \mathrm{Hom}(\mathcal{O}_X, M^\ell) = \Gamma^\nabla(X, M^\ell)$, $V \mapsto V \otimes \omega_X$.[15] Every $\mathcal{D}$-module M admits the maximal constant submodule $\Gamma^\nabla(X, M^\ell) \otimes \omega_X \subset M$.

For any $M \in \mathcal{M}(X)$ all its constant quotients form a directed projective system of constant $\mathcal{D}$-modules M_{const}. The functor $M \mapsto M_{const}$ is right exact.

Suppose X is proper. Then M admits the maximal constant quotient; i.e., M_{const} is a plain $\mathcal{D}_X$-module.[16]

The adjunction formula for the projection from X to a point (see 2.1.2) yields a canonical identification $M^\ell_{const} \xrightarrow{\sim} H_0^{DR}(X, M) \otimes \mathcal{O}_X$. For any point $s \in X$ the composition $H^0 Ri^!_s M[n] = M^\ell_s \xrightarrow{\pi_s} (M_{const})_s = H_0^{DR}(X, M)$ comes from the canonical morphism $i_{s*} Ri^!_s M \to M$.

Let $S \subset X$ be any finite non-empty subset, $j_S : U \hookrightarrow X$ its complement. The standard exact triangle $\underset{S}{\oplus} i_{s*} Ri^!_s M \to M \to Rj_{S*} j^*_S M \to \underset{S}{\oplus} i_{s*} Ri^!_s M[1]$ shows that the kernel of the morphism $\pi_S = \oplus \pi_s : \underset{S}{\oplus} M^\ell_s \to H_0^{DR}(X, M[n])$ is equal to the image of the "residue" map $\mathrm{Res}_S = \oplus \mathrm{Res}_s : H^{n-1}_{DR}(U, M) \to \underset{S}{\oplus} M^\ell_s$.

2.1.13. Modifications at a point. Let $x \in X$ be a (closed) point. Denote by i_x the embedding $\{x\} \hookrightarrow X$; let $j_x : U_x := X \smallsetminus \{x\} \hookrightarrow X$ be its complement. For a $\mathcal{D}$-module M on X let $\Xi_x(M)$ be the set of all submodules $M_\xi \subset M$ such that M/M_ξ is supported at x. This is a subfilter in the ordered set of all submodules of M, so $\Xi_x(M)$ is a topology on M.

[15] To prove that these functors are mutually inverse, notice that for a vector space V all sub-$\mathcal{D}$-modules of $V \otimes \mathcal{O}_X$ are of the form $W \otimes \mathcal{O}_X$, $W \subset V$; this follows from the irreducibility of the $\mathcal{D}$-module $\mathcal{O}_X$.

[16] A direct way to see this: reduce to the case where M^ℓ is coherent; then notice that $\dim \mathrm{Hom}_{\mathcal{D}}(M^\ell, \mathcal{O}_X) < \infty$.

Consider the vector space $h(M)_x$. For every $M_\xi \in \Xi_x(M)$ the sequence $0 \to h(M_\xi)_x \to h(M)_x \to h(M/M_\xi) \to 0$ is exact (since $H^{-1}_{DR}(M/M_\xi) = 0$). Therefore the map $N \mapsto h(N)_x \subset h(M)_x$ is a bijection between the set of $\mathcal{D}$-submodules $N \subset M$ containing M_ξ and that of vector subspaces of $h(M)_x$ containing $h(M_\xi)_x$ (since $\mathcal{D}$-submodules of M/M_ξ are the same as vector subspaces of $h(M/M_\xi)_x$). The subspaces $h(M_\xi)_x$, $M_\xi \in \Xi_x(M)$, form a topology on $h(M)_x$ called the Ξ_x-*topology.* Thus $M_\xi \mapsto h(M_\xi)_x$ is a bijection between $\Xi_x(M)$ and the set of open vector subspaces for the Ξ_x-topology.

Denote by $\hat{h}_x(M)$ the completion of $h(M)_x$ with respect to the Ξ_x-topology, i.e., the projective limit of the system of vector spaces $i^!_x(M/M_\xi) = h(M/M_\xi)_x$, $M_\xi \in \Xi_x(M)$. We consider $\hat{h}_x$ as a functor from $\mathcal{M}(X)$ to the category of complete separated topological vector spaces.[17] For any discrete vector space F one has[18]

$$\mathrm{Hom}_{cont}(\hat{h}_x(M), F) = \mathrm{Hom}(M, i_{x*}F), \tag{2.1.13.1}$$

so the functor $\hat{h}_x$ is right exact.

REMARKS. (i) Let $O_x := \varprojlim \mathcal{O}_x/\mathrm{m}_x^n$ be the formal completion of the local ring at x. Set $M_{O_x} := M \underset{\mathcal{O}_X}{\otimes} O_x$; this is a $\mathcal{D}_{O_x}$-module. Then the above picture depends only on M_{O_x}. Indeed, the map $M_\xi \mapsto M_{\xi O_x}$ is a bijection between $\Xi_x(M)$ and the set of $\mathcal{D}_{O_x}$-submodules of M_{O_x} with the quotients supported at x.[19] So $\Xi_x(M)$ can be understood as the Ξ_x-topology on M_{O_x}. The vector space $h(M_{O_x}) = M \underset{\mathcal{D}_X}{\otimes} O_x$ carries the Ξ_x-topology formed by the subspaces $h(M_{\xi O_x})$. For every $M_\xi \in \Xi_x(M)$ one has $M/M_\xi = M_{O_x}/M_{\xi O_x}$; hence $h(M/M_\xi) = h(M_{O_x}/M_{\xi O_x}) = h(M_{O_x})/h(M_{\xi O_x})$. Therefore $\hat{h}_x(M)$ is the completion of $h(M_{O_x})$ with respect to the Ξ_x-topology.

(ii) If M is a coherent $\mathcal{D}$-module, then $\hat{h}_x(M)$ is a profinite-dimensional vector space. Indeed, every M/M_ξ is a coherent $\mathcal{D}$-module supported at x; hence $i^!_x(M/M_\xi)$ is a finite-dimensional vector space.

We refer to any topology $\Xi^?_x(M)$ weaker than $\Xi_x(M)$ as a *topology at x on M*. As above, it amounts to a topology on $h(M)_x$ which is weaker than the Ξ_x-topology. The corresponding completion $\hat{h}^?_x(M)$ equals $\varprojlim i^!_x(M/M_\xi)$, $M_\xi \in \Xi^?_x(M)$. As in Remark (i) we can consider $\Xi^?_x(M)$ as a topology on M_{O_x} and $\hat{h}^?_x(M)$ as the completion of $h(M_{O_x})$.

EXAMPLES. (i) Assume that M is an induced $\mathcal{D}$-module, $M = B_{\mathcal{D}}$ (see 2.1.8), so $h(M) = B$. Denote by $\Xi_x(B)$ the topology on B formed by all quasi-coherent $\mathcal{O}$-submodules $P \subset B$ such that B/P is supported set-theoretically at x. Then $\Xi_x(M)$ is generated by submodules $P_{\mathcal{D}}$, $P \in \Xi_x(B)$.[20] Therefore $\hat{h}_x(M)$ is the completion of B with respect to the topology $\Xi_x(B)$. If B is $\mathcal{O}$-coherent, then this is just the formal completion of B at x. We will prove in 2.1.17 that for arbitrary B the $\Xi_x(B)$-completion of B equals $B \otimes O_x$.

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[17]By "topological vector space" we always mean the vector space equipped with a *linear* topology (there is a base of neighborhoods of 0 formed by linear subspaces).

[18]We would denote $\hat{h}_x(M)$ by $i^*_x M$ but, unfortunately, the notation was already used in 2.1.2.

[19]Since the $\mathcal{O}$-modules supported at x are the same as the O_x-modules supported at x.

[20]For $M_\xi \in \Xi_x(M)$ set $P := M_\xi \cap B$. Then $P \in \Xi_x(B)$ and $M_\xi \supset P_{\mathcal{D}}$.

(ii) Assume that M is any coherent $\mathcal{D}$-module. To compute $\hat{h}_x(M)$, one can represent M as a cokernel of a morphism $\varphi : \mathcal{D}_X^m \to \mathcal{D}_X^n$. We have seen that $\hat{h}_x(\mathcal{D}_X) = O_x$, so $\hat{h}_x(M)$ is the cokernel of the corresponding matrix of differential operators[21] $\varphi : O_x^m \to O_x^n$.

2.1.14. Lemma. *If the restriction of M to $U_x := X \smallsetminus \{x\}$ is a coherent $\mathcal{D}$-module, then the Ξ_x-topology on $h(M_{O_x})$[22] is complete and separated; i.e.,*

$$h(M_{O_x}) \xrightarrow{\sim} \hat{h}_x(M). \tag{2.1.14.1}$$

Proof. We want to show that the morphism of vector spaces $h(M_{O_x}) \to \hat{h}_x(M)$ is an isomorphism. Our M is an extension of a $\mathcal{D}$-module M' supported at x by a coherent $\mathcal{D}$-module M''. We have already know the statement for M' (see Remark (i) in 2.1.13) and M'' (see Example (ii) of 2.1.13). Since the sequences $0 \to h(M''_{O_x}) \to h(M_{O_x}) \to h(M'_{O_x}) \to 0$ and $0 \to \hat{h}_x(M'') \to \hat{h}_x(M) \to \hat{h}_x(M') \to 0$ are exact, we are done. □

Remarks. (i) If M is a $\mathcal{D}$-module as in 2.1.14, then $\hat{h}_x(M)$ admits an open profinite-dimensional subspace.[23] Such topological vector spaces are discussed in 2.7.9 under the name of *Tate vector spaces*.

(ii) If M is an arbitrary $\mathcal{D}$-module, then the canonical map $\phi_M : h(M_{O_x}) \to \hat{h}_x(M)$ need not be an isomorphism. Here is an example. Assume that $X = \operatorname{Spec} R$ is an affine curve, so $R = H^0_{DR}(X, \mathcal{D}_X) = \operatorname{Ext}^1_{\mathcal{M}(X)}(\omega_X, \mathcal{D}_X)$. Let M be the universal extension of $R \otimes \omega_X$ by $\mathcal{D}_X$. Then $M \underset{\mathcal{D}_X}{\otimes} O_x = O_x/R \neq 0$ and $\hat{h}_x(M) = 0$ is the completion of O_x/R with respect to the quotient topology; i.e., $\hat{h}_x(M) = 0$.

However ϕ_M is always an isomorphism if M is either an induced $\mathcal{D}$-module (see Example (i) of 2.1.13 and see 2.1.17) or isomorphic to a direct sum of coherent $\mathcal{D}$-modules. In fact, the argument used in the proof of the above lemma shows that it suffices to assume that the restriction of M to U_x satisfies either of these conditions.

2.1.15. Remark. Sometimes it is convenient to consider a finer object than a plain topological vector space $\hat{h}_x(M)$. Namely, consider the directed set of all $\mathcal{D}$-coherent submodules $M_\alpha \subset M$ and set $\tilde{h}_x(M) := \varinjlim \hat{h}_x(M_\alpha) = \varinjlim M_\alpha \underset{\mathcal{D}_X}{\otimes} O_x$. Thus $\tilde{h}_x(M)$ equals $M \underset{\mathcal{D}_X}{\otimes} O_x$ as a plain vector space, but we consider it as an ind-object of the category of profinite-dimensional vector spaces. Recall that such an animal is the same as a left exact contravariant functor on the category of profinite-dimensional vector spaces with values in vector spaces. So $\tilde{h}_x(M)$ "represents" the functor $F \mapsto \varinjlim \operatorname{Hom}(F, \hat{h}_x(M_\alpha))$, F a profinite-dimensional vector space. Notice that the dual object $\tilde{h}_x(M)^*$ – which is a pro-object of the category of vector spaces – equals $\varprojlim \operatorname{Hom}(M_\alpha, i_{x*}k)$.

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[21]We use the fact that a morphism of profinite-dimensional vector spaces has closed image.

[22]See Remark (i) in 2.1.13.

[23]Namely, the image of $\hat{h}_x(M_\xi)$ where $M_\xi \in \Xi_x(M)$ is a coherent $\mathcal{D}_X$-module (as explained in 2.1.13, $\hat{h}_x(M_\xi)$ is profinite-dimensional).

2.1.16. Moving the point. We will also need a version of 2.1.13 in the case where x depends on parameters. So let Y be a scheme which we assume to be quasi-compact and quasi-separated,[24] $x \in X(Y)$, $i_x : Y \hookrightarrow X \times Y$ the graph of x. Denote by $\mathcal{M}(X \times Y/Y)$ the category of right $\mathcal{D}_{X\times Y/Y} := \mathcal{D}_X \boxtimes \mathcal{O}_Y$-modules which are quasi-coherent as $\mathcal{O}_{X\times Y}$-modules. Let $\mathcal{M}(X \times Y/Y)_x$ be the full subcategory of $\mathcal{D}_{X\times Y/Y}$-modules supported (set-theoretically) at the graph of x. A version of Kashiwara's lemma (see 2.1.3) says that the functor $i_x^!$ identifies $\mathcal{M}(X \times Y/Y)_x$ with the category of quasi-coherent $\mathcal{O}_Y$-modules.

For $M \in \mathcal{M}(X \times Y/Y)$ we denote by $\Xi_x(M)$ the set of submodules $M_\xi \subset M$ such that $M/M_\xi \in \mathcal{M}(X \times Y/Y)_x$. This is a subfilter in the ordered set of all submodules of M. We have a $\Xi_x(M)$-projective system of quasi-coherent $\mathcal{O}_Y$-modules $i_x^!(M/M_\xi)$ connected by surjective maps.

For $N \in \mathcal{M}(X \times Y/Y)$ set $h(N) := N \underset{\mathcal{D}_{X\times Y/Y}}{\otimes} \mathcal{O}_{X\times Y}$; this is a $p_Y^{\cdot}\mathcal{O}_Y$-module.[25] Since $i_x^!(M/M_\xi) = i_x^{\cdot} h(M/M_\xi)$, the projective limit of $i_x^!(M/M_\xi)$ equals the completion of $i_x^{\cdot} h(M)$ with respect to the topology defined by the images of the $i_x^{\cdot} h(M_\xi)$. We denote it by $\hat{h}_x(M)$.

If M is an induced module, $M = B \otimes \mathcal{D}_{X\times Y/Y}$ where B is an $\mathcal{O}_{X\times Y}$-module, then $h(M) = B$ and $\hat{h}_x(M)$ is the completion of B with respect to the topology formed by all quasi-coherent $\mathcal{O}$-submodules $P \subset B$ which coincide with B outside of (the graph of) x. In particular, if B is $\mathcal{O}$-coherent, then $\hat{h}_x(M)$ is the formal completion of B at x.

As in 2.1.14 for a coherent $\mathcal{D}_{X\times Y/Y}$-module M one identifies $\hat{h}_x(M)$ with the $\mathcal{D}_{X\times Y/Y}$-tensor product of M and the formal completion of $\mathcal{O}_{X\times Y}$ at x.

Remarks. (i) The above construction is compatible with the base change by arbitrary morphisms $f : Y' \to Y$. Namely, the evident morphism of pro-$\mathcal{O}_{Y'}$-modules $\hat{h}_{xf}((f \times id_X)^* M) \to f^* \hat{h}_x(M)$ is an isomorphism.

(ii) If Y is a smooth variety and M is a right $\mathcal{D}_{X\times Y}$-module, then the differential operators on Y act on M in a continuous way with respect to the Ξ_x-topology. Therefore $\hat{h}_x(M)$ is a sheaf of topological right $\mathcal{D}_Y$-modules. Similarly, if $\mathcal{D}_Y$ acts on M from the left, then $\hat{h}_x(M)$ is a sheaf of topological left $\mathcal{D}_Y$-modules.

(iii) If F is a $\mathcal{D}$-module on Y and $\varphi : \hat{h}_x(M) \to F$ is a continuous $\mathcal{D}_Y$-morphism, then it vanishes on the image of $\hat{h}_x(M_\xi)$ where $M_\xi \in \Xi_x(M)$ is a sub-$\mathcal{D}_{X\times Y}$-module of M. In particular, assume that $Y = X$, $x = id_X$, so $i_x = \Delta$, and $M = L \boxtimes \mathcal{O}_X$ for a $\mathcal{D}$-module L on X. Then every morphism φ as above is trivial (if $\dim X \neq 0$).

2.1.17. *I*-topology. In this section we prove a result in commutative algebra we referred to in Example (i) of 2.1.13 and Remark (ii) in 2.1.14. It will not be used in the rest of this work.

Let A be a noetherian ring, $I \subset A$ an ideal such that A is complete in the I-adic topology. Let C be the category of A-modules and $C_0 \subset C$ the full subcategory of A-modules M such that each element of M is annihilated by some power of I. The *I-topology* on an A-module M is the weakest topology such that every A-linear map from M to a module from C_0 equipped with the discrete topology

[24] The assumption is needed to assure that topological $\mathcal{O}_Y$-modules have a local nature, i.e., pro-objects in the category of quasi-coherent $\mathcal{O}$-modules form a sheaf of categories on $Y_{\mathcal{Z}ar}$, and our constructions are compatible with the localization of Y.

[25] Here $p_Y : X \times Y \to Y$ is the projection.

is continuous. The set B_M of all submodules $\tilde{M} \subset M$ such that $M/\tilde{M} \in C_0$ is a base of neighborhoods of 0 for the I-topology. If M is finitely generated, then the I-topology coincides with the I-adic topology.

THEOREM. (a) *If N is a submodule of an A-module M then the I-topology on M induces the I-topology on N.*

(b) *If A has finite Krull dimension then every A-module M is complete and separated for the I-topology.*

Proof. To prove (a), we have to show that for every $\tilde{N} \in B_N$ there is an $\tilde{M} \in B_M$ such that $\tilde{M} \cap N = \tilde{N}$. Let P be the set of pairs $(L, \tilde{L})$ where L is a submodule of M and $\tilde{L} \in B_L$. Equip P with the following ordering: $(L, \tilde{L}) \le (L', \tilde{L}')$ if $L \subset L'$ and $\tilde{L} = L \cap \tilde{L}'$. By Zorn's lemma there is a maximal $(L, \tilde{L}) \in P$ such that $(L, \tilde{L}) \ge (N, \tilde{N})$. It remains to show that $L = M$. Indeed, if $x \in M$, $x \notin L$, then one can construct $(L', \tilde{L}') > (L, \tilde{L})$ by putting $L' := L + Ax$, $\tilde{L}' := L + U$ where $U \subset Ax$ is an open submodule of Ax such that $L \cap U = \tilde{L} \cap Ax$ (U exists because the I-topology on $L \cap Ax$ is induced by the I-topology on Ax).

Using (a) we reduce the proof of separatedness of M to the well-known case where M is finitely generated (A need not have finite Krull dimension for this property).

Denote by $\hat{M}$ the completion of M with respect to the I-topology. Let $C^? \subset C$ be the full subcategory of A-modules M such that the map $M \to \hat{M}$ is bijective. We will show that $C^? = C$ if A has finite Krull dimension.

LEMMA. *$C^?$ has the following properties:*

(i) *Finitely generated modules belong to $C^?$.*

(ii) *Suppose that the sequence $0 \to M' \to M'' \to M \to 0$ is exact and $M \in C^?$. Then $M' \in C^? \Leftrightarrow M'' \in C^?$.*

(iii) *If $M_j \in C^?$, $j \in J$, then $\bigoplus_j M_j \in C^?$.*

(iv) *Suppose $M = \bigcup_i M_i$, $i = 1, 2, \dots$, $M_i \subset M_{i+1}$. If $M_i \in C^?$ for all i, then $M \in C^?$.*

Proof of Lemma. (i) is well known.

(ii) is a corollary of the left exactness of the functor $M \mapsto \hat{M}$, which is proved as follows. If the sequence $0 \to M_1 \to M_2 \to M_3$ is exact, then $\operatorname{Ker}(\hat{M}_2 \to \hat{M}_3)$ is the closure of M_1 in $\hat{M}_2$, i.e., the completion of M_1 with respect to the topology induced from the I-topology on M_2. We have proved that the induced topology is the I-topology on M_1, so $\operatorname{Ker}(\hat{M}_2 \to \hat{M}_3) = \hat{M}_1$.

To prove (iii) it suffices to show that if $M = \bigoplus_j M_j$, then the image of the natural embedding $f : \hat{M} \to \prod_j \hat{M}_j$ equals $\bigoplus_j \hat{M}_j$. We only have to prove that $\operatorname{Im} f \subset \bigoplus_j \hat{M}_j$. Assume that $x = (x_j) \in \prod_j \hat{M}_j$ and $J' := \{j \in J | x_j \ne 0\}$ is infinite. Choose open submodules $U_j \subset M_j$ so that for every $j \in J'$ the image of x_j in M_j/U_j is non-zero. Then the image of x in $\prod_j M_j/U_j$ does not belong to $\bigoplus_j M_j/U_j$, so $x \notin \operatorname{Im} f$.

To prove (iv), it suffices to show that every $x \in \hat{M}$ belongs to the closure of $M_i \subset \hat{M}$ for some i (this closure equals $\hat{M}_i = M_i$). If this is not true, then for every i there is an open submodule $U_i \subset M$ such that the image of x in $M/(M_i + U_i)$ is non-zero. We can assume that $U_1 \supset U_2 \supset \cdots$. The submodule $U := \bigcup_i (M_i \cap U_i) \subset M$ is open and $U + M_i \subset U_i + M_i$. So for every i the image of x in $M/(M_i + U)$ is non-zero; i.e., the image of x in M/U does not belong to the image of M_i in M/U. This is impossible because $M = \bigcup_i M_i$. $\square$

We assume now that A has finite Krull dimension. It remains to show that if a full subcategory $C^? \subset C$ satisfies properties (i)–(iv) from the lemma, then $C^? = C$. Let $C_i \subset C$ be the full subcategory consisting of modules M such that $M_{\mathfrak{p}} = 0$ for every prime ideal $\mathfrak{p} \subset A$ with $\dim A/\mathfrak{p} > i$. We will prove by induction that $C_i \subset C^?$. Suppose we know that $C_{i-1} \subset C^?$. Let $M \in C_i$. Denote by P_i the set of all primes $\mathfrak{p} \subset A$ such that $\dim A/\mathfrak{p} = i$. The map $M \to \prod_{\mathfrak{p}\in P_i} M_{\mathfrak{p}}$ factors through $\sum_{\mathfrak{p}\in P_i} M_{\mathfrak{p}}$, and the kernel and cokernel of the morphism $M \to \sum_{\mathfrak{p}\in P_i} M_{\mathfrak{p}}$ belong to C_{i-1}. By (ii),(iii), and the induction assumption, to prove that $M \in C^?$ it suffices to show that $M_{\mathfrak{p}} \in C^?$ for each $\mathfrak{p} \in P_i$. Let $\mathfrak{p} \in P_i$. Our $M_{\mathfrak{p}}$ is an $A_{\mathfrak{p}}$-module such that each element of $M_{\mathfrak{p}}$ is annihilated by a power of $\mathfrak{p}A_{\mathfrak{p}}$. By (ii) and (iv) we are reduced to proving that vector spaces over $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ belong to $C^?$. By (iii) it suffices to show that $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \in C^?$. The morphism $A/\mathfrak{p} \to A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ is injective and its cokernel belongs to $C_{i-1} \subset C^?$. So $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \in C^?$ by (i) and (ii). $\square$

Remark. If A is a 1-dimensional local domain, then the proof of statement (b) of the above theorem is very simple. In this case every A-module M has a free submodule N such that $M/N \in C_0$, so we can assume that M is free. Then apply statement (iii) of the lemma.

2.2. The compound tensor structure

In this section we describe a canonical augmented compound tensor structure on the category of $\mathcal{D}_X$-modules $\mathcal{M}(X)$ where X is a smooth scheme. We define the tensor structure $\otimes^!$ in 2.2.1, the pseudo-tensor structure P^* in 2.2.3, the compound tensor product maps $\otimes^I_{S,T}$ in 2.2.6, and the augmentation functor h in 2.2.7; h is non-degenerate and reliable (see 2.2.8). We show in 2.2.9 that the translation functor $M \to M[-\dim X]$ admits a canonical homotopy pseudo-tensor extension $C\mathcal{M}(X)^! \to C\mathcal{M}(X)^*$. In 2.2.10 it is shown that DR is naturally a homotopy pseudo-tensor functor; another construction, that works in case $\dim X = 1$, is presented in 2.2.11. We explain in 2.2.12 that our compound tensor structure can be encoded into a compound $\mathcal{D}_X$-operad, discuss duality and inner Hom in 2.2.15–2.2.17, explain how $*$ operations can be seen on the level of polylinear operations between the h sheaves in 2.2.18, and establish their continuity properties in 2.2.19.

2.2.1. The ! tensor structure. The category $\mathcal{M}^\ell$ is a tensor category (see 2.1.1). We use the standard equivalence $\mathcal{M}(X) \xrightarrow{\sim} \mathcal{M}^\ell(X)$, $M \mapsto M^\ell := M\omega_X^{-1}$, to define the tensor product $\otimes^!$ on $\mathcal{M}(X)$, so we have $M_1 \otimes^! M_2 = M_1^\ell \underset{\mathcal{O}_X}{\otimes} M_2$. The object ω_X is a unit object for $\otimes^!$.

2.2.2. Consider the derived tensor product $\overset{L}{\otimes}{}^!$ on $D\mathcal{M}(X)$.[26] According to (2.1.2.3) for $M_i \in D\mathcal{M}(X)$ one has

$$\overset{L}{\otimes}{}^!_I M_i = \Delta^{(I)!}(\boxtimes(M_i[\dim X]))[-\dim X] = \Delta^{(I)!}(\boxtimes M_i) \otimes \lambda_I^{\otimes \dim X}[(|I|-1)\dim X]$$

[26]Notice that $\overset{L}{\otimes}{}^!$ is identified with the tensor product on $D\mathcal{M}^\ell(X)$ by the above "standard" t-exact equivalence, and *not* by the canonical equivalence (2.1.1.1).

where $\lambda_I := \det(k^I)[|I|] = (k[1])^{\otimes I}[-|I|]$ (the last equality comes since $k[\dim X]^{\otimes I} = \lambda_I^{\otimes \dim X}[|I| \dim X]$). Therefore for $M_i \in \mathcal{M}(X)$ there is a canonical isomorphism

$$\otimes^!_I M_i = H^{(|I|-1)\dim X} \Delta^{(I)!}(\boxtimes M_i) \otimes \lambda_I^{\otimes \dim X}. \tag{2.2.2.1}$$

EXAMPLE. Assume that $M_i = \omega_X$. One has $\omega_X^{\otimes^! I} = \omega_X$ and $\omega_X^{\boxtimes I} = \omega_{X^I} \otimes \lambda_I^{\otimes \dim X}$, so (2.2.2.1) comes from the identification $\omega_X = H^{(|I|-1)\dim X}\Delta^{(I)!}(\omega_{X^I})$.

2.2.3. The $*$ pseudo-tensor structure. For an I-family L_i of $\mathcal{D}$-modules, $I \in \mathcal{S}$, and a $\mathcal{D}$-module M set

$$P^*_I(\{L_i\}, M) := \operatorname{Hom}(\boxtimes L_i, \Delta^{(I)}_* M). \tag{2.2.3.1}$$

The elements of P^*_I are called $*$ *I-operations*. They have étale local nature: we have the sheaf $\underline{P}^*_I(\{L_i\}, M)$, $U \mapsto P^*_I(\{L_i|_U\}, M|_U)$, on the étale topology of X. For a surjective map $\pi\colon J \twoheadrightarrow I$ the composition map

$$P^*_I(\{L_i\}, M) \otimes (\otimes_I P^*_{J_i}(\{K_j\}, L_i)) \longrightarrow P^*_J(\{K_j\}, M) \tag{2.2.3.2}$$

sends $\varphi \otimes (\otimes \psi_i)$ to $\varphi(\psi_i)$ defined as the composition

$$\boxtimes_J K_j \xrightarrow{\boxtimes \psi_i} \boxtimes_I \Delta^{(J_i)}_* L_i = \Delta^{(\pi)}_* (\boxtimes_I L_i) \xrightarrow{\Delta^{(\pi)}_*(\varphi)} \Delta^{(\pi)}_* \Delta^{(I)}_* M = \Delta^{(J)}_* M$$

where $\Delta^{(\pi)} = \prod_I \Delta^{(J_i)}\colon X^I \hookrightarrow X^J$. The composition is associative, so P^*_I define on $\mathcal{M}(X)$ an abelian pseudo-tensor structure. We denote this pseudo-tensor category by $\mathcal{M}(X)^*$. The pseudo-tensor categories $\mathcal{M}(U)^*$, $U \in X_{\text{ét}}$, form a sheaf of pseudo-tensor categories on the étale topology of X.

For $\varphi \in P^*_I(\{L_i\}, M)$ we will sometimes denote its image considered (by Kashiwara's lemma) as a submodule of M by $\varphi(\{L_i\})$.

REMARKS. (i) The sheaf-theoretic restriction to the diagonal identifies our $*$ I-operations with $\mathcal{D}_X^{\otimes I}$-module morphisms $\underset{k}{\otimes} L_i \to \Delta^{(I)\cdot}\Delta^{(I)}_* M$.

(ii) One has $P^*_I(\{L_i\}, M) = \operatorname{Hom}(\otimes^* L_i, M)$ where $\otimes^* L_i := \Delta^{(I)*}(\boxtimes L_i)$ is the projective system of all the $\mathcal{D}_{X^I}$-module quotients of $\boxtimes L_i$ supported on the diagonal, considered as a projective system of $\mathcal{D}_X$-modules via the $\Delta^{(I)!}$-equivalence.

(iii) The category $\mathcal{M}_{\mathcal{O}}(X)$ of quasi-coherent $\mathcal{O}_X$-modules carries a natural $\mathcal{M}(X)^*$-action (see 1.2.11). Namely, the vector space of operations $P^*_{\tilde{I}}(\{M_i, F\}, G)$, where F, G are (quasi-coherent) $\mathcal{O}_X$-modules and M_i, $i \in I$, are $\mathcal{D}_X$-modules, is defined as $P^*_{\tilde{I}}(\{M_i, F\}, G) := \operatorname{Hom}_{\mathcal{D}_X^{\boxtimes I} \boxtimes \mathcal{O}_X}((\boxtimes M_i) \boxtimes F, \Delta^{(\tilde{I})}_* G)$. Here $\Delta^{(\tilde{I})}_* G := (\Delta^{(\tilde{I})}_{\cdot} G) \underset{\mathcal{O}_{X^{\tilde{I}}}}{\otimes} (\mathcal{D}_X^{\boxtimes I} \boxtimes \mathcal{O}_X)$. The composition of these operations with usual $*$ operations between $\mathcal{D}$-modules and morphisms of $\mathcal{O}$-modules (as needed in 1.2.11) is clear.

Consider the induction functor $\mathcal{M}_{\mathcal{O}}(X) \to \mathcal{M}(X)$, $F \mapsto F_{\mathcal{D}}$ (see 2.1.8). One has an obvious natural map $P^*_{\tilde{I}}(\{M_i, F\}, G) \hookrightarrow P^*_{\tilde{I}}(\{M_i, F_{\mathcal{D}}\}, G_{\mathcal{D}})$. On the other hand, if P, Q are $\mathcal{D}$-modules, then there is an obvious map $P^*_{\tilde{I}}(\{M_i, P\}, Q) \hookrightarrow P^*_{\tilde{I}}(\{M_i, P_{\mathcal{O}}\}, Q_{\mathcal{O}})$, where $P_{\mathcal{O}}$, $Q_{\mathcal{O}}$ are P, Q considered as $\mathcal{O}$-modules. Therefore both induction and restriction functors $\mathcal{M}_{\mathcal{O}}(X) \rightleftarrows \mathcal{M}(X)$ are compatible with the $\mathcal{M}(X)^*$-actions. Here the $\mathcal{M}(X)^*$ acts on $\mathcal{M}(X)$ in the standard way; see 1.2.12(i).

2.2.4. EXAMPLES. (i) Let us describe the $*$ pseudo-tensor structure on the subcategory of induced modules (see 2.1.8). For $\mathcal{O}$-modules $\{F_i\}_{i\in I}, G$ denote by $\mathrm{Diff}_I(\{F_i\},G)\subset \mathrm{Hom}_k(\underset{k}{\otimes} F_i, G)$ the subspace of differential I-operations (those φ's for which the maps $F_i\to G$, $f_i\mapsto \varphi(f_i\otimes(\underset{j\neq i}{\otimes} f_j))$, are differential operators for any i and any fixed local sections $f_j\in F_j$, $j\neq i$). Equivalently, $\mathrm{Diff}_I(\{F_i\},G) = \mathrm{Diff}(\boxtimes F_i, \Delta_\cdot^{(I)}G)$.

The differential operations are stable with respect to composition so they define a pseudo-tensor structure on the category $\mathcal{D}iff(X)$; denote this pseudo-tensor category by $\mathcal{D}iff(X)^*$. One has $P_I^*(\{F_{i\mathcal{D}}\},G_{\mathcal{D}}) = \mathrm{Hom}((\boxtimes F_i)_{\mathcal{D}},(\Delta_\cdot G)_{\mathcal{D}}) = \mathrm{Diff}_I(\{F_i\},G)$ (see 2.1.8, in particular Example (iii) there); i.e., $\cdot_{\mathcal{D}}$ extends to a fully faithful pseudo-tensor functor

$$\cdot_{\mathcal{D}}\colon \mathcal{D}iff(X)^*\hookrightarrow \mathcal{M}(X)^*. \tag{2.2.4.1}$$

(ii) Assume that $|I|\geq 2$ and one of the $\mathcal{D}$-modules L_i is lisse (e.g., equal to ω_X). Then $P_I^*(\{L_i\},M)=0$.

(iii) The de Rham DG algebra $DR := (\mathcal{O}_X\to\Omega_X^1\to\cdots)$ is a commutative DG algebra in $\mathcal{D}iff(X)^*$, so the corresponding induced complex $DR_{\mathcal{D}}$ is a commutative DG algebra in $\mathcal{M}(X)^*$. Notice that $DR_{\mathcal{D}}$ is naturally a resolution of $\omega_X[-\dim X]$ (see, e.g., (2.1.9.1)), so $\omega_X[-\dim X]$ is naturally a homotopy commutative DG algebra in $\mathcal{M}(X)^*$.

EXERCISE. Show that the cone of the product morphism $DR_{\mathcal{D}}\boxtimes DR_{\mathcal{D}}\to \Delta_* DR_{\mathcal{D}}$ is naturally quasi-isomorphic to $j_!\omega_U[-2\dim X+1]$ where $j_!$ is the usual functor on the derived category of holonomic $\mathcal{D}$-modules.

2.2.5. Let $i\colon X\hookrightarrow Y$ be an embedding of smooth schemes. The left exact functors $H^0 i_*$, $H^0 i^!$ between $\mathcal{M}(X)$ and $\mathcal{M}(Y)$ act on P_I^* in the obvious way; i.e., they are pseudo-tensor functors. If Y is a closed subscheme, then $H^0 i^!$ is right adjoint to $i_* = H^0 i_*$ as pseudo-tensor functors (which means that $P_I^*(\{i_* N_i\},M) = P_I^*(\{N_i\},H^0 i^! M)$ for any $N_i\in\mathcal{M}(X)$, $M\in\mathcal{M}(Y)$), so i_* is a fully faithful embedding of pseudo-tensor categories.

2.2.6. The tensor product maps. We have to define the canonical maps (see 1.3.12)

$$\otimes_{S,T}^I\colon\ \otimes_S P_{I_s}^*(\{L_i\},M_s)\to P_T^*(\{\otimes_{I_t}^! L_i\},\otimes_S^! M_s). \tag{2.2.6.1}$$

According to the first formula in 2.2.2, one has canonical isomorphisms in $D\mathcal{M}(X^T)$ (here $\alpha := k[-\dim X]$, so $\alpha^{-1} = k[\dim X]$):

$$\begin{aligned}\Delta^{(\pi_T)!}(\boxtimes_I L_i) &= \boxtimes_T(\Delta^{(I_t)!}(\boxtimes_{I_t} L_i)) = \boxtimes_T((\overset{L}{\otimes}{}^!_{I_t} L_i)\otimes\alpha^{-1}\otimes\alpha^{\otimes I_t})\\ &= (\boxtimes_T(\overset{L}{\otimes}{}^!_{I_t} L_i))\otimes(\alpha^{-1})^{\otimes T}\otimes\alpha^{\otimes I},\\ \Delta^{(\pi_T)!}(\boxtimes_S(\Delta_*^{(I_s)} M_s)) &= \Delta^{(\pi_T)!}\Delta_*^{(\pi_S)}(\boxtimes_S M_s) = \Delta_*^{(T)}\Delta^{(S)!}(\boxtimes_S M_s)\\ &= \Delta_*^{(T)}\left(\overset{L}{\otimes}{}^!_S M_s\right)\otimes\alpha^{-1}\otimes\alpha^{\otimes S}.\end{aligned}$$

In the second line we used the base change canonical isomorphism $\Delta^{(\pi_T)!}\Delta_*^{(\pi_S)} = \Delta_*^{(T)}\Delta^{(S)!}$ (recall that $X^S, X^T\subset X^I$ are transveral and $X^S\cap X^T = X$; see 1.3.2(i)).

Consider the diagonal embeddings $\Delta^{(S)}\colon k \hookrightarrow k^S$, $\Delta^{(\pi_T)}\colon k^T \hookrightarrow k^I$. The embedding $\Delta^{(\pi_S)}\colon k^S \to k^I$ induces an isomorphism between the quotients $k^S/k \xrightarrow{\sim} k^I/k^T$. One has $\alpha^{-1}\otimes\alpha^{\otimes S} = \det((k^S/k)[-\dim X])$ and $(\alpha^{-1})^{\otimes T}\otimes\alpha^{\otimes I} = \det((k^I/k)^T[-\dim X])$, so we get the identification $\alpha^{-1}\otimes\alpha^{\otimes S} = (\alpha^{-1})^{\otimes T}\otimes\alpha^{\otimes I}$. Now $\otimes^I_{S,T}$ is the composition

$$\otimes_S P^*_{I_s}(\{L_i\},M_s) \to \mathrm{Hom}(\boxtimes_I L_i, \boxtimes_S \Delta^{(I_s)}_* M_s) \to \mathrm{Hom}(\boxtimes_T(\otimes^!_{I_t}L_i), \Delta^{(T)}_*(\otimes^!_S M_s))$$

where the first arrow is $\boxtimes_S$ and the second one is $H^{(|I|-|T|)\dim X}\Delta^{(\pi_T)!}$.

The commutativity of diagrams (i) and (ii) in 1.3.12 is clear. Therefore we have defined the compound tensor category structure on $\mathcal{M}(X)$. This compound tensor category $\mathcal{M}(X)^{*!}$ is abelian (see 1.3.14) with a strong unit object ω_X (see 1.3.16, 1.3.17, 2.2.4(ii)).

The above constructions are compatible with étale localization, so $\mathcal{M}(U)^{*!}$, $U \in X_{\text{ét}}$, form a sheaf of compound tensor categories $\mathcal{M}(X_{\text{ét}})^{*!}$ over $X_{\text{ét}}$.

2.2.7. The augmentation functor. Let us show that the de Rham functor h (see 2.1.6) is a non-degenerate $*$-augmentation functor on $\mathcal{M}(X_{\text{ét}})^{*!}$ (see 1.2.5, 1.3.10). We have to define for a finite set I of order ≥ 2 and $i_0 \in I$ a canonical morphism of sheaves[27]

$$h_{I,i_0}\colon P^*_I(\{L_i\},M)\otimes h(L_{i_0}) \to P^*_{I\smallsetminus\{i_0\}}(\{L_i\},M), \quad \varphi\otimes\bar{l} \mapsto \varphi_{\bar{l}}. \tag{2.2.7.1}$$

Namely, for $\varphi \in P^*_I(\{L_i\},M)$ and $l_i \in L_i$ one has

$$\varphi_{\bar{l}_{i_0}}\Big(\underset{I\smallsetminus\{i_0\}}{\boxtimes} l_i\Big) = Tr_{i_0}\varphi\Big(\underset{I}{\boxtimes} l_i\Big). \tag{2.2.7.2}$$

Here $Tr_{i_0}\colon pr_{I,i_0\cdot}\Delta^{(I)}_* M \to \Delta^{(I\smallsetminus\{i_0\})}_* M$ is the trace morphism for the projection $pr_{I,i_0}\colon X^I \to X^{I\smallsetminus\{i_0\}}$. It is easy to see that h_{I,i_0} is well-defined by (2.2.7.2). Compatibilities (i), (ii) in 1.2.5 and 1.3.10 are straightforward, so h makes $\mathcal{M}(X)^{*!}$ an augmented compound tensor category.

Let $Sh(X)^{\otimes}$ be the tensor category of sheaves of k vector spaces. According to 1.2.7, h extends to a pseudo-tensor functor

$$h:\mathcal{M}(X)^* \to Sh(X)^{\otimes}. \tag{2.2.7.3}$$

The maps $h_I\colon P^*_I(\{L_i\},M) \to \mathrm{Hom}(\otimes_I h(L_i), h(M))$ are $h_I(\varphi)(\otimes\bar{l}_i) = \overline{\varphi(\boxtimes l_i)}$; here we consider $\overline{\varphi(\boxtimes l_i)}$ as a section of $\Delta^{(I)\cdot}h(\Delta^{(I)}_* M) = h(M)$ (see 2.1.7).

If $i\colon X \hookrightarrow Y$ is a closed embedding, then $i_*\colon \mathcal{M}(X)^* \hookrightarrow \mathcal{M}(Y)^*$ is compatible with the augmentation functors; i.e., it is an augmented pseudo-tensor functor (see 1.2.8).

Remark. The map $h_I\colon P^*_I(\{L_i\},M) \to \mathrm{Hom}(\otimes_I h(L_i), h(M))$ need not be injective. However it is injective if M is either an induced $\mathcal{D}$-module (e.g., if M is supported at a single point) or a coherent $\mathcal{D}_X$-module without $\mathcal{O}_X$-torsion.[28]

2.2.8. The next proposition shows that these difficulties disappear if we consider $\tilde{h}$ from 1.4.7 instead of h:

[27] Here we consider P^* as a sheaf on the étale topology of X.

[28] *Proof:* In the induced case by left exactness of h we can assume that the L_i are also induced (replace L_i by $L_{i\mathcal{D}} \twoheadrightarrow L_i$ if necessary), and then use 2.2.4(i). In the coherent case use statement (ii) of the lemma from 2.1.6.

PROPOSITION. *Our h is non-degenerate (see 1.2.5)*[29] *and reliable (see 1.4.7).*

Proof. Let us show that h is non-degenerate. Consider the map $\alpha : P_I^*(\{L_i\}, M) \to \mathrm{Hom}(h(L_{i_0}), P^*_{I\smallsetminus\{i_0\}}(\{L_i\}, M))$ coming from (2.2.7.1). We want to show that α is injective. Take a non-zero $\varphi \in P_I^*(\{L_i\}, M)$. Replacing M by the image of φ, we can assume that φ is surjective. To check that $\alpha(\varphi) \neq 0$, we will show that the sum of images of all operations $\varphi_{\bar{l}} \in P^*_{I\smallsetminus\{i_0\}}(\{L_i\}, M)$, $l \in L_{i_0}$, equals M. Take $m \in \Delta_*^{(I\smallsetminus\{i_0\})} M$. Choose $\tilde{m} \in \Delta_*^{(I)} M$ such that $m = Tr_{i_0}\tilde{m}$. Since φ is surjective, one can write $\tilde{m} = \sum\limits_a \varphi(\boxtimes l_i^a)$. Then $m = \sum\limits_a \varphi_{\bar{l}^a_{i_0}}(\boxtimes\limits_{i\neq i_0} l_i^a)$; q.e.d.

Let us show that h is reliable. A commutative unital algebra $R \in \mathcal{M}(X)^!$ is the same as a commutative unital algebra R^ℓ in the tensor category $\mathcal{M}^\ell(X)$, i.e., a quasi-coherent $\mathcal{O}_X$-algebra equipped with an integrable connection along X. We call (see 2.3.1) such R^ℓ a *commutative* $\mathcal{D}_X$*-algebra* and denote the corresponding category by $\mathcal{C}omu^!(X)$. Every $M \in \mathcal{M}(X)$ yields a functor $\tilde{h}(M) : \mathcal{C}omu^!(X) \to Sh(X)$, $R^\ell \mapsto \tilde{h}(M)_{R^\ell} := h(M \otimes R^\ell) = M \underset{\mathcal{D}_X}{\otimes} R^\ell$. We want to show that the map (see 1.4.7) $\tilde{h} : P_I^*(\{M_i\}, N) \to \mathrm{Mor}(\underset{I}{\otimes}\tilde{h}(M_i), \tilde{h}(N))$ is bijective.

Let us construct the inverse map $\kappa : \mathrm{Mor}(\otimes\tilde{h}(M_i), \tilde{h}(N)) \to P_I^*(\{M_i\}, N)$ to $\tilde{h}$. Let $\psi : \otimes\tilde{h}(M_i) \to \tilde{h}(N)$ be a morphism of functors. Let A^ℓ be the commutative $\mathcal{D}_X$-algebra freely generated by I sections a_i, so $A^\ell = \mathrm{Sym}(\bigoplus \mathcal{D}_X \cdot a_i) = \mathrm{Sym}(D^{\ell I})$ where D^ℓ is $\mathcal{D}_X$ considered as a left $\mathcal{D}_X$-module. Notice that $\mathrm{End}\, A^\ell \supset \mathrm{End}(D^{\ell I}) = \mathrm{Mat}_I(\mathcal{D}_X^{opp}) \supset \mathcal{D}_X^{opp\, I}$ (the diagonal matrices), so the morphism $\psi_{A^\ell} : \otimes\tilde{h}(M_i)_{A^\ell} \to \tilde{h}(N)_{A^\ell}$ commutes with the right action of $\mathcal{D}_X^I$.[30] The obvious $\mathbb{Z}^I$-grading on A^ℓ yields a $\mathbb{Z}^I$-grading on $\tilde{h}(?)_{A^\ell}$; since it comes from the action of diagonal matrices $k^I \subset \mathrm{End} A^\ell$, our ψ_{A^ℓ} is compatible with the grading. The component of degree $1^I = (1, \dots, 1)$ of A^ℓ is $D^{\otimes I}$, so the corresponding component of $\tilde{h}(N)_{A^\ell}$ equals $N \underset{\mathcal{D}_X}{\otimes} D^{\otimes I} = \Delta^{(I)\cdot}\Delta_*^{(I)} N$.[31] The component of the same degree of $\otimes\tilde{h}(M_i)_{A^\ell}$ contains $\underset{k}{\otimes} M_i$: we identify $\otimes m_i$ with $\otimes(m_i \otimes a_i)$. Restricting ψ to this subsheaf, we get a morphism $\kappa(\psi) : \underset{k}{\otimes} M_i \to \Delta^{(I)\cdot}\Delta_*^{(I)} N$. Notice that for every $(\partial_i) \in \mathcal{D}_X^I \subset \mathrm{End}\, A^\ell$, the restriction of its action to $\underset{k}{\otimes} M_i \subset \otimes\tilde{h}(M_i)_{A^\ell}$ and $\Delta^{(I)\cdot}\Delta_*^{(I)} N \subset \tilde{h}(N)_{A^\ell}$ coincides with the obvious action of $\otimes\partial_i \in \mathcal{D}_X^{\otimes I}$ coming from the $\mathcal{D}$-module structure on the M_i and N. Since $\kappa(\psi)$ commutes with this action, one has $\kappa(\psi) \in P_I^*(\{M_i\}, N)$ according to Remark (i) in 2.2.3.

It is clear that κ is left inverse to $\tilde{h}$. So it remains to show that κ is injective; i.e., ψ is uniquely determined by $\kappa(\psi)$. Take any commutative $\mathcal{D}_X$-algebra R^ℓ and consider the morphism $\psi_{R^\ell} : \otimes\tilde{h}(M_i)_{R^\ell} \to \tilde{h}(N)_{R^\ell}$. Take a local section $\gamma = \otimes(m_i \otimes r_i) \in \tilde{h}(M_i)_{R^\ell}$; let us compute $\psi_{R^\ell}(\gamma) \in \tilde{h}(N)_{R^\ell}$ in terms of $\kappa(\psi)$. We can assume that the r_i's are global sections of R^ℓ (replacing R^ℓ by $j_*j^*R^\ell$ where $j : U \hookrightarrow X$ is an open subset on which all r_i's are defined).[32] Consider

[29] See 2.2.15 for a more precise statement.

[30] $\mathcal{D}_X^I$ acts as a semigroup with respect to multiplication.

[31] The latter equality comes from (2.1.3.1) for $i = \Delta^{(I)}$ since $\Delta^{(I)*}\mathcal{D}_{X^I} = \Delta^{(I)*}D^{\ell\boxtimes I} = D^{\ell\otimes I}$ by (2.1.2.4). See also 2.2.9 below.

[32] We use the fact that the morphism of $\mathcal{D}_X$-algebras $R^\ell \to j_*j^*R^\ell$ yields a morphism $\tilde{h}(N)_{R^\ell} \to \tilde{h}(N)_{j_*j^*R^\ell}$ which is an isomorphism over U.

the morphism $\nu : A^\ell \to R^\ell$ of $\mathcal{D}_X$-algebras which sends a_i to r_i. Then $\psi_{R^\ell}(\gamma) = \nu\psi_{A^\ell}(\otimes(m_i \otimes a_i)) = \nu\kappa(\psi)(\boxtimes m_i)$. We are done. $\square$

2.2.9. According to Remark in 1.3.15, the tensor category $\mathcal{M}(X)^!$ acts on $\mathcal{M}(X)^*$, so the tensor product extends canonically to a pseudo-tensor functor

$$\otimes : \mathcal{M}(X)^! \otimes \mathcal{M}(X)^* \to \mathcal{M}(X)^*. \tag{2.2.9.1}$$

Therefore, by 1.1.6(vi), any commutative algebra $F \in \mathcal{M}(X)^*$ yields a pseudo-tensor functor $\mathcal{M}(X)^! \to \mathcal{M}(X)^*$, $M \mapsto M \otimes F$.

For example, $DR_{\mathcal{D}} \in C\mathcal{M}(X)^*$ (see 2.2.4(iii)) yields a pseudo-tensor DG functor

$$C\mathcal{M}(X)^! \to C\mathcal{M}(X)^*, \quad M \mapsto M \otimes DR_{\mathcal{D}}, \tag{2.2.9.2}$$

which is a resolution of the functor $M \mapsto M[-\dim X]$.

Similarly, by 1.1.6(vi), any commutative (not necessary unital) algebra $P \in \mathcal{M}(X)^!$ yields a pseudo-tensor functor $\mathcal{M}(X)^* \to \mathcal{M}(X)^*$, $M \mapsto M \otimes P$. This fact can be used as follows.

2.2.10. Enhanced de Rham complexes. The de Rham functor (see 2.1.7) $DR : C\mathcal{M}(X) \to CSh(X)$ is *not* a pseudo-tensor functor, as opposed to h (see (2.2.7.3)). It is naturally a *homotopy* pseudo-tensor functor though. In fact, there is a natural family of mutually homotopically equivalent pseudo-tensor functors which are homotopically equivalent to DR as mere DG functors. Below we suppose that X is quasi-projective.

Let $\epsilon_{\mathcal{P}} : \mathcal{P} \to \mathcal{O}_X$ be any homotopically $\mathcal{D}_X$-flat non-unital commutative$^!$ algebra resolution of $\mathcal{O}_X$. So $\mathcal{P}$ is a commutative DG algebra in the tensor category of left $\mathcal{D}_X$-modules, $\epsilon_{\mathcal{P}}$ a quasi-isomorphism of such algebras, and we assume that $\mathcal{P}$ is homotopically $\mathcal{D}_X$-flat as a mere complex of $\mathcal{D}_X$-modules (see 2.1.1).

EXAMPLE. If X is a curve, then a simplest $\mathcal{P}$ is a two term resolution of $\mathcal{O}_X$ with $\mathcal{P}^0 = \mathrm{Sym}^{>0}\mathcal{D}_X$, $\epsilon_{\mathcal{P}}$ sending $1 \in \mathcal{D}_X \subset \mathcal{P}^0$ to $1 \in \mathcal{O}_X$, and $\mathcal{P}^{-1} := \mathrm{Ker}\epsilon_{\mathcal{P}}$.

REMARK. For an arbitrary X one can take for $\mathcal{P}$ any resolution of $\mathcal{O}_X$ which is isomorphic as a mere graded commutative$^!$ algebra to $\mathrm{Sym}^{>0}T$, where T is an X-locally projective graded $\mathcal{D}_X$-module having degrees ≤ 0. For a quasi-projective X, it can be choosen so that T is isomorphic to a direct sum of $\mathcal{D}$-modules of type $\mathcal{D}_X \otimes \mathcal{L}$ where $\mathcal{L}$ is a line bundle (if needed, one can assume that all the $\mathcal{L}$'s are negative powers of a given ample line bundle).

By (2.2.7.3) and 2.2.9, we have a pseudo-tensor DG functor

$$C\mathcal{M}(X)^* \to CSh(X)^{\otimes}, \quad M \mapsto h(M \otimes \mathcal{P}). \tag{2.2.10.1}$$

The canonical quasi-isomorphisms $h(M \otimes \mathcal{P}) \leftarrow DR(M \otimes \mathcal{P}) \xrightarrow{\epsilon_{\mathcal{P}}} DR(M)$ for $M \in C\mathcal{M}(X)$ make (2.2.10.1) a homotopy version of the de Rham functor.

The rest of the section can be skipped by the reader. Consider the category of all $\mathcal{P} = (\mathcal{P}, \epsilon_{\mathcal{P}})$ as above. It is a tensor category (without unit) in the obvious way.

LEMMA. *Every functor from our category to a groupoid is isomorphic to a trivial one.*

Proof. Denote our functor by $^-$. For two objects $\mathcal{P}$, $\mathcal{P}'$ let $\pi_{\mathcal{P}} : \mathcal{P} \otimes \mathcal{P}' \to \mathcal{P}$ be the morphism $\pi_{\mathcal{P}}(p \otimes p') := \epsilon_{\mathcal{P}'}(p')p$.

(i) Let $\nu, \mu : \mathcal{P}_1 \to \mathcal{P}_2$ be two morphisms and let $\mathcal{P}$ be any object. Set $\nu_{\mathcal{P}} := \nu \otimes id_{\mathcal{P}}$, and the same for $\mu_{\mathcal{P}}$. Then $\bar{\nu} = \bar{\mu}$ if $\bar{\nu_{\mathcal{P}}} = \bar{\mu_{\mathcal{P}}}$. This follows since $\nu\pi_{\mathcal{P}_1} = \pi_{\mathcal{P}_2}\nu_{\mathcal{P}}$, $\mu\pi_{\mathcal{P}_1} = \pi_{\mathcal{P}_2}\mu_{\mathcal{P}}$.

(ii) Take any $\mathcal{P}$. We have the morphisms $\pi_{\mathcal{P}}, \pi'_{\mathcal{P}} : \mathcal{P} \otimes \mathcal{P} \to \mathcal{P}$ where $\pi'_{\mathcal{P}} := \pi_{\mathcal{P}}\sigma$ where σ is the transposition of multiples. These morphisms produce the same arrow in the groupoid. Indeed, by (i) it suffices to check that the images of $(\pi_{\mathcal{P}})_{\mathcal{P}}, (\pi'_{\mathcal{P}})_{\mathcal{P}} : (\mathcal{P} \otimes \mathcal{P}) \otimes \mathcal{P} \to \mathcal{P} \otimes \mathcal{P}$ in the groupoid coincide, which is clear since $\pi'_{\mathcal{P}}(\pi_{\mathcal{P}})_{\mathcal{P}} = \pi'_{\mathcal{P}}(\pi'_{\mathcal{P}})_{\mathcal{P}}$.

(iii) Let $\zeta, \eta : \mathcal{P}_1 \to \mathcal{P}_2$ be any two morphisms. Then $\bar{\zeta} = \bar{\eta}$. This follows from (ii) since $\zeta\pi_{\mathcal{P}_1} = \pi_{\mathcal{P}_2}(\zeta \otimes \eta)$ and $\eta\pi'_{\mathcal{P}_1} = \pi'_{\mathcal{P}_2}(\zeta \otimes \eta)$.

(iv) For any finite collection of objects $\{\mathcal{P}_\alpha\}$ one can find $\mathcal{P}$ such that for every α there is a morphism $\phi_\alpha : \mathcal{P} \to \mathcal{P}_\alpha$. Namely, one can take $\mathcal{P} := \otimes \mathcal{P}_\alpha$ and $\pi_\alpha := id_{\mathcal{P}_\alpha} \otimes (\underset{\alpha' \neq \alpha}{\otimes} \epsilon_{\mathcal{P}_{\alpha'}})$.

(v) To finish the proof, let us show that for a pair of objects $\mathcal{P}_0$, $\mathcal{P}_n$ and a chain of morphisms $\mathcal{P}_0 \xrightarrow{\chi_1} \mathcal{P}'_1 \xleftarrow{\psi_1} \mathcal{P}_1 \xrightarrow{\chi_2} \cdots \xrightarrow{\chi_n} \mathcal{P}'_n \xleftarrow{\psi_n} \mathcal{P}_n$ the composition $\bar{\chi}_n \bar{\psi}_n^{-1} \cdots \bar{\psi}_1^{-1} \bar{\chi}_1 : \bar{\mathcal{P}}_0 \to \bar{\mathcal{P}}_n$ does not depend on the chain. To see this, choose $\phi_i : \mathcal{P} \to \mathcal{P}_i$ as in (iv). Since $\bar{\chi}_i \bar{\phi}_{i-1} = \bar{\psi}_i \bar{\phi}_i$ by (iii), our composition equals $\bar{\phi}_n \bar{\phi}_0^{-1}$, and we are done. □

Remarks. (i) It is clear that one can find $\mathcal{P}$ such that $\mathcal{P}^{>0} = 0$ and each $\mathcal{P}^a$ is $\mathcal{D}_X$-flat. Since the homological dimension of $\mathcal{D}_X$ equals $\dim X$, for every such $\mathcal{P}$ the truncated DG algebra $\mathcal{P}^{-\dim X}/d(\mathcal{P}^{-\dim X-1}) \to \mathcal{P}^{-\dim X+1} \to \cdots \to \mathcal{P}^0$ satisfies the same conditions. So one can find $\mathcal{P}$ supported in degrees $[-\dim X, 0]$ and such that every $\mathcal{P}^a$ is $\mathcal{D}_X$-flat.

(ii) The categories of these two kinds of $\mathcal{P}$ satisfy the above lemma as well (the proof does not change).

2.2.11. *This section will not be used in the text. The reader can skip it.*

In case $\dim X = 1$, there is another way to make DR a homotopy pseudo-tensor functor. It is convenient to consider DR as a functor with values in $C\mathcal{D}iff(X)$. Let us show that it extends naturally to a pseudo-tensor functor (see 2.2.4(i))

$$DR : C\mathcal{M}(X)^* \otimes \bar{\mathcal{E}} \to C\mathcal{D}iff(X)^* \qquad (2.2.11.1)$$

where $\bar{\mathcal{E}}$ is a certain DG operad homotopically equivalent to the unit k-operad $\mathcal{C}om$.

Consider the operad of projections (see 1.1.4(ii)) as an operad of groupoids with the contractible groupoid structure. Passing from groupoids to the classifying simplicial sets (see 4.1.1(ii)), we get a simplicial operad E. Let $\mathcal{E} := \mathcal{N}orm\, k[E]$ be the corresponding DG operad of normalized k-chains. Finally, set $\bar{\mathcal{E}} := \mathcal{E}/\tau_{\leq -1}\mathcal{E}$; this is a quotient DG operad of $\mathcal{E}$. Notice that both $\mathcal{E}$ and $\bar{\mathcal{E}}$ are naturally resolutions of $\mathcal{C}om$. The operad $\mathcal{E}^0 = \bar{\mathcal{E}}^0$ is the k-linear envelope of the operad of projections (see 1.1.4), the projection $\mathcal{E}^0 \to \mathcal{C}om$ is $\mathcal{E}^0_I \to k$, $e_i \mapsto 1$, where e_i are the base vectors of $\mathcal{E}^0_I := k[I]$. Our $\bar{\mathcal{E}}$ is a length 2 resolution of $\mathcal{C}om$; it is uniquely determined by the above description of $\bar{\mathcal{E}}^0$. We define $c_{ii'} \in \bar{\mathcal{E}}_I^{-1}$ by the condition $dc_{ii'} = e_i - e_{i'}$.

For $M \in C\mathcal{M}(X)$, $I \in \mathcal{S}$ the complexes $DR(M)$ and $DR(\Delta_*^{(I)}M)$ can be considered naturally as objects $C\mathcal{D}iff(X)$ (see 2.1.8, especially Example (i) there). One has a canonical embedding $DR(M) \hookrightarrow DR(\Delta_*^{(I)}M)$ (see 2.1.7(b)) which is a quasi-isomorphism (see 2.1.10). We will define in a moment a natural morphism of complexes $\pi = \pi_M^{(I)} : DR(\Delta_*^{(I)}M) \otimes \bar{\mathcal{E}}_I \to DR(M)$ such that $\pi(m \otimes e_i)$ is the integral of m along the fibers of the ith projection $X^I \to X$. Such π is unique, since the de Rham differential $DR(M)^{-1} \to DR(M)^0$ is an injective morphism in $\mathcal{D}iff(X)$.

To construct π, consider first the case $M = \mathcal{D}_X$. We need to specify the component $DR(\Delta_*^{(I)}\mathcal{D}_X)^0 \otimes \bar{\mathcal{E}}_I^{-1} \to DR(\mathcal{D}_X)^{-1}$, i.e., $\Delta_*^{(I)}\mathcal{D}_X \otimes \bar{\mathcal{E}}_I^{-1} \to \mathcal{D}_X \otimes \Theta_X$, of $\pi_{\mathcal{D}_X}^{(I)}$. Since the differential in $DR(\mathcal{D}_X)$ is injective as a usual map of sheaves, we determine it from the formula $d\pi(m\otimes c) = \pi(m\otimes dc)$. Notice that $\mathcal{D}_X$ is actually a $\mathcal{D}_X$-bimodule, and the above construction used only the right $\mathcal{D}_X$-module structure on it, so $\pi_{\mathcal{D}_X}^{(I)}$ just defined is a morphism of left $\mathcal{D}_X$-modules. Now for arbitrary M one has $DR(M) = M \underset{\mathcal{D}_X}{\otimes} DR(\mathcal{D}_X)$, $DR(\Delta_*^{(I)}M) = M \underset{\mathcal{D}_X}{\otimes} DR(\Delta_*^{(I)}\mathcal{D}_X)$, and $\pi_M^{(I)} := id_M \otimes \pi_{\mathcal{D}_X}^{(I)}$.

One defines (2.2.11.1) as follows. For a $*$ operation $\varphi : \boxtimes L_i \to \Delta_*^{(I)}M$ and $c \in \bar{\mathcal{E}}_I$ the operation $DR(\varphi\otimes c) : \otimes DR(L_i) \to DR(M)$ is $\otimes \ell_i \mapsto \pi_M^{(I)}\varphi(\boxtimes \ell_i)\otimes c$. The compatibility with composition of operations is immediate.

The rest of this section contains auxiliary technical material; we suggest the reader skip it, returning when necessary.

2.2.12. We return to arbitrary $\dim X$. Let us show that our augmented compound tensor structure comes from a certain augmented compound strict $\mathcal{D}_X$-operad $\mathcal{B}^{*!}$ in the way explained in 1.3.18, 1.3.19.

Let us define $\mathcal{B}^{*!} = (\mathcal{B}^*, \mathcal{B}^!, <\ >^I_{S,T})$. For a finite set I let $\mathcal{B}^*_I$ be the tensor product of I copies of $\mathcal{D}_X$ considered as *left* $\mathcal{D}$-modules. This is a left $\mathcal{D}$-module on X, and the right $\mathcal{D}_X$-action on each copy of $\mathcal{D}_X$ defines the right $\mathcal{D}_X^{\otimes I}$-action on $\mathcal{B}^*_I$ (here $\mathcal{D}_X^{\otimes I}$ is the Ith tensor power of $\mathcal{D}_X$ considered as a sheaf of k-algebras). So $\mathcal{B}^*_I$ is a $(\mathcal{D}_X - \mathcal{D}_X^{\otimes I})$-bimodule. Similarly, let $\mathcal{B}^!_I$ be the $\otimes^!$-product of I copies of $\mathcal{D}_X$ considered as *right* $\mathcal{D}_X$-modules; this is a $(\mathcal{D}_X^{\otimes I} - \mathcal{D}_X)$-bimodule.

Note that $\mathcal{B}^*_I$ is generated, as right $\mathcal{D}_X^{\otimes I}$-module, by the subsheaf $\mathcal{O}_X = \mathcal{O}_X 1^{\otimes I}$ on which $\mathcal{O}_X^{\otimes I} \subset \mathcal{D}_X^{\otimes I}$ acts via the product map $\mathcal{O}_X^{\otimes I} \to \mathcal{O}_X$. In fact $\mathcal{B}^*_I = \mathcal{O}_X \underset{\mathcal{O}_X^{\otimes I}}{\otimes} \mathcal{D}_X^{\otimes I}$. Similarly, $\mathcal{B}^!_I = \mathcal{D}_X^{\otimes I} \underset{\mathcal{O}_X^{\otimes I}}{\otimes} (\omega_X^{(1-|I|)})$ as left $\mathcal{D}_X^{\otimes I}$-modules where $\omega_X^{(1-|I|)}$ is the $\mathcal{O}_X$-tensor power.

For a map $J \to I$ the composition morphisms for our augmented operads $\mathcal{B}^*_I \underset{\mathcal{D}_X^{\otimes I}}{\otimes} (\mathcal{B}^*_{J_i}) \to \mathcal{B}^*_J$, $(\otimes \mathcal{B}^!_{J_i}) \underset{\mathcal{D}_X^{\otimes I}}{\otimes} \mathcal{B}^!_I \to \mathcal{B}^!_J$ are the isomorphisms of, respectively, $(\mathcal{D}_X - \mathcal{D}_X^{\otimes J})$- and $(\mathcal{D}_X^{\otimes J} - \mathcal{D}_X)$-bimodules, uniquely determined by the property that $f_I \otimes (\otimes f_{J_i}) \mapsto f_I \Pi f_{J_i} \in \mathcal{O}_X \subset \mathcal{B}^*_J$, $(\otimes \nu_{J_i}) \otimes \nu_I \mapsto (\Pi \nu_{J_i})\nu_I \in \omega_X^{(1-|J|)} \subset \mathcal{B}^!_J$ for $f_I \in \mathcal{O}_X \subset \mathcal{B}^*_I$, $\nu_I \in \omega_X^{(1-|I|)} \subset \mathcal{B}^!_I$, etc.

It remains to define $<\ >^I_{S,T}$: $(\underset{S}{\otimes} \mathcal{B}^*_{I_s}) \underset{\mathcal{D}_X^{\otimes I}}{\otimes} (\underset{T}{\otimes} \mathcal{B}^!_{I_t}) \to \mathcal{B}^!_S \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_T$. The arrow is, in fact, an isomorphism, and we will construct its inverse. One has $(\underset{S}{\otimes} \mathcal{B}^*_{I_s}) \underset{\mathcal{D}_X^{\otimes I}}{\otimes} (\underset{T}{\otimes} \mathcal{B}^!_{I_t})$ $= \mathcal{O}_X^{\otimes S} \underset{\mathcal{O}_X^{\otimes I}}{\otimes} (\mathcal{D}_X \omega_X^{-1})^{\otimes I} \underset{\mathcal{O}_X^{\otimes I}}{\otimes} \omega_X^{\otimes T}$ where the $\mathcal{O}_X^{\otimes I}$-module structure on $\mathcal{O}_X^{\otimes S}$, $\omega_X^{\otimes T}$ comes from the π_S and π_T product morphisms $\mathcal{O}_X^{\otimes I} \to \mathcal{O}_X^{\otimes S}$, $\mathcal{O}_X^{\otimes I} \to \mathcal{O}_X^{\otimes T}$. This is a $(\mathcal{D}_X^{\otimes S} - \mathcal{D}_X^{\otimes T})$-bimodule containing an $\mathcal{O}_X^{\otimes S \sqcup T}$-submodule $\mathcal{O}_X^{\otimes S} \underset{\mathcal{O}_X^{\otimes I}}{\otimes} \omega_X^{-|I|} \underset{\mathcal{O}_X^{\otimes I}}{\otimes} \omega_X^{\otimes T}$

$= \omega_X^{-|S|+1}$. On the other hand, $\mathcal{B}^!_S \underset{\mathcal{O}_X}{\otimes} \mathcal{B}^*_T$ is a $(\mathcal{D}_X^{\otimes S}, \mathcal{D}_X^{\otimes T})$-bimodule freely generated by its $\mathcal{O}_X^{\otimes S\sqcup T}$-module $\omega_X^{-|S|+1}$. Therefore we get a morphism $\beta : \mathcal{B}^!_S \underset{\mathcal{O}_X}{\otimes} \mathcal{B}^*_T \to (\underset{S}{\otimes} \mathcal{B}^*_{I_s}) \underset{\mathcal{D}_X^{\otimes I}}{\otimes} (\underset{T}{\otimes} \mathcal{B}^!_{I_t})$. Notice that the action of Θ_X on $\mathcal{O}_X^{\otimes S} \underset{\mathcal{O}_X^{\otimes I}}{\otimes} (\mathcal{D}_X \omega_X^{-1})^{\otimes I} \underset{\mathcal{O}_X^{\otimes I}}{\otimes} \omega_X^{\otimes T}$ by transport of structure coincides with the adjoint action for the $(\mathcal{D}_X^{\otimes S} - \mathcal{D}_X^{\otimes T})$-bimodule structure of the diagonal embedded $\Theta_X \hookrightarrow \mathcal{D}_X^{\otimes S}, \mathcal{D}_X^{\otimes T}$. In particular, the latter action of Θ_X on $\omega_X^{-|S|+1} \subset (\underset{S}{\otimes} \mathcal{B}^*_{I_s}) \underset{\mathcal{D}_X^{\otimes I}}{\otimes} (\underset{T}{\otimes} \mathcal{B}^!_{I_t})$ coincides with the evident action; hence β factors through $\mathcal{B}^!_S \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_T \to (\underset{S}{\otimes} \mathcal{B}^*_{I_s}) \underset{\mathcal{D}_X^{\otimes I}}{\otimes} (\underset{T}{\otimes} \mathcal{B}^!_{I_t})$. A computation on the level of symbols shows that this arrow is an isomorphism; our $< \ >^I_{S,T}$ is its inverse.

The axioms of augmented compound operad are immediate.

Let us check that the augmented compound tensor structure defined on $\mathcal{M}(X)$ by $\mathcal{B}^{*!}$ (see 1.3.18 and 1.3.19) coincides with that defined in 2.2.1 – 2.2.7. It is clear that $\otimes^!_I L_i = (\otimes L_i) \underset{\mathcal{D}_X^{\otimes I}}{\otimes} \mathcal{B}^!_I$. The sheaf-theoretic restriction of the $\mathcal{D}_{X^I}$-module $\Delta^{(I)}_* M$ to the diagonal is a $\mathcal{D}_X^{\otimes I}$-module. There is a unique isomorphism of $\mathcal{D}_X^{\otimes I}$-modules

$$M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_I \xrightarrow{\sim} \Delta^{(I)\cdot} \Delta^{(I)}_* M \tag{2.2.12.1}$$

which induces the identity isomorphism between the subsheaves $M = M \otimes 1^{\otimes I}$ and $M = \Delta^{(I)!}\Delta^{(I)}_* M$. This yields the isomorphism

$$\mathrm{Hom}_{\mathcal{D}_X^{\otimes I}}(\otimes L_i, M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_I) \xrightarrow{\sim} P^*_I(\{L_i\}, M). \tag{2.2.12.2}$$

We leave it to the reader to identify the maps $< \ \ >^I_{S,T}$.

2.2.13. We finish this section with several remarks about the $*$ pseudo-tensor structure. The operad $\mathcal{B}^*_I$ from 2.2.12 satisfies conditions (i) and (ii) of 1.2.3. Therefore, for any *coherent* $\mathcal{D}$-modules L_i and an arbitrary $\mathcal{D}$-module M, the object $\mathcal{P}_I(\{L_i\}, M) \in \mathcal{M}(X)$ exists. Explicitly, one has

$$\mathcal{P}_I(\{L_i\}, M) = \underline{\mathrm{Hom}}_{\mathcal{D}_X^{\otimes I}}(\underset{k}{\otimes} L_i, M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_{\widetilde{I}}). \tag{2.2.13.1}$$

Consider the morphism of $(\mathcal{O}_X, \mathcal{D}_X^{\otimes I})$-bimodules $\mathcal{B}^*_I \to \mathcal{B}^*_{\widetilde{I}}$, $b \mapsto 1. \otimes b$ (recall that $\widetilde{I} = \{\ \cdot\ \} \sqcup I$). The corresponding morphism $\mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} \mathcal{B}^*_I \to \mathcal{B}^*_{\widetilde{I}}$ of $(\mathcal{D}_X, \mathcal{D}_X^{\otimes I})$-bimodules is an isomorphism. It yields the identification $M \underset{\mathcal{O}_X}{\otimes} \mathcal{B}^*_I = M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_{\widetilde{I}}$; hence also

$$\mathcal{P}_I(\{L_i\}, M) = \underline{\mathrm{Hom}}_{\mathcal{D}_X^{\otimes I}}(\underset{k}{\otimes} L_i, M \underset{\mathcal{O}_X}{\otimes} \mathcal{B}^*_I). \tag{2.2.13.2}$$

Note that the right $\mathcal{D}_X$-action on $M \underset{\mathcal{O}_X}{\otimes} \mathcal{B}^*_{\widetilde{I}}$ (used to define the $\mathcal{D}$-module structure on the $\underline{\mathrm{Hom}}$ sheaf) identifies with the right $\mathcal{D}$-module structure on $M \underset{\mathcal{O}_X}{\otimes} \mathcal{B}^*_I$ that comes from the right $\mathcal{D}_X$-module structure on M and the left $\mathcal{D}_X$-module structure on $\mathcal{B}^*_I$ (see 2.1.1).

Remark. The functor $\mathcal{P}_I^*$ obviously commutes with inverse images for étale morphisms; i.e., it is defined on the sheaf of categories $\mathcal{M}(X_{\text{ét}})$. It also commutes with direct images for closed embeddings (as follows immediately from 2.2.5).

2.2.14. If we deal with induced $\mathcal{D}_X$-modules, i.e., $L_i = F_{i\mathcal{D}}$, $M = G_{\mathcal{D}}$, where F_i are coherent $\mathcal{O}$-modules, then the corresponding $\mathcal{D}$-module $\mathcal{P}_I^*$ can be described as follows. Consider the sheaf of differential I-operations $\mathrm{Diff}_I(\{F_i\}, G)_X$.[33] This is an object of $\mathcal{D}iff(X)$ by Example (i) in 2.1.8, so we have the corresponding $\mathcal{D}$-module $\mathrm{Diff}_I(\{F_i\}, G)_{\mathcal{D}} = \mathrm{Diff}_I(\{F_i\}, G)_{X\mathcal{D}}$. Now the canonical $\tilde{I}$-operation $\in \mathrm{Diff}_{\tilde{I}}(\{\mathrm{Diff}_I(\{F_i\}, G)_X, F_i\}, G) = P_{\tilde{I}}^*(\{\mathrm{Diff}_I(\{F_i\}, G)_{\mathcal{D}}, F_{i\mathcal{D}}\}, G_{\mathcal{D}})$ yields an isomorphism[34]

$$\mathrm{Diff}_I(\{F_i\}, G)_{\mathcal{D}} \xrightarrow{\sim} \mathcal{P}_I^*(\{F_{i\mathcal{D}}\}, G_{\mathcal{D}}). \tag{2.2.14.1}$$

2.2.15. According to 2.2.13, for a $\mathcal{D}_X$-coherent L and any $M \in \mathcal{M}(X)$ we have the $\mathcal{H}om$ $\mathcal{D}_X$-module $\mathcal{H}om^*(L, M) = \underline{\mathrm{Hom}}_{\mathcal{D}_X}(L, M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X) \in \mathcal{M}(X)$ equipped with a universal "evaluation" $*$ pairing $\varepsilon \in P_2^*$ $(\{\mathcal{H}om^*(L, M), L\}, M)$. If K is another coherent $\mathcal{D}$-module, then the composition $*$ pairing

$$c \in P_2^*(\{\mathcal{H}om^*(L, M), \mathcal{H}om^*(K, L)\}, \mathcal{H}om^*(K, M)) \tag{2.2.15.1}$$

is defined; the composition is associative and compatible with the evaluation (see 1.2.2).

Here is a more detailed description. Recall that $M \otimes \mathcal{D}_X := M \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$ carries two commuting right $\mathcal{D}_X$-actions: the first one comes from the right $\mathcal{D}_X$-action on $\mathcal{D}_X$; the second one comes from the right $\mathcal{D}_X$-module structure on M and the left $\mathcal{D}_X$-module structure on $\mathcal{D}_X$. Since[35] $M \otimes \mathcal{D}_X = M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_{\{1,2\}}$, one has a canonical automorphism of $M \otimes \mathcal{D}_X$ interchanging the two $\mathcal{D}$-module structures.

Now a section of $\mathcal{H}om^*(L, M)$ is a morphism of $\mathcal{D}_X$-modules $L \to M \otimes \mathcal{D}_X$ with respect to the first $\mathcal{D}$-module structure on $M \otimes \mathcal{D}_X$; the $\mathcal{D}$-module structure on $\mathcal{H}om^*(L, M)$ comes from the second $\mathcal{D}$-module structure on $M \otimes \mathcal{D}_X$. Here ε is just the usual evaluation morphism $\underline{\mathrm{Hom}}(L, M \otimes \mathcal{D}_X) \underset{k}{\otimes} L \to M \otimes \mathcal{D}_X$. To describe c explicitly, it is convenient to understand $\underline{\mathrm{Hom}}_{\mathcal{D}_X}(L, M \otimes \mathcal{D}_X)$ in a way different from that used above: the morphisms are understood with respect to the second $\mathcal{D}$-module structure on $M \otimes \mathcal{D}_X$, and the action of $\mathcal{D}_X$ on $\underline{\mathrm{Hom}}_{\mathcal{D}_X}(L, M \otimes \mathcal{D}_X)$ comes from the first one. Let us identify $\mathcal{H}om^*(L, M)$ with $\underline{\mathrm{Hom}}_{\mathcal{D}_X}(L, M \otimes \mathcal{D}_X)$ using the above symmetry of $M \otimes \mathcal{D}_X$ interchanging the $\mathcal{D}$-module structures. Then the $*$ pairing c becomes the usual composition

$$\underline{\mathrm{Hom}}(L, M \otimes \mathcal{D}_X) \underset{k}{\otimes} \underline{\mathrm{Hom}}(K, L \otimes \mathcal{D}_X) \to \underline{\mathrm{Hom}}(K, (M \otimes \mathcal{D}_X) \otimes \mathcal{D}_X) = \underline{\mathrm{Hom}}(K, M \otimes \mathcal{D}_X) \otimes \mathcal{D}_X.$$

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[33] Which is the X-sheafified version of the vector space of differential I-operations from 2.2.4(i).

[34] To check that our arrow is an isomorphism, one reduces (using the left exactness of our functors) to the case $F_i = \mathcal{O}_X$ where the statement is obvious.

[35] See 2.2.13 for notation.

2.2.16. We see that for any coherent $M \in \mathcal{M}(X)$ the $\mathcal{D}$-module $\mathcal{E}nd^*(M) = \mathcal{H}om^*(M,M)$ is an associative algebra in the $*$ sense acting on M. This action is universal: for any associative aglebra A in $\mathcal{M}(X)^*$ an A-module structure on M is the same as a morphism of associative algebras $A \to \mathcal{E}nd^* M$.

Set $M^\circ := \mathcal{H}om^*(M, \omega_X) = \underline{\mathrm{Hom}}_{\mathcal{D}_X}(M, \omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X)$; this is again a coherent module. We have a canonical pairing $\langle\ \rangle \in P_2^*(\{M^\circ, M\}, \omega_X)$.

If M is a vector $\mathcal{D}_X$-bundle (see 2.1.5), then $\langle\ \rangle$ is non-degenerate (see 1.4.2); in particular, $M \otimes M^\circ \xrightarrow{\sim} \mathcal{E}nd^* M$.

Remarks. (i) Notice that $\omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$ considered as a right $\mathcal{D}_X$-module with respect to the second $\mathcal{D}$-module structure (see 2.2.15) equals $(\mathcal{D}_X)^r$ where we consider $\mathcal{D}_X$ as a left $\mathcal{D}$-module. So, by 2.2.15, one has $M^\circ = \underline{\mathrm{Hom}}_{\mathcal{D}_X}(M^\ell, \mathcal{D}_X)$.

(ii) Let F be a vector bundle. It yields the corresponding left and right induced $\mathcal{D}_X$-modules ${}_{\mathcal{D}}F := \mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} F \in \mathcal{M}^\ell(X)$, $F_{\mathcal{D}} := F \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \in \mathcal{M}(X)$. We see that[36]

$$(F_{\mathcal{D}})^\circ = (F^* \otimes \omega_X)_{\mathcal{D}}, \quad ({}_{\mathcal{D}}F)^\circ = (F^*)_{\mathcal{D}}. \tag{2.2.16.1}$$

2.2.17. Global duality. Suppose X is proper of dimension n. Let V, V° be mutually dual vector $\mathcal{D}_X$-bundles. The canonical pairing $\in P_2^*(\{V^\circ, V\}, \omega_X)$ together with the trace morphism $\mathrm{tr} : R\Gamma_{DR}(X, \omega_X) \to k[-n]$ yields a canonical pairing

$$R\Gamma_{DR}(X, V^\circ) \otimes R\Gamma_{DR}(X, V) \to k[-n]. \tag{2.2.17.1}$$

Lemma. *This pairing is non-degenerate; i.e.,* $R\Gamma_{DR}(X, V^\circ)$ *is dual to the complex* $R\Gamma_{DR}(X, V)[n]$.

Proof. Using appropriate resolutions, one is reduced to the induced situation where the above duality is the usual Serre duality. □

If X is not proper, the above lemma holds if one replaces one of the cohomology groups by cohomology with compact supports. We do not need the statement in whole generality; consider just the case when X is affine. Then $H^i_{DR}(X, V) = 0$ for $i \neq 0$, and $H^0_{DR}(X, V) = \Gamma(X, h(V))$. Now the morphism $H^0_{DR}(X, V) \to \mathrm{Hom}(V^\circ, \omega_X)$, defined by the canonical pairing, is an isomorphism.[37] One can rewrite it as a canonical isomorphism (see 2.1.12 for notation)[38]

$$(V^\circ)_{const} \xrightarrow{\sim} H^0_{DR}(X, V)^* \otimes \omega_X. \tag{2.2.17.2}$$

Remark. We can consider families of vector $\mathcal{D}_X$-bundles; i.e., locally projective $F \otimes \mathcal{D}_X$-modules V of finite rank where F is any commutative k-algebra. For such V its dual V° is $\mathcal{H}om_{F \otimes \mathcal{D}_X}(V^\ell, F \otimes \mathcal{D}_X)$. The above statements (together with the proofs) remain valid with the following modifications: If X is projective, then $R\Gamma_{DR}(X, V)$ is a perfect F-complex, and the canonical pairing identifies $R\Gamma_{DR}(X, V^\circ)$ with the F-complex dual to $R\Gamma_{DR}(X, V)[\dim X]$. If X is affine, then $H^i_{DR}(X, V) = 0$ for $i \neq 0$, $H^0_{DR}(X, V) = \Gamma(X, h(V))$ is a projective F-module, and (2.2.17.2) holds with $H^0_{DR}(X, V)^* := \mathrm{Hom}_F(H^0_{DR}(X, V), F)$.

[36] In the second equality we identified left and right $\mathcal{D}$-modules in the usual way.

[37] It suffices to check this statement for $V = \mathcal{D}_X$ where it is obvious.

[38] Here $H^0_{DR}(X, V)^*$ is considered as a profinite-dimensional vector space.

2.2.18. According to 1.3.11, the duality functor $^\circ\colon \mathcal{M}(X)^\circ_{coh} \to \mathcal{M}(X)_{coh}$ extends canonically to a pseudo-tensor functor $^\circ\colon \mathcal{M}(X)^{\circ!}_{coh} \to \mathcal{M}(X)^*_{coh}$. It induces an equivalence between the full pseudo-tensor subcategories of vector $\mathcal{D}_X$-bundles.

On $\mathcal{M}(X)^{\circ!}_{coh}$ we have a standard augmentation functor $h^!(M) := \underline{\mathrm{Hom}}(M, \omega_X)$ and a canonical morphism $h(M^\circ) \to h^!(M)$ of augmentation functors; see (1.3.11.3). If M is a vector $\mathcal{D}_X$-bundle, then this is an isomorphism.[39]

REMARK. If F_i, G are vector bundles on X, then the duality $\mathrm{Hom}(G^\circ_{\mathcal{D}}, \otimes^! F^\circ_{i\mathcal{D}}) \xrightarrow{\sim} P^*_I(\{F_{i\mathcal{D}}\}, G_{\mathcal{D}})$ is the composition of the following identifications (see 2.2.12) $\mathrm{Hom}({}_{\mathcal{D}}G^*, \otimes({}_{\mathcal{D}}F^*_i)) = \mathrm{Hom}_{\mathcal{O}_X}(G^*, \mathcal{B}^*_I \underset{\mathcal{O}_X^{\otimes I}}{\otimes} (\otimes F^*_i)) = \mathrm{Hom}_{\mathcal{O}_{X^I}}(\boxtimes F_i, G \underset{\mathcal{O}_X}{\otimes} \mathcal{B}^*_I) = \mathrm{Hom}(\boxtimes F_{i\mathcal{D}}, (\Delta^{(I)}_\cdot G)_{\mathcal{D}}) = \mathrm{Hom}(\boxtimes F_{i\mathcal{D}}, \Delta^{(I)}_* G_{\mathcal{D}})$.

2.2.19. A remark about the augmentation functor. Take $I, J \in \mathcal{S}$, the families of objects $\{L_i\}$, $\{A_j\}$, M and consider the composition morphism for $I \hookrightarrow I \sqcup J$ in our augmented pseudo-tensor structure $\underline{P}^*_{I\sqcup J}(\{L_i, A_j\}, M) \otimes \underset{J}{\otimes} h(A_j) \to \underline{P}^*_I(\{L_i\}, M)$. We can rewrite it as an embedding[40]

$$h_{IJ}\colon P^*_{I\sqcup J}(\{L_i, A_j\}, M) \hookrightarrow \mathrm{Hom}\,(\underset{J}{\otimes} h(A_j), \underline{P}^*_I(\{L_i\}, M))\,. \tag{2.2.19.1}$$

LEMMA. *The image of h_{IJ} is the subspace of those $\varphi\colon \underset{J}{\otimes} h(A_j) \to P^*_I(\{L_i\}, M)$ for which the map $\widetilde{\varphi}\colon (\otimes L_i) \otimes (\otimes A_j) \to M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_I$, $(\otimes \ell_i) \otimes (\otimes a_j) \mapsto \varphi(\otimes \bar{a}_j)(\otimes \ell_i)$, is a differential operator with respect to any of the variables A_j (see Example (iii) in 2.1.8).*

In other words, φ satisfies the following condition (here $\mathcal{I}$ is the kernel of $\mathcal{O}_X^{\otimes I\sqcup J} \to \mathcal{O}_X$):

(∗) *For any local section $s \in (\otimes L_i) \otimes (\otimes A_j)$ one has $\widetilde{\varphi}(\mathcal{I}^N s) = 0$ for $N \gg 0$.*

Proof. Let $Tr\colon M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_{I\sqcup J} \to M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_{I\sqcup J} \underset{\mathcal{D}_X^{\otimes J}}{\otimes} \mathcal{O}_X^{\otimes J} = M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_I$ be the trace along J-variables. The morphism $M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_{I\sqcup J} \to \mathrm{Hom}_{\mathcal{D}_X^{\otimes I}}(\mathcal{D}_X^{\otimes I} \otimes \mathcal{O}_X^{\otimes J}, M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_I)$ of $\mathcal{D}_X^{\otimes I\sqcup J}$-modules which assigns to $b \in M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_{I\sqcup J}$ the map $\alpha \mapsto Tr(b\alpha)$ is injective. Its image is the subspace of those $\psi\colon \mathcal{D}_X^{\otimes I} \otimes \mathcal{O}_X^{\otimes J} \to M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_I$ that kill some $\mathcal{I}^N \subset \mathcal{O}_X^{\otimes I\sqcup J} \subset \mathcal{D}_X^{\otimes I} \otimes \mathcal{O}_X^{\otimes J}$. This morphism maps $P^*_{I\sqcup J}(\{L_i, A_j\}, M)$ injectively to

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{D}_X^{\otimes I\sqcup J}}((\otimes L_i) \otimes (\otimes A_j), \mathrm{Hom}_{\mathcal{D}_X^{\otimes I}}(\mathcal{D}_X^{\otimes I} \otimes \mathcal{O}_X^{\otimes J}, M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_I))\\ &\quad = \mathrm{Hom}_{\mathcal{D}_X^{\otimes I}}((\otimes L_i) \otimes (\otimes h(A_j)), M \underset{\mathcal{D}_X}{\otimes} \mathcal{B}^*_I) = \mathrm{Hom}(\otimes h(A_j), \underline{P}^*_I(\{L_i\}, M)).\end{aligned}$$

The image coincides with the subspace defined by condition (∗). Since this map coincides with $h_{I,J}$, we are done. □

2.2.20. The following technical lemma will be of use. Let L_i, M be $\mathcal{D}$-modules on X, $\varphi \in P^*_I(\{L_i\}, M)$, and let $x \in X$ be a closed point. Set $U_x := X \smallsetminus \{x\}$. We use the terminology of 2.1.13.

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[39] It suffices to check this statement for $M = \mathcal{D}_X$ where it is obvious.

[40] Our h is non-degenerate by 2.2.8.

LEMMA. (i) *The polylinear map* $h(\varphi) : \otimes h(L_i)_x \to h(M)_x$ *is* Ξ_x*-continuous with respect to each variable.*

(ii) *Assume that the restriction of each* L_i *to* U_x *is a countably generated* $\mathcal{D}$*-module. Then* $h(\varphi)$ *is* Ξ_x*-continuous.*

(iii) *Assume that for certain* $i_0 \in I$ *every* L_i*, but, possibly* L_{i_0}*, is a coherent* $\mathcal{D}_X$*-module. Then for each* $M_\xi \in \Xi_x(M)$ *there exists* $L_{i_0\xi} \in \Xi_x(L_{i_0})$ *such that*

$$\varphi(L_{i_0\xi} \boxtimes (\underset{i\neq i_0}{\boxtimes} L_i)) \subset \Delta_*^{(I)} M_\xi. \tag{2.2.20.1}$$

Proof. (i) Pick $i_0 \in I$ and $\bar{l}_i \in h(L_i)_x$ for $i \neq i_0$. We have to show that the map $h(L_{i_0})_x \to h(M)_x$, $\bar{l}_{i_0} \mapsto h(\varphi)(\bar{l}_{i_0} \otimes (\otimes \bar{l}_i))$, is Ξ_x-continuous. This is clear since this map comes from a morphism of $\mathcal{D}$-modules $L_{i_0} \to M$ that comes from φ and the $\bar{l}_i$'s by the augmentation functor structure on h.

(ii) An induction by $|I|$ shows (use the augmentation functor structure on h) that to prove continuity of $h(\varphi)$, it suffices to find for every $M_\xi \in \Xi_x(M)$ some $L_{i\xi} \in \Xi_x(L_i)$ such that $h(\varphi)(\otimes L_{i\xi}) \subset M_\xi$. Shrinking X if necessary, we can assume that X is affine and we have functions t_a, $a = 1, \dots, n$, on it which are local coordinates at x and have no other common zero. Let $l_{i1}, l_{i2}, \dots$ be sections of L_i that generate the restriction of L_i to U_x as a $\mathcal{D}_{U_x}$-module. The quotient $\Delta_*^{(I)} M / \Delta_*^{(I)} M_\xi = \Delta_*(M/M_\xi)$ is a $\mathcal{D}_{X^I}$-module supported at $\Delta^{(I)}(x)$, so every its section is killed by sufficiently high power of each t_a lifted from either copy of X. Therefore, using induction by $b = 1, 2, \dots$, we can choose integers $n(a, b, i) \geq 0$ such that $h(\varphi)(\underset{I}{\boxtimes} t_a^{n(a,b_i,i)} l_{ib_i}) \in \Delta_*^{(I)} M_\xi \subset \Delta_*^{(I)} M$ for every $1 \leq b_i \leq b$. Our $L_{i\xi}$ is the $\mathcal{D}$-submodule of L_i generated by $\{t_a^{n(a,b,i)} l_{ib}\}$.

(iii) Choose for $i \neq i_0$ a finite set $\{l_{ij}\}$ of local sections of L_i at x that generates L_i. Now take for $L_{i_0\xi}$ the intersection of the preimages of $\Delta_*^{(I)} M_\xi$ by all maps $L_{i_0} \to \Delta_*^{(I)} M$, $l \mapsto \varphi(l \boxtimes (\boxtimes l_{ij}))$. □

We see that in situation (ii) the morphism $h_I(\varphi)$: $\otimes h(L_i)_x \to h(M)_x$ extends by continuity to the map $\hat{h}_I(\varphi)_x$: $\widehat{\otimes}\, \hat{h}_x(L_i) \to \hat{h}_x(M)$. Here $\widehat{\otimes}\, \hat{h}_x(L_i) := \varprojlim \otimes h_x(L_i/L_{i\,\xi_i})$.

If we decide to play with objects $\tilde{h}_x$ from 2.1.15 instead of plain topological vector spaces, then the countability condition of (ii) becomes irrelevant.

2.3. $\mathcal{D}_X$-schemes

A commutative algebra in $\mathcal{M}^!(X)$ can be considered as a "coordinate-free" version of a system of non-linear differential equations (just as $\mathcal{D}$-modules provide a coordinate-free language for systems of linear differential equations). In this section we deal with basic algebro-geometric properties of these objects.

The bulk of the literature on foundational subjects of the geometric theory of non-linear differential equations is huge (take [**G**] and [**V**] to estimate the span). The Euler-Lagrange equations (the classical calculus of variations) are treated in the nice review articles [**DF**], [**Z**].

We consider jet schemes in 2.3.2–2.3.3, quasi-coherent $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules on a $\mathcal{D}_X$-scheme $\mathcal{Y}$ in 2.3.5, and prove in 2.3.6–2.3.9 that for affine $\mathcal{Y}$ projectivity of such modules is a local property, which is a $\mathcal{D}$-version of a theorem of Raynaud-Gruson [**RG**]. The $*$ operations between quasi-coherent $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules are considered in 2.3.11–2.3.12, the notion of smoothness for $\mathcal{D}_X$-schemes in 2.3.13–2.3.16. We show that smooth $\mathcal{D}_X$ do not necessary admit local coordinate systems (see 2.3.17–2.3.18), and discuss the notion of vector $\mathcal{D}_X$-scheme in 2.3.19. The setting of the calculus of variations is briefly described in 2.3.20.

2.3.1. A *commutative* $\mathcal{D}_X$*-algebra* is a commutative unital $\mathcal{O}_X$-algebra equipped with an integrable connection along X. Unless stated explicitly otherwise, our $\mathcal{D}_X$-algebras are assumed to be $\mathcal{O}_X$-quasi-coherent. A $\mathcal{D}_X$*-scheme* is an X-scheme equipped with an integrable connection along X. Denote by $\mathcal{C}omu_{\mathcal{D}}(X)$, $\mathcal{S}ch_{\mathcal{D}}(X)$ the corresponding categories. We have a fully faithful functor Spec: $\mathcal{C}omu_{\mathcal{D}}(X)^{\circ} \hookrightarrow \mathcal{S}ch_{\mathcal{D}}(X)$; its essential image is the category $\mathcal{A}ff\mathcal{S}ch_{\mathcal{D}}(X)$ of $\mathcal{D}_X$-schemes affine over X.

Replacing schemes by algebraic spaces, we get the notion of an *algebraic* $\mathcal{D}_X$*-space*. For a $\mathcal{D}_X$-algebra R^{ℓ} and an algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ a morphism of algebraic spaces $\operatorname{Spec} R^{\ell} \to \mathcal{Y}$ is sometimes referred to as a $\mathcal{D}_X$*-algebra* R^{ℓ}*-point* of $\mathcal{Y}$.

Remark. Let x_a be a coordinate system on X. Assume we have a system of non-linear differential equations $P_\alpha(u_i, \partial_a u_i, \dots) = 0$ where P_α are polynomial functions with coefficients in functions on X. It yields a $\mathcal{D}_X$-algebra A^{ℓ} defined as the quotient of the free $\mathcal{D}_X$-algebra $\operatorname{Sym}(\oplus \mathcal{D}_X \cdot u_i)$ modulo the relations P_α. A solution of our system with values in a (not necessarily quasi-coherent) $\mathcal{D}_X$-algebra R^{ℓ} is a collection of $\phi_i \in R^{\ell}$ such that $P_\alpha(\phi_i, \partial_a \phi_i, \dots) = 0$. This is the same as a morphism of $\mathcal{D}_X$-algebras $A \to R$, $u_i \mapsto \phi_i$, i.e., a $\mathcal{D}_X$-algebra R^{ℓ}-point of $\operatorname{Spec} A^{\ell}$. Therefore $\mathcal{D}_X$-algebras provide a "coordinate-free" language for non-linear differential equations.

Set $\mathcal{C}omu^!(X) := \mathcal{C}omu(\mathcal{M}^!(X))$ (we use the notation of 1.4.6). The equivalence of tensor categories $\mathcal{M}^!(X) \xrightarrow{\sim} \mathcal{M}^{\ell}(X)$, $A \mapsto A^{\ell} = A\omega_X^{-1}$, yields the equivalence

$$\mathcal{C}omu^!(X) \xrightarrow{\sim} \mathcal{C}omu_{\mathcal{D}}(X). \tag{2.3.1.1}$$

2.3.2. Jet schemes. Denote by $\mathcal{C}omu(X)$ the category of commutative $\mathcal{O}_X$-algebras quasi-coherent as $\mathcal{O}_X$-modules and by $\mathcal{S}ch(X)$ the category of X-schemes. We have the obvious forgetting of connection functors

$$o\colon \mathcal{C}omu_{\mathcal{D}}(X) \to \mathcal{C}omu(X), \quad \mathcal{S}ch_{\mathcal{D}}(X) \to \mathcal{S}ch(X). \tag{2.3.2.1}$$

They admit the left, resp. right, adjoint functors

$$\mathcal{C}omu(X) \to \mathcal{C}omu_{\mathcal{D}}(X), \quad \mathcal{S}ch(X) \to \mathcal{S}ch_{\mathcal{D}}(X). \tag{2.3.2.2}$$

We denote them both by $\mathcal{J}$. For $R \in \mathcal{C}omu(X)$, $\mathcal{J}R$ is generated by R. Explicitly, it is the $\mathcal{D}_X$-algebra quotient of the symmetric algebra $\operatorname{Sym}^{\cdot}\Big(\mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} R\Big)$ modulo the ideal generated by elements $\partial(1\otimes r_1 \cdot 1\otimes r_2 - 1\otimes r_1 r_2) \in \operatorname{Sym}^2\Big(\mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} R\Big) + \mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} R$ and $\partial(1\otimes 1_R - 1) \in \mathcal{D}_X \otimes R + \mathcal{O}_X$ where $r_i \in R$, $\partial \in \mathcal{D}_X$, 1_R is the unit of R. This construction is compatible with localization. The functor $\mathcal{J}\colon \mathcal{S}ch(X) \to \mathcal{S}ch_{\mathcal{D}}(X)$ is $\operatorname{Spec} R \mapsto \operatorname{Spec} \mathcal{J}R$ for affine schemes; for arbitrary schemes use gluing. Note that for any $Z \in \mathcal{S}ch(X)$ the canonical projection $\mathcal{J}Z \to Z$ induces a bijection $(\mathcal{J}Z)_{\mathcal{D}}(X) \xrightarrow{\sim} Z(X)$ where $(\mathcal{J}Z)_{\mathcal{D}}(X)$ is the set of all horizontal sections $X \to Z$.

2.3.3. Here is a different description of $\mathcal{J}R$. Denote by $\mathcal{I}$ the kernel of the product map $\mathcal{O}_X^{\otimes 2} = \mathcal{O}_X \underset{k}{\otimes} \mathcal{O}_X \to \mathcal{O}_X$; for $n \geq 1$ set $\mathcal{O}_{X^{(n)}} := \mathcal{O}_X^{\otimes 2}/\mathcal{I}^n$. These $\mathcal{O}_{X^{(n)}}$ form a projective system of algebras; there are two obvious morphisms $\mathcal{O}_X \rightrightarrows \mathcal{O}_{X^{(n)}}$. Therefore any $\mathcal{O}_X$-algebra B yields two $\mathcal{O}_{X^{(n)}}$-algebras: $B \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{X^{(n)}}$ and $\mathcal{O}_{X^{(n)}} \underset{\mathcal{O}_X}{\otimes} B$.

Lemma. *For any $R, B \in \mathcal{C}omu(X)$ one has*

$$\operatorname{Hom}(\mathcal{J}R, B) = \varprojlim \operatorname{Hom}\left(\mathcal{O}_{X^{(n)}} \underset{\mathcal{O}_X}{\otimes} R, B \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{X^{(n)}}\right) \tag{2.3.3.1}$$

where the left and right Hom *mean, respectively, the morphisms of $\mathcal{O}_X$- and $\mathcal{O}_{X^{(n)}}$-algebras.*

Proof. Our statement is X-local, so we can assume that X is affine, $X = \operatorname{Spec} C$. Thus $\mathcal{O}_X$-algebras are the same as C-algebras. For a C-algebra B denote by J_B the kernel of the map $B \underset{k}{\otimes} C \to B$, $b \otimes c \mapsto bc$. Set $\mathcal{T}B := \varprojlim B \otimes C/J_B^n$. Let us consider $\mathcal{T}B$ as a C-algebra via the morphism $C \to \mathcal{T}B$, $c \mapsto 1 \otimes c$. The projection $\mathcal{T}B \to C$, $b \otimes c \mapsto bc$, is a morphism of C-algebras. Note that $\mathcal{T}B$ is a $\mathcal{D}_X$-algebra in the obvious way (the elments $b \otimes 1 \in \mathcal{T}B$ are horizontal). It is easy to see that the functor $\mathcal{C}omu(X) \to \mathcal{C}omu_{\mathcal{D}}(X)$, $B \mapsto \mathcal{T}B$, is *right* adjoint to the functor o: $\mathcal{C}omu_{\mathcal{D}}(X) \to \mathcal{C}omu(X)$. Therefore we have $\operatorname{Hom}(o\mathcal{J}R, B) = \operatorname{Hom}(\mathcal{J}R, \mathcal{T}B) = \operatorname{Hom}(R, o\mathcal{T}B)$. The latter set coincides with $\varprojlim$ Hom in the statement of the lemma; we are done. □

So for any X-scheme Y the k-points of $\mathcal{J}Y$ are the same as pairs (x, γ) where $x \in X$ and γ is a section of Y on the formal neighborhood of x. We call $\mathcal{J}Y$ the *jet scheme of* Y.

Remark. The above constructions are compatible with étale localization, so we may replace schemes by algebraic $\mathcal{D}_X$-spaces.

2.3.4. For $A^\ell \in \mathcal{C}omu_{\mathcal{D}}(X)$ we denote by $A^\ell[\mathcal{D}_X]$ the ring of differential operators with coefficients in A^ℓ. By definiton, this is a sheaf of associative algebras equipped with a morphism of algebras $A^\ell \to A^\ell[\mathcal{D}_X]$ and that of Lie algebras $\Theta_X \to A^\ell[\mathcal{D}_X]$ which satisfy the relations $\tau \cdot a - a \cdot \tau = \tau(a)$, $f \cdot \tau = f\tau$ for $a \in A^\ell$, $f \in \mathcal{O}_X \subset A^\ell$, $\tau \in \Theta_X$, and universal with respect to these data in the obvious sense. The embedding of Θ_X extends to the morphism of associative algebras $\mathcal{D}_X \to A^\ell[\mathcal{D}_X]$. One checks easily that the morphism of sheaves $A^\ell \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \to A^\ell[\mathcal{D}_X]$, $a \otimes \partial \mapsto a \cdot \partial$ is an isomorphism.

For any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ we have the corresponding sheaf $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$ on $\mathcal{Y}_{\text{ét}}$.

2.3.5. An A^ℓ-module in the tensor category $\mathcal{M}^\ell(X)$ is the same as a left $A^\ell[\mathcal{D}_X]$-module quasi-coherent as an $\mathcal{O}_X$-module. Similarly, an A-module in $\mathcal{M}^!(X)$ is the same as a right $A^\ell[\mathcal{D}_X]$-module quasi-coherent as an $\mathcal{O}_X$-module. We denote by $\mathcal{M}^\ell(X, A^\ell)$, $\mathcal{M}(X, A^\ell) = \mathcal{M}^r(X, A^\ell)$ the categories of left, resp. right, $A^\ell[\mathcal{D}_X]$-modules. These are tensor categories (canonically identified) with unit object equal to A^ℓ, resp. A.

Every morphism of commutative $\mathcal{D}_X$-algebras $f : A^\ell \to B^\ell$ yields an adjoint pair of functors $f^* : \mathcal{M}(X, A^\ell) \to \mathcal{M}(X, B^\ell)$, $f_\cdot : \mathcal{M}(X, B^\ell) \to \mathcal{M}(X, A^\ell)$ where

$f^*M := B^\ell \underset{A^\ell}{\otimes} M$, $f.N$ is N considered as an $A^\ell[\mathcal{D}_X]$-module. Notice that f^* is a tensor functor.

The faithfully flat descent works in any abelian tensor category, so $A^\ell[\mathcal{D}_X]$-modules are local objects with respect to the flat topology.[41] So for any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ we have the corresponding categories $\mathcal{M}^\ell(\mathcal{Y})$, $\mathcal{M}^r(\mathcal{Y}) = \mathcal{M}(\mathcal{Y})$. Their objects are sheaves of left, resp. right, $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-modules which are quasi-coherent as $\mathcal{O}_Y$-modules. The categories $\mathcal{M}^\ell(\mathcal{U})$, $\mathcal{M}(\mathcal{U})$ for $\mathcal{U} \in \mathcal{Y}_{\text{ét}}$ form sheaves of categories $\mathcal{M}^\ell(\mathcal{Y}_{\text{ét}})$ on $\mathcal{Y}_{\text{ét}}$. One has $\mathcal{M}^\ell(\operatorname{Spec} A^\ell) = \mathcal{M}^\ell(X, A)$, $\mathcal{M}(\operatorname{Spec} A^\ell) = \mathcal{M}(X, A^\ell)$.

Most of the facts mentioned in 2.1 render easily to the $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-modules setting. E.g., if G, L are left $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-modules and M is right one, then $G \underset{\mathcal{O}_\mathcal{Y}}{\otimes} L$ is a left $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-module in a natural way, and $L \underset{\mathcal{O}_\mathcal{Y}}{\otimes} M$ is a right one. We have the standard equivalence $\mathcal{M}^\ell(\mathcal{Y}) \to \mathcal{M}^r(\mathcal{Y})$, $L \mapsto L\omega_X$. Any $N \in \mathcal{M}(\mathcal{Y})$ admits a canonical finite left resolution $(DR(N))_\mathcal{D}$ whose terms are induced $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-modules.

2.3.6. Theorem. *Assume that $\mathcal{Y}$ is a $\mathcal{D}_X$-scheme which is affine (as a k-scheme). Then projectivity of an $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-module is a local property for the Zariski or étale topology.*

Proof. It is found in 2.3.7–2.3.9.

2.3.7. Let us first prove the theorem for a finitely generated $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-module P (which is sufficient for the most of applications). Suppose that P is projective over some étale covering of $\mathcal{Y}$. We need to show that P is projective. Take $N \in \mathcal{M}(\mathcal{Y})$. Consider the sheaf $\underline{\operatorname{Hom}}(P, N)$ on $\mathcal{Y}_{\text{ét}}$. It depends on N in an exact way, so projectivity of P follows if we prove that $H^i(\mathcal{Y}_{\text{ét}}, \underline{\operatorname{Hom}}(P, N)) = 0$ for $i > 0$.

This is true if N is an induced module, $N = F \underset{\mathcal{O}_\mathcal{Y}}{\otimes} \mathcal{D}_X$. Indeed, in this situation $\underline{\operatorname{Hom}}(P, N)$ admits a structure of a quasi-coherent $\mathcal{O}_\mathcal{Y}$-module. Namely, the $\mathcal{O}_\mathcal{Y}$-module structure on $\underline{\operatorname{Hom}}(P, N)$ comes from an $\mathcal{O}_\mathcal{Y}$-action on N inherited from F. Quasi-coherence is an étale local property, so it suffices to check it under the assumption that P is projective. Then $\underline{\operatorname{Hom}}(P, N)$ is isomorphic to a direct summand of a sum of several copies of N, which is obviously quasi-coherent.

To prove acyclicity of $\underline{\operatorname{Hom}}(P, N)$ for arbitrary N, use the fact that N admits a finite left resolution by induced modules (see 2.1.9 and 2.3.5). We are done.

2.3.8. Without the finite generatedness assumption the theorem is nontrivial even if X is a point. In this case it was conjectured by Grothendieck (see Remark 9.5.8 from [**Gr1**]) and proved by Raynaud and Gruson (even for the fpqc topology); see [**RG**] 3.1.4. We will use the method of [**RG**]. It is based on the following fact: a flat module M over a ring R is projective if and only if M has the following properties:

(i) M is a direct sum of countably generated modules;
(ii) M is a Mittag-Leffler module

(the implication projectivity ⇒(i) is due to Kaplansky [**Ka**]; the fact that (i) and (ii) imply projectivity is proved in [**RG**], p. 74). A flat module M is said to be Mittag-Leffler if for some (or for every) representation of M as a direct limit of finitely generated projective modules P_i (i belongs to a directed ordered set I) the

[41] f is faithfully flat as a morphism of commutative algebras in the tensor category $\mathcal{M}^\ell(X)$ if and only if it is faithfully flat as a morphism of plain $\mathcal{O}_X$-algebras.

projective system (P_i^*) satisfies the Mittag-Leffler condition (i.e., for every $i \in I$ there exists $j \geq i$ such that $\mathrm{Im}(P_j^* \to P_i^*) = \mathrm{Im}(P_k^* \to P_i^*)$ for all $k \geq j$). Here $P_i^* := \mathrm{Hom}_R(P_i, R)$.

Now the theorem is easily reduced to the following lemma.

2.3.9. Lemma. *Assume that X is affine and $\mathcal{Y}$ is an affine $\mathcal{D}_X$-scheme. Then a flat $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-module M is globally flat; i.e., $\Gamma(\mathcal{Y}, M)$ is flat over $\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X])$.*

Proof. M has a finite left resolution consisting of globally flat $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules (indeed, M is $\mathcal{O}_{\mathcal{Y}}$-flat, so the canonical resolution $(DR(M))_{\mathcal{D}}$ has the required property). Therefore it suffices to show that if there is an exact sequence $0 \to F_1 \to F_0 \to M \to 0$ with F_0, F_1 globally flat and M flat, then M is globally flat; i.e., for every left $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-module L the morphism $\Gamma(\mathcal{Y}, F_1) \otimes_{\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X])} \Gamma(\mathcal{Y}, L) \to \Gamma(\mathcal{Y}, F_0) \otimes_{\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X])} \Gamma(\mathcal{Y}, L)$ is injective. The map $\Gamma(\mathcal{Y}, F_i) \otimes_{\Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X])} \Gamma(\mathcal{Y}, L) \to \Gamma(\mathcal{Y}, F_i \otimes_{\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]} L)$ is an isomorphism (represent F_i as a direct limit of finitely generated free $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules). Finally, the morphism $F_1 \otimes_{\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]} L \to F_0 \otimes_{\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]} L$ is injective because M is flat. □

2.3.10. We see that for any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ there is a notion of $\mathcal{Y}$-locally projective $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules. Notice that for $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules on an affine $\mathcal{Y}$ the property of being finitely generated is obviously étale local. So we know what $\mathcal{Y}$-locally finitely generated $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules on any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ are. We call a $\mathcal{Y}$-locally finitely generated and projective $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-module a *vector $\mathcal{D}_X$-bundle on $\mathcal{Y}$.*

2.3.11. According to 1.4.6 (and 2.2) for a commutative $\mathcal{D}_X$-algebra A^ℓ the category $\mathcal{M}(X, A^\ell)$ carries a canonical augmented compound tensor structure. As in 1.4.6 for $L_i, M \in \mathcal{M}(X, A^\ell)$ we denote the corresponding space of A^ℓ-polylinear $*$ operations by $P^*_{A\,I}(\{L_i\}, M)$.

In fact, this structure comes from an augmented compound strict $A^\ell[\mathcal{D}_X]$-operad $\mathcal{B}_A^{*!}$. We leave the explicit construction of $\mathcal{B}_A^{*!}$ to the reader (repeat 2.2.12 replacing $\mathcal{D}_X$ by $A^\ell[\mathcal{D}_X]$ everywhere). The discussion in 2.2.13–2.2.18 remains literally valid for $A^\ell[\mathcal{D}_X]$-modules if we replace the words "coherent $\mathcal{D}_X$-module" by "finitely presented $A^\ell[\mathcal{D}_X]$-module". The lemma from 2.2.19 also remains valid.

According to Remark (i) of 1.4.6 for every morphism of commutative $\mathcal{D}_X$-algebras $f : B^\ell \to A^\ell$ the corresponding tensor functor f^* from 2.3.5 extends to a compound tensor functor (base change)

$$f^* : \mathcal{M}(X, A^\ell)^{*!} \to \mathcal{M}(X, B^\ell)^{*!}. \tag{2.3.11.1}$$

We also have a forgetful pseudo-tensor functor

$$f_\cdot : \mathcal{M}(X, B^\ell)^{*} \to \mathcal{M}(X, A^\ell)^{*} \tag{2.3.11.2}$$

right adjoint to f^* considered as a $*$ pseudo-tensor functor (see Remark (ii) of 1.4.6).

Lemma. *$\mathcal{M}(X, A^\ell)^{*!}$ has a local nature with respect to the flat topology of* $\mathrm{Spec}\, A^\ell$.

Proof. We already discussed the flat descent of $A^\ell[\mathcal{D}_X]$-modules in 2.3.5, so it remains to check that the A-polylinear $*$ operations satisfy the flat descent property. For $L_i, M \in \mathcal{M}(X, A^\ell)$ the above adjunction identifies the descent data with operations $\psi \in P^*_{A\,I}(\{L_i\}, B^\ell \underset{A^\ell}{\otimes} M)$ whose composition with the two arrows $B^\ell \underset{A^\ell}{\otimes} M \rightrightarrows B^\ell \underset{A^\ell}{\otimes} B^\ell \underset{A^\ell}{\otimes} M$, $b\otimes m \mapsto b\otimes 1\otimes m, 1\otimes b\otimes m$, coincide. Now the exact sequence $0 \to M \to B^\ell \underset{A^\ell}{\otimes} M \to B^\ell \underset{A^\ell}{\otimes} B^\ell \underset{A^\ell}{\otimes} M$ (the latter arrow is the difference of the above two standard arrows) together with the left exactness of P^*_A implies the desired descent property. □

2.3.12. For L_i, M as above a $*$ operation $\varphi \in P^*_I(\{L_i\}, M)$ is *A-polydifferential* if for every finitely generated $A^\ell[\mathcal{D}_X]$-submodules $L'_i \subset L_i$ the restriction $\varphi|_{\{L'_i\}} \in P^*_I(\{L'_i\}, M)$ is an A-polydifferential $*$ operation of finite order (see 1.4.8). The composition of A-polydifferential operations is A-polydifferential, so they define another pseudo-tensor structure on $\mathcal{M}(X, A^\ell)$.

LEMMA. *A-polydifferential $*$ operations have a local nature with respect to the étale topology of* Spec A^ℓ.

Proof. The case of arbitrary A-polydifferential operations reduces immediately to that of A-polydifferential operations of finite order, and then to that of A-polylinear ones (replace L_i by $L_i^{(n)}$ as in 1.4.8 and notice that the $L_i^{(n)}$ have an étale local nature). Then use the previous lemma. □

Therefore for any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ we have the compound tensor category $\mathcal{M}(\mathcal{Y})^{*!}$ and the sheaf of compound tensor categories $\mathcal{M}(\mathcal{Y}_{\text{ét}})^{*!}$ on $\mathcal{Y}_{\text{ét}}$, the pull-back functors f^* are compound tensor functors, etc. The polydifferential $*$ operations also make sense in this setting.

REMARKS. (i) One has the following immediate analog of 2.2.13. Let $\{L_i\}$ be a finite collection of finitely presented $A^\ell[\mathcal{D}_X]$-modules, M any $A^\ell[\mathcal{D}_X]$-module. Then the $A^\ell[\mathcal{D}_X]$-module $\mathcal{P}^*_{A\,I}(\{L_i\}, M)$ is well defined. The functor $\mathcal{P}^*_{A\,I}$ is compatible with flat pull-backs. So the functor $\mathcal{P}^*_I$ makes sense on any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$.

(ii) In particular, for every finitely presented $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-module L its dual $L^\circ := \mathcal{H}om^*(L, \mathcal{O}_\mathcal{Y}\omega_X)$ is well defined. In particular, if the $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-module of differentials $\Omega_\mathcal{Y} = \Omega_{\mathcal{Y}/X}$ is finitely presented,[42] then $\Theta_\mathcal{Y} := \Omega^\circ_\mathcal{Y}$ is well defined.

(iii) If L is a vector $\mathcal{D}_X$-bundle on $\mathcal{Y}$, then L° is also a vector $\mathcal{D}_X$-bundle on $\mathcal{Y}$, and the canonical pairing is non-degenerate (see 1.4.2).

2.3.13. We are going to discuss the notion of smoothness for algebraic $\mathcal{D}_X$-spaces. First let us recall the definition of formally smooth algebras.

Let $\mathcal{M}$ be any abelian tensor k-category with unit object (so $\otimes$ is right exact). A commutative algebra A in $\mathcal{M}$ is *formally smooth* if for any (commutative) algebra R and an ideal $I \subset R$ such that $I^2 = 0$, any morphism $f\colon A \to R/I$ may be lifted to a morphism $\tilde{f}\colon A \to R$. It suffices to check this in the case when f is an isomorphism; i.e., $R/I = A$, $f = id$ (replace arbitrary R, I, f by $\widetilde{R} := A \underset{R/I}{\times} R$, $\widetilde{I} := \operatorname{Ker}(\widetilde{R} \to A) = I$, $f = id$). Note that the symmetric algebra Sym V is formally smooth if and only if V is a projective object of $\mathcal{M}$.

Assume that $\mathcal{M}$ has many projective objects. We may represent A as a quotient B/J, $B = \operatorname{Sym} V$ for a projective V. Then A is formally smooth iff the projection $B/J^2 \to B/J = A$ has a section $\sigma\colon A \to B/J^2$ (if σ exists, then for any R, I,

[42]This happens if $\mathcal{Y}$ is (locally) finitely $\mathcal{D}_X$-presented or if $\Omega_\mathcal{Y}$ is a vector $\mathcal{D}_X$-bundle.

f as above f lifts to $\hat{f}: B/J^2 \to R$ and one may set $\tilde{f} = f\sigma$). Note that the sections σ correspond bijectively to derivations $\partial: B \to J/J^2$ such that $\partial(x) = x \bmod J^2$ for $x \in J$ (namely, $\sigma(b \bmod J) = b - \partial(b)$). We may consider instead of ∂ the corresponding morphism of B-modules φ: $\Omega_B/J \cdot \Omega_B \to J/J^2$; the condition on ∂ means that $\varphi\nu = id_{J/J^2}$ where ν: $J/J^2 \to \Omega_B/J \cdot \Omega_B$ is the usual map $\nu(x \bmod J^2) = dx \bmod J \cdot \Omega_B$. Therefore $A = B/J$ is formally smooth iff ν is a split injection. Set $\Omega_A^{(-1)} := \operatorname{Ker} \nu$. Since $\operatorname{Coker} \nu = \Omega_A$ and $\Omega_B/J \cdot \Omega_B$ is a projective A-module, we see (as in [**Gr1**] 9.5.7) that

A is formally smooth iff $\Omega_A^{(-1)} = 0$ and Ω_A is a projective A-module.

Remark. The above modules Ω_A, $\Omega_A^{(-1)}$ are cohomology of the *cotangent complex* ${}^L\Omega_A$ (see [**Gr1**], [**Il**], or [**H**] §7) which is the left derived functor of the functor $A \mapsto \Omega_A$. To define it, one extends the category of commutative algebras to that of commutative unital DG algebras where the standard homotopy theory formalism is available (see, e.g., [**H**]). Now for a cofibrant DG algebra R its cotangent complex coincides with the R-module Ω_R considered as the object of the derived category $D(R)$ of R-modules. For arbitrary DG algebra R one considers its cofibrant resolution $\tilde{R} \to R$ and defines ${}^L\Omega_R$ as the image of $\Omega_{\tilde{R}}$ under the equivalence $D(\tilde{R}) \xrightarrow{\sim} D(R)$; it does not depend on the choice of $\tilde{R}$. If A is a plain algebra, then ${}^L\Omega_A$ is acyclic in degrees > 0 and $H^0({}^L\Omega_A) = \Omega_A$, $H^{-1}({}^L\Omega_A) = \Omega_A^{(-1)}$.

2.3.14. Let us return to our tensor category $\mathcal{M}^\ell(X)$ assuming for the moment that X is affine. Let A^ℓ be a commutative $\mathcal{D}_X$-algebra. One check immediately that Ω_A considered as a mere A^ℓ-module (we forget about the $\mathcal{D}_X$-action) coincides with the module of relative differentials $\Omega_{A^\ell/\mathcal{O}_X}$. The same is true for the cotangent complex.

Proposition. (i) *A^ℓ is formally smooth iff it is formally smooth as an $\mathcal{O}_X$-algebra and Ω_{A^ℓ} is a projective $A^\ell[\mathcal{D}_X]$-module.*

(ii) *A^ℓ is formally smooth as an $\mathcal{O}_X$-algebra iff it is formally smooth as a k-algebra.*

Proof. (i) Use the remark above, together with the fact that every projective $A^\ell[\mathcal{D}_X]$-module is a projective A^ℓ-module.

(ii) Follows from the next lemma (the morphism $\Omega_{A^\ell} \to \Omega_X \underset{\mathcal{O}_X}{\otimes} A^\ell$ left inverse to μ corresponds to the canonical derivation ∇: $A^\ell \to \Omega_X \underset{\mathcal{O}_X}{\otimes} A^\ell$).

Lemma. *If A is an $\mathcal{O}_X$-algebra, then A is a formally smooth $\mathcal{O}_X$-algebra iff A is formally smooth as a k-algebra and the morphism of A-modules $\mu: \Omega_X \underset{\mathcal{O}_X}{\otimes} A \to \Omega_A$ is a split injection.*

Proof of Lemma. Since X is smooth over k, the standard long exact sequence reduces to $0 \to \Omega_A^{(-1)} \to \Omega_{A/\mathcal{O}_X}^{(-1)} \to \Omega_X \underset{\mathcal{O}_X}{\otimes} A \to \Omega_A \to \Omega_{A/\mathcal{O}_X} \to 0$; this implies the lemma. Here is another ad hoc proof. Let us represent A as B/J, $B = \operatorname{Sym} V$, V a free $\mathcal{O}_X$-module. The formal smoothness of A over $\mathcal{O}_X$ means that the morphism of A-modules ν: $J/J^2 \to \Omega_{B/\mathcal{O}}/J \cdot \Omega_{B/\mathcal{O}}$ admits a left inverse. The formal smoothness over k means that the morphism $\tilde{\nu}$: $J/J^2 \to \Omega_{B/k}/J \cdot \Omega_{B/k}$ admits a left inverse. It remains to apply the following easy sublemma to $P = J/J^2$, $Q = \Omega_{B/k}/J \cdot \Omega_{B/k}$, $R = \Omega_X \underset{\mathcal{O}_X}{\otimes} A$.

Easy Sublemma. *Suppose we have morphisms* $P \xrightarrow{f} Q \xleftarrow{g} R$ *where* g *is a split injection. Then* $P \to Q/Im\, g$ *is a split injection iff both* $f: P \to Q$ *and* $R \to Q/Im f$ *are such.*

Proof of Easy Sublemma. Write Q as $R \oplus S$, so $g(r) = (r, 0)$, $f(p) = (\varphi(p), \psi(p))$. The morphism $R \to Q/f(P)$ is a split injection iff there is a $\gamma: S \to R$ such that $\varphi = \gamma\psi$, i.e., when $\varphi \in \mathrm{Hom}(S, R)\psi$; the morphism f is a split injection iff $id_P \in \mathrm{Hom}(R, P)\varphi + \mathrm{Hom}(S, P)\psi$. We need to show that $id_P \in \mathrm{Hom}(S, P)\psi$ if and only if $id_P \in \mathrm{Hom}(R, P)\varphi + \mathrm{Hom}(S, P)\psi$ and $\varphi \in \mathrm{Hom}(S, R)\psi$, which is clear. □

2.3.15. From now on we do *not* assume that X is affine.

Let A^ℓ be a $\mathcal{D}_X$-algebra. Back in 2.3.13 we defined Ω_A, $\Omega_A^{(-1)}$ assuming that X is affine. The construction is compatible with the localization of X, and we define these $A^\ell[\mathcal{D}_X]$-modules for arbitrary X by gluing. More generally, we have all $H^a({}^L\Omega_A) \in \mathcal{M}(X, A^\ell)$.[43]

Definition. We say that A^ℓ is

– *formally smooth* if Ω_A is a projective $A^\ell[\mathcal{D}_X]$-module locally on X (see 2.3.6) and $\Omega_R^{(-1)}$ vanishes;

– *smooth* if, in addition, A^ℓ is finitely generated as a $\mathcal{D}_X$-algebra;

– *very smooth* if it is smooth and $H^{\neq 0}({}^L\Omega_A) = 0$.

Notice that the $H^a({}^L\Omega_A)$ are compatible with the étale localization of Spec A^ℓ, so the above notions make sense for an arbitrary algebraic $\mathcal{D}_X$-space $\mathcal{Y}$. If $\mathcal{Y}$ is smooth, then $\Omega_\mathcal{Y}$ is a vector $\mathcal{D}_X$-bundle on $\mathcal{Y}$.

Remarks. (i) $\mathcal{Y}$ is formally smooth if and only if for every affine scheme S equipped with a $\mathcal{D}_X$-scheme structure and every closed $\mathcal{D}_X$-subscheme $S_0 \subset S$ defined by an ideal I with $I^2 = 0$, the map $\mathrm{Mor}_{\mathcal{D}_X}(S, \mathcal{Y}) \to \mathrm{Mor}_{\mathcal{D}_X}(S_0, \mathcal{Y})$ is surjective.[44]

(ii) We do *not* know if there exists a smooth A^ℓ which is not very smooth.[45] We also do not know if a formally smooth A^ℓ is necessary $\mathcal{O}_X$-flat and reduced.

(iii) Suppose that A^ℓ is smooth and, as a mere $\mathcal{O}_X$-algebra, it can be represented locally on X as an inductive limit of smooth $\mathcal{O}_X$-algebras. Then A^ℓ is very smooth (see the remark that opens 2.3.14). In particular, the jet $\mathcal{D}_X$-scheme $\mathcal{J}Z$ for a smooth X-scheme Z is very smooth.

2.3.16. A morphism $\varphi : \mathcal{Y} \to \mathcal{Z}$ of algebraic $\mathcal{D}_X$-spaces is *formally smooth* if for every (S, S_0) as in Remark (i) from 2.3.15, any pair of φ-compatible morphisms $f : S_0 \to \mathcal{Y}$, $g : S \to \mathcal{Z}$ lifts to a morphism $\tilde{f} : S \to \mathcal{Y}$. The reformulation of this property in terms of relative differentials is left to the reader. One says that φ is *formaly étale* if, in addition, $\tilde{f}$ is unique.

[43] In fact, at least when X is quasi-projective, one has a well-defined object ${}^L\Omega_A$ of the derived category $D\mathcal{M}(X, A^\ell)$ defined by means of a global semi-free resolution of A^ℓ; see 4.6.1 and 4.6.4.

[44] To prove the "if" statement, notice that locally a $\mathcal{D}_X$-morphism $f : S_0 \to \mathcal{Y}$ can be extended to a morphism $S \to \mathcal{Y}$ and these extensions form a torsor over $\mathcal{H}om(f^*\Omega_\mathcal{Y}, I)$. Such a torsor is the same as an extension of $f^*\Omega_\mathcal{Y}$ by I. Now $f^*\Omega_\mathcal{Y}$ is locally projective and therefore projective (see 2.3.6), so $\mathrm{Ext}(f^*\Omega_\mathcal{Y}, I) = 0$ and our torsor is trivial.

[45] There seems to be no known example of a formally smooth algebra over a field k of characteristic 0 such that $H^{\neq 0}({}^L\Omega_A)$ does not vanish. For char $k \neq 0$ such an example was communicated to us by O. Gabber.

2.3.17. Coordinate systems. For a $\mathcal{D}_X$-scheme $\mathcal{Y}$ a *coordinate system* on $\mathcal{Y}$ is an étale morphism of $\mathcal{D}_X$-schemes $\nu\colon \mathcal{Y} \to \operatorname{Spec}\operatorname{Sym}^{\cdot}(\mathcal{D}_X^n)$, $n \geq 0$. Any $\mathcal{Y}$ that admits locally a coordinate system is very smooth.[46] The jet scheme $\mathcal{J}Z$ for Z/X smooth admits a local coordinate system (for any étale $\varphi\colon Z \to \mathbb{A}_X^n$ the morphism $\mathcal{J}\varphi\colon \mathcal{J}Z \to \mathcal{J}\mathbb{A}_X^n = \operatorname{Spec}\operatorname{Sym}^{\cdot}(\mathcal{D}_X^n)$ is a coordinate system on $\mathcal{J}Z$). This is *not* true (even at a generic point!) for arbitrary smooth $\mathcal{D}_X$-schemes.

Here is the reason. Suppose that our $\mathcal{Y} = \operatorname{Spec} A^\ell$ is a smooth reduced $\mathcal{D}_X$-scheme such that $\Omega_{A^\ell} \simeq A^\ell[\mathcal{D}_X]$. Notice that any invertible element of $A^\ell[\mathcal{D}_X]$ lies in $A^\ell \subset A^\ell[\mathcal{D}_X]$. Denote by $\mathcal{L}$ the image of A^ℓ by any isomorhism $A^\ell[\mathcal{D}_X] \xrightarrow{\sim} \Omega_{A^\ell}$. This is a canonically defined line subbundle of Ω_A. If $\mathcal{Y}$ admits locally a coordinate system, then $\mathcal{L}$ is an integrable subbundle; i.e., $d(\mathcal{L}) \subset \mathcal{L} \wedge \Omega_{A^\ell} \subset \Omega^2_{A^\ell}$.

2.3.18. Example. Let us construct a smooth A^ℓ with non-integrable $\mathcal{L}$. Take $X = \operatorname{Spec} k[t]$, $B^\ell = k[t, u, v, u', v', \dots, v'^{-1}]$ (here $u' := \partial_t u$, etc., so B^ℓ is the localization of $\operatorname{Sym}^{\cdot}(\mathcal{D}_X^2) = k[t, u, v, u', \dots]$ by v'). Let $I \subset B^\ell$ be the $\mathcal{D}_X$-ideal generated by $\varphi = v + u'v'$; set $A^\ell := B^\ell/I$. Consider the morphism of $B^\ell[\mathcal{D}_X]$-modules

$$\chi\colon B^\ell[\mathcal{D}_X] \oplus B^\ell[\mathcal{D}_X] \to \Omega_{B^\ell}, \quad (\partial_1, \partial_2) \mapsto \partial_1 d\varphi + \partial_2 \xi,$$

where $\xi := du + (u'/v')dv \in \Omega_{B^\ell}$. Localizing B^ℓ (and A^ℓ) further by $v' - u''v' + u'v''$, we see that χ becomes an isomorphism. Now A^ℓ is smooth, and the map $A^\ell[\mathcal{D}_X] \to \Omega_{A^\ell}$, $\partial \mapsto \partial\xi_{A^\ell}$, is an isomorphism. Here ξ_{A^ℓ} is the restriction of ξ to $\operatorname{Spec} A^\ell$; this is a generator of $\mathcal{L}$. One has $\xi_{A^\ell} \wedge d\xi_{A^\ell} = \xi_{A^\ell} \wedge d(u'/v') \wedge dv = -\xi_{A^\ell} \wedge d(v/v'^2) \wedge dv = \alpha\xi_{A^\ell} \wedge dv \wedge dv'$, $\alpha \neq 0$, where u and v are considered as elements of A^ℓ. It is easy to check that $dv = \beta\partial_t\xi_{A^\ell}$, $\beta \neq 0$, so $\xi_{A^\ell} \wedge d\xi_{A^\ell} \neq 0$ and $\mathcal{L}$ is non-integrable.

2.3.19. Vector $\mathcal{D}_X$-schemes. By definition, these are vector space objects in the category $\mathcal{A}ff\mathcal{S}ch_{\mathcal{D}}(X)$ of $\mathcal{D}_X$-schemes affine over X. For a left $\mathcal{D}$-module N^ℓ and $R^\ell \in \mathcal{C}omu_{\mathcal{D}}(X)$ morphisms of $\mathcal{D}_X$-algebras $\operatorname{Sym}(N^\ell) \to R^\ell$ are the same as morphisms of $\mathcal{D}$-modules $N^\ell \to R^\ell$, so they form a vector space. Thus $\mathbb{V}(N^\ell) := \operatorname{Spec}\operatorname{Sym}(N^\ell)$ is a vector $\mathcal{D}_X$-scheme.[47] It is easy to see that the functor

$$\mathbb{V} : \mathcal{M}^\ell(X)^\circ \to \{\text{vector } \mathcal{D}_X\text{-schemes}\} \qquad (2.3.19.1)$$

is an equivalence of categories. The inverse functor assigns to a vector $\mathcal{D}_X$-scheme V the pull-back of Ω_V by the zero section morphism $X \to V$.

Consider the category $\Phi = \Phi(X)$ of all functors $F : \mathcal{C}omu_{\mathcal{D}}(X) \to Sh(X)$;[48] this is a tensor category in the obvious way. A vector $\mathcal{D}_X$-scheme V can be considered as an object of Φ: for an open $j : U \to X$ one has $V(R^\ell)(U) := V|_U(R^\ell|_U) = V(j_*j^*R^\ell)$. It is clear that the category of vector $\mathcal{D}_X$-schemes is a full subcategory of Φ, so we have a fully faithful embedding $\mathbb{V} : \mathcal{M}^\ell(X)^\circ \hookrightarrow \Phi$. On the other hand, according to 2.2.8, we have a fully faithful embedding $\tilde{h} : \mathcal{M}(X) \to \Phi$.

Lemma. *The intersection of these two subcategories is the category of vector $\mathcal{D}_X$-bundles.*

[46] Use the remark that opens 2.3.14.

[47] For a closed point $x \in X$ the fiber $\mathbb{V}(N^\ell)_x$ is the profinite-dimensional vector space $N_x^{\ell *}$ dual to N_x^ℓ.

[48] Recall that $Sh(X)$ is the category of sheaves of k-vector spaces on the étale topology of X.

Proof. Let M be a vector $\mathcal{D}_X$-bundle; $M^\circ := \mathcal{H}om_{\mathcal{D}_X}(M^\ell, \mathcal{D}_X)$ is its dual (see 2.2.16). One has $\tilde{h}(M^\circ)(R^\ell) = h(R^\ell \otimes M^\circ) = M^\circ \underset{\mathcal{D}_X}{\otimes} R^\ell \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}_X}(M^\ell, R^\ell) = \mathbb{V}(M^\ell)$; hence $\tilde{h}(M^\circ) = \mathbb{V}(M^\ell)$.

It remains to show that if for some $N^\ell \in \mathcal{M}(X)^\ell$ there exists $M \in \mathcal{M}(X)$ such that $\tilde{h}(M) = \mathbb{V}(N^\ell)$, then N is a vector $\mathcal{D}_X$-bundle. We can assume that X is affine.

We need some notation. For $L^\ell \in \mathcal{M}^\ell(X)$ the multiplicative group k^* acts on L^ℓ by homotheties, so for any $F \in \Phi$ it acts on the vector space $F(\mathrm{Sym}L^\ell)$. Denote by $F^1(L^\ell) \subset F(\mathrm{Sym}L^\ell)$ the subsheaf on which k^* acts by the standard character. Therefore F yields a functor $F^1 : \mathcal{M}^\ell(X) \to Sh(X)$.

One has $\mathbb{V}(N^\ell)^1(L^\ell) = \underline{\mathrm{Hom}}_{\mathcal{D}_X}(N^\ell, L^\ell)$ and $\tilde{h}(M)^1(L^\ell) = M \underset{\mathcal{D}_X}{\otimes} L^\ell = h(M \otimes L^\ell)$. The latter functor commutes with direct limits and is right exact. If the functors are equal, then N^ℓ is a finitely presented and projective $\mathcal{D}$-module, i.e., a vector $\mathcal{D}_X$-bundle. We are done. □

2.3.20. The calculus of variations. To end the section let us explain, as a mere illustration, the first notions of the calculus of variations in the present context.

Suppose we have a $\mathcal{D}_X$-scheme $\mathcal{Y} = \mathrm{Spec}\, R^\ell$ such that $\Theta_\mathcal{Y}$ is well defined (see Remark (ii) of 2.3.12), e.g., $\mathcal{Y}$ is a smooth $\mathcal{D}_X$-scheme. Any $\nu \in h(\Omega^1_{\mathcal{Y}/X})$ defines a morphism of $R^\ell[\mathcal{D}_X]$-modules $i_\nu : \Theta_\mathcal{Y} \to R$; its image is the *Euler-Lagrange* ideal I_ν. One gets the *Euler-Lagrange* $\mathcal{D}_X$-scheme $\mathcal{Y}_\nu := \mathrm{Spec}\, R^\ell/I^\ell_\nu$. One usually considers the situation when $\nu = d\bar{f}$ for certain $f \in R$ called the *lagrangian*; $\bar{f} \in h(R)$ is called the *action*.

Remark. A more natural object is the DG $\mathcal{D}_X$-algebra $\mathrm{Sym}_{R^\ell}(\Theta^\ell_\mathcal{Y}[1])$ with the differential d_ν which equals i_ν on the degree -1 component $\Theta^\ell_\mathcal{Y}[1]$. If ν is closed (in particular, if it comes from a lagrangian), then it is an odd coisson DG algebra (see Exercise in 1.4.18).

Example. Suppose that $\mathcal{Y} = \mathcal{J}Z$ for a smooth X-scheme Z. Let $\pi : \mathcal{Y} \to Z$ be the canonical morphism. Then $d\pi : \pi^*\Omega^1_{Z/X} \to \Omega^1_{\mathcal{Y}/X}$ identifies $\Omega^1_{\mathcal{Y}/X}$ with the left $R^\ell[\mathcal{D}_X]$-module induced from $\pi^*\Omega^1_{Z/X}$. Thus $h(\Omega^1_{\mathcal{Y}/X}) = \pi^*\Omega^1_{Z/X} \otimes \omega_X = R \underset{\mathcal{O}_Z}{\otimes} \Omega^1_{Z/X}$. The de Rham differential $h(d) : h(R) \to h(\Omega^1_{\mathcal{Y}/X}) = R \underset{\mathcal{O}_Z}{\otimes} \Omega^1_{Z/X}$ is called the *variational derivative*. By (2.2.16.1) there is a canonical isomorphism

$$(\pi^*\Theta_{Z/X})_\mathcal{D} \xrightarrow{\sim} \Theta_\mathcal{Y} \qquad (2.3.20.1)$$

of right $R^\ell[\mathcal{D}_X]$-modules.[49] Thus for a local coordinate system x_i on Z the Euler-Lagrange ideal is generated, as a $\mathcal{D}_X$-ideal, by the $d\bar{f}$-images of $\partial_{x_i} \in \Theta_{Z/X} \subset \Theta_\mathcal{Y}$ in R. We leave it to the reader to check that these are indeed the usual Euler-Lagrange equations.

Remark. Here is another way to define (2.3.20.1). Consider $\Theta_{Z/X}$ as a Lie algebra acting on Z/X. It acts on $\mathcal{Y} = \mathcal{J}Z$ by transport of structure. Since $\Theta_{Z/X}$ is a Lie $\mathcal{O}_Z$-algebroid, this action permits us to define a Lie $\mathcal{O}_\mathcal{Y}$-algebroid structure on

[49] Explicitly, it is the composition of the identifications $\Theta_\mathcal{Y} := \mathcal{H}om_{R^\ell[\mathcal{D}_X]}(\Omega^1_{\mathcal{Y}/X}, R^\ell[\mathcal{D}_X]) = \mathcal{H}om_{R^\ell}(\pi^*\Omega^1_{Z/X}, R^\ell[\mathcal{D}_X]) = \mathcal{H}om_{R^\ell}(\pi^*\Omega^1_{Z/X}, R^\ell) \otimes \mathcal{D}_X = (\pi^*\Theta_{Z/X})_\mathcal{D}$.

$\pi^*\Theta_{Z/X}$. Thus $(\pi^*\Theta_{Z/X})_{\mathcal{D}}$ is naturally a Lie* algebra acting on $\mathcal{Y}$, and (2.3.20.1) is the action morphism.

Suppose we have a Lie* algebra L acting on $\mathcal{Y}$ (see 1.4.9, 2.5.3, 2.5.6(c)). We say that our action is *L-invariant* if the $\tilde{h}(L)$-action on $\tilde{h}(R)$ leaves $\bar{f}$ invariant. In other words, the Lie algebra $h(L)$ fixes $\bar{f} \in h(R)$, and this happens also after any base change, i.e., replacing L by $L \otimes F^\ell$, R^ℓ by $R^\ell \otimes F^\ell$ where F^ℓ is any commutative $\mathcal{D}_X$-algebra. Since h is reliable (see 2.2.8) this happens if and only if the map $h(R) \to \mathrm{Hom}_{\mathcal{D}_X}(L, R)$ defined by the action map (see (2.2.7.1)) $L \boxtimes R \to \Delta_* R$ sends $\bar{f}$ to 0. This is essentially Noether's theorem.

Remark. The above map is equal to the composition $h(R^\ell) \xrightarrow{d} h(\Omega_{\mathcal{Y}/X}) \to \mathrm{Hom}(\Theta_{\mathcal{Y}}, R) \to \mathrm{Hom}(L, R)$ where the last arrow comes from the action morphism $L \to \Theta_{\mathcal{Y}}$. So the property of being L-invariant actually depends on $d\bar{f}$.

2.4. The spaces of horizontal sections

In this section we assume that $X \neq \emptyset$ is connected of dimension n.

We consider the space of global horizontal sections of a $\mathcal{D}_X$-scheme Spec, R^ℓ affine over X, which is the same as the space of global solutions of the corresponding system of differential equations. This is an ind-affine ind-scheme (see 2.4.1) which is a true affine scheme if X is proper (see 2.4.2); the corresponding algebra of functions is denoted by $\langle R \rangle$. The non-proper situation can be reduced to the proper situation by considering all possible extensions of R^ℓ to a compactification (see 2.4.3). In 2.4.4–2.4.6 we give some explicit descriptions of $\langle R \rangle$ which will be of use in Chapter 4. In 2.4.7 we explain how one checks the smoothness of $\langle R \rangle$ assuming that R^ℓ is smooth. In the case of $\dim X = 1$ we consider the ind-scheme of horizontal sections of $\mathrm{Spec}\, R^\ell$ over the formal punctured disc at a point $x \in X$, describe its (topological) algebra of functions R_x^{as} in 2.4.8–2.4.11, and relate the local and global pictures in 2.4.12.

For the (more manageable) derived version of the functor $R^\ell \mapsto \langle R \rangle$, see 4.6.

2.4.1. We will play with ind-schemes in the strict sense calling them simply ind-schemes. So an ind-scheme Y is a functor on the category of commutative algebras, $R \mapsto Y(R)$, which can be represented as the inductive limit of a directed family of quasi-compact schemes and closed embeddings. Some basic material about ind-schemes can be found in [**BD**] 7.11–7.12.

When discussing a topological commutative algebra Q, we always tacitly assume that the topology is complete and has a base formed by open ideals $\mathcal{I}_\alpha$, so $Q = \varprojlim Q/\mathcal{I}_\alpha$. For more details, see 3.6.1. We write $\mathrm{Spf}\, Q := \bigcup \mathrm{Spec}\, Q/\mathcal{I}_\alpha$; this is an ind-affine ind-scheme.

Denote by $\mathcal{C}omu(k)$ the category of commutative unital k-algebras. For any $F \in \mathcal{C}omu(k)$ the constant left $\mathcal{D}$-module $F \otimes \mathcal{O}_X$ is a $\mathcal{D}_X$-algebra in the obvious way. This functor $\mathcal{C}omu(k) \to \mathcal{C}omu_{\mathcal{D}}(X)$ identifies $\mathcal{C}omu(k)$ with the full subcategory $\mathcal{C}omu_{\mathcal{D}}^{\nabla}(X)$ of constant $\mathcal{D}_X$-algebras.[50] It has a right adjoint functor $\Gamma^\nabla \colon \mathcal{C}omu_{\mathcal{D}}(X) \to \mathcal{C}omu(k)$, $\Gamma^\nabla(R^\ell) = \Gamma^\nabla(X, R^\ell) = \mathrm{Hom}(\mathcal{O}_X, R^\ell) = H^0_{DR}(X, R[-n])$.

Let us see if our embedding admits a left adjoint. So for $R^\ell \in \mathcal{C}omu_{\mathcal{D}}(X)$ consider the functor $F \mapsto \mathrm{Hom}(R^\ell, F \otimes \mathcal{O}_X)$ on $\mathcal{C}omu(k)$.

[50] I.e., those $\mathcal{D}_X$-algebras which are constant as a $\mathcal{D}_X$-module; see 2.1.12.

LEMMA. (i) *This functor is representable by a topological commutative algebra* $\langle R\rangle = \langle R\rangle(X)$. *For any* $x \in X(k)$, $\langle R\rangle$ *is naturally a completion of* R^ℓ_x.

(ii) $R \mapsto \langle R\rangle$ *is a tensor functor: one has* $\langle \otimes R_\alpha\rangle = \hat{\otimes}\langle R_\alpha\rangle$.

Proof. (i) Consider the set $\mathcal{C}_R$ of $\mathcal{D}_X$-ideals $I^\ell \subset R^\ell$ such that R^ℓ/I^ℓ is a constant $\mathcal{D}$-algebra. Since any subquotient of a constant $\mathcal{D}$-module is constant, we see that $\mathcal{C}_R$ is a subfilter in the ordered set of all ideals in R^ℓ. For every $I^\ell \in \mathcal{C}_R$ we have a commutative algebra $\langle R\rangle_I := \Gamma^\nabla(X, R^\ell/I^\ell)$, and for $I' \supset I$ a surjective map $\langle R\rangle_I \twoheadrightarrow \langle R\rangle_{I'}$. Therefore $\mathrm{Hom}(R^\ell, F\otimes \mathcal{O}_X) = \bigcup \mathrm{Hom}(R^\ell/I^\ell, F\otimes\mathcal{O}_X) = \bigcup \mathrm{Hom}(\langle R\rangle_I, F) = \mathrm{Hom}_{cont}(\langle R\rangle, F)$ where $\langle R\rangle$ is the $\mathcal{C}_R$-projective limit of the $\langle R\rangle_I$'s.

For any $I^\ell \in \mathcal{C}_R$ one has $\langle R\rangle_I = R^\ell_x/I^\ell_x$, so we have a natural morphism $R^\ell_x \to \langle R\rangle$ with dense image; q.e.d.

(ii) Evident. □

By construction, the ind-affine ind-scheme $\mathrm{Spf}\,\langle R\rangle := \bigcup \mathrm{Spec}\langle R\rangle_I$ is the space of horizontal sections $X \to \mathrm{Spec}\, R^\ell$. It is naturally an ind-subscheme of any fiber $\mathrm{Spec}\, R^\ell_x$. We denote it also by $\langle \mathrm{Spec}\, R^\ell\rangle$.

EXERCISE. If R is formally smooth and X is affine, then $\langle R\rangle$ is formally smooth.

Consider the category $\langle R\rangle$mod of discrete $\langle R\rangle$-modules. Since $\langle R\rangle \otimes \mathcal{O}_X$ is a completion of R^ℓ, we have a fully faithful embedding

(2.4.1.1) $$\langle R\rangle\mathrm{mod} \to \mathcal{M}(X, R^\ell), \quad V \mapsto V\otimes\omega_X.$$

EXERCISE. Show that its essential image is equal to the subcategory of $R^\ell[\mathcal{D}_X]$-modules which are constant as $\mathcal{D}_X$-modules.

Notice that $\langle R\rangle(X)$ is covariantly functorial with respect to the étale localization of X. Precisely, for any morphism $U' \to U$ in $X_{\acute{e}t}$, where U', U are connected, there is a natural morphism $\langle R\rangle(U') \twoheadrightarrow \langle R\rangle(U)$. It is immediate that $U \mapsto \mathrm{Spf}\langle R\rangle(U)$ is a sheaf of ind-schemes on $X_{\acute{e}t}$; we denote it by $\mathrm{Spf}\langle R\rangle_X = \langle \mathrm{Spec}\, R^\ell\rangle_X$.

2.4.2. Suppose that X is proper.

PROPOSITION. (i) $\langle R\rangle$ *is a discrete algebra.*

(ii) *If* R *is finitely generated as a* $\mathcal{D}_X$*-algebra, then* $\langle R\rangle$ *is finitely generated.*

Proof. (i) We want to show that R^ℓ admits the maximal constant $\mathcal{D}$-algebra quotient R^ℓ/I^ℓ_0. By 2.1.12, R^ℓ admits the maximal constant $\mathcal{D}$-module quotient $\sigma_R : R^\ell \twoheadrightarrow R^\ell_{const}$ (see 2.1.12). Then I^ℓ_0 is the ideal generated by the kernel of σ_R.

(ii) Clear, since the equivalence $\mathcal{C}omu^\nabla_{\mathcal{D}}(X) \xrightarrow{\sim} \mathcal{C}omu(k)$ identifies constant $\mathcal{D}_X$-algebras which are finitely generated as $\mathcal{D}_X$-algebras with finitely generated algebras. □

2.4.3. Let $j : U \hookrightarrow X$ be a non-empty Zariski open subset. For $R^\ell \in \mathcal{C}omu_{\mathcal{D}}(U)$ the left $\mathcal{D}$-module $j.R^\ell := H^0 j_* R^\ell$ is a $\mathcal{D}_X$-algebra in the obvious way. Let $\Xi^c(R)$ denotes the set of $\mathcal{D}_X$-subalgebras $R^\ell_\xi \subset j.R^\ell$ such that $j^* R^\ell_\xi = R^\ell$. The intersection of two subalgebras from $\Xi^c(R)$ belongs to $\Xi^c(R)$, so this is a subfilter in the ordered set of all $\mathcal{D}_X$-subalgebras of $j.R^\ell$. We have a $\Xi^c(R)$-projective system of topological algebras[51] $\langle R_\xi\rangle(X)$ connected by epimorphisms. The canonical maps

[51] If X is proper, then the $\langle R_\xi\rangle(X)$ are discrete; see 2.4.2.

$\langle R\rangle(U) \to \langle R_\xi\rangle(X)$ yield a morphism (the limit is taken over $\Xi^c(R)$)

$$\langle R\rangle(U) \to \varprojlim \langle R_\xi\rangle(X). \tag{2.4.3.1}$$

PROPOSITION. *This is an isomorphism.*

Proof. It suffices to show that any morphism[52] $\varphi\colon R^\ell \to F\otimes\mathcal{O}_U$ is the restriction to U of some morphism $\varphi_X\colon R^\ell_\xi \to F\otimes\mathcal{O}_X$, $R^\ell_\xi \in \Xi^c(R)$. Note that φ defines a morphism of $\mathcal{D}_X$-algebras $j_\cdot(\varphi)\colon j_\cdot R^\ell \to j_\cdot(F\otimes\mathcal{O}_U)$, and $F\otimes\mathcal{O}_X$ is a $\mathcal{D}_X$-subalgebra of $j_\cdot(F\otimes\mathcal{O}_U)$. Set $R^\ell_\xi := j_\cdot(\varphi)^{-1}(F\otimes\mathcal{O}_X)$, $\varphi_X := j_\cdot(\varphi)\big|_{R^\ell_\xi}$. We are done. □

COROLLARY. *If R is a finitely generated $\mathcal{D}_X$-algebra, then* $\mathrm{Spf}\langle R\rangle_X$ *is a sheaf of ind-schemes of ind-finite type.* □

2.4.4. In 2.4.4–2.4.7 we assume that X is proper. Here are some "explicit" descriptions of $\langle R\rangle$.

We saw in 2.1.12 that $R^\ell_{const} = H_0^{DR}(X,R)\otimes\mathcal{O}_X$, so $\langle R\rangle$ is a quotient of $H_0^{DR}(X,R)$.

Let $\mathcal{I}^\ell \subset R^\ell\otimes R^\ell$ be the kernel of the multiplication map $R^\ell\otimes R^\ell \to R^\ell$. Consider the morphism $\sigma_R^{\otimes 2} : R^\ell\otimes R^\ell \to H_0^{DR}(X,R)^{\otimes 2}\otimes\mathcal{O}_X$. Restricting it to $\mathcal{I}$, passing to the de Rham cohomology, and using $\mathrm{tr} : H_0^{DR}(X,\omega_X) \xrightarrow{\sim} k$, we get a morphism

$$\nu : H_0^{DR}(X,\mathcal{I}) \to H_0^{DR}(X,R)^{\otimes 2}. \tag{2.4.4.1}$$

The projection $H_0^{DR}(X,R) \twoheadrightarrow \langle R\rangle$ together with the product on $\langle R\rangle$ identifies $\langle R\rangle$ with a quotient of $H_0^{DR}(X,R)^{\otimes 2}$.

LEMMA. *One has $\langle R\rangle = \mathrm{Coker}\,\nu$.*

Proof. Consider a commutative diagram

$$\begin{array}{ccc} R^\ell & \longrightarrow & \langle R\rangle\otimes\mathcal{O}_X \\ \uparrow & & \uparrow \\ R^\ell\otimes R^\ell & \longrightarrow & H_0^{DR}(X,R)^{\otimes 2}\otimes\mathcal{O}_X \end{array}$$

According to the proof of 2.4.2(i), $\langle R\rangle\otimes\mathcal{O}_X$ is the quotient of R^ℓ modulo the ideal $R^\ell\cdot\mathrm{Ker}\,\sigma_R$, or, equivalently, the quotient of $R^\ell\otimes R^\ell$ modulo $\mathcal{I} + \mathrm{Ker}\,\sigma_R\otimes R^\ell + R^\ell\otimes\mathrm{Ker}\,\sigma_R$. This is $\mathrm{Coker}\nu\otimes\mathcal{O}_X$; q.e.d. □

2.4.5. Here is another description of $\langle R\rangle$. Let $\Delta\colon X \hookrightarrow X\times X$ be the diagonal embedding, $j\colon U := X\times X \smallsetminus \Delta(X) \hookrightarrow X\times X$ its complement. Consider the morphism $Rj_*j^*R\boxtimes R \to \Delta_*R[-n+1]$ in $D\mathcal{M}(X\times X)$ defined as the composition of the canonical "residue" morphism $Rj_*j^*(R\boxtimes R) \to \Delta_*\Delta^!(R\boxtimes R)[1] \xrightarrow{\sim} \Delta_*(R\overset{L}{\otimes}{}^!R)[-n+1]$ and the product $R\overset{L}{\otimes}{}^!R \to R\otimes^! R \to R$. Passing to de Rham cohomology,[53] we get a morphism of vector spaces

$$\mu :\ H^0_{DR}(U, R\boxtimes R[2n-1]) \to H_0^{DR}(X,R). \tag{2.4.5.1}$$

[52]Here F is a commutative algebra.

[53]One has $H^\cdot_{DR}(X\times X, j_*j^*B) = H^\cdot_{DR}(U,B)$, $H^\cdot_{DR}(X\times X,\Delta_*C) = H^\cdot_{DR}(X,C)$.

Lemma. *One has* $\langle R\rangle = \operatorname{Coker}\mu$.

Proof. Denote by C the cone of our morphism $j_*j^*R\boxtimes R[-1]\to \Delta_* R[-n]$; this is an object of $D\mathcal{M}(X\times X)$.[54] One has[55] $\operatorname{Coker}\mu = H^{2n}_{DR}(X\times X, C)$.

Consider the obvious morphism $R\boxtimes R\to C$. Denote by D its cone, so we have the exact triangle $R\boxtimes R\to C\to D$. The complex D is supported on the diagonal; its top non-zero cohomology is $H^{n-1}D = \Delta_*\mathcal{I}$. Therefore $H^{2n-1}_{DR}(X\times X, D) = H^n_{DR}(X,\mathcal{I})$ and all the higher de Rham cohomologies vanish. So the long exact sequence of de Rham cohomology identifies $H^{2n}_{DR}(X\times X, C)$ with the cokernel of the arrow $H^{2n-1}_{DR}(X\times X, D)\to H^{2n}_{DR}(X\times X, R\boxtimes R)$ which may be rewritten as $\nu' : H^n_{DR}(X,\mathcal{I})\to H^n_{DR}(X,R)^{\otimes 2}$.

So we see that $\operatorname{Coker}\mu = \operatorname{Coker}\nu'$. We leave it to the reader to check that our ν' equals the map ν from (2.4.4.1). Now use 2.4.4. □

2.4.6. Now let $\{x_s\}\subset X$, $s\in S$, be a finite non-empty subset. We have the canonical residue map $\operatorname{Res}_S = (\operatorname{Res}_{x_s})\colon H^{n-1}_{DR}(X\smallsetminus S, R)\to \prod R^\ell_{x_s}$ (see 2.1.12); denote its image by V_S. Let $\otimes R^\ell_{x_s}\twoheadrightarrow \langle R\rangle$ be the ring morphism that combines the surjections $R^\ell_{x_s}\twoheadrightarrow\langle R\rangle$, $s\in S$ (see 2.4.1 and 2.4.2). As follows from the definition of $\langle R\rangle$ (and 2.1.12), its kernel contains $f(V_S)$ where the map $f:\prod R^\ell_{x_s}\to \otimes R^\ell_{x_s}$ sends $(a_s)\in\prod R^\ell_{x_s}$ to $\Sigma a_s\otimes 1^{\otimes S\smallsetminus\{s\}}$. Hence it contains the ideal I_S generated by $f(V_S)$. We have defined $\alpha_S : (\otimes R^\ell_{x_s})/I_S\twoheadrightarrow\langle R\rangle$.

Lemma. *One has* $\alpha_S : (\otimes R^\ell_{x_s})/I_S \xrightarrow{\sim} \langle R\rangle$.

Proof. Consider the natural projections $\pi_{x_s} : R^\ell_{x_s}\twoheadrightarrow (R^\ell_{const})_{x_s} = H^{DR}_0(X,R)$. According to 2.1.12, the kernel of $\pi_S = \Sigma\pi_{x_s} : \prod R^\ell_{x_s}\twoheadrightarrow H^{DR}_0(X,R)$ equals V_S.

(a) Assume that $|S|=1$, so we have a single point $x_s = x\in X$. As follows from the proof of 2.4.2(i), $\langle R\rangle\otimes\mathcal{O}_X$ is the quotient of R^ℓ modulo the ideal generated by the kernel of the projection $\pi : R^\ell\twoheadrightarrow R^\ell_{const}$. Therefore, passing to the x-fibers, we see that $\langle R\rangle$ is the quotient of R^ℓ_x modulo the ideal I_x generated by $\operatorname{Ker}\pi_x = V_x$, and we are done.

(b) Assume that $|S|\geq 2$. Take any $0\in S$. Then $R^\ell_{x_0}\cap V_S = V_{x_0}$ (indeed, both subspaces are equal to the kernel of π_{x_0}). Therefore the morphism $R^\ell_{x_0}\to \underset{S}{\otimes} R^\ell_{x_s}$ yields a morphism $\zeta : R^\ell_{x_0}/I_{x_0}\to (\underset{S}{\otimes} R^\ell_{x_s})/I_S$. Since $R^\ell_{x_0} + V_S = \prod R^\ell_{x_s}$ (which follows from the surjectivity of π_{x_0}), we see that ζ is surjective. The composition of ζ with α_S is the map $\alpha_{x_0} : R^\ell_{x_0}/I_{x_0}\to\langle R\rangle$ for the 1-point set $\{x_0\}$ which we know to be an isomorphism by (a). Thus α_S is an isomorphism. □

Remark. One may also deduce 2.4.6 in the case $|S|=1$ (i.e., (a) above) from 2.4.5. Namely, set $U_x := X\smallsetminus\{x\}$; denote by i_x, $\tilde{i}_x$ the embeddings $\{x\}\hookrightarrow X$, $X=\{x\}\times X\hookrightarrow X\times X$. Then $\tilde{i}^!_x(R\boxtimes R) = (i^!_s R)\otimes R$, so we have a commutative diagram

$$
\begin{array}{ccc}
H^0_{DR}(U, R\boxtimes R[2n-1]) & \xrightarrow{\ \mu\ } & H^{DR}_0(X,R)\\
\uparrow & & \uparrow\\
R^\ell_x\otimes H^0_{DR}(U_x, R[n-1]) & \xrightarrow{\ \mu_x\ } & R^\ell_x
\end{array}
\tag{2.4.6.1}
$$

[54] In the most important case of $n=1$ our C is actually a $\mathcal{D}$-module, not merely a complex.

[55] Since U is non-compact, $H^{2n}_{DR}(U,B)=0$ for every $B\in\mathcal{M}(U)$.

the vertical arrows come from canonical morphisms $\tilde{i}_{x*}\tilde{i}^!_x(R\boxtimes R) \to R\boxtimes R$, $i_{x*}i^!_x R \to R$, and μ_x sends $a \otimes b$ to $a\,\mathrm{Res}_x b$. The standard long exact sequences show that the vertical arrows are surjective; the kernel of the right one is $\mathrm{Im}\,(\mathrm{Res}_x)$. Since $\mathrm{Res}_x(b) = \mu_x(1\otimes b)$, one has $\mathrm{Im}\,(\mathrm{Res}_x) \subset \mathrm{Im}(\mu_x)$. Therefore, $\mathrm{Coker}\,\mu = R^\ell_x/\mathrm{Im}(\mu_x)$ $= R^\ell_x/I_x$, so, by 2.4.5, we have $R^\ell_x/I_x \xrightarrow{\sim} \langle R\rangle$. One checks immediately that this is the morphism of 2.4.6.

2.4.7. For $M \in \mathcal{M}(X, R^\ell)$ the quotient $M_{\langle R\rangle} := M/I^\ell_0 M$ is an $\langle R\rangle \otimes \mathcal{D}_X$-module; therefore $H^{DR}_0(X, M_{\langle R\rangle})$ is an $\langle R\rangle$-module. It follows from 2.1.12 that the functor $M \mapsto H^{DR}_0(X, M_{\langle R\rangle})$ is left adjoint to the above embedding.

PROPOSITION. (i) *One has a canonical isomorphism* $\Omega_{\langle R\rangle} \xrightarrow{\sim} H^{DR}_0(X, (\Omega_R)_{\langle R\rangle})$.

(ii) *Suppose R^ℓ is smooth, $H^{DR}_1(X, (\Omega_R)_{\langle R\rangle}) = 0$, and $H^{DR}_0(X, (\Omega_R)_{\langle R\rangle})$ is a projective $\langle R\rangle$-module. Then $\langle R\rangle$ is smooth. The same is true for "smooth" replaced by "formally smooth".*

Proof. (i) Follows from the adjunction.

(ii) By 2.4.2 it suffices to consider formal smoothness. We want to show that every extension P of $\langle R\rangle$ by an ideal N of square 0 (which is an $\langle R\rangle$-module) splits. Let $\tilde{R}^\ell$ be the pull-back of $P \otimes \mathcal{O}_X$ by the morphism $R^\ell \to \langle R\rangle \otimes \mathcal{O}_X$; this is a $\mathcal{D}_X$-algebra extension of R^ℓ by the ideal $N \otimes \mathcal{O}_X$ of square 0. By adjunction, it suffices to show that this extension splits. It happens locally on X since R^ℓ is formally smooth. The obstruction to the existence of global splitting lies in $\mathrm{Ext}^1_{R^\ell[\mathcal{D}_X]}(\Omega_R, N \otimes \omega_X) = \mathrm{Ext}^1_{\langle R\rangle\otimes\mathcal{D}_X}((\Omega_R)_{\langle R\rangle}, N \otimes \omega_X)$ (the latter equality comes since Ω_R is a locally projective $R^\ell[\mathcal{D}_X]$-module). By duality, it equals $\mathrm{Hom}_{\langle R\rangle}(R\Gamma_{DR}(X, \Omega_R)_{\langle R\rangle}[n-1], N)$ which vanishes by our conditions. □

REMARK. Suppose $\dim X = 1$. For smooth R the conditions of (ii) amount to the smoothness of $\langle R\rangle = H^{ch}_0(X, R)$ and the vanishing of $H^{ch}_1(X, R)$ (see 4.6).

2.4.8. For the rest of the section, X is a curve. Let $x \in X$ be a (closed) point, $j_x : U_x \hookrightarrow X$ the complement to x. For $R^\ell \in \mathcal{C}omu_{\mathcal{D}}(U_x)$ consider the topology $\Xi^{as}_x = \Xi^{as}_x(j_{x*}R)$ on $j_{x*}R$ at x (see 2.1.13 for terminology) formed by all $\mathcal{D}_X$-subalgebras R^ℓ_ξ such that $j^*_x R_\xi = R$. We have

$$R^\ell_{\xi x} = i^!_x((j_{x*}R)/R_\xi) = h((j_{x*}R)/R_\xi)_x. \tag{2.4.8.1}$$

We see that for $R^\ell_{\xi'} \subset R^\ell_\xi$ the morphism of fibers $R^\ell_{\xi' x} \to R^\ell_{\xi x}$ is surjective. Denote by R^{as}_x the Ξ^{as}_x-projective limit of the commutative algebras $R^\ell_{\xi x}$.[56] As a plain topological vector space it is equal to the Ξ^{as}_x-completion of the vector space $h(j_{x*}R)_x$ (see 2.1.13).

REMARK. The functor $R^\ell \mapsto R^{as}_x$ is *not* compatible with the localization of R^ℓ, so it cannot be extended to non-affine $\mathcal{D}_{U_x}$-schemes.

Let Q be any topological commutative algebra. Following [**BD**] 7.11.1, we call an ideal $I \subset Q$ (and the quotient Q/I) *reasonable* if I is open and for every open ideal $J \subset I$ the ideal $I/J \subset Q/J$ is finitely generated. We say that Q (and the ind-scheme $\mathrm{Spf}\,Q$) is *reasonable* if the reasonable ideals form a base of the topology.

[56]The superscript "*as*" means "associated" or "associative"; this notation will be clearified in 3.6.

Example. The topological algebras $k[t_0, t_1, \cdots]] := \varprojlim k[t_0, t_1, \cdots, t_n]$ and $k[\cdots, t_{-1}, t_0, t_1, \cdots]] := \varprojlim k[\cdots, t_{-1}, t_0, t_1, \cdots, t_n]$ are reasonable, while the algebra $k[[t_0, t_1, \cdots]] := \varprojlim k[t_0, t_1, \cdots]/\mathfrak{m}^n$ is unreasonable (here $\mathfrak{m} \subset k[t_0, t_1, \cdots]$ is the ideal generated by all t_i's).

Lemma. *For any finitely generated $\mathcal{D}$-algebra R^ℓ the topological algebra R_x^{as} is reasonable. More precisely, if $R_\xi^\ell \in \Xi_x^{as}(X)$ is a finitely generated $\mathcal{D}_X$-algebra, then $R_{\xi x}^\ell$ is a reasonable quotient of R_x^{as}.*

Proof. It suffices to consider the case of a free finitely generated $\mathcal{D}_{U_x}$-algebra; here the statement is obvious. □

2.4.9. Let us show that the ind-affine ind-scheme $\operatorname{Spf} R_x^{as} := \bigcup \operatorname{Spec} R_{\xi x}^\ell$, $\xi \in \Xi_x^{as}$, is the space of horizontal sections of $\operatorname{Spec} R^\ell$ over the formal punctured disc at x. Denote by K_x the field of fractions of $O_x := \varprojlim \mathcal{O}_x/\mathrm{m}_x^n$. For a vector space H set $H \widehat{\otimes} O_x := \varprojlim H \otimes O_x/\mathrm{m}_{O_x}^n$, $H \widehat{\otimes} K_x := \varprojlim H \otimes K_x/\mathrm{m}_{O_x}^n$. So if t is a parameter at x, then $H \widehat{\otimes} O_x = H[\![t]\!]$, $H \widehat{\otimes} K_x = H(\!(t)\!)$. Let us consider $H \widehat{\otimes} O_x \subset H \widehat{\otimes} K_x$ as sheaves on X supported at x; these are *non-quasi-coherent* sheaves of $\mathcal{D}_X$-modules ($\mathcal{D}_X$-sheaves in the terminology of 3.5.1) in the obvious way. If H is an algebra, then $H \widehat{\otimes} O_x$, $H \widehat{\otimes} K_x$ are naturally (non-quasi-coherent) $\mathcal{D}_X$-algebras.

Proposition. *For a commutative algebra H there is a canonical identification*

$$\operatorname{Hom}(R_x^{as}, H) = \operatorname{Hom}(j_{x*}R^\ell, H \widehat{\otimes} K_x). \tag{2.4.9.1}$$

Here the former Hom *is the set of continuous morphisms of commutative algebras; the latter one is that of morphisms of commutative $\mathcal{D}_X$-algebras.*

Proof (cf. the proof of the proposition in 2.4.3). Suppose we have a morphism of commutative $\mathcal{D}_X$-algebras $\phi : j_{x*}R^\ell \to H \widehat{\otimes} K_x$. Since $H \widehat{\otimes} O_x$ is a $\mathcal{D}_X$-subalgebra of $H \widehat{\otimes} K_x$ and the quotient $H \widehat{\otimes} K_x / H \widehat{\otimes} O_x$ is quasi-coherent, we see that $R_\phi^\ell := \phi^{-1}(H \widehat{\otimes} O_x) \subset j_{x*}R^\ell$ belongs to $\Xi_x^{as}(j_{x*}R)$ and $\phi_x : R_{\phi x}^\ell \to H$ is a morphism of algebras. We have defined a map $\operatorname{Hom}(j_{x*}R^\ell, H \widehat{\otimes} K_x) \to \bigcap \operatorname{Hom}(R_{\xi x}^\ell, H) = \operatorname{Hom}(R_x^{as}, H)$, $\phi \mapsto \phi_x$. The verification of its bijectivity reduces easily to the case when R^ℓ is a free $\mathcal{D}_{U_x}$-algebra, then to that of $R^\ell = \operatorname{Sym}(\mathcal{D}_{U_x})$ where it is clear. □

Remark. The above proposition also follows easily from its linear counterpart (3.5.4.2).

2.4.10. Corollary. (i) *For $N^\ell \in \mathcal{M}^\ell(U_x)$ one has*

$$(\operatorname{Sym} N^\ell)_x^{as} = \operatorname{Sym} \hat{h}_x(j_{x*}N) \tag{2.4.10.1}$$

where the latter Sym *is the topological symmetric algebra, i.e., the completion of the plain symmetric algebra with respect to the topology formed by ideals generated by open vector subspaces of $\hat{h}_x(j_{x*}N)$. In particular, $(\operatorname{Sym} \mathcal{D}_{U_x})_x^{as} = \operatorname{Sym}(\omega_{K_x})$.*

(ii) *If R is formally smooth, then the topological algebra R_x^{as}, or the ind-scheme $\operatorname{Spf} R_x^{as}$, is formally smooth.*

(iii) *$R \mapsto R_x^{as}$ is a tensor functor: one has $(\otimes R_\alpha)_x^{as} = \hat{\otimes} R_{\alpha x}^{as}$.* □

Remark. For an explicit description of formally smooth topological algebras, see [**BD**] 7.12.20–7.12.23.

2.4.11. For a commutative $\mathcal{D}$-algebra R^ℓ on X and an $R^\ell[\mathcal{D}_X]$-module N we denote by $\Xi_x^R(N)$ the topology on N at x (see 2.1.13) formed by all $N_\eta \in \Xi_x(N)$ which are R^ℓ-submodules of N. Since each $N_{\eta x}$ is an R_x^ℓ-module, the completion $\hat{h}_x^R(N)$ is a topological R_x^ℓ-module.

For R^ℓ as in 2.4.8 and an R^ℓ-module M let $\Xi_x^R(j_{x*}M)$ be the topology at x on $j_{x*}M$ formed by all $M_\eta \subset j_{x*}M$ such that $M_\eta \in \Xi_x^{R_\xi}(j_{x*}M)$ for some $R_\xi \in \Xi_x^{as}(j_{x*}R)$. The completion $\hat{h}_x^R(M) := \hat{h}_x^R(j_{x*}M)$ is a topological R_x^{as}-module.

Remarks. (i) Suppose M is generated by finitely many sections $\{m_\alpha\}$. Then the submodules $\sum R_\xi^\ell[\mathcal{D}_X] \cdot t^n m_\alpha$, where $R_\xi^\ell \in \Xi_x^{as}(R^\ell)$ and n a positive integer, form a base of the topology $\Xi_x^R(j_{x*}M)$. Here t is a parameter at x, and we restricted X to make t invertible on U_x.

(ii) Since $\hat{h}_x^R(M) = \varprojlim M_{\eta x}^\ell$, we see that there is a natural continuous transformation $\hat{h}_x^R(\otimes M_i) \to \hat{\otimes}\hat{h}_x^R(M_i)$. Thus $\hat{h}_x^R$ transforms !-coalgebras to topological coalgebras.

2.4.12. Let $\{x_s\} \subset X$, $s \in S$, be a finite set, $j_S : U_S \hookrightarrow X$ its complement. For a $\mathcal{D}$-algebra R on U_S set $R_S^{as} = \widehat{\otimes}\, R_{x_s}^{as} := \varprojlim \otimes R_{\xi_{x_s} x_s}^\ell$. Assume now that X is proper. Subalgebras from $\Xi^c(R)$ (see 2.4.3) are just intersections of subalgebras from $\Xi_{x_s}^{as}(j_{S*}R)$, $s \in S$, so, by 2.4.3, we have the canonical maps $R_{x_s}^{as} \to \langle R\rangle(U_S)$. They define a continuous morphism

$$v\colon R_S^{as} \to \langle R\rangle(U_S). \tag{2.4.12.1}$$

According to 2.4.6, it identifies $\langle R\rangle(U_S)$ with the quotient of R_S^{as} modulo the closed ideal generated by the image of the map

$$r_S = \varprojlim \mathrm{Res}_S^{R_\xi}\colon\ H_{DR}^0(U_S, R) \to \Pi R_{x_s}^{as} \to R_S^{as}. \tag{2.4.12.2}$$

Remark. r_S evidently factors thru the quotient $\Gamma(U_S, h(R))$ of $H_{DR}^0(U_S, R)$.

In geometric language, the embedding $\mathrm{Spf}\,\langle R\rangle(U_S) \hookrightarrow \mathrm{Spf}\,R_S^{as}$ assigns to a horizontal section of $\mathrm{Spec}\,R^\ell$ over U_S the collection of its restrictions to the formal punctured discs at x_s. Notice that $\mathrm{Spec}\,\langle R\rangle(X)$ is the intersection of $\mathrm{Spf}\,\langle R\rangle(U_S)$ and $\prod \mathrm{Spec}\,R_{x_s}^\ell$ in $\mathrm{Spf}\,R_S^{as}$.

2.5. Lie* algebras and algebroids

The notion (if not the name) of the Lie* algebra has been, undoubtedly, well known in mathematical physics for quite a long time. Since the first version of this chapter was available as a preprint in 1995, Lie* algebras migrated to mathematical literature. For example, they appear (in the translation equivariant setting; cf. 0.15) in [**K**], [**DK**] under the alias of "conformal algebras", which further mutated into "pseudoalgebras" of [**BAK**], and in [**P**], [**DLM**], [**FBZ**] as "vertex Lie algebras" (while "conformal algebras" of [**FBZ**] are translation equivariant Lie* algebras equipped with a "Virasoro vector").

We begin with remarks about $\mathcal{B}^*$ algebras for any operad $\mathcal{B}$ and the corresponding h-sheaves of $\mathcal{B}$ algebras (see 2.5.1 and 2.5.2). Passing to Lie* algebras, we describe Lie* brackets in terms of the corresponding adjoint actions in 2.5.5. First examples of Lie* algebras are in 2.5.6; in 2.5.7 we discuss the relation between Lie$^!$ coalgebras and Lie* algebras. The examples of Kac-Moody and Virasoro algebras (here X is a curve) are in 2.5.8–2.5.10. In 2.5.12–2.5.15 we play with a natural topology on the Lie algebra $h(L)_x$, L a Lie* algebra, $x \in X$. The rest of the section deals with Lie* algebroids. They are introduced in 2.5.16; the equivariant setting is discussed in 2.5.17. We show that a Lie* algebroid yields (topological) Lie algebroids on ind-schemes of sections over formal punctured discs in 2.5.18–2.5.21. For a global theory see 4.6.10 and 4.6.11. Elliptic Lie* algebroids are considered in 2.5.22; a geometric example (suggested by D. Gaitsgory) is discussed in 2.5.23.

The reader is referred to [**K**], [**DK**], [**BAK**] for solutions to some natural classification problems (e.g. a description of simple translation equivariant Lie* algebras on $\mathbb{A}^1$ which are coherent $\mathcal{D}$-modules), and to [**BKV**] for some computations of the Lie* algebra cohomology. A certain $*$ version of the Wedderburn theorem can be found in [**Re**].

2.5.1. Having at hand the $*$ pseudo-tensor structure, we may consider for any k-operad $\mathcal{B}$ the category $\mathcal{B}^*(X)$ of $\mathcal{B}$ algebras in $\mathcal{M}(X)^*$, and for any $L \in \mathcal{B}^*(X)$ (and a $\mathcal{B}$-module operad $\mathcal{C}$) the corresponding category $\mathcal{M}(X, L) = \mathcal{M}(X)(L, \mathcal{C})$ of L-modules. We have the sheaves of categories $\mathcal{B}^*(X_{\acute{e}t})$ and $\mathcal{M}(X_{\acute{e}t}, L)$ on $X_{\acute{e}t}$.

For a locally closed embedding $i: X \hookrightarrow Y$ one has (see 2.2.5) the adjoint functors

$$\mathcal{B}^*(X) \underset{i^!}{\overset{i_*}{\rightleftarrows}} \mathcal{B}^*(Y). \tag{2.5.1.1}$$

If X is a closed subscheme, then i_* identifies $\mathcal{B}^*(X)$ with the full subcategory of $\mathcal{B}^*(Y)$ that consists of algebras supported on X.

2.5.2. The functor h sends $\mathcal{B}^*$ algebras on X to $\mathcal{B}$ algebras in the tensor category $Sh^{\otimes}(X)$ of sheaves of k-vector spaces on X.

If R^ℓ is a commutative $\mathcal{D}_X$-algebra, then a $\mathcal{B}^*$ algebra structure on a $\mathcal{D}$-module L yields a $\mathcal{B}^*$ algebra structure on $R^\ell \otimes L$; hence a $\mathcal{B}$ algebra structure on $\tilde{h}(L)(R^\ell) := h(L \otimes R^\ell) = L \underset{\mathcal{D}_X}{\otimes} R^\ell$ (see 1.4.6). Since h is a reliable augmentation functor (see 1.4.7 and 2.2.8), this gives a bijective correspondence between $\mathcal{B}^*$ algebra structures on L and data which assign to every commutative $\mathcal{D}_X$-algebra R^ℓ a $\mathcal{B}$ algebra structure on $h(L \otimes R^\ell)$ in a way compatible with morphisms of $\mathcal{D}_X$-algebras.

For $L \in \mathcal{B}^*(X)$ an L-module structure on a $\mathcal{D}$-module M defines on M the $\mathcal{C}(h(L))$-module structure (see 1.2.17). Since h is non-degenerate (see 2.2.8), we get a fully faithful embedding

$$\mathcal{M}(X, L) \hookrightarrow \mathcal{C}(h(L))\text{-modules in } \mathcal{M}(X). \tag{2.5.2.1}$$

Its image can be described using 2.2.19; we leave this description to the reader.

Due to Remark (iii) in 2.2.3 and 1.2.13, we can also consider L-module structures on $\mathcal{O}_X$-modules.

2.5.3. We will be mainly interested in Lie* algebras. Therefore a Lie* algebra is a $\mathcal{D}$-module L equipped with a Lie* bracket, i.e., a $*$-pairing $[\ ] \in P_2^*(\{L, L\}, L)$ which is skew-symmetric and satisfies the Jacobi identity. Then the sheaf $h(L)$ is a Lie algebra with bracket $[\bar{\ell}_1, \bar{\ell}_2] = \overline{[\ ](\ell_1 \boxtimes \ell_2)}$; moreover, $\tilde{h}(L)(R^\ell) := h(L \otimes R^\ell)$ is a Lie algebra for every commutative $\mathcal{D}_X$-algebra R^ℓ. Recall that $\mathcal{C}(h(L))$ is the universal enveloping algebra of $h(L)$ and a $\mathcal{C}(h(L))$-module is the same as an $h(L)$-module.

Consider the category Φ^{Lie} of all functors F on $\mathcal{C}omu_{\mathcal{D}}(X)$ with values in the category of sheaves of Lie k-algebras on X (i.e., Φ^{Lie} is the category of Lie algebras in the tensor category Φ from 2.3.19). A Lie* algebra L yields $\tilde{h}(L) \in \Phi^{Lie}$; the functor

$$\tilde{h} : \mathcal{L}ie^*(X) \to \Phi^{Lie} \tag{2.5.3.1}$$

is a fully faithful embedding.

For $\mathcal{D}$-submodules $M, N \subset L$, following the notation of 2.2.3, we define $[M, N] \subset L$ by $\Delta_*[M, N] := [\ ](M \boxtimes N)$.

2.5.4. An L-action on $M \in \mathcal{M}(X)$ yields an $h(L)$-action on M given by

$$\bar{\ell}(m) = Tr_2(\,\cdot\,(\ell \boxtimes m)). \tag{2.5.4.1}$$

Here $\cdot \in P_2^*(\{L, M\}, M)$ is the L-action and $Tr_2 \colon \Delta^{\cdot}\Delta_* M = pr_{2\cdot}\Delta_* M \to M$ is the trace map for the projection $pr_2 \colon X \times X \to X$.

According to 2.2.19, this way we get a bijective correspondence between the L-module structures on $M \in \mathcal{M}(X)$ and such $h(L)$-actions on M (we call them *good* actions) that for every $m \in M$ the map $L \to M$, $\ell \mapsto \bar{\ell}m$, is a differential operator.

Remarks. (i) Let M be an L-module. For a section $\bar{m}$ of $h(M)$ its *stabilizer* $L^{\bar{m}} \subset L$ is the kernel of the morphism $L \to M$, $l \mapsto l\bar{m}$; this is a Lie* subalgebra of L. For $R^\ell \in \mathcal{C}omu_{\mathcal{D}}(X)$ there is an obvious morphism from $h(L^{\bar{m}} \otimes R^\ell)$ to the stabilizer of $\bar{m} \in h(M \otimes R^\ell)$ in $h(L \otimes R^\ell)$; it need not be an isomorphism (see Example in 2.5.7).

(ii) Any lisse $\mathcal{D}$-submodule of M is killed by the L-action (see 2.2.4(ii)).

Examples. (i) Let V be a coherent $\mathcal{D}$-module. Then the $h(\mathcal{E}nd^* V)$-action on V coincides with the map $h(\mathcal{H}om(V, V \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X)) \to \operatorname{End} V$, which sends $\bar{\varphi}$, $\varphi \in \mathcal{H}om(V, V \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X)$ to the composition $V \xrightarrow{\varphi} V \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \to V$; the second arrow is $v \otimes \partial \mapsto v\partial$

(ii) The adjoint action of L yields an $h(L)$-action on L denoted by $\bar{\ell} \mapsto ad_{\bar{\ell}} \in \operatorname{End} L$.

The next technical proposition may be useful when one wants to check if a given $*$ operation is actually a Lie* bracket. The reader can skip it, returning when necessary.

2.5.5. Proposition. *For a $\mathcal{D}$-module L* Lie* *brackets on L are in bijective correspondence with the morphisms of sheaves $h(L) \to \operatorname{End} L$, $\bar{\ell} \mapsto ad_{\bar{\ell}}$, which satisfy the following properties:*

(i) $\left[ad_{\bar{\ell}_1}, ad_{\bar{\ell}_2}\right] = ad_{ad_{\bar{\ell}_1}(\bar{\ell}_2)}$ *for any* $\bar{\ell}_1, \bar{\ell}_2 \in h(L)$.

(ii) $ad_{\bar{\ell}}(\ell) = 0$ *for any* $\ell \in L$.

(iii) *For any $\ell' \in L$ the map $L \to L$, $\ell \mapsto ad_{\bar{\ell}}(\ell')$, is a differential operator (with respect to the $\mathcal{O}$-module structure on L).*

Proof. We use 2.2.19. The condition (iii) is exactly the condition $(*)$ of 2.2.19; hence it means that ad comes from a $*$-pairing $[\ ] \in P_2^*(\{L, L\}, L)$. The property (i) is equivalent to the Jacobi identity by 2.2.19. The property (ii) means that $Tr_2[\ ] \colon L^{\otimes 2} \to L$ vanishes on the symmetric part $L^{\otimes 2+} \subset L^{\otimes 2}$. Set $N := \Delta^{\cdot}\Delta_* L$. Since $[\ ] \colon L^{\otimes 2} \to N$ is $\mathcal{D}_X^{\otimes 2}$-linear, $M := [\ ](L^{\otimes 2+})$ is a $\mathcal{D}_X^{\otimes 2+}$-submodule of N such that $Tr_2(M) = 0$. So the skew-symmetry of $[\ ]$ follows from the next lemma.

LEMMA. *If $M \subset N$ is a $\mathcal{D}_X^{\otimes 2+}$-submodule such that $Tr_2(M) = 0$, then $M = 0$.*

Proof. Take $m \in M$. For any $f \in \mathcal{O}_X$, $\theta \in \Theta_X$ we have $Tr_2(m\cdot(f\otimes\theta+\theta\otimes f)) = 0$. Since $Tr_2(n\cdot(\theta\otimes 1)) = 0$ and $Tr_2(n\cdot(1\otimes\theta)) = Tr_2(n)\cdot\theta$ for any $n \in N$, we obtain $(Tr_2(m\cdot(f\otimes 1)))\theta = 0$. Since $\Theta_X\mathcal{D}_X = \mathcal{D}_X$, we have $Tr_2(m\cdot(f\otimes 1)) = 0$ for any $f \in \mathcal{O}_X$. Choose a coordinate system $t_1,\dots,t_n$ on X and for $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{Z}_+^k$ set $t^\alpha := t_1^{\alpha_1}\dots t_k^{\alpha_k} \in \mathcal{O}_X$, $\partial^\alpha := \partial_{t_1}^{\alpha_1}\dots\partial_{t_k}^{\alpha_k} \in \mathcal{D}_X$. Since $N = L\cdot(\mathcal{D}_X\otimes 1)$, we can write m as $\sum_\alpha \ell_\alpha\cdot(\partial^\alpha\otimes 1)$, $\ell_\alpha \in L$. Then $Tr_2(m\cdot(f\otimes 1)) = \sum_\alpha \ell_\alpha\partial^\alpha(f)$. The equality $\sum_\alpha \ell_\alpha\partial^\alpha(f) = 0$ for all $f \in \mathcal{O}_X$ implies $\ell_\alpha = 0$ (set $f = t^\beta$). □

2.5.6. Basic examples. (a) For any coherent $\mathcal{D}_X$-module M we have an associative*, hence Lie*, algebra $\mathcal{E}nd^*(M)$ (see 2.2.15 and 2.2.16). If M is a vector $\mathcal{D}_X$-bundle, then $\mathcal{E}nd^*(M)$ considered as a Lie* algebra is denoted by $\mathfrak{gl}(M)$.

(b) By 2.2.4(i) for an $\mathcal{O}$-module P a Lie bracket on P which is a bidifferential operator is the same as a Lie* bracket on the induced $\mathcal{D}$-module $P_\mathcal{D}$. Notice that $\tilde{h}(P)(R^\ell) = P \underset{\mathcal{O}_X}{\otimes} R^\ell$, and the bracket on this sheaf is just the extension of the Lie bracket on P defined by the $\mathcal{D}_X$-algebra structure on R^ℓ.

EXAMPLES. (i) The Lie bracket of vector fields defines a Lie* algebra structure on $\Theta_\mathcal{D} := \Theta_{X\mathcal{D}}$. For every $\mathcal{O}$-module F equipped with a Θ_X-action such that the action map is a differential operator with respect to Θ_X[57] the corresponding $\mathcal{D}$-module $F_\mathcal{D}$ is a $\Theta_\mathcal{D}$-module. E.g., the $(\omega_X^{\otimes j})_\mathcal{D}$ are $\Theta_\mathcal{D}$-modules.

(ii) If $\mathfrak{g}$ is a Lie algebra, then $\mathfrak{g}_\mathcal{O} = \mathfrak{g}\otimes\mathcal{O}_X$ is a Lie $\mathcal{O}_X$-algebra; hence $\mathfrak{g}_\mathcal{D} := \mathfrak{g}\otimes\mathcal{D}_X = (\mathfrak{g}_\mathcal{O})_\mathcal{D}$ is a Lie* algebra. Notice that $\tilde{h}(\mathfrak{g}_\mathcal{D})(R^\ell) = \mathfrak{g}\otimes R^\ell$; the bracket extends the bracket on $\mathfrak{g}$ by R^ℓ-linearity.

(iii) If F is a coherent $\mathcal{O}$-module, then the sheaf $\mathrm{Diff}(F,F)_X$ of differential operators is an associative algebra in $\mathcal{D}iff(X)^*$ which acts on F. So $\mathrm{Diff}(F,F)_\mathcal{D} := \mathrm{Diff}(F,F)_{X\mathcal{D}}$ is an associative* (hence Lie*) algebra that acts on $F_\mathcal{D}$. This action yields a canonical isomorphism of associative* algebras $\mathrm{Diff}(F,F)_\mathcal{D} \xrightarrow{\sim} \mathcal{E}nd^*(F_\mathcal{D})$ (see 2.2.16 and (2.2.14.1)).

REMARK. As follows from 2.2.10, any Lie* algebra L admits a functorial left Lie* algebra resolution $\tilde{L}$ whose terms are induced $\mathcal{D}$-modules locally on X. Namely, one can take $\tilde{L} = L\otimes\mathcal{P}$ where $\mathcal{P}$ is a resolution of $\mathcal{O}_X$ from Remark in 2.2.10.

(c) Let L be a Lie* algebra. We know what its action on any $\mathcal{D}_X$-algebra is (see 1.4.9). Such an action has the étale local nature, so we know what the L-action on any algebraic $\mathcal{D}_X$-space is.

For example, let R^ℓ be a $\mathcal{D}_X$-algebra such that Ω_R is a finitely-presented $R^\ell[\mathcal{D}_X]$-module, so Θ_R is well defined (see Remark (ii) in 2.3.12). Then Θ_R is a Lie* algebra acting on R^ℓ (see 1.4.16). On the other hand, we have the Lie algebra $E := \mathrm{Der}_{\mathcal{D}_X} R^\ell$ of horizontal $\mathcal{O}_X$-derivations of R^ℓ (experts in non-linear differential equations sometimes call E the *Lie-Bäcklund* algebra). We consider E as a sheaf on X. One has the canonical morphism of Lie algebras $h(\Theta_R) \to E$. If Ω_{R^ℓ} is a projective $R^\ell[\mathcal{D}_X]$-module (which happens if R^ℓ is $\mathcal{D}_X$-smooth), then this is an isomorphism (see Remark (iii) in 2.3.12).

Θ_R is the first example of a Lie* R-algebroid (see 1.4.11 and 2.5.16 below).

[57] We assume that Θ_X acts on F as on an $\mathcal{O}$-module, so the action is automatically a differential operator with respect to F.

REMARK. If Ω_R is a free $\mathcal{R}^\ell[\mathcal{D}_X]$-module, then Θ_R has the following description. Choosing generators $\nu_1, \ldots, \nu_n \in \Omega_R$, we get an identification $E \xrightarrow{\sim} R^{\ell n}$, $\tau \mapsto (\langle \nu_i, \tau \rangle)$. It yields, in particular, an $\mathcal{O}_X$-module structure on E. With respect to this structure the bracket on E is a bidifferential operator, so we have the Lie* algebra $E_{\mathcal{D}}$. Clearly $\Theta_R = E_{\mathcal{D}}$.

(d) If F is a commutative* algebra and P is a Lie$^!$ algebra, then $P \otimes F$ is naturally a Lie* algebra (see 2.2.9). Taking $F = DR_{\mathcal{D}}$ (see 2.2.4(iii)), we find that the complex $P \otimes DR_{\mathcal{D}}$, which is a resolution of $P[-\dim X]$, is naturally a Lie* DG algebra. Take $P = \mathfrak{g}_{\mathcal{O}}$, where $\mathfrak{g}$ is a Lie k-algebra; we see that $\mathfrak{g} \otimes F$ is a Lie* algebra. Thus $\mathfrak{g} \otimes \omega_X[-\dim X]$ is naturally a homotopy Lie* algebra.

2.5.7. Lie$^!$ coalgebras versus Lie* algebras. Some Lie* algebras arise as duals of natural Lie$^!$ coalgebras. If the $\mathcal{D}$-modules we play with are vector $\mathcal{D}_X$-bundles, then the two structures are equivalent. Otherwise, as we will see, they are pretty different.[58]

We are basically interested in Lie* algebras (and not in Lie$^!$ algebras) because of their connection with chiral algebras.

Let us start with some generalities:

(i) A Lie$^!$ coalgebra is just a Lie algebra in the tensor category $\mathcal{M}(X)^{!\circ}$. Equivalently, this is a left $\mathcal{D}_X$-module N^ℓ equipped with a morphism $N^\ell \to N^\ell \otimes N^\ell$ which satisfies the Lie cobracket property.

If R^ℓ is a commutative $\mathcal{D}_X$-algebra and N^ℓ is a Lie$^!$ coalgebra, then $\mathbb{V}(N^\ell)(R^\ell) = \mathrm{Hom}_{\mathcal{D}_X}(N^\ell, R^\ell) = \mathrm{Hom}(\mathrm{Sym} N^\ell, R^\ell)$ has a natural structure of a Lie k-algebra. Therefore the vector $\mathcal{D}_X$-scheme (see 2.3.19) $\mathbb{V}(N^\ell)$ carries a canonical Lie algebra structure. It is easy to see that the functor

$$\mathbb{V} : \{\text{Lie}^! \text{ coalgebras}\}^\circ \to \{\text{Lie } k\text{-algebras in } \mathcal{A}ff\mathcal{S}ch_{\mathcal{D}}(X)\} \tag{2.5.7.1}$$

is an equivalence of categories.

We have a fully faithful embedding $\mathbb{V}$ of the category dual to that of Lie$^!$ coalgebras to the category Φ^{Lie} (see 2.5.3), $\mathbb{V}(N^\ell)(R^\ell) = \underline{\mathrm{Hom}}(L^\ell, R^\ell)$. We also have the fully faithful embedding $\tilde{h} : \mathcal{L}ie^*(X) \hookrightarrow \Phi^{Lie}$; see (2.5.3.1). The next lemma describes the intersection of these subcategories:

LEMMA. (a) *A Lie* algebra L corresponds to some Lie$^!$ coalgebra if and only if L is a vector $\mathcal{D}_X$-bundle.*

(b) *A Lie$^!$ coalgebra N^ℓ corresponds to some Lie* coalgebra if and only if N^ℓ is a vector $\mathcal{D}_X$-bundle.*

(c) *If a Lie* algebra L and a Lie$^!$ coalgebra N^ℓ correspond to each other, then L and N^ℓ are dual $\mathcal{D}$-modules; i.e., $N^\ell = \underline{\mathrm{Hom}}_{\mathcal{D}_X}(L, \mathcal{D}_X)$ and $L = \underline{\mathrm{Hom}}_{\mathcal{D}_X}(N^\ell, \mathcal{D}_X)$.*

Proof. Use the lemma in 2.3.19. □

If F is a vector bundle on X, then $F_{\mathcal{D}} := F \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$ is dual to $N^\ell := {}_{\mathcal{D}}F^* := \mathcal{D}_X \underset{\mathcal{O}_X}{\otimes} F^*$, so a Lie* bracket on $F_{\mathcal{D}}$ is the same as a Lie cobracket ${}_{\mathcal{D}}F^* \to {}_{\mathcal{D}}F^* \otimes {}_{\mathcal{D}}F^*$, i.e., a section of $F \otimes \Lambda^2({}_{\mathcal{D}}F^*)$ satisfying a certain condition. Sometimes it is convenient to write Lie* brackets on $F_{\mathcal{D}}$ in this format.

[58]Of course, in the $\mathcal{D}_X$-coherent situation the difference disappears if one passes to derived categories.

EXAMPLE. Consider the standard Lie* algebra structure on $\Theta_{\mathcal{D}}$ from (b)(i) in 2.5.6. The corresponding section of[59] $\Theta_X \otimes \Lambda^2({}_{\mathcal{D}}\Omega)$ is $\sum_{i,j} \partial_i \otimes ((1 \otimes dx^j) \wedge (\partial_j \otimes dx^i))$. Similarly, in the situation of (b)(ii) in 2.5.6 (we assume that $\dim \mathfrak{g} < \infty$) the corresponding section is the usual cobracket element in $\mathfrak{g}^* \otimes \Lambda^2 \mathfrak{g}^* \subset \mathfrak{g}^* \otimes \Lambda^2({}_{\mathcal{D}}\mathfrak{g}^*)$.

(ii) For a Lie coalgebra N^ℓ let $\mathcal{M}(X, N^\ell)$ be the category of N^ℓ-comodules. Notice that for $M \in \mathcal{M}(X, N^\ell)$ the Lie algebra $\mathbb{V}(N^\ell)(R^\ell)$ acts on $M \otimes R^\ell$ as on an $R^\ell[\mathcal{D}_X]$-module in a way compatible with the morphisms of the R^ℓ's. Therefore $\mathbb{V}(N^\ell) \in \Phi^{Lie}$ acts on $\tilde{h}(M) \in \Phi$.

Assume that N is a vector $\mathcal{D}_X$-bundle, so its dual $N^\circ := \underline{\mathrm{Hom}}_{\mathcal{D}_X}(N^\ell, \mathcal{D}_X)$ is a Lie* algebra, and $\mathbb{V}(N^\ell) = \tilde{h}(N^\circ) \in \Phi^{Lie}$. The above $\tilde{h}(N^\circ)$-action on $\tilde{h}(M)$ amounts to a $*$ action of N° on M.[60] Therefore we have an equivalence of tensor categories

$$\mathcal{M}(X, N^\ell) \xrightarrow{\sim} \mathcal{M}(X, N^\circ). \tag{2.5.7.2}$$

(iii) Assume that N^ℓ is a coherent $\mathcal{D}$-module, but not necessarily a vector $\mathcal{D}_X$-bundle. The above constructions can be partially extended to this situation as follows. According to 2.2.18, the dual to a Lie cobracket $N^\ell \to N^\ell \otimes N^\ell$ is a Lie* bracket on the dual $\mathcal{D}$-module N°. We have a canonical morphism $\tilde{h}(N^\circ) \to \mathbb{V}(N^\ell)$ in Φ^{Lie}. Namely, the morphism $h(N^\circ \otimes R^\ell) \to \underline{\mathrm{Hom}}_{\mathcal{D}_X}(N^\ell, R^\ell)$ comes from the pairing $id_{R^\ell} \otimes \langle\rangle \in P_2^*(\{R^\ell \otimes N^\circ, N\}, R)$ where $\langle\rangle \in P_2^*(\{N^\circ, N\}, \omega_X)$ is the canonical pairing. In particular, for $R^\ell = O_x$, $x \in X$,[61] we get a canonical morphism of profinite-dimensional Lie algebras $\hat{h}_x(N^\circ) = \Gamma(X, h(N^\circ \otimes O_x)) \to \mathrm{Hom}_{\mathcal{D}_X}(N^\ell, O_x) = N_x^{\ell *}$.

We see that any N^ℓ-comodule M is automatically an N°-module, so we have a faithful functor $\mathcal{M}(X, N^\ell) \to \mathcal{M}(X, N^\circ)$. Therefore an N^ℓ-coaction on an algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ induces an action of N° on $\mathcal{Y}$.

(iv) If G is a group $\mathcal{D}_X$-scheme, then the restriction of $\Omega_G = \Omega_{G/X}$ to the unit section $X \subset G$ has a natural structure of Lie$^!$ coalgebra. We denote it by CoLie(G). If G is smooth, then CoLie(G) is a vector $\mathcal{D}_X$-bundle, so we have the dual Lie* algebra Lie(G) called the Lie* algebra of G. A G-action on an algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ yields a CoLie(G)-coaction on $\mathcal{Y}$, hence an action of Lie(G) on $\mathcal{Y}$.

(v) The next example illustrates the fact that the world of Lie$^!$ coalgebras differs from that of Lie* algebras if one considers not only vector $\mathcal{D}_X$-bundles. We will show that on a symplectic variety X the Lie algebra $\mathcal{O}_X$ of hamiltonians comes naturally from a Lie* algebra, while the Lie algebra of symplectic vector fields comes from a Lie$^!$ coalgebra and *not* a Lie* algebra.

EXAMPLE. Assume that our X is a symplectic variety with symplectic form ν. For every commutative $\mathcal{D}_X$-algebra R^ℓ one has the Lie algebra Sympl(R^ℓ) := the stabilizer of ν with respect to the action of the Lie algebra $R^\ell \otimes \Theta_X$ on $R^\ell \otimes \Omega_X^2$ (e.g., if R^ℓ is the sheaf of holomorphic functions, then Sympl(R^ℓ) is the sheaf of holomorphic symplectic vector fields). Symplectic vector fields on X can be

[59]Here $\Omega = \Omega_X^1 = \Theta_X^*$.

[60]It can be constructed explicitly as the composition of $\langle\ \rangle \otimes id_M \in P_2^*(\{N^\circ, N \otimes M\}, M)$ with the coaction morphism $M \to N \otimes M$.

[61]Here O_x is the formal completion of $\mathcal{O}_x$ considered as a $\mathcal{D}_X$-algebra (i.e., O_x = the direct image of the structure sheaf by the morphism Spec$O_x \to X$).

identified with closed 1-forms, i.e., $\mathcal{D}_X$-module morphisms $K^\ell \to \mathcal{O}_X$ where K^ℓ is the cokernel of the differential ${}_{\mathcal{D}}\Lambda^2\Theta \to {}_{\mathcal{D}}\Theta$ from the de Rham complex $DR(\mathcal{D}_X)$ of the right $\mathcal{D}_X$-module $\mathcal{D}_X$ (see 2.1.7). Moreover, one has a canonical isomorphism $\underline{\mathrm{Hom}}_{\mathcal{D}_X}(K^\ell, R^\ell) = \mathrm{Sympl}(R^\ell)$ functorial in the $\mathcal{D}_X$-algebra R^ℓ. So K^ℓ is a Lie$^!$ coalgebra. K^ℓ does not come from a Lie* algebra because the $\mathcal{D}_X$-module K^ℓ is not locally projective ($DR(\mathcal{D}_X)$ is a locally projective resolution of the left $\mathcal{D}_X$-module $\mathcal{O}_X$, so if K^ℓ were locally projective, then $H^{-1}_{DR}(M) = \mathrm{Tor}_1(M, \mathcal{O}_X)$ would vanish for all left $\mathcal{D}_X$-modules M). The following realization of K^ℓ may be convenient. We have the Poisson bracket on $\mathcal{O}_X$, the Lie bracket on $\Theta_X = \Omega_X$, and the hamiltonian action morphism $f : \mathcal{O}_X \to \Theta_X = \Omega_X$; quite similarly, for every commutative $\mathcal{D}_X$-algebra R^ℓ one has the Lie algebra structures on R^ℓ and $R^\ell \otimes_{\mathcal{O}_X} \Omega_X$ as well as the Lie algebra morphism $f_R : R^\ell \to R^\ell \otimes_{\mathcal{O}_X} \Omega_X$, which factors through $\mathrm{Sympl}(R^\ell)$. Since $R^\ell = \underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{D}_X, R^\ell)$ and $R^\ell \otimes_{\mathcal{O}_X} \Omega_X = \underline{\mathrm{Hom}}_{\mathcal{D}_X}({}_{\mathcal{D}}\Theta, R^\ell)$, we get Lie$^!$ coalgebra structures on $\mathcal{D}_X = {}_{\mathcal{D}}\mathcal{O}$ and ${}_{\mathcal{D}}\Theta$ as well as the Lie$^!$ coalgebra morphisms $\varphi : {}_{\mathcal{D}}\Theta \to K^\ell$, $\psi : K^\ell \to {}_{\mathcal{D}}\mathcal{O}$. It is easy to see that φ is the canonical projection ${}_{\mathcal{D}}\Theta \to \mathrm{Coker}({}_{\mathcal{D}}\Lambda^2\Theta \to {}_{\mathcal{D}}\Theta)$ and $\psi\varphi$ is the differential from the complex $DR(\mathcal{D}_X)$. Since $DR(\mathcal{D}_X)$ is a resolution of $\mathcal{O}_X$, we see that ψ induces an isomorphism $K^\ell \xrightarrow{\sim} \mathrm{Ker}({}_{\mathcal{D}}\mathcal{O} \to \mathcal{O}_X)$.

Now let us look at the Lie* algebras associated to our symplectic variety (X, ν). The Poisson bracket on $\mathcal{O}_X$ and the Lie bracket on Θ_X induce Lie* structures on the corresponding induced $\mathcal{D}$-modules $\mathcal{O}_{\mathcal{D}} = \mathcal{D}_X$ and $\Theta_{\mathcal{D}}$. The morphism $a : \mathcal{O}_{\mathcal{D}} \to \Theta_{\mathcal{D}}$ corresponding to the hamiltonian action morphism $\mathcal{O}_X \to \Theta_X$ is *injective* (because a non-zero morphism from $\mathcal{D}_X$ to a vector $\mathcal{D}_X$-bundle is always injective). Denote by $Stab_\nu$ the stabilizer[62] of $\nu \in \Omega^2_X = h(\Omega^2_X \otimes_{\mathcal{O}_X} \mathcal{D}_X)$ with respect to the action of $\Theta_{\mathcal{D}}$ on $\Omega^2_{\mathcal{D}} := \Omega^2_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$. Then $Stab_\nu = \mathrm{Im}\, a$. Indeed, $Stab_\nu = \mathrm{Ker}(\Theta_{\mathcal{D}} \xrightarrow{\alpha_{\mathcal{D}}} \Omega^2_{\mathcal{D}}) = \mathrm{Ker}(\Omega^1_{\mathcal{D}} \xrightarrow{d_{\mathcal{D}}} \Omega^2_{\mathcal{D}})$ where $d_{\mathcal{D}}$, $\alpha_{\mathcal{D}}$ are morphisms of induced $\mathcal{D}$-modules corresponding to $d : \Omega^1_X \to \Omega^2_X$ and $\alpha : \Theta_X \to \Omega^2_X$, $\alpha(\tau) := \tau(\nu)$. Thus $Stab_\nu = \mathrm{Im}\, a$ by 2.1.9.

2.5.8. ω-extensions. Let L be a Lie* algebra. We will consider Lie* algebra extensions $L^\flat$ of L by $\omega = \omega_X$. Notice that any such extension is automatically central (indeed, one has $P^*_2(\{L, L\}, L^\flat) \xrightarrow{\sim} P^*_2(\{L^\flat, L^\flat\}, L^\flat)$, see Remark (ii) in 2.5.4). All $L^\flat$ form a Picard groupoid[63] in the usual way; we denote it by $\mathcal{P}(L)$.

From now until 2.5.10 we assume that $\dim X = 1$.

We will define canonical ω-extensions of some natural Lie* algebras of the type considered in 2.5.6(b). So these are examples of Lie* algebras which do not come from Lie$^!$ coalgebras.[64]

The constructions follow the same pattern. Namely, to define our ω-extension $L^\flat$ of L we first construct an L-module extension $L^\natural$ of the adjoint representation by an L-module which equals $\omega_{\mathcal{D}} := \omega \otimes \mathcal{D}_X$ as a plain $\mathcal{D}$-module and has the property that the canonical morphism $\omega_{\mathcal{D}} \twoheadrightarrow \omega$ is L-invariant. Now $L^\flat$ is the push-out of $L^\natural$ by this arrow. The L-action on $L^\flat$ yields a binary $*$ operation $[\ ]$ on $L^\flat$,

[62] See Remark (i) from 2.5.4 for the definition of stabilizer

[63] Recall that a Picard groupoid is a tensor category whose morphisms and objects are invertible, see [SGA 4] Exp. XVIII 1.4.

[64] Notice that due to the presence of ω the corresponding $\mathcal{D}$-modules are *not* induced, nor even quasi-induced in the sense of 2.1.11. For the same reason they cannot be represented as duals.

$[a,b] := \pi(a)b$ (where $\pi : L^\flat \to L$ is the projection), which lifts the Lie* bracket on L. It satisfies automatically the Jacobi identity; skew-symmetry follows for separate reasons.

2.5.9. The Kac-Moody extension. We are in the situation of Example (ii) in 2.5.6(b). Let κ be an ad-invariant symmetric bilinear form on $\mathfrak{g}$. We will define an ω-extension $\mathfrak{g}^\kappa_{\mathcal{D}}$ of $\mathfrak{g}_{\mathcal{D}}$ called the *(affine) Kac-Moody* Lie* algebra; if $\mathfrak{g}$ is commutative, it is the *Heisenberg* Lie* algebra.

Consider the pairing $\tilde\phi : \mathfrak{g}_{\mathcal{O}} \times \mathfrak{g}_{\mathcal{O}} \to \omega$, $\tilde\phi(a,b) := \kappa(da, b)$ and the corresponding $*$ pairing $\tilde\phi_{\mathcal{D}} \in P^*_2(\{\mathfrak{g}_{\mathcal{D}}, \mathfrak{g}_{\mathcal{D}}\}, \omega_{\mathcal{D}})$; let $\phi \in P^*_2(\{\mathfrak{g}_{\mathcal{D}}, \mathfrak{g}_{\mathcal{D}}\}, \omega)$ be the composition of $\tilde\phi_{\mathcal{D}}$ with the canonical morphism $\omega_{\mathcal{D}} = \omega \otimes \mathcal{D}_X \to \omega$. Since κ is ad-invariant, one has $\tilde\phi([a,b],c) = \tilde\phi(a,[b,c]) - \tilde\phi(b,[a,c])$. Since κ is symmetric, one has $\tilde\phi(a,b)+\tilde\phi(b,a) = d\kappa(a,b)$; hence ϕ is skew-symmetric. Therefore ϕ is a 2-cocycle of the Lie* algebra $\mathfrak{g}_{\mathcal{D}}$. Our $\mathfrak{g}^\kappa_{\mathcal{D}}$ is the extension corresponding to this cocycle.

In the $\tilde h$ language (see 2.5.3) the definition is even simpler. Namely, for a test $\mathcal{D}_X$-algebra R^ℓ the corresponding Lie algebra $\tilde h(\mathfrak{g}^\kappa_{\mathcal{D}})(R^\ell)$ is the extension $\mathfrak{g}(R^\ell)^\kappa$ of $\mathfrak{g}(R^\ell) = \mathfrak{g} \otimes R^\ell$ by $h(R^\ell)$ defined by the cocycle $a, b \mapsto \overline{\kappa(da,b)}$.

Notice that $\mathfrak{g}_{\mathcal{D}}$ acts on $\mathfrak{g}_{\mathcal{D}} \oplus \omega_{\mathcal{D}}$ so that the only non-zero components of the action $\in P^*_2(\{\mathfrak{g}_{\mathcal{D}}, \mathfrak{g}_{\mathcal{D}} \oplus \omega_{\mathcal{D}}\}, \mathfrak{g}_{\mathcal{D}} \oplus \omega_{\mathcal{D}})$ are $[\ ,\ ]_{\mathfrak{g}_{\mathcal{D}}}$ and $\tilde\phi_{\mathcal{D}}$. Denote this $\mathfrak{g}_{\mathcal{D}}$-module by $\mathcal{P}^\kappa$; this is an extension of the adjoint representation by the trivial $\mathfrak{g}_{\mathcal{D}}$-module $\omega_{\mathcal{D}}$. The adjoint action of $\mathfrak{g}^\kappa_{\mathcal{D}}$ factors through $\mathfrak{g}_{\mathcal{D}}$; as a $\mathfrak{g}_{\mathcal{D}}$-module, $\mathfrak{g}^\kappa_{\mathcal{D}}$ is the push-out of $\mathcal{P}^\kappa$ by the projection $\omega_{\mathcal{D}} \twoheadrightarrow \omega$.

Remark. The Lie algebra Θ_X acts on $\mathfrak{g}_{\mathcal{O}}$ and ω in the obvious way, and $\tilde\phi$ is invariant with respect to this action. Thus $\mathcal{P}^\kappa$ and $\mathfrak{g}^\kappa_{\mathcal{D}}$ carry a canonical action of the Lie* algebra $\Theta_{\mathcal{D}}$.

Suppose that $\mathfrak{g}$ is the Lie algebra of an algebraic group G and κ is Ad_G-invariant. We have the corresponding jet group $\mathcal{D}_X$-scheme $\mathcal{J}G = \mathcal{J}G_X$ (see 2.3.2); its Lie* algebra equals $\mathfrak{g}_{\mathcal{D}}$. Notice that $\mathcal{P}^\kappa$, considered as a mere vector $\mathcal{D}_X$-bundle, carries a natural $\mathcal{J}G$-action. Namely, for a test $\mathcal{D}_X$-algebra R^ℓ a $\mathcal{D}_X$-scheme R^ℓ-point of $\mathcal{J}G$ is the same as $g \in G(R^\ell)$. Such g acts on $\mathcal{P}^\kappa \otimes R^\ell$ so that the corresponding action on $h(\mathcal{P}^\kappa \otimes R^\ell) = \mathfrak{g} \otimes R^\ell \oplus \omega \otimes R^\ell$ is

$$g(a+\nu) := \mathrm{Ad}_g(a) + \kappa((g^{-1}dg, a) + \nu). \tag{2.5.9.1}$$

Here $g^{-1}dg \in \mathfrak{g} \otimes \omega$ and we differentiate g using the structure connection on R^ℓ. Now the corresponding action of $\mathrm{Lie}(\mathcal{J}G) = \mathfrak{g}_{\mathcal{D}}$ coincides with the action we defined previously.

The above formula can be interpreted as follows. Consider the space of left G-invariant 1-forms on $G_X = G \times X$ as a vector bundle on X; denote by $\mathcal{P}^*$ the induced vector $\mathcal{D}_X$-bundle. For a test $\mathcal{D}_X$-algebra R^ℓ one can interpret $h(\mathcal{P}^* \otimes R^\ell)$ as the space of left G-invariant 1-forms μ on $G \times \mathrm{Spec}\, R^\ell$ along the preimage of the horizontal foliation of $\mathrm{Spec}\, R^\ell$ defined by the structure connection on R^ℓ. Then $G(R^\ell)$ acts on this space by right translations. Therefore $\mathcal{J}G$ acts on $\mathcal{P}^*$. Our $\mathcal{P}^*$ is an extension of $\mathfrak{g}^*_{\mathcal{D}}$ (1-forms relative to the projection $G_X \to X$) by $\omega_{\mathcal{D}}$, and the $\mathcal{J}G$-action preserves the projection. Our κ defines a morphism of vector $\mathcal{D}_X$-bundles $\mathfrak{g}_{\mathcal{D}} \to \mathfrak{g}^*_{\mathcal{D}}$. It lifts canonically to a morphism of $\omega_{\mathcal{D}}$-extensions of

$$\alpha_{\mathcal{P}} : \mathcal{P}^\kappa \to \mathcal{P}^* \tag{2.5.9.2}$$

which commutes with the $\mathcal{J}G$-actions. Namely, our $\mathcal{P}^*$ admits an evident decomposition $\mathcal{P}^* = \mathfrak{g}^*_{\mathcal{D}} \oplus \omega_{\mathcal{D}}$, and $\alpha_{\mathcal{P}}$ identifies it with the corresponding decomposition of $\mathcal{P}$. The compatibility with the $\mathcal{J}G$-action follows from (2.5.9.1), since the $G(R^\ell)$-action on $\mathcal{P}^*$ in terms of the above decomposition is $g(a^* + \nu) = \mathrm{Ad}^*_g(a) + a(g^{-1}dg) + \nu$.

We have seen that $G(\mathcal{O}_X)$ acts on our picture, so we can twist it by any G-torsor $\mathfrak{F}$ getting the *twisted* Kac-Moody extension $\mathfrak{g}(\mathfrak{F})^\kappa_{\mathcal{D}}$, etc. Notice that $G(\mathfrak{F})$ is the group X-scheme $\mathrm{Aut}(\mathfrak{F})$ of automorphisms of $\mathfrak{F}$, $\mathcal{J}G(\mathfrak{F}) = \mathcal{J}\mathrm{Aut}(\mathfrak{F})$ is the group $\mathcal{D}_X$-scheme $\mathrm{Aut}(\mathcal{J}\mathfrak{F})$ of automorphisms of the $\mathcal{J}G$-torsor $\mathcal{J}\mathfrak{F}$, and $\mathfrak{g}(\mathfrak{F})_{\mathcal{D}}$ its Lie* algebra. Our $\mathcal{P}^*(\mathfrak{F})$ is the induced vector $\mathcal{D}_X$-bundle corresponding to the vector bundle of G-invariant 1-forms on $\mathfrak{F}$ which is an $\omega_{\mathcal{D}}$-extension of $\mathfrak{g}^*(\mathfrak{F})_{\mathcal{D}}$; a section $\mu \in h(\mathcal{P}^*(\mathfrak{F}))$ is the same as a G-invariant 1-form on $\mathfrak{F}_R$ (:= the pull-back of $\mathfrak{F}$ to $\mathrm{Spec}\, R^\ell$) along the horizontal foliation on $\mathrm{Spec}\, R^\ell$. Our $\alpha_{\mathcal{P}}$ identifies $\mathcal{P}(\mathfrak{F})$ with the pull-back of the $\mathcal{P}^*(\mathfrak{F})$ by the morphism $\mathfrak{g}(\mathfrak{F})_{\mathcal{D}} \to \mathfrak{g}^*(\mathfrak{F})_{\mathcal{D}}$ defined by κ.

Here is a convenient interpretation of $\mathfrak{g}(\mathfrak{F})^\kappa_{\mathcal{D}}$. Consider the $\mathcal{D}_X$-scheme $\mathcal{C}onn(\mathfrak{F})$ of connections on $\mathfrak{F}$. By definition, for a test $\mathcal{D}_X$-algebra R^ℓ a $\mathcal{D}_X$-algebra R^ℓ-point $\mathrm{Spec}\, R^\ell \to \mathcal{C}onn(\mathfrak{F})$ is the same as a horizontal connection ∇_R, i.e., a connection along the horizontal foliation on $\mathrm{Spec}\, R^\ell$, on the G-torsor $\mathfrak{F}_R$. Horizontal connections form a torsor with respect to $\mathfrak{g}(\mathfrak{F}) \otimes \omega \otimes R^\ell = \mathfrak{g}(\mathfrak{F}) \otimes R$, so $\mathcal{C}onn(\mathfrak{F})$ is a torsor for the vector $\mathcal{D}_X$-scheme $\mathrm{Spec}\,\mathrm{Sym}(\mathfrak{g}^*(\mathfrak{F})^\ell_{\mathcal{D}})$. Thus $\mathcal{C}onn(\mathfrak{F}) = \mathrm{Spec}\, A^\ell$ where the $\mathcal{D}_X$-algebra $A^\ell = A^\ell(\mathcal{C}onn(\mathfrak{F}))$ carries a natural filtration $A^\ell_\cdot$ such that $\mathrm{gr} A^\ell = \mathrm{Sym}(\mathfrak{g}^*(\mathfrak{F})^\ell_{\mathcal{D}})$. In particular, A_1 is an ω-extension of $\mathfrak{g}^*(\mathfrak{F})_{\mathcal{D}}$. The group $\mathcal{D}_X$-scheme $\mathrm{Aut}(\mathcal{J}\mathfrak{F}) = \mathcal{J}\mathrm{Aut}(\mathfrak{F})$ acts naturally on $\mathcal{C}onn(\mathfrak{F})$: namely, a $\mathcal{D}_X$-scheme R^ℓ-point of $\mathrm{Aut}(\mathcal{J}\mathfrak{F})$ is the same as an element of $\mathrm{Aut}(\mathfrak{F})(R^\ell)$, and it acts on ∇_R by transport of structure.

LEMMA. *A_1 identifies canonically with the push-out of the $\omega_{\mathcal{D}}$-extension $\mathcal{P}^*(\mathfrak{F})$ by the projection $\omega_{\mathcal{D}} \twoheadrightarrow \omega$. The identification commutes with the Aut($\mathcal{J}\mathfrak{F}$)-action.*

Proof. The promised identification is the same as a morphism of $\mathcal{D}_X$-modules $\varphi : \mathcal{P}^*(\mathfrak{F}) \to A_1$ compatible with projections to $\mathfrak{g}^*(\mathfrak{F})_{\mathcal{D}}$ and equal to the projection $\omega_{\mathcal{D}} \twoheadrightarrow \omega$ on $\omega_{\mathcal{D}} \subset \mathcal{P}^*(\mathfrak{F})$. Such φ amounts to a rule that assigns to a G-invariant 1-form μ on $\mathfrak{F}_R$ along the preimage of the horizontal foliation on R^ℓ and ∇_R as above an element $\varphi(\mu)_{\nabla_R} \in R$. Our ϕ should be R^ℓ-linear with respect to μ, functorial with respect to morphisms of R^ℓ, and should satisfy the following properties:

(i) $\varphi(\mu)_{\nabla_R} = \mu$ if μ came from $\mathrm{Spec}\, R^\ell$; i.e., μ is a 1-form along the horizontal leaves on $\mathrm{Spec}\, R^\ell$ = an element of $R^\ell \otimes \omega = R$.

(ii) $\varphi(\mu)_{\nabla_R + \psi} = \varphi(\mu)_{\nabla_R} + \mu(\psi)$ for any $\psi \in \mathfrak{g}(\mathfrak{F}) \otimes \omega \otimes R^\ell = \mathfrak{g}(\mathfrak{F}) \otimes R$.

(iii) $\varphi(g\mu)_{g\nabla_R} = \varphi(\mu)_{\nabla_R}$ for any $g \in G(R^\ell)$.

Set $\varphi(\nu)_{\nabla_R} := \nabla^*_R(\nu)$. Here ∇_R is considered as a lifting of a horizontal vector field on $\mathrm{Spec}\, R^\ell$ to a G-invariant vector field on $\mathfrak{F}_R$. The above properties are evident. □

Combining the above identification with (2.5.9.2), we get a canonical Aut($\mathcal{J}\mathfrak{F}$)-equivariant morphism

(2.5.9.3) $$\alpha_\kappa : \mathfrak{g}(\mathfrak{F})^\kappa_{\mathcal{D}} \to A_1$$

which is the identity on ω and lifts the morphism $\mathfrak{g}(\mathfrak{F})_{\mathcal{D}} \to \mathfrak{g}^*(\mathfrak{F})_{\mathcal{D}}$ defined by κ. It is an isomorphism if κ is non-degenerate.

Notice that every connection ∇_R as above defines a splitting of the $h(R)$-extension $h(A_1 \otimes R)$ of $\mathfrak{g}^*(\mathfrak{F}) \otimes R^\ell$ whose image is the kernel of the retraction

$h(A_1^\ell \otimes R) \to h(R)$ defined by the "value at ∇" morphism $A_1^\ell \otimes R^\ell \to R^\ell$. It yields, via (2.5.9.3), a splitting $s_{\nabla_R} : \mathfrak{g}(\mathfrak{F}) \otimes R^\ell \to h(\mathfrak{g}(\mathfrak{F})^\kappa_{\mathcal{D}} \otimes R^\ell)$. It is clear that splittings s_{∇_R} are functorial with respect to the morphisms of the R^ℓ's (base change) and satisfy the following properties:

(a) For any $\psi \in \mathfrak{g}(\mathfrak{F}) \otimes R$ one has $s_{\nabla_R + \psi}(a) = s_{\nabla_R}(a) - \kappa(\psi, a)$.

(b) For $g \in \mathrm{Aut}(\mathfrak{F})(R^\ell)$ one has $g(s_{\nabla_R}) = s_{g(\nabla_R)}$.

COROLLARY. *$\mathfrak{g}(\mathfrak{F})^\kappa_{\mathcal{D}}$ is a unique ω-extension of $\mathfrak{g}(\mathfrak{F})_{\mathcal{D}}$ equipped with an action of Aut($\mathcal{J}\mathfrak{F}$) that integrates the adjoint action of $\mathfrak{g}(\mathfrak{F})_{\mathcal{D}}$ on $\mathfrak{g}(\mathfrak{F})^\kappa_{\mathcal{D}}$ and a map s from $\mathcal{C}onn(\mathfrak{F})$ to splittings of $\tilde{h}(\mathfrak{g}(\mathfrak{F})^\kappa_{\mathcal{D}})$ which satisfies (a), (b).* □

REMARKS. (i) The construction of $\mathfrak{g}(\mathfrak{F})^\kappa$ in terms of connections immediately generalizes to the situation when $\mathfrak{F}$ is a G-torsor on algebraic $\mathcal{D}_X$-space $\mathcal{Y}$. Here $\mathcal{C}onn(\mathfrak{F})$ is the space of horizontal connections ∇^h on $\mathfrak{F}$, i.e., connections along the horizontal foliation of $\mathcal{Y}$. Notice that any "vertical" (i.e., relative to X) connection ∇^v on $\mathfrak{F}$ provides a connection $\nabla^v_{\mathcal{C}onn} : \mathcal{C}onn(\mathfrak{F}) \to \Omega^1_{\mathcal{Y}/X} \otimes \mathfrak{g}(\mathfrak{F})\omega$ on the torsor $\mathcal{C}onn(\mathfrak{F})$, a derivation d_{∇^v} of $\Omega^{\cdot}_{\mathcal{Y}/X} \otimes \mathfrak{g}(\mathfrak{F})$, and its lifting $d^\kappa_{\nabla^v}$ to $\Omega^{\cdot}_{\mathcal{Y}/X} \otimes \mathfrak{g}(\mathfrak{F})^\kappa$ such that $d^\kappa_{\nabla^v} s_{\nabla^h}(a) = s_{\nabla^h} d_{\nabla^v}(a) - (\nabla^v_{\mathcal{C}onn}(\nabla^h), a)$.

(ii) $\mathfrak{g}(\mathfrak{F})$ is the $\mathcal{O}_X$-linear part of the Lie $\mathcal{O}_X$-algebroid $\mathcal{A}(\mathfrak{F})$ of all infinitesimal symmetries of $(\mathfrak{F}, X)$. Our $\mathfrak{g}(\mathfrak{F})^\kappa_{\mathcal{D}}$ comes from a canonical ω-extension of $\mathcal{A}(\mathfrak{F})_{\mathcal{D}}$. For every connection ∇ on $\mathfrak{F}$ this extension canonically splits over $\nabla(\Theta_{\mathcal{D}})$. See [**BS**] where a more general setting is discussed.

2.5.10. The Virasoro extension. This is an ω-extension of the Lie* algebra $\Theta_{\mathcal{D}}$. Its construction is similar to that of the Kac-Moody extension with the torsor of connections on $\mathfrak{F}$ replaced by the torsor of projective connections on X.

(a) Let $X^{(n)}$ be the nth infinitesimal neighborhood of the diagonal $X \hookrightarrow X \times X$ for some $n \geq 0$, so $\mathcal{O}_{X^{(n)}} := \mathcal{O}_{X\times X}/\mathcal{J}^{n+1}$ where $\mathcal{J}$ is the ideal of the diagonal. The scheme $X^{(n)}$ carries a canonical involution σ (transposition of coordinates); the action of the Lie algebra Θ_X on $X = X^{(0)}$ extends in the obvious (diagonal) way to an action on $X^{(n)}$ commuting with σ. The filtration $\mathcal{J}^a\mathcal{O}_{X^{(n)}}$ is (Θ_X, σ)-invariant. One has $\mathrm{gr}_a \mathcal{O}_{X^{(n)}} = \omega_X^{\otimes a}$ (for $a \leq n$) with the standard action of Θ_X; σ acts by multiplication by $(-1)^a$.

An $\mathcal{O}$-module E on $X^{(n)}$ yields, by Example (i) in 2.1.8, the $\mathcal{D}$-module $E_{\mathcal{D}}$ on X. Our E carries a canonical filtration $\mathcal{J}^a E$; if E is a locally free $\mathcal{O}_{X^{(n)}}$-module, then $\mathrm{gr}_a E = E_X \otimes \omega_X^{\otimes a}$ for $a \leq n$,[65] so the corresponding filtration on $E_{\mathcal{D}}$ has successive quotients $(E_X \otimes \omega_X^{\otimes a})_{\mathcal{D}}$. If E is σ-equivariant, then σ acts on $E_{\mathcal{D}}$. We have the plain sheaf of σ-invariants E^σ and the $\mathcal{D}$-module $E^\sigma_{\mathcal{D}} := (E_{\mathcal{D}})^\sigma$; one has $h(E^\sigma_{\mathcal{D}}) = E^\sigma$. An action of Θ_X on E which is a bidifferential operator yields an action of the Lie* algebra $\Theta_{\mathcal{D}}$ on $E_{\mathcal{D}}$ (see 2.2.4(i)). If the actions of σ and Θ_X on E commute, then the actions of σ and $\Theta_{\mathcal{D}}$ on $E_{\mathcal{D}}$ also commute; thus $\Theta_{\mathcal{D}}$ acts on $E^\sigma_{\mathcal{D}}$.

REMARK. The projection $\mathcal{O}_X^* \twoheadleftarrow \mathcal{O}^*_{X^{(n)}}$ has a canonical section $\mathfrak{s} : \mathcal{O}_X^* \to \mathcal{O}^*_{X^{(n)}}$ uniquely defined by the properties that it has an étale local nature and $\mathfrak{s}(f^2) = f \boxtimes f|_{X^{(n)}}$. It is clear that $\mathfrak{s}$ is σ-invariant and commutes with the action of Θ_X. Explicit formula: $\mathfrak{s}(f)(x,y) = f(x) + \frac{1}{2}f'(x)(y-x) + \frac{1}{4}(f''(x) - f'(x)^2/f(x))(y-x)^2 + \ldots$. So a line bundle $\mathcal{L}$ on X yields a σ-equivariant line bundle $\mathcal{L}^{\mathfrak{s}}$ on $X^{(n)}$. To construct

[65] Here $E_X := E/\mathcal{J}E$.

$\mathcal{L}^{\mathfrak{s}}$ one should choose (locally) a square root $\mathcal{L}^{\otimes 1/2}$; then $\mathcal{L}^{\mathfrak{s}} = \mathcal{L}^{\otimes 1/2} \boxtimes \mathcal{L}^{\otimes 1/2}|_{X^{(n)}}$. If Θ_X acts on $\mathcal{L}$, then $\mathcal{L}^{\mathfrak{s}}$ is a (σ, Θ_X)-equivariant line bundle.

(b) We will use (a) with $n = 2$ to construct an ω_X-extension of the Lie* algebra $\Theta_{\mathcal{D}}$. We follow the format of 2.5.8, so our Lie* algebra extension comes from a $\mathcal{D}$-module extension of $\Theta^{\natural}_{\mathcal{D}}$ by $\omega_{\mathcal{D}}$ equipped with a Lie* action of $\Theta_{\mathcal{D}}$ which equals the usual action on the subquotients. There are two ways to define such $\Theta^{\natural}_{\mathcal{D}}$:

(i) Assume we have a (σ, Θ_X)-equivariant line bundle ν on $X^{(2)}$ together with an equivariant identification $\nu_X := \nu/\mathcal{I}\nu = \Theta_X$.[66] Our extension is $\Theta^{\natural}_{\mathcal{D}}(\nu) := \nu^{\sigma}_{\mathcal{D}}$.

(ii) Assume we have a (σ, Θ_X)-equivariant line bundle λ on $X^{(2)}$ together with an equivariant identification $\lambda_X = \mathcal{O}_X$.[67] Then λ^{σ} is a Θ_X-equivariant extension of $\mathcal{O}_X$ by $\omega_X^{\otimes 2}$, so the preimage $\mathfrak{P}(\lambda)$ of $1 \in \mathcal{O}_X$ is a Θ_X-equivariant $\omega_X^{\otimes 2}$-torsor. Tensoring it by Θ_X, we get a Θ_X-equivariant $\mathcal{O}_X$-module extension $\mathcal{P}(\lambda)$ of Θ_X by ω_X (so $\mathfrak{P}(\lambda)$ is the torsor of $\mathcal{O}_X$-linear splittings of $\mathcal{P}(\lambda)$). Set $\Theta^{\natural}_{\mathcal{D}}(\lambda) := \mathcal{P}(\lambda)_{\mathcal{D}}$.

Remark. For λ, ν as above the extension $\Theta^{\natural}_{\mathcal{D}}(\lambda \otimes \nu)$ is the Baer sum of extensions $\Theta^{\natural}_{\mathcal{D}}(\lambda)$ and $\Theta^{\natural}_{\mathcal{D}}(\nu)$. Similarly, $\Theta^{\natural}_{\mathcal{D}}(\lambda \otimes \lambda')$ is the Baer sum of $\Theta^{\natural}_{\mathcal{D}}(\lambda)$ and $\Theta^{\natural}_{\mathcal{D}}(\lambda')$.

As was explained in 2.5.8, any $\Theta^{\natural}_{\mathcal{D}}$ as above yields (by push-out by the canonical morphism $\omega_{\mathcal{D}} \to \omega_X$) a $\Theta_{\mathcal{D}}$-module extensions $\Theta^{\flat}_{\mathcal{D}}$ of $\Theta_{\mathcal{D}}$ by ω_X; the $\Theta_{\mathcal{D}}$-action can be rewritten as a binary $*$ operation on $\Theta^{\flat}_{\mathcal{D}}$ which lifts the Lie* bracket on $\Theta_{\mathcal{D}}$. If this operation happens to be skew-symmetric, then this is a Lie* bracket, so $\Theta^{\flat}_{\mathcal{D}}$ is a Lie* algebra extension of $\Theta_{\mathcal{D}}$.

(c) We apply the format of (b) to the following (σ, Θ_X)-equivariant line bundles. Our ν is the restriction of $\mathcal{O}_{X\times X}(\Delta)$ to $X^{(2)}$; σ is the transposition action multiplied by -1, and Θ_X acts in the obvious way. Set $\lambda := \nu \otimes \omega_X^{\mathfrak{s}}$.

We will see at the end of (d) below that the bracket on $\Theta^{\flat}_{\mathcal{D}}(\lambda)$ is skew-symmetric, so $\Theta^{\flat}_{\mathcal{D}}(\lambda)$ is a Lie* algebra extension of $\Theta_{\mathcal{D}}$ by ω_X.

Definition. $\Theta^{\flat}_{\mathcal{D}}(\lambda)$ is called the *Virasoro* Lie* algebra. For $c \in k$ the *Virasoro extension of central charge c* is the c-multiple of the extension $\Theta^{\flat}_{\mathcal{D}}(\lambda)$; we denote it by $\Theta^{(c)}_{\mathcal{D}}$.

Remark. Since $\Theta_{\mathcal{D}}$ is a perfect Lie* algebra,[68] any its ω-extension is rigid.

Lemma. $\Theta^{\flat}_{\mathcal{D}}(\nu)$ *coincides with the Virasoro extension of central charge* -2.

Proof. We will show that the extension $\Theta^{\natural}_{\mathcal{D}}(\nu)$ is canonically isomorphic to the -2-multiple of $\Theta^{\natural}_{\mathcal{D}}(\lambda)$. Such an identification amounts to a canonical splitting of $\Theta^{\natural}_{\mathcal{D}}(\lambda^{\otimes 2} \otimes \nu)$. To define this splitting, notice that $\lambda^{\otimes 2} \otimes \nu$ is the restriction of $\omega_X \boxtimes \omega_X(3\Delta)$ to $X^{(2)}$. Let $\mathrm{Res}^1, \mathrm{Res}^2 : \omega_X \boxtimes \omega_X(\infty\Delta) \to \omega_X$ be the residue around the diagonal along the first or the second variable. The restriction of Res^i to $\omega_X \boxtimes \omega_X(\Delta)$ is the obvious projection $\omega_X \boxtimes \omega_X(\Delta) \to \omega_X \boxtimes \omega_X(\Delta)/\omega_X \boxtimes \omega_X = \omega_X$, so the Res^i yield Θ_X-equivariant retractions of $\lambda^{\otimes 2} \otimes \nu$ to $\omega_X \subset \lambda^{\otimes 2} \otimes \nu$. The restrictions of the corresponding $\mathcal{D}$-module retractions $\mathrm{Res}^i_{\mathcal{D}} : \lambda^{\otimes 2} \otimes \nu \to \omega_{\mathcal{D}}$ to $(\lambda^{\otimes 2} \otimes \nu)^{\sigma}_{\mathcal{D}} = \Theta^{\natural}_{\mathcal{D}}(\lambda^{\otimes 2} \otimes \nu)$ coincide; this retraction defines the desired splitting. □

[66]Here σ acts on Θ_X trivially.

[67]Here σ acts on $\mathcal{O}_X$ trivially.

[68]A proof of this fact is contained in the proof of the sublemma in 2.7.3.

(d) Elements of the $\omega_X^{\otimes 2}$-torsor $\mathfrak{P} := \mathfrak{P}(\lambda)$ (see (b)(ii), (c)) are called *projective connections.*

Remarks. (i) A local coordinate t on X yields a projective connection ρ_t defined as the restriction of $(x-y)^{-1}dx^{1/2}dy^{1/2}$ to $X^{(2)}$.[69] An immediate calculation shows that $(f(t)\partial_t)(\rho_t) = \frac{1}{12}f'''(t)dt^2$.

(ii) We have $\lambda^{\otimes m} = (\omega_X^{\otimes m/2} \boxtimes \omega_X^{\otimes m/2})(m\Delta)/((m-3)\Delta)$. So for $m = 3, 4$ one has embeddings $m\mathfrak{P} \hookrightarrow (j_* j^* \omega_X^{\otimes m/2} \boxtimes \omega_X^{\otimes m/2})/\omega_X^{\otimes m/2} \boxtimes \omega_X^{\otimes m/2} = \mathrm{Diff}(\omega_X^{\otimes 1-m/2}, \omega_X^{\otimes m/2})$ that identify projective connections[70] with symmetric differential operators $\omega_X^{\otimes -1/2} \to \omega_X^{\otimes 3/2}$ of order 2 and principal symbol 1 and, respectively, skew-symmetric differential operators $\omega_X^{\otimes -1} \to \omega_X^{\otimes 2}$ of order 3 and principal symbol 1. For an interpretation of projective connections as $\mathfrak{sl}_2$-opers, see sect. 3.1.6 and 3.1.7 in [**BD**].

As follows from Remark (i), a coordinate t yields a $\mathcal{D}$-module splitting of the extension $\Theta_X^{\natural}(\lambda)$. In terms of this splitting the $\Theta_{\mathcal{D}}$-action is given by the cocycle $f(t)\partial_t, g(t)\partial_t \mapsto \frac{1}{12}f'''(t)g(t)dt$. Thus the bracket on $\Theta_X^{\flat}(\lambda)$ is given by the classical Virasoro 2-cocycle $f(t)\partial_t, g(t)\partial_t \mapsto \frac{1}{12}f'''(t)g(t)dt$ (modulo exact forms). It is obviously skew-symmetric, which fulfills the promise made in (c) above.

2.5.11. Let L be a Lie* algebra, $x \in X$ a (closed) point, $j_x : U_x := X \smallsetminus \{x\} \hookrightarrow X$. The vector space $h(L)_x$ carries the Ξ_x-topology (see 2.1.13).

Lemma. (i) *The Lie bracket on $h(L)_x$ is Ξ_x-continuous with respect to each variable.*

(ii) *If $j_x^* L$ is a countably generated $\mathcal{D}_{U_x}$-module, then the Lie bracket is Ξ_x-continuous.*

Proof. See (i) and (ii) in 2.2.20. □

Remark. If we decide to play with the better objects $\tilde{h}_x$ from 2.1.15, then the countability condition becomes irrelevant.

Examples. If we are in the situation of 2.5.6(b) and the restriction of P to the complement of x is $\mathcal{O}$-coherent, then $\hat{h}_x(P_{\mathcal{D}}) = P \otimes O_x$ (see 2.1.14) and the Lie bracket comes from the bidifferential operator defining the Lie bracket on P.

In the situation of 2.5.9 the Lie algebra $\hat{h}(j_{x*}j_x^* \mathfrak{g}_{\mathcal{D}}^{\kappa})$ is the central extension of $\mathfrak{g}(K_x) = \hat{h}(j_{x*}j_x^*\mathfrak{g}_{\mathcal{D}})$ by $k = \hat{h}(j_{x*}j_x^*\omega_X)$ defined by the 2-cocycle $a, b \mapsto \mathrm{Res}_x(da, b)$, i.e., the usual affine Kac-Moody algebra.

2.5.12. Denote by $\Xi_x^{Lie}(L)$ the topology on L at x (see 2.1.13 for terminology) whose base consists of those $L_\xi \in \Xi_x(L)$ which are Lie* subalgebras of L. As follows from the Remark in 2.2.7, the latter condition amounts to the fact that $h(L_\xi)_x$ is a Lie subalgebra of $h(L)_x$. Therefore one can describe the Ξ_x^{Lie}-topology on $h(L)_x$ as the topology whose base is formed by all Ξ_x-open Lie subalgebras of $h(L)_x$.

The Lie bracket on $h(L)_x$ is automatically continuous with respect to the Ξ_x^{Lie}-topology. Indeed, the base of our topology is formed by Lie subalgebras, and for every open Lie subalgebra V the adjoint action of V on $h(L)_x/V$ is continuous by 2.5.11(i). Therefore the completion $\hat{h}_x^{Lie}(L)$ is a topological Lie algebra.

[69] Here x, y are coordinates on $X \times X$ corresponding to t.

[70] Here we identify $\mathfrak{P}$ with $3\mathfrak{P}, 4\mathfrak{P}$ by means of multiplications by 3 and 4.

REMARK. As in Remark (i) in 2.1.13 we can also consider Ξ_x^{Lie} as a topology on the Lie algebra $h(L_{O_x})$; the corresponding completion coincides with $\hat{h}_x^{Lie}(L)$.

Similarly, $\hat{h}_x(L)$ is a topological Lie algebra if L satisfies the condition of 2.5.11(ii). We have a canonical continuous morphism $\hat{h}_x(L) \to \hat{h}_x^{Lie}(L)$ of Lie algebras.

The two topologies do not differ if L satisfies a coherency condition:

2.5.13. LEMMA. (i) *Let M be an L-module such that its restriction to U_x is a coherent $\mathcal{D}$-module. Then for any $M_\xi, M_{\xi'} \in \Xi_x(M)$ such that $M_{\xi'}$ is a coherent $\mathcal{D}_X$-module, there exists $L_\zeta \in \Xi_x^{Lie}(L)$ such that $L_\zeta(M_{\xi'}) \subset M_\xi$.*

(ii) *If the restriction of L to U_x is a coherent $\mathcal{D}$-module, then the topologies $\Xi_x(L)$ and $\Xi_x^{Lie}(L)$ coincide.*

REMARK. Using the terminology of 2.7.9 the above statements can be reformulated as follows. Recall that $F := \hat{h}_x(M)$ is a Tate vector space (see 2.1.13 and 2.7.9 for terminology), and (i) just says that the action morphism $h(L)_x \to \operatorname{End} \hat{h}_x(M)$ is continuous with respect to the Ξ_x^{Lie}-topology and the topology on $\operatorname{End} F$ introduced in 2.7.9. In the situation of (ii), $\hat{h}_x(L)$ is a topological Lie algebra which is a Tate vector space. The argument below actually proves that any such object has a base of the topology formed by open Lie subalgebras.

Proof. (i) Replacing M_ξ by $M_\xi \cap M_{\xi'}$ we may assume that $M_\xi \subset M_{\xi'}$. According to 2.2.20(iii), the maximal sub-$\mathcal{D}$-module $L_\zeta \subset L$ such that $L_\zeta(M_{\xi'}) \subset M_\xi$ belongs to $\Xi_x(L)$. It is also a Lie* algebra, so we are done.

(ii) We want to show that every $L_\xi \in \Xi_x(L)$ contains some $L_\zeta \in \Xi_x^{Lie}(L)$. We can assume that L_ξ is a coherent $\mathcal{D}$-module. According to the proof of (i) the normalizer N of L_ξ is open. Set $L_\zeta := N \cap L_\xi$. □

2.5.14. In the non-coherent situation the topologies Ξ_x and Ξ_x^{Lie} can be very different:

EXAMPLE. Consider the Lie* algebra $L = \mathfrak{gl}(\mathcal{D}_X)$. Then $h(L)$ is $\mathcal{D}_X$ considered as a sheaf of Lie algebras. For every $x \in X$ the completion $\hat{h}_x(L)$ is a Lie algebra because L satisfies the condition of 2.5.11(ii). Our L is an induced $\mathcal{D}_X$-module, so, by Example (i) of 2.1.13, the completion $\hat{h}_x(L)$ is the Lie algebra $D = \mathcal{D}_{O_x}$ of differential operators with coefficients in O_x (the formal completion of the local ring at x). Denote by $D^{\leq n}$ the subspace of differential operators of order $\leq n$; this is a free O_x-module of rank $n+1$. A vector subspace of D is open in the Ξ_x-topology if its intersection with every $D^{\leq n}$ is open. The Ξ_x^{Lie}-topology is defined by Ξ_x-open Lie subalgebras. It is easy to see that for every open subspace $P \subset D^{\leq 2}$ the Lie subalgebra generated by P is open. Such subalgebras form a base of the topology Ξ_x^{Lie}; in particular, the topology Ξ_x^{Lie}, as opposed to Ξ_x, has a countable base.

Let us describe the Ξ_x^{Lie}-topology on D explicitly assuming that $\dim X = 1$. Let t be a local parameter at x, so $O_x = k[[t]]$. Set $\partial = \partial_t$. For $a \geq 0$, $b \geq 1$ let $D^{a,b} \subset D$ be the space of differential operators $\Sigma \phi_i(t)\partial^i$, $\phi_i(t) \in t^{a+bi} O_x$.

LEMMA. *The $D^{a,b}$ form a basis of the Ξ_x^{Lie}-topology on D.*

Proof. Indeed, the $D^{a,b}$ are open in the Ξ_x-topology and they are associative, hence Lie, subalgebras of D. It remains to check that any open Lie subalgebra

$Q \subset D$ contains some $D^{a,b}$. Set $D_i^{a,b} := D^{a,b} \cap D^{\leq i}$. A simple calculation shows that the operator $A_b := ad_{t^{b+1}\partial^2} \in \operatorname{End} D$ preserves $D^{a,b}$. It sends $D^{\leq i}$ to $D^{\leq i+1}$ and the corresponding operator $\mathrm{gr}_i D^{a,b} \to \mathrm{gr}_{i+1} D^{a,b}$ is an isomorphism if $a \geq 1$, $i \geq 0$. Therefore $D^{a,b} = \underset{i \geq o}{\oplus} A_b^i(t^a O_x)$. Now choose $a, b \geq 1$ such that $t^a O_x \subset Q$ and $t^{b+1}\partial^2 \in Q$. Then $D^{a,b} \subset Q$. □

We see that the completion $\hat{h}_x^{Lie}(L)$ consists of "differential operators of infinite order" $\Sigma \phi_i(t)\partial^i$ such that the number of the first non-vanishing coefficient of $\phi_i(t)$ grows faster than any linear function of i.

Remarks. (i) We will see in 2.7.13 that the above L carries an important topology which is weaker than Ξ_x^{Lie}.

(ii) The above wonders seem to be artificial: they disappear if we consider instead of $\hat{h}_x$ or $\hat{h}_x^{Lie}$ the more sophisticated object $\tilde{h}_x(L)$ from 2.1.15.

2.5.15. Let L be any Lie* algebra. Notice that for an $h(L)_x$-module V the action map $h(L)_x \times V \to V$ is continuous with respect to the Ξ_x^{Lie}-topology (and the discrete topology on V) if and only if it is continuous with respect to the Ξ_x-topology (since the stabilizer of every $v \in V$ is a Lie subalgebra). Such $h(L)_x$-modules are called *discrete*; the corresponding category is denoted by $\hat{h}_x(L)$mod.

Let $\mathcal{M}(X, L)_x$ be the category of L-modules supported at x. For $M \in \mathcal{M}(X, L)_x$ the Lie algebra $h(L)_x$ acts canonically on M; hence $i_x^! M = h(M)$ is an $h(L)_x$-module. By 2.5.13(i) it is discrete.

Lemma. *The functor $h : \mathcal{M}(X, L)_x \to \hat{h}_x(L)mod$ is an equivalence of categories.*

Proof. Follows from 2.5.4. □

2.5.16. Lie* algebroids. Let R^ℓ be a commutative $\mathcal{D}_X$-algebra. According to 1.4.11, we have the notion of Lie* R-algebroid. We denote by $\mathcal{M}(X, R, \mathcal{L})$ the category of $\mathcal{L}$-modules (see 1.4.12). As follows from the lemma in 2.3.12, these objects have an étale local nature. Therefore we know what a Lie* algebroid $\mathcal{L}$ on any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ is and what are $\mathcal{L}$-modules are. Then $h(\mathcal{L})$ is a sheaf of Lie algebras (on $\mathcal{Y}_{ét}$) acting on $\mathcal{O}_\mathcal{Y}$ by horizontal $\mathcal{O}_X$-derivations.

The constructions of 1.4.14 are étale local as well (for the same reason). So for a Lie* algebroid $\mathcal{L}$ on $\mathcal{Y}$ which is a vector $\mathcal{D}_X$-bundle on $\mathcal{Y}$ (see 2.3.10) we have a commutative DG (super)algebra $\mathcal{C}(\mathcal{L})_\mathcal{Y}$ called the *de Rham-Chevalley complex* of $\mathcal{L}$, and for an $\mathcal{L}$-module M we have a DG $\mathcal{C}(\mathcal{L})_\mathcal{Y}$-module $\mathcal{C}(\mathcal{L}, M)_\mathcal{Y}$. If we forget about the differential, then $\mathcal{C}(\mathcal{L})_\mathcal{Y} = \mathrm{Sym}(\mathcal{L}^\circ[-1])$, $\mathcal{C}(\mathcal{L}, M)_\mathcal{Y} = \mathrm{Sym}(\mathcal{L}^\circ[-1]) \otimes M$ where $\mathcal{L}^\circ$ is the vector $\mathcal{D}_X$-bundle on $\mathcal{Y}$ dual to $\mathcal{L}$.

For example, if $\Omega_\mathcal{Y}$ is finitely presented, then the tangent Lie algebroid $\Theta_\mathcal{Y}$ is well defined (see 1.4.16 and Remarks in 2.3.12). If $\Omega_\mathcal{Y}$ is a vector $\mathcal{D}_\mathcal{Y}$-bundle, which happens if $\mathcal{Y}$ is smooth (see 2.3.15), then $h(\Theta_\mathcal{Y})$ is the Lie algebra of all horizontal $\mathcal{O}_X$-derivations of $\mathcal{O}_\mathcal{Y}$. One has $\mathcal{C}(\Theta_\mathcal{Y})_\mathcal{Y} = DR_{\mathcal{Y}/X}$ (the relative de Rham complex).

More generally, if V is a vector $\mathcal{D}_X$-bundle on such $\mathcal{Y}$, then the Lie* algebroid $\mathcal{E}(V)$ on $\mathcal{Y}$ (see Remark (iii) of 1.4.16) is well defined. The morphism τ: $\mathcal{E}(V) \to \Theta_\mathcal{Y}$ is surjective, so $\mathcal{E}(V)$ is an extension of $\Theta_\mathcal{Y}$ by $\mathfrak{g}l(V)$.

2.5.17. Equivariant Lie* algebroids. The format of equivariant Lie* algebroids is very parallel to the one of the usual equivariant Lie algebroids (see, e.g., [**BB**] 1.8). Below we denote by S a test $\mathcal{D}_X$-algebra, $\mathcal{Y}_S := \mathcal{Y} \underset{X}{\times} \operatorname{Spec} S$; for $M \in \mathcal{M}(Y)$ set $M_S := M \otimes S \in \mathcal{M}(\mathcal{Y}_S)$.

Suppose we have a group $\mathcal{D}_X$-scheme G acting on $\mathcal{Y}$ and a Lie* algebroid $\mathcal{L}$ on $\mathcal{Y}$. A *weak G-action* on $\mathcal{L}$ is an action of G on $\mathcal{L}$ as on an $\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]$-module which preserves the Lie* bracket and the $\mathcal{L}$-action on $\mathcal{O}_\mathcal{Y}$. In more details, for any test $\mathcal{D}_X$-algebra S the $\mathcal{O}_{\mathcal{Y}_S}[\mathcal{D}_X]$-module $\mathcal{L}_S$ is naturally a Lie* algebroid on $\mathcal{Y}_S$ (acting along the fibers of the projection $\mathcal{Y}_S \to \mathcal{Y}$); the group of $\mathcal{D}_X$-scheme points $\operatorname{Spec} S \to G$ acts on $\mathcal{Y}_S$. Now a weak G-action on $\mathcal{L}$ is a rule which assigns to each S the action of the above group on the Lie* $\mathcal{O}_{\mathcal{Y}_S}$-algebroid $\mathcal{L}_S$ in a way compatible with the base change.

Suppose G is smooth; set $L := \operatorname{Lie}(G)$. A *strong G-action* on $\mathcal{L}$ is a weak G-action together with a morphism of Lie* algebras $\alpha : L \to \mathcal{L}$ which satisfies the following conditions:

(i) α is compatible with the Lie* actions on $\mathcal{O}_\mathcal{Y}$ and with the G-actions (the G-action on L is the adjoint one).

(ii) The L-action on $\mathcal{L}$ coming from the G-action coincides with the L-action via α and the adjoint action of $\mathcal{L}$.

We refer to a Lie* algebroid equipped with a weak or strong G-action as a *weakly* or *strongly G-equivariant* Lie* algebroid.

Example. If V is a G-equivariant vector $\mathcal{D}_X$-bundle then $\mathcal{E}(V)$, if well defined, is a strongly G-equivariant Lie* algebroid.

As in the setting of the usual Lie algebroids, the strong G-action can be naturally interpreted as follows (we give a brief sketch of constructions leaving the details to the reader).

If $\pi : \mathcal{Z} \to \mathcal{Y}$ is a smooth morphism of algebraic $\mathcal{D}_X$-spaces which is flat as a morphism of the usual schemes, then the pull-back $\pi^\dagger(\mathcal{L})$ of $\mathcal{L}$ is defined. This is a Lie* algebroid on $\mathcal{Z}$ constructed as follows (see 2.9.3 for a parallel construction in the setting of the usual Lie algebroids). Consider the right $\mathcal{O}_\mathcal{Z}[\mathcal{D}_X]$-module $\pi^*\mathcal{L} := \mathcal{O}_\mathcal{Z} \underset{\pi^{-1}\mathcal{O}_\mathcal{Y}}{\otimes} \pi^{-1}\mathcal{L}$. The action of $\mathcal{L}$ on $\mathcal{O}_\mathcal{Y}$ is a morphism of right $\mathcal{O}_\mathcal{Z}[\mathcal{D}_X]$-modules $\mathcal{L} \to \mathcal{H}om_{\mathcal{O}_\mathcal{Y}[\mathcal{D}_X]}(\Omega_\mathcal{Y}, \mathcal{O}_\mathcal{Y}[\mathcal{D}_X])$; pulling it back to $\mathcal{Z}$, we get a morphism $\pi^*\mathcal{L} \to \mathcal{H}om_{\mathcal{O}_\mathcal{Z}[\mathcal{D}_X]}(\pi^*\Omega_\mathcal{Y}, \mathcal{O}_\mathcal{Z}[\mathcal{D}_X])$. A section of $\pi^\dagger\mathcal{L}$ is a pair (ℓ, θ) where $\ell \in \pi^*\mathcal{L}$ and $\theta \in \mathcal{H}om_{\mathcal{O}_\mathcal{Z}[\mathcal{D}_X]}(\Omega_\mathcal{Z}, \mathcal{O}_\mathcal{Z}[\mathcal{D}_X])$ a morphism whose restriction to $\pi^*\Omega_\mathcal{Y} \subset \Omega_\mathcal{Z}$ coincides with the morphism defined by ℓ. As a mere $\mathcal{O}_\mathcal{Z}[\mathcal{D}_X]$-module, $\pi^\dagger\mathcal{L}$ is an extension of $\pi^*\mathcal{L}$ by $\Theta_{\mathcal{Z}/\mathcal{Y}}$.

Now $\pi^\dagger\mathcal{L}$ is naturally a Lie* algebroid on $\mathcal{Z}$; for an $\mathcal{L}$-module M its pull-back π^*M to $\mathcal{Z}$ is naturally a $\pi^\dagger\mathcal{L}$-module. The functors $\pi^\dagger$ are compatible with the composition of π's and satisfy the descent property.

Therefore, if a smooth group $\mathcal{D}_X$-scheme G acts on $\mathcal{Y}$, then we know what a G-action on $\mathcal{L}$ is. We leave it to the reader to check that this is the same as a strong G-action on $\mathcal{L}$ as defined above.

2.5.18. Here is a local version of the above picture (cf. 2.5.11–2.5.13).

The following terminology will be of use. Let Q be a topological commutative algebra and P a Q-algebroid equipped with a linear topology. One says that P is a *topological Lie Q-algebroid* if the bracket on P, the Q-action on P, and the P-action on Q are continuous.[71] For the rest of the section we assume that $\dim X = 1$. Let

[71] We assume, as always, that the topologies on P and Q are complete.

$x \in X$ be a point, $U_x := X \smallsetminus \{x\}$ its complement. Let R^ℓ be a commutative $\mathcal{D}$-algebra on U_x, $\mathcal{L}$ a Lie* R-algebroid. Since $\mathcal{L}$ is an R^ℓ-module, $j_{x*}\mathcal{L}$ carries the topology Ξ_x^R at x (see 2.4.11).

LEMMA. (i) *The action of $h_x(j_{x*}\mathcal{L})$ on $h_x(j_{x*}R)$ is Ξ_x^R-Ξ_x^{as}-continuous with respect to each variable. The Lie bracket on $h_x(j_{x*}\mathcal{L})_x$ is Ξ_x^R-continuous with respect to each variable.*

(ii) *If R^ℓ is a countably generated $\mathcal{D}_{U_x}$-algebra and $\mathcal{L}$ is a countably generated $R^\ell[\mathcal{D}_{U_x}]$-module, then the action is Ξ_x^R-Ξ_x^{as}-continuous and the bracket is Ξ_x^R-continuous.*

Proof. (i) Take $R_\xi \in \Xi_x^{as}(j_{x*}R)$. For $\bar{l} \in h_x(j_{x*}\mathcal{L})$ one has $\{r \in R_\xi : \bar{l}^{-1}(r) \in R_\xi\} \in \Xi_x^{as}$. For $\bar{r} \in h_x(R)$ one has $\{l \in \mathcal{L} : \bar{l}(\bar{r}) \in h_x(R_\xi)\} \in \Xi_x^R$.

(ii) Take R_ξ as above and $\mathcal{L}_\zeta \in \Xi_x^R$ which is an R_ξ-module. It suffices to find $\mathcal{L}_\eta \in \Xi_x^R$, $R_\nu \in \Xi_x^{as}$ such that the action map on $\mathcal{L}_\eta$, R_ν takes values in R_ξ and the bracket on $\mathcal{L}_\eta$ takes values in $\mathcal{L}_\zeta$. Together with (i) this proves the continuity.

Let $P \subset R_\xi$, $Q \subset \mathcal{L}_\zeta$ be countably generated $\mathcal{D}$-submodules such that $P|_{U_x}$ generates R^ℓ as a $\mathcal{D}_{U_x}$-algebra and $Q|_{U_x}$ generates $\mathcal{L}$ as an $R^\ell[\mathcal{D}_{U_x}]$-module. According to 2.2.20(ii), there exists $P' \in \Xi_x(P)$, $Q' \in \Xi_x(Q)$ such that the action map on Q', P' takes values in R_ξ and the bracket on Q' takes values in $\mathcal{L}_\zeta$. Now let R_ν be the $\mathcal{D}_X$-subalgebra of $j_{x*}R$ generated by P', and let $\mathcal{L}_\eta$ be the R_ν-submodule of $j_{x*}\mathcal{L}$ generated by Q'. □

We see that the countability conditions of (ii) in the lemma imply that $\hat{h}_x^R(\mathcal{L}) := \hat{h}_x^R(j_{x*}\mathcal{L})$ is a topological Lie R_x^{as}-algebroid.

REMARK. The countability condition is irrelevant in the setting of objects $\tilde{h}_x$ from 2.1.15.

2.5.19. For arbitrary R, $\mathcal{L}$ let $\Xi_x^{R\,\mathcal{L}}$ be the set of pairs $(R_\xi, \mathcal{L}_\xi) \in \Xi_x^{as}(j_{x*}R) \times \Xi_x(j_{x*}\mathcal{L})$ such that $\mathcal{L}_\xi$ is a Lie* R_ξ-subalgebroid of $j_{x*}\mathcal{L}$.[72] The corresponding R_ξ and $\mathcal{L}_\xi$ form topologies on $j_{x*}R$ and $j_{x*}\mathcal{L}$ at x which we denote by $\Xi_x^{as\mathcal{L}}(R)$, $\Xi_x^{LieR}(\mathcal{L})$. Let $R_x^{as\mathcal{L}} := \hat{h}_x^{as\mathcal{L}}(j_{x*}R)$, $\hat{h}_x^{LieR}(\mathcal{L}) := \hat{h}_x^{LieR}(j_{x*}\mathcal{L})$ be the completions. Then $R_x^{as\mathcal{L}}$ is a commutative topological algebra and $\hat{h}_x^{LieR}(\mathcal{L})$ a topological Lie $R_x^{as\mathcal{L}}$-algebroid.

LEMMA. *Suppose that R^ℓ is a finitely generated $\mathcal{D}_{U_x}$-algebra and $\mathcal{L}$ is a finitely generated $R^\ell[\mathcal{D}_{U_x}]$-module. Then the above topologies on $j_{x*}R$ and $j_{x*}\mathcal{L}$ coincide: one has $\Xi_x^{as} = \Xi_x^{as\mathcal{L}}$ and $\Xi_x^R = \Xi_x^{LieR}$. Thus $R_x^{as\mathcal{L}} = R_x^{as}$, $\hat{h}_x^{LieR}(\mathcal{L}) = \hat{h}_x^R(\mathcal{L})$.*

Proof. Take $R_\xi \in \Xi_x^{as}(j_{x*}R)$ and $\mathcal{L}_\xi \in \Xi_x(j_{x*}\mathcal{L})$ which is an R_ξ^ℓ-submodule of $j_{x*}\mathcal{L}$. We want to find $(R_\zeta, \mathcal{L}_\zeta) \in \Xi_x^{R\,\mathcal{L}}$ such that $R_\zeta \subset R_\xi$, $\mathcal{L}_\zeta \subset \mathcal{L}_\xi$. Shrinking R_ξ, $\mathcal{L}_\xi$ if necessary, we can assume that R_ξ is a finitely generated $\mathcal{D}_X$-algebra and $\mathcal{L}_\xi$ is a finitely generated $R_\xi^\ell[\mathcal{D}_X]$-module. Set $R_\zeta = R_\xi$ and define $\mathcal{L}_\zeta$ to be the intersection of $\mathcal{L}_\xi$ and the normalizers of R_ξ and $\mathcal{L}_\xi$ in $j_{x*}\mathcal{L}$. We leave it to the reader to show that $(R_\zeta, \mathcal{L}_\zeta) \in \Xi_x^{R\mathcal{L}}$. □

2.5.20. Suppose that R^ℓ is a finitely generated $\mathcal{D}_{U_x}$-algebra and our $\mathcal{L}$ is a projective $R^\ell[\mathcal{D}_{U_x}]$-module of finite rank. By 1.4.14, the dual module $\mathcal{L}^\circ$ is naturally

[72] I.e., $\mathcal{L}_\xi$ is a Lie* subalgebra and an R_ξ^ℓ-submodule of $j_{x*}\mathcal{L}$, and the action of $\mathcal{L}_\xi$ on $j_{x*}R$ preserves R_ξ.

a Lie$^!$ coalgebra in $\mathcal{M}(X, R^\ell)$ which yields (see 1.4.15) a formal $\mathcal{D}_X$-scheme groupoid $\mathcal{G}$ on $\operatorname{Spec} R^\ell$. Then (see Remark in 2.4.11 and 2.4.10(iii)) $\hat{h}^R_x(\mathcal{L}^\circ)$ is a topological Lie R^{as}_x-coalgebroid and $\mathcal{G}^{as}_x$ is a formal groupoid on $\operatorname{Spec} R^{as}_x$.

On the other hand, the duality $*$-pairing between $\mathcal{L}$ and $\mathcal{L}^\circ$ yields a continuous R^{as}_x-bilinear pairing $\hat{h}^R_x(\mathcal{L}) \times \hat{h}^R_x(\mathcal{L}^\circ) \to R^{as}_x$.

Exercise. Show that the latter pairing is non-degenerate, i.e., identifies either of the topological R^{as}_x-modules $\hat{h}^R_x(\mathcal{L})$, $\hat{h}^R_x(\mathcal{L}^\circ)$ with the topological dual to another. Then the topological Lie R^{as}_x-coalgebroid structure on $\hat{h}^R_x(\mathcal{L}^\circ)$ is dual to the topological Lie R^{as}_x-algebroid structure on $\hat{h}^R_x(\mathcal{L})$, and $\hat{h}^R_x(\mathcal{L}^\circ)$ identifies canonically with the Lie R^{as}_x-coalgebroid of the formal groupoid $\mathcal{G}^{as}_x$.

2.5.21. Let Q be a reasonable topological commutative algebra (see 2.4.8), Θ_Q the Lie Q-algebroid Θ_Q of continuous derivations of Q. Then Q carries a natural topology whose base is formed by all subalgebroids $\Theta_{Q;I,S} \subset \Theta_Q$; here $I \subset Q$ is a reasonable ideal, $S \subset Q/I$ a finite set, and $\Theta_{Q;I,S} := \{\tau \in \Theta_Q : \tau(I), \tau(S) \subset I\}$. This topology makes Θ_Q a topological Lie Q-algebroid. It is a final object in the category of topological Lie Q-algebroids.

Suppose R^ℓ is smooth (see 2.3.15). Then R^{as}_x is reasonable and formally smooth (see Lemma in 2.4.8 and 2.4.10(ii)), so we have a topological Lie R^{as}_x-algebroid $\Theta_{R^{as}_x}$. The Lie* R-algebroid Θ_R satisfies the conditions of 2.5.19; it defines a topological Lie R^{as}_x-algebroid $\hat{h}^R_x(\Theta_R)$.

Proposition. *One has* $\hat{h}^R_x(\Theta_R) \xrightarrow{\sim} \Theta_{R^{as}_x}$.

Proof. Consider the canonical morphism $\psi : \hat{h}^R_x(\Theta_R) \to \Theta_{R^{as}_x}$ of topological Lie R^{as}_x-algebroids. We want to prove that ψ is an isomorphism. It suffices to show that it is an isomorphism of abstract vector spaces. This implies automatically that ψ is a homeomorphism since ψ is continuous and our topological vector spaces admit a countable base of open linear subspaces of countable codimension.

Take any $R^\ell_\xi \in \Xi^{as}_x(R)$. By 2.4.9, there is a canonical morphism of $\mathcal{D}_X$-algebras $j_{x*}R^\ell \to R^\ell_{\xi x} \widehat{\otimes} K_x$ (corresponding to the projection $R^{as}_x \to R^\ell_{\xi x}$) and an identification $\operatorname{Der}_{cont}(R^{as}_x, R^\ell_{\xi x}) \xrightarrow{\sim} \operatorname{Der}_{\mathcal{D}}(j_{x*}R^\ell, R^\ell_{\xi x} \widehat{\otimes} K_x) = h(j_{x*}\Theta_R \underset{j_{x*}R^\ell}{\otimes} (R^\ell_{\xi x} \widehat{\otimes} K_x))$.

If V is any finitely generated projective $R^\ell[\mathcal{D}_{U_x}]$-module, then the vector space $h(j_{x*}V \underset{j_{x*}R^\ell}{\otimes} (R^\ell_{\xi x} \widehat{\otimes} K_x))$ equals the completion of $h_x(j_{x*}V)$ with respect to the $\Xi^{R_\xi}_x$-topology (see 2.4.11). Indeed, each submodule $N \in \Xi^{R_\xi}_x(j_{x*}V)$ yields a projection $j_{x*}V \underset{j_{x*}R^\ell}{\otimes} (R^\ell_{\xi x} \widehat{\otimes} K_x) = N \underset{R^\ell_\xi}{\otimes} (R^\ell_{\xi x} \widehat{\otimes} K_x) \twoheadrightarrow N \underset{R^\ell_\xi}{\otimes} (R^\ell_{\xi x} \widehat{\otimes} K_x/O_x) = i_{x*}N^\ell_x$, hence a projection $h(j_{x*}V \underset{j_{x*}R^\ell}{\otimes} (R^\ell_{\xi x} \widehat{\otimes} K_x)) \twoheadrightarrow N^\ell_x$. Passing to the limit, we get a linear map $h(j_{x*}V \underset{j_{x*}R^\ell}{\otimes} (R^\ell_{\xi x} \widehat{\otimes} K_x)) \to \hat{h}^{R_\xi}_x(j_{x*}V)$. To see that this is an isomorphism, it suffices (since both sides are additive in V) to consider the case $V = R^\ell[\mathcal{D}_{U_x}]$; here both our vector spaces equal $R^\ell_\xi \widehat{\otimes} K_x$.

Taking $V = \Theta_R$, we get an isomorphism $\phi_\xi : \operatorname{Der}_{cont}(R^{as}_x, R^\ell_{\xi x}) \xrightarrow{\sim} \hat{h}^{R_\xi}_x(j_{x*}\Theta_R)$. Passing to the projective limit with respect to the R^ℓ_ξ's, we get an isomorphism of vector spaces $\phi : \Theta_{R^{as}_x} \xrightarrow{\sim} \hat{h}^R_x(\Theta_R)$ which is left inverse to ψ. We are done. □

2.5.22. Elliptic algebroids. As above, we assume that X is a curve. Suppose we have a morphism $\phi : M \to N$ of vector $\mathcal{D}_X$-bundles on an algebraic $\mathcal{D}_X$-space $\mathcal{Y}$. We say that ϕ is *elliptic* if it is injective and $\operatorname{Coker}\tau$ is a locally projective $\mathcal{O}_{\mathcal{Y}}$-module of finite rank. This property is stable under duality: ϕ is elliptic if and only if $\phi^\circ : N^\circ \to M^\circ$ is. Notice that $(\operatorname{Coker}\phi)^\ell$ and $(\operatorname{Coker}\phi^\circ)^\ell$ are mutually dual vector bundles with horizontal connections on $\mathcal{Y}$.

Suppose $\mathcal{Y}$ is smooth, and let $\mathcal{L}$ be a Lie* algebroid on $\mathcal{Y}$ which is a vector $\mathcal{D}_X$-bundle. We say that $\mathcal{L}$ is *elliptic* if the anchor morphism $\tau : \mathcal{L} \to \Theta_{\mathcal{Y}}$ is elliptic.

Let $\mathcal{G}$ be the formal $\mathcal{D}_X$-scheme groupoid $\mathcal{G}$ on $\mathcal{Y}$ that corresponds to $\mathcal{L}$ (see 1.4.15). Then $\mathcal{L}$ is elliptic if and only if $\mathcal{G}$ is formally transitive and the corresponding formal group $\mathcal{D}_X$-scheme on $\mathcal{Y}$ – the restriction of $\mathcal{G}$ to the diagonal – is smooth as a mere formal $\mathcal{Y}$-scheme.

The Lie algebra $\mathfrak{g}^\ell_{\mathcal{L}}$ of the latter formal group scheme equals $(\operatorname{Coker}\tau)^\ell$ as a mere vector bundle with a horizontal connection on $\mathcal{Y}$. The Lie bracket on $(\operatorname{Coker}\tau)^\ell$ comes from the cobracket on the dual vector bundle $(\operatorname{Coker}\tau^\circ)^\ell$ which is the quotient of the Lie$^!$ coalgebroid $\mathcal{L}^\circ$ (see 1.4.14).

Notice that $\mathcal{G}$ acts on $\mathfrak{g}^\ell_{\mathcal{L}}$ by adjoint action, and this action is compatible with the Lie bracket. Thus $\mathfrak{g}_{\mathcal{L}}$ is a Lie algebra in the tensor category of $\mathcal{L}$-modules. The projection $\Theta_{\mathcal{Y}} \twoheadrightarrow \operatorname{Coker}\tau = \mathfrak{g}_{\mathcal{L}}$ is compatible with the $\mathcal{L}$-actions where $\mathcal{L}$ acts on $\Theta_{\mathcal{Y}}$ in the adjoint way.

2.5.23. Some important elliptic algebroids arise in the following way. Let G be an algebraic group with Lie algebra $\mathfrak{g}$, $\mathcal{Y}$ a smooth algebraic $\mathcal{D}_X$-space, $\mathfrak{F}_{\mathcal{Y}}$ a G-torsor on $\mathcal{Y}$ equipped with a horizontal connection ∇^h; i.e., $(\mathfrak{F}_{\mathcal{Y}}, \nabla^h)$ is a $\mathcal{D}_X$-scheme G-torsor. Let $\mathfrak{A}$ be the universal groupoid that acts on $(\mathfrak{F}, \mathcal{Y})$; i.e., a groupoid on $\mathcal{Y}$ whose arrows connecting two points y, y' of $\mathcal{Y}$ are isomorphisms of G-torsors $\mathfrak{F}_{\mathcal{Y}y} \xrightarrow{\sim} \mathfrak{F}_{\mathcal{Y}y'}$. The connection on $\mathfrak{F}_{\mathcal{Y}}$ yields a horizontal connection on the groupoid, so it is a $\mathcal{D}_X$-scheme groupoid. In other words, if y, y' were $\mathcal{D}_X$-scheme points, then a $\mathcal{D}_X$-scheme point of our groupoid connecting them is an isomorphism of G-torsors $\mathfrak{F}_{\mathcal{Y}y} \xrightarrow{\sim} \mathfrak{F}_{\mathcal{Y}y'}$ compatible with the connections. Let $\mathfrak{A}^\wedge$ be the formal completion of $\mathfrak{A}$ and let $\mathcal{L}^\circ$ be its Lie $\mathcal{O}_{\mathcal{Y}}$-coalgebroid.

We say that $(\mathfrak{F}_{\mathcal{Y}}, \nabla^h)$ is *non-degenerate* if $\mathfrak{A}^\wedge$ is formally smooth over $\mathcal{Y}$ in the $\mathcal{D}_X$-scheme sense or, equivalently, if $\mathcal{L}^\circ$ is a vector $\mathcal{D}_X$-bundle on $\mathcal{Y}$. If this happens, then the dual Lie* algebroid $\mathcal{L}$ is defined (see 1.4.15 and 1.4.14). It is automatically elliptic. The corresponding Lie algebra $\mathfrak{g}^\ell_{\mathcal{L}}$ equals $\mathfrak{g}(\mathfrak{F}_{\mathcal{Y}})$ (the $\mathfrak{F}_{\mathcal{Y}}$-twist of $\mathfrak{g}$ with respect to the adjoint action).

The non-degeneracy condition can be checked as follows. Our $\mathcal{L}^\circ$ consists of G-invariant 1-forms on $\mathfrak{F}_{\mathcal{Y}}$ relative to X; it is an extension of $\mathfrak{g}^*(\mathfrak{F}_{\mathcal{Y}})$ by $\Omega^1_{\mathcal{Y}/X}$. Such a left $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-module extension amounts to an extension of $\mathcal{O}_{\mathcal{Y}}$ by $\mathfrak{g}(\mathfrak{F}_{\mathcal{Y}}) \otimes \Omega^1_{\mathcal{Y}/X}$; let $\alpha \in \Gamma(\mathcal{Y}, h(\mathfrak{g}(\mathfrak{F}_{\mathcal{Y}}) \otimes \Omega^1_{\mathcal{Y}/X}))$ be its (local) class. To compute it explicitly, choose a vertical connection ∇^v on $\mathfrak{F}_{\mathcal{Y}}$ and consider the component of the curvature of $\nabla^v + \nabla^h$ which is a section of $\mathfrak{g}(\mathfrak{F}_{\mathcal{Y}}) \otimes \Omega^1_{\mathcal{Y}/X} \otimes \omega_X$; our α is the class of this form. Now α amounts to a morphism of $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules $\Theta^\ell_{\mathcal{Y}} \to \mathfrak{g}(\mathfrak{F}_{\mathcal{Y}})$. Then $(\mathfrak{F}_{\mathcal{Y}}, \nabla^h)$ is non-degenerate if and only if the latter morphism is surjective.

Example. We follow the notation of 2.5.9. Let $\mathfrak{F}$ be a G-torsor on X. Set $\mathcal{Y} := \mathcal{C}onn(\mathfrak{F})$; let $\mathfrak{F}_{\mathcal{Y}}$ be the pull-back of $\mathfrak{F}$ to $\mathcal{Y}$ and let ∇^h be the universal horizontal connection. Then $(\mathfrak{F}_{\mathcal{Y}}, \nabla^h)$ is non-degenerate. To see this, consider the gauge action of the group $\mathcal{D}_X$-scheme $\mathcal{J}\mathrm{Aut}(\mathfrak{F})$ on $\mathcal{Y}$ and $\mathfrak{F}_{\mathcal{Y}}$. The corresponding groupoid equals $\mathfrak{A}(\mathfrak{F}_{\mathcal{Y}})$. Therefore $\mathcal{L}^\circ$ equals the pull-back of $\mathfrak{g}^*(\mathfrak{F})^\ell_{\mathcal{D}}$ to $\mathcal{Y}$, which is evidently a vector $\mathcal{D}_X$-bundle on $\mathcal{Y}$, and we are done. Notice that $\mathcal{L}$ is the Lie* algebroid defined by the gauge action of the Lie* algebra $\mathfrak{g}(\mathfrak{F})_{\mathcal{D}}$ on $\mathcal{Y}$ (see 1.4.13).

For another example see 2.6.8.

2.6. Coisson algebras

The coisson algebra structure appeared in the study of Hamiltonian formalism in the calculus of variations at the end of 1970s, see [**DG1**], [**DG2**], [**Ku**], [**KuM**], [**Ma**], and the recent book [**Di**] and references therein. The reader may also look in Chapter 15 of [**FBZ**] or [**DLM**] (where the term "vertex Poisson algebra" is used).

We discussed coisson algebras in the general setting of compound tensor categories in 1.4. Below we show that the functor $R \mapsto R^{as}_x$ transforms coisson algebras to topological Poisson algebras. For a global version of this statement see 4.3.1(iii) and 4.7.3. We consider then elliptic coisson structures and treat briefly two geometric examples, namely, the space of connections on a given G-bundle and the space of opers; for the latter subject see [**DS**], [**BD**], [**Fr**], or [**FBZ**] 15.6, 15.7.

2.6.1. According to 1.4.18 we have the category $\mathrm{Cois}(X) = \mathrm{Cois}(\mathcal{M}^{*!}(X))$ of coisson algebras on X. Explicitly, a coisson algebra is a commutative $\mathcal{D}_X$-algebra A^ℓ together with a Lie* bracket $\{\ \}$ on A (the coisson bracket) such that for any local sections $a, b, c \in A^\ell$ and $\nu \in \omega_X$ one has $\{ab\nu, c\nu\} = p_1^* a \cdot \{b\nu, c\nu\} + p_1^* b \cdot \{a\nu, c\nu\} \in \Delta_* A$ (we use the left $p_1^* A^\ell$-module structure on $\Delta_* A$).

Coisson brackets on a $\mathcal{D}_X$-algebra A^ℓ are A-bidifferential operators, so they have the étale local nature (see the lemma in 2.3.12). Thus they make sense on any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$. According to 1.4.18 a coisson structure on $\mathcal{Y}$ yields a Lie* algebroid structure on $\Omega_{\mathcal{Y}}$ (which determines the coisson structure uniquely).

2.6.2. For the rest of the section we assume that X is a curve.

Let $x \in X$ be a (closed) point, $j_x\colon U_x := X \smallsetminus \{x\} \hookrightarrow X$ its complement. Let A^ℓ be a coisson algebra on U_x. Consider the commutative topological algebra A^{as}_x defined in 2.4.8. Recall that A^{as}_x is the completion of the vector space $h(j_{x*}A)_x$ with respect to the Ξ^{as}_x-topology. The coisson bracket defines a Lie algebra structure on $h(j_{x*}A)_x$.

LEMMA. (i) *The Lie bracket is Ξ^{as}_x-continuous with respect to each variable.*

(ii) *If A^ℓ is a countably generated $\mathcal{D}_{U_x}$-algebra,*[73] *then the Lie bracket is Ξ^{as}_x-continuous. Its continuous extension to A^{as}_x is a Poisson bracket.*

Proof (cf. 2.5.11). (i) We want to show that for every $\bar{l} \in h(j_{x*}A)_x$ the operator $ad_{\bar{l}}$ is Ξ^{as}_x-continuous. By 2.2.20(i) it is Ξ_x-continuous; hence for every $A_\xi \in \Xi^{as}_x$ its preimage $ad_{\bar{l}}^{-1}(A_\xi)$ is Ξ_x-open. Since $ad_{\bar{l}}$ is a derivation, $ad_{\bar{l}}^{-1}(A_\xi) \cap A_\xi \in \Xi^{as}_x$; q.e.d.

(ii) We will find for every $A_\xi \in \Xi^{as}_x$ some $A_\zeta \in \Xi^{as}_x$ such that $\{A_\zeta, A_\zeta\} \subset A_\xi$.[74] Together with (i) it proves the Ξ^{as}_x-continuity of the bracket. Our A_ξ is a countably generated $\mathcal{D}$-module. According to 2.2.20(ii) the morphism $h(A_\xi)_x \otimes h(A_\xi)_x \to h(j_{x*}A)_x$ coming from the coisson bracket is Ξ_x-continuous. So for some $L \in \Xi_x(A_\xi)$ one has $\{h(L)_x, h(L)_x\} \subset h(A_\xi)_x$; i.e., $\{L, L\} \subset A_\xi$. Our A_ζ is the subalgebra generated by L. □

[73] Equivalently, this means that A is a countably generated $\mathcal{D}$-module.

[74] See 2.5.3 for the definition of $\{A_\zeta, A_\zeta\}$.

2.6.3. Assume that A satisfies the countability condition of (ii) in the above lemma. We define the bracket on A_x^{as} extending the bracket on $h_x(j_{x*}A)$ by continuity. It is automatically a Poisson bracket since $h(j_{x*}A)$ acts on $j_{x*}A$ by derivations.

Remark. It is easy to see that $\hat{h}_x^A(\Omega_A) = \varprojlim \Omega_{A^\ell_{\xi x}}$, $A_\xi \in \Xi_x^{as}(A)$; i.e., it is the module $\hat{\Omega}_{A_x^{as}}$ of the topological Kahler differentials of A_x^{as}. Now a coisson structure on A yields the Lie A-algebroid structure on Ω_A, hence, according to 2.5.18, the topological Lie A_x^{as}-algebroid structure on $\hat{h}_x^A(\Omega_A)$. On the other hand, $\hat{\Omega}_{A_x^{as}}$ carries the topological Lie A_x^{as}-algebroid structure coming from the Poisson bracket on A_x^{as}. The two structures coincide.

Remark. The countability condition becomes irrelevant if we consider instead of plain topologies the finer structure from 2.1.15.

Let A^ℓ be an arbitrary (not necessarily countably generated) coisson algebra. Denote by Ξ_x^{cois} the topology at x on $j_{x*}A$ formed by all $A_\xi \in \Xi_x^{as}$ which are coisson subalgebras of $j_{x*}A$. The coisson bracket on $h_x(j_{x*}A)$ is continuous with respect to this topology,[75] so the corresponding completion is a topological Poisson algebra which we denote by A_x^{cois}.

If A^ℓ is countably generated, then A_x^{cois} is the completion of A_x^{as} with respect to the topology formed by those open ideals which are Lie subalgebras of A_x^{as}.

2.6.4. Lemma. *If A^ℓ is a finitely generated $\mathcal{D}_X$-algebra, then the topologies Ξ_x^{cois} and Ξ_x^{as} on $j_{x*}A$ coincide, so $A_x^{cois} = A_x^{as}$.*

Proof (cf. 2.5.13). We want to show that every $A_\xi \in \Xi_x^{as}$ contains $A_\zeta \in \Xi_x^{as}$ which is a coisson subalgebra. We can assume that A_ξ is generated as a $\mathcal{D}_X$-algebra by a $\mathcal{D}_X$-coherent submodule $M \subset A_\xi$. Let $A_\zeta \subset A_\xi$ be the maximal $\mathcal{D}_X$-submodule such that $\{A_\zeta, A_\xi\} \subset A_\xi$. We will show that $A_\zeta \in \Xi_x^{cois}$. Our A_ζ is the maximal $\mathcal{D}_X$-submodule of A_ξ such that $\{A_\zeta, M\} \subset A_\xi$, so, by 2.2.20, $A_\zeta \in \Xi_x$. Clearly A_ζ is a Lie* subalgebra. One has $\{A_\zeta \cdot A_\zeta, A_\xi\} \subset A_\zeta \cdot \{A_\zeta, A_\xi\} \subset A_\xi \cdot A_\xi \subset A_\xi$, so $A_\zeta \cdot A_\zeta \subset A_\zeta$. □

Remark. If A^ℓ is not finitely generated, then the topologies Ξ_x^{cois} and Ξ_x^{as} can be different. This happens, e.g., for $A^\ell = \mathrm{Sym}(\mathfrak{gl}(\mathcal{D}_X))$ (see 2.5.14).[76]

2.6.5. Let $S \subset X$ be a finite subset, $j_S : U_S \hookrightarrow X$ the complementary open embedding. Let A be a countably generated $\mathcal{D}_{U_S}$-algebra equipped with a coisson structure, so, by 2.6.2, $A_S^{as} = \widehat{\bigotimes}_{s\in S} A_s^{as}$ (see 2.4.12) is a topological Poisson algebra. The canonical morphism $\Gamma(U_S, h(A)) \to A_S^{as}$ from 2.4.12 commutes with brackets; i.e., it is a hamiltonian action of the Lie algebra $\Gamma(U_S, h(A))$ on A_S^{as}. If X is proper and S is non-empty, then, according to 2.4.6 and 2.4.12, the zero fiber of the momentum map coincides with $\mathrm{Spf}\,\langle A\rangle(U_S) \subset \mathrm{Spf}\, A_S^{as} = \prod \mathrm{Spf}\, A_s^{as}$.

Remark. The hamiltonian reduction appears when one looks at the chiral homology of a chiral quantization (mod t^2) of our coisson algebra.

[75]Indeed, for every $A_\xi \in \Xi_x^{cois}$ the action of the Lie algebra $h_x(A_\xi)$ on $A^\ell_{\xi x}$ is continuous with respect to the topology on $h_x(A_\xi)$ formed by $h_x(A_\zeta)$ where $A_\zeta \in \Xi_x^{cois}(A)$, $A_\zeta \subset A_\xi$.

[76]Indeed, for any Lie* algebra L the topology Ξ_x^{as} on the coisson algebra $\mathrm{Sym}\, L$ induces on $L \subset \mathrm{Sym}\, L$ the topology Ξ_x, and Ξ_x^{cois} induces on L the topology Ξ_x^{Lie}.

2.6.6. Elliptic brackets. Suppose that R^ℓ is a smooth $\mathcal{D}_X$-algebra, $\{\ \}$ a coisson bracket.

We say that $\{\ \}$ is *non-degenerate* or *symplectic* if the anchor morphism $\tau = \tau_{\{\ \}} : \Omega_R \to \Theta_R$ for the Lie* algebroid structure on Ω_R is an isomorphism. Symplectic brackets occur quite rarely though. Here is a more relevant notion:

We say that our bracket is *elliptic* if Ω_R is an elliptic Lie* algebroid. By 2.5.22, an elliptic $\{\ \}$ yields a Lie algebra $\mathfrak{g}_{\{\ \}} := \mathfrak{g}_{\Omega_R}$ in the tensor category of coisson R-modules (see 1.4.20) which is a locally projective R^ℓ-module. This $\mathfrak{g}_{\{\ \}}$ vanishes if and only if $\{\ \}$ is symplectic.

PROPOSITION. *$\mathfrak{g}$ carries a natural non-degenerate ad- and Ω_R-invariant symmetric bilinear form, i.e., a morphism of coisson modules* $(\ ,\ ) : \mathrm{Sym}^2_{R^\ell}\mathfrak{g}^\ell_{\{\ \}} \to R^\ell$.

Proof. The anchor morphism $\tau : \Omega_R \to \Theta_R$ has property $\tau^\circ = \tau$ (we use the identifications $\Theta_R = \Omega_R^\circ$, $\Omega_R = \Theta_R^\circ$). Thus we have a non-degenerate skew-symmetric pairing $\in P_2^*(\{\mathcal{C}one(\tau), \mathcal{C}one(\tau)\}, R)$ which yields a tensor symmetric pairing $\mathfrak{g}^{\ell\otimes 2}_{\{\ \}} \to R^\ell$. We leave it to the reader to check the properties. □

In the rest of the section we discuss briefly two geometric examples of elliptic brackets.

2.6.7. The coisson structure on $\mathcal{C}onn(\mathfrak{F})$. We follow the notation of 2.5.9. Consider the twisted Kac-Moody extension $\mathfrak{g}(\mathfrak{F})^\kappa_{\mathcal{D}}$. Let A be its twisted symmetric algebra; i.e., A^ℓ is the quotient of $\mathrm{Sym}(\mathfrak{g}(\mathfrak{F})^\kappa_{\mathcal{D}})^\ell$ modulo the relation $1^\kappa = 1$ where 1^κ is the generator of $\mathcal{O}_X \subset \mathfrak{g}(\mathfrak{F})^\kappa_{\mathcal{D}}$. According to Example (iii) in 1.4.18, A^ℓ is a coisson algebra; the bracket is defined by the condition that $\mathfrak{g}(\mathfrak{F})^\kappa_{\mathcal{D}} \hookrightarrow A$ is a morphism of Lie* algebras. The coisson bracket is denoted by $\{\ \}^\kappa$.

Suppose that the form κ is non-degenerate. Then, by 2.5.9, one has a canonical identification of the $\mathcal{D}_X$-schemes $\mathrm{Spec}\, A^\ell = \mathcal{C}onn(\mathfrak{F})$.

PROPOSITION. *The Lie* algebroid Ω_A defined by $\{\ \}^\kappa$ identifies canonically with the Lie* algebroid $\mathcal{L}$ from Example in 2.5.23. Therefore the bracket $\{\ \}^\kappa$ is elliptic and the Lie $A^\ell[\mathcal{D}_X]$-algebra $\mathfrak{g}_{\{\ \}^\kappa}$ equals $\mathfrak{g}(\mathfrak{F})_A$.*

Proof. Back in 2.5.23 we identified $\mathcal{L}$ with the Lie* algebroid $A^\ell \otimes \mathfrak{g}(\mathfrak{F})_{\mathcal{D}}$ defined by the gauge action of the Lie* algebra $\mathfrak{g}(\mathfrak{F})_{\mathcal{D}}$ on $\mathcal{C}onn(\mathfrak{F})$. It remains to identify the latter Lie* algebroid with Ω_A. Since $\mathcal{C}onn(\mathfrak{F})$ is a torsor for the vector $\mathcal{D}_X$-scheme of $\mathfrak{g}(\mathfrak{F})$-valued forms, there is a canonical identification $A^\ell \otimes \mathfrak{g}^*(\mathfrak{F})_{\mathcal{D}} \xrightarrow{\sim} \Omega_A$. Using $\kappa : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$, we can rewrite it as an identification $A^\ell \otimes \mathfrak{g}(\mathfrak{F})_{\mathcal{D}} \xrightarrow{\sim} \Omega_A$. By 2.5.9, the latter is an isomorphism of Lie* A-algebroids, and we are done. □

REMARK. The resolution $\Omega_A \xrightarrow{\tau} \Theta_A$ of $\mathfrak{g}_{\{\ \}^\kappa}$ identifies naturally with canonical resolution (2.1.9.1) of $\mathfrak{g}(\mathfrak{F})_A$. The form $(\ ,\ )$ on $\mathfrak{g}_{\{\ \}^\kappa}$ from 2.6.6 becomes the form κ on $\mathfrak{g}(\mathfrak{F})_A$.

2.6.8. The coisson structure on the moduli of opers. For all details and proofs of the statements below, see the references at the beginning of the section.

(a) Let G be a semi-simple group; for simplicity, we assume it to be adjoint. Let $N \subset B \subset G$ be a Borel subgroup and its nilradical, $T = B/N$ the Cartan torus, $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{g}$, $\mathfrak{t} = \mathfrak{b}/\mathfrak{n}$ the corresponding Lie algebras, $\psi : \mathfrak{n} \to k$ a non-degenerate character.[77] Let $\mathfrak{F}_B$ be a B-torsor on X and $\mathfrak{F}_G$, $\mathfrak{F}_T$ the induced G- and T-torsors;

[77] The vector space $\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$ is the direct sum of lines corresponding to simple roots; "non-degenerate" means that $\psi : \mathfrak{n}/[\mathfrak{n},\mathfrak{n}] \to k$ does not vanish on each of the lines.

the adjoint action yields the corresponding twisted Lie $\mathcal{O}_X$-algebras $\mathfrak{n}(\mathfrak{F}) \subset \mathfrak{b}(\mathfrak{F}) \subset \mathfrak{g}(\mathfrak{F})$. For a character $\gamma : B \to \mathbb{G}_m$ we denote by $\mathfrak{F}_B^\gamma$ the corresponding induced $\mathcal{O}_X^*$-torsor = line bundle on X. Suppose that for every simple root $\gamma : B \to \mathbb{G}_m$ we are given an identification of the line bundles $\mathfrak{F}_B^\gamma \xrightarrow{\sim} \omega_X$. Then ψ yields a morphism $\mathfrak{n}(\mathfrak{F}) \to \omega_X$ which extends to a morphism of Lie* algebras $\mathfrak{n}(\mathfrak{F})_{\mathcal{D}} \to \omega_X$; we denote it again by ψ.

Let κ be an ad-invariant symmetric bilinear form on $\mathfrak{g}$, $\mathfrak{g}(\mathfrak{F})^\kappa_{\mathcal{D}}$ the Kac-Moody extension of $\mathfrak{g}_{\mathcal{D}}$ twisted by $\mathfrak{F}_G$, and A the coisson algebra from 2.6.7.

The restriction of κ to $\mathfrak{n}$ vanishes, so we have a canonical embedding of Lie* algebras $\alpha : \mathfrak{n}(\mathfrak{F})_{\mathcal{D}} \hookrightarrow \mathfrak{g}(\mathfrak{F})^\kappa_{\mathcal{D}} \subset A$. Consider the corresponding BRST reduction $\mathcal{C}_{\text{BRST}}(\mathfrak{n}(\mathfrak{F}), A)_c$ (see 1.4.21).

Proposition. *The BRST reduction is regular, i.e.,* $H^{\neq 0}\, \mathcal{C}_{\text{BRST}}(\mathfrak{n}(\mathfrak{F}), A)_c$ *vanishes, and the coisson algebra* $W^\kappa_c := H^0_{\text{BRST}}(\mathfrak{n}(\mathfrak{F})_{\mathcal{D}}, A)$ *coincides with the hamiltonian* $\mathfrak{n}(\mathfrak{F})_{\mathcal{D}}$*-reduction of* A *(see 1.4.26).* □

The above hamiltonian reduction was introduced in [**DS**], and W^κ_c is known as the *Gelfand-Dikii* algebra. It can be interpreted geometrically as follows.

(b) For a test $\mathcal{D}_X$-algebra R^ℓ an $\mathfrak{g}$-*oper* on $\operatorname{Spec} R^\ell$ is a triple $(\mathfrak{F}_G, \nabla, \mathfrak{F}_B)$ where $\mathfrak{F}_G$ is a G-torsor on $\operatorname{Spec} R^\ell$, ∇ is a horizontal connection on $\mathfrak{F}_G$, and $\mathfrak{F}_B \subset \mathfrak{F}_G$ is a reduction of $\mathfrak{F}_G$ to B. We demand that:

(i) ∇ satisfies the Griffith transversality condition with respect to $\mathfrak{F}_B$; i.e., the horizontal form $c(\nabla) := \nabla \bmod \mathfrak{b}(\mathfrak{F}_B) \in R^\ell \otimes \omega_X \otimes (\mathfrak{g}/\mathfrak{b})(\mathfrak{F}_B)$ takes values in $\mathfrak{n}^\perp/\mathfrak{b}(\mathfrak{F}_B)$;

(ii) $c(\nabla)$ is non-degenerate; i.e., its components with respect to the simple root decomposition $\mathfrak{n}^\perp/\mathfrak{b} \xrightarrow{\sim} \prod k_\gamma$ are all invertible.

The functor which assigns to R the set of isomorphism classes[78] of $\mathfrak{g}$-opers is representable by a smooth $\mathcal{D}_X$-scheme $\mathcal{O}p_{\mathfrak{g}}$.

(c) Suppose that κ is non-degenerate. Then there is a natural morphism of $\mathcal{D}_X$-schemes

(2.6.8.1) $$\operatorname{Spec} W^\kappa_c \to \mathcal{O}p_{\mathfrak{g}}$$

defined as follows. Let $I \subset A$ be the ideal generated by $\alpha(\mathfrak{n}(\mathfrak{F})) \subset A$. The canonical connection on the restriction of $\mathfrak{F}_{G\,A}$ to $\operatorname{Spec} A/I \subset \mathcal{C}onn(\mathfrak{F}_G)$ together with the B-structure $\mathfrak{F}_B$ form a $\mathfrak{g}$-oper on $\operatorname{Spec} A/I$. As follows from Proposition in 2.6.7, the fibers of the smooth projection $\operatorname{Spec} A/I \to \operatorname{Spec} W^\kappa_c$ are orbits of the gauge action of the group $\mathcal{D}_X$-scheme $\operatorname{Ker}(\mathcal{J}\mathrm{Aut}(\mathfrak{F}_B) \to \mathcal{J}\mathrm{Aut}(\mathfrak{F}_T)) \subset \mathcal{J}\mathrm{Aut}(\mathfrak{F}_G)$ whose Lie* algebra equals $\mathfrak{n}(\mathfrak{F}_B)$. This action is free; it lifts to $\mathfrak{F}_B$ preserving the canonical connection. So our $\mathfrak{g}$-oper descends to $\operatorname{Spec} W^\kappa_c$; this is (2.6.8.1).

The proposition from 2.6.7 implies that the Lie* algebroid $\Omega_{W^\kappa_c}$ defined by the coisson structure acts naturally on $\mathfrak{F}_G$ preserving ∇. Let $\mathcal{L}^\circ$ be the Lie coalgebroid on $\operatorname{Spec} W^\kappa_c$ defined by the pair $(\mathfrak{F}_G, \nabla)$ as in 2.5.23.[79] The above action can be rewritten as a canonical morphism of Lie coalgebroids

(2.6.8.2) $$\mathcal{L}^\circ \to \Theta^\ell_{W^\kappa_c}.$$

[78] $\mathfrak{g}$-opers are rigid, i.e., admit no non-trivial automorphisms.

[79] In 2.5.23 we used the notation $(\mathfrak{F}_{\mathcal{Y}}, \nabla^h)$.

Proposition. *Both (2.6.8.1) and (2.6.8.2) are isomorphisms. So the Gelfand-Dikii coisson structure is elliptic and the pair $(\mathfrak{F}_G, \nabla)$ on $\mathcal{O}p_{\mathfrak{g}}$ is non-degenerate (see 2.5.23). The form $(\ ,\ )$ from 2.6.6 identifies with the form κ on $\mathfrak{g}(\mathfrak{F}_G)$.* □

2.7. The Tate extension

In this section $\dim X = 1$.

We will discuss a canonical central extension of a matrix Lie* algebra on a curve. A linear algebra version of this ubiquitous extension first appeared (implicitly) in Tate's remarkable note [**T**].[80] It was rediscovered as a Lie algebra of symmetries of a Heisenberg or Clifford module (metaplectic representation) and studied under various names[81] in the beginning of the 1980s; see [**Sa**], [**DJKM**], [**DJM**], [**KP**], [**SW**], [**PS**], and [**Kap2**]. For this point of view see section 3.8.

As was pointed out in [**Kap2**], the linear algebra objects called Tate vector spaces here were, in fact, introduced by Lefschetz ([**Lef**], pp. 78–79) under the name of locally linearly compact spaces and were used by Chevalley (these authors did not perceive the Tate extension though).

A group-theoretic version of the Tate extension was studied in the L^2 setting in [**PS**];[82] for an algebraic variant see [**Kap2**], [**Dr2**] and [**BBE**].

We begin in 2.7.1 with a general format for the construction of central extensions of Lie algebras. The Tate extension in the $\mathcal{D}$-module setting is defined in 2.7.2 and 2.7.3; the situation with extra parameters is considered in 2.7.6. Tate's original linear algebra construction is presented in 2.7.7–2.7.9; the two constructions are related in 2.7.10–2.7.14. The Kac-Moody and Virasoro extensions can be embedded naturally in the Tate extension; see 2.7.5.

The exposition of 2.7.1–2.7.14 essentially follows [**BS**]; the meaning of the constructions will become clear in the chiral context of 3.8.5–3.8.6.

We do not discuss representation theory of the Tate extension; for this subject see [**FKRW**] and [**KR**].

2.7.1. Some concrete nonsense. We work in an abelian pseudo-tensor k-category. For simplicity of notation formulas are written in terms of "symbolic elements" of our objects.

(i) Let L be a Lie algebra. Denote by $\mathcal{E}(L)$ the category of pairs $(L^\natural, \pi)$ where $L^\natural$ is an L-module, $\pi : L^\natural \to L$ a surjective morphism of L-modules (we consider L as an L-module with respect to the adjoint action). We often write $L^\natural$ for $(L^\natural, \pi)$ and set $L_0^\natural := \operatorname{Ker} \pi$.

Any $L^\natural \in \mathcal{E}(L)$ yields a symmetric L-invariant pairing $(\ ) \in P_2(\{L^\natural, L^\natural\}, L_0^\natural)$, $(a, b) := \pi(a)b + \pi(b)a$. Notice that the L-action on $L_0^\natural$ is trivial if and only if $(\ ) \in P_2(\{L, L\}, L_0^\natural) \subset P_2(\{L^\natural, L^\natural\}, L_0^\natural)$. We say that $L^\natural$ is *central* if $(\ ) = 0$.

[80] Which was, apparently, the first work on the subject of algebraic conformal field theory.

[81] E.g., in Moscow they used to call it "the Japanese extension"; cf. A. Brehm's discourse in "The Life of Animals" on the names of the common cockroach *Blatta germanica* in different tongues: Russians call the fellow Prussian, in Austrian highlands he is known as Russian, etc.

[82] [**PS**] deals with the "semi-global" setting of function spaces on a fixed circle; a simpler picture for the space of germs of holomorphic functions with (possibly essential) singularities at the origin (i.e., the case of an infinitely small circle) is discussed in a sequel to [**BBE**].

Denote by $\mathcal{CE}(L)$ the category of central Lie algebra extensions of L. Notice that for every $L^\flat \in \mathcal{CE}(L)$ the adjoint action of $L^\flat$ factors through L. So $L^\flat$ is an L-module; the structure projection $L^\flat \to L$ makes it an object of $\mathcal{E}(L)$.

LEMMA. *The functor $\mathcal{CE}(L) \to \mathcal{E}(L)$ we have defined is fully faithful. It identifies $\mathcal{CE}(L)$ with the category of central objects in $\mathcal{E}(L)$.* □

Assume we have $L^\natural \in \mathcal{E}(L)$ and an L-invariant morphism $\mathrm{tr} : L_0^\natural \to F$ (the L-action on F is trivial). We say that tr is *central* if $\mathrm{tr}(\ ) \in P_2(\{L, L\}, F)$ equals 0. Denote by $L^\natural_{\mathrm{tr}}$ the push-forward of $L^\natural$ by tr; this is an object of $\mathcal{E}(L)$ which is an extension of L by F. It is clear that $L^\natural_{\mathrm{tr}} \in \mathcal{CE}(L)$ if and only if tr is central.

REMARK. If every L-invariant symmetric F-valued pairing on L is trivial, then every tr as above is automatically central.

We suggest that the reader skip the rest of this section, returning to it when necessary.

(ii) Here is an example of extensions that arise in the above manner:

LEMMA. *Let L be a Lie algebra, $L_c, L_d \subset L$ ideals such that $L_c + L_d = L$, and* $\mathrm{tr} : L_f := L_c \cap L_d \to F$ *a morphism such that* $\mathrm{tr}([l_c, l_d]) = 0$ *for every $l_c \in L_c$, $l_d \in L_d$. Then there exists a central extension $L^\flat$ of L by F together with sections $s_c : L_c \to L^\flat$, $s_d : L_d \to L^\flat$ such that the images of s_c, s_d are ideals in $L^\flat$*[83] *and for any $l \in L_f$ one has $(s_c - s_d)l = \mathrm{tr}(l) \in F \subset L^\flat$. Such $(L^\flat, s_c, s_d)$ is unique (up to a unique isomorphism).*

Proof. The uniqueness of $(L^\flat, s_c, s_d)$ is clear. Let us construct $L^\flat$. Set $L^\natural := L_c \oplus L_d$, and consider the extension of L-modules $0 \to L_f \xrightarrow{\alpha} L^\natural \xrightarrow{\beta} L \to 0$ where $\alpha(l) = (l, -l)$, $\beta(l_c, l_d) = l_c + l_d$ (the action of L is the adjoint one). Then $L^\natural \in \mathcal{E}(L)$ and tr is a central morphism (the L-invariance of tr and the vanishing of $\mathrm{tr}(\ )$ follow from the property $\mathrm{tr}([L_c, L_d]) = 0$). Now set $L^\flat := L^\natural_{\mathrm{tr}}$. □

REMARK. In the situation of the above lemma one has $L_c/L_f \xrightarrow{\sim} L/L_d$. The Lie algebra L_c/L_f has a central extension $(L_c/L_f)^\flat$ defined as the push-out of the extension $0 \to L_f \to L_c \to L_c/L_f \to 0$ by $\mathrm{tr} : L_f \to F$. Now s_c yields an isomorphism $(L_c/L_f)^\flat \xrightarrow{\sim} L^\flat/s_d(L_d)$.

Similarly, s_d yields an isomorphism $(L_d/L_f)^\flat \xrightarrow{\sim} L^\flat/s_c(L_c)$ where the central extension $(L_d/L_f)^\flat$ of L_d/L_f is the push-out of the extension $0 \to L_f \to L_d \to L_d/L_f \to 0$ by $-\mathrm{tr} : L_f \to F$.

(iii) Let A be an associative algebra. Denote by A^{Lie} our A considered as a Lie algebra.

If $A^\flat$ is a central extension of A^{Lie} by F, then its bracket comes from a skew-symetric pairing $[\ ]^\flat \in P_2(\{A, A\}, A^\flat)$. We say that $A^\flat$ satisfies the *cyclic property* if for every $a, b, c \in A$ one has $[ab, c]^\flat + [bc, a]^\flat + [ca, b]^\flat = 0$.

Denote by $\mathcal{E}(A)$ the category of pairs $(A^\natural, \pi)$ where $A^\natural$ is an A-bimodule, $\pi : A^\natural \to A$ a surjective morphism of A-bimodules. Notice that any A-bimodule is an A^{Lie}-module with respect to the commutator action. Thus every object of

[83] Or, equivalently, that s_c, s_d are morphisms of L-modules (with respect to the adjoint actions).

$\mathcal{E}(A)$ can be considered as an object of $\mathcal{E}(A^{Lie})$; i.e., we have a faithful functor $\mathcal{E}(A) \to \mathcal{E}(A^{Lie})$.

Take $A^\natural \in \mathcal{E}(A)$ and let $\mathrm{tr} : A^\natural_f := \operatorname{Ker}\pi \to F$ be a central A^{Lie}-invariant morphism, so $A^\flat := A^\natural_{\mathrm{tr}}$ is a central Lie algebra extension of A^{Lie} by F.

LEMMA. *$A^\flat$ satisfies the cyclic property.*

Proof. Let $\tilde{c}$ be a lifting of c. We have a tautological identity $[ab, \tilde{c}] + [b\tilde{c}, a] + [\tilde{c}a, b] = 0$ in $A^\natural$ (it holds in any A-bimodule).[84] Now notice that $\mathrm{tr}([bc, \tilde{a}] - [b\tilde{c}, a]) = \mathrm{tr}(a, bc) = 0$ (since $b\tilde{c}$ is a lifting of bc) and similarly $\mathrm{tr}([ca, \tilde{b}] - [\tilde{c}a, b]) = \mathrm{tr}(ca, b) = 0$. □

For example, if we have two-sided ideals $A_c, A_d \subset A$, $A_c + A_d = A$, and a morphism $\mathrm{tr} : A_f := A_c \cap A_d \to F$ which vanishes on $[A_c, A_d] \subset A_f$, then the central extension $A^\flat$ of A^{Lie} defined in (ii) above satisfies the cyclic property.

REMARK. Assume that our pseudo-tensor category is a tensor unital category (say, the category of k-modules) and A is a commutative unital algebra. Consider the A-module of differentials Ω_A, so we have a universal derivation $d : A \to \Omega_A$. For every object F the map $\operatorname{Hom}(\Omega_A, F) \to P_2(\{A, A\}, F)$, which sends $\varphi : \Omega_A \to F$ to the operation $a, b \mapsto \varphi(adb)$ is injective, and its image consists of those operations ψ that $\psi(a, bc) = \psi(ab, c) + \psi(ac, b)$. Notice that φ vanishes on the image of d if and only if the corresponding ψ is skew-symmetric. Now for $(A^\natural, \mathrm{tr})$ as above one has $[\ ]^\flat \in P_2(\{A, A\}, F)$ (since A is commutative), and our lemma shows that $[\ ]^\flat$ yields a morphism $\Omega_A/d(A) \to F$. This is one of the principal ideas of Tate's work [**T**], a portent of the cyclic homology formalism.

(iv) Suppose our pseudo-tensor category is augmented. For A, $A^\natural$ as above $h(A)$ is an associative algebra and $A^\natural$ is an $h(A)$-bimodule (see 1.2.8). The A-actions also yield morphisms $A \otimes h(A^\natural) \to A^\natural$, $h(A^\natural) \otimes A \to A^\natural$ which are morphisms of, respectively, $(A, h(A))$- and $(h(A), A)$-bimodules.

If $z^\natural \in h(A)$ is such that $z := \pi(z^\natural) \in h(A)$ is an $A^\natural$-central element (i.e., left and right multiplications by z on $A^\natural$ coincide), then the morphism $\partial_{z^\natural} : A \to A^\natural_0$, $a \mapsto az^\natural - z^\natural a$, is a derivation. If $\mathrm{tr} : A^\natural_f \to F$ is an A^{Lie}-invariant morphism, then the composition $\mathrm{tr}\,\partial_{z^\natural} : A \to F$ depends only on z, so we denote it by $\mathrm{tr}\,\partial_z$. It is clear that $\mathrm{tr}\,\partial_z$ kills the commutant $[A, A]$.

Let us assume that A is a unital associative algebra (see 1.2.8), $A^\natural$ is a unital bimodule, and $1 = 1_A \in h(A)$ lifts to $h(A^\natural)$. We get a canonical morphism $\mathrm{tr}\,\partial_1 : A \to F$.

LEMMA. *One has* $\mathrm{tr}(a, b) = \mathrm{tr}\,\partial_1(ab)$. *Therefore* tr *is central if and only if* $\mathrm{tr}\,\partial_1 = 0$.

Proof. Choose $\tilde{1} \in h(A^\natural)$. We have two sections $A \to A^\natural$, $a \mapsto 1^\natural a, a1^\natural$. One has $(1^\natural a, b1^\natural) = ab1^\natural - 1^\natural ab$. Applying tr, we get the lemma. □

In particular, if A^{Lie} is perfect, i.e., $[A, A] = A$, then every tr is central.

[84] This identity can be interpreted as follows. For any A-bimodule M and $a \in A$ denote by l_a, r_a the left, resp. right, multiplication by a; set $ad_a := l_a - r_a$. The identity says that $ad_{ab} = ad_a l_b + ad_b r_a$. Define the A-bimodule structure on $\operatorname{End}(M)$ by $af := fr_a$, $fb := fl_b$. Then the identity says that $ad_{ab} = ad_a b + a ad_b$; i.e., the map $A \to \operatorname{End}(M)$, $a \mapsto ad_a$, is a derivation.

2.7.2. The Tate extension in the $\mathcal{D}$-module setting. Let V, V' be $\mathcal{D}$-modules on X, $\langle\ \ \rangle \in P_2^*(\{V,V'\},\omega_X) = P_2^*(\{V',V\},\omega_X)$ a $*$ pairing. By 1.4.2 the $\mathcal{D}$-module $\mathcal{G} := V\otimes V'$ is an associative* algebra which acts on V, V'. We have the corresponding Lie* algebra $\mathcal{G}^{Lie}$; the pairing is $\mathcal{G}^{Lie}$-invariant.

The Tate extension is a canonical central extension of Lie* algebras

$$0\to\omega_X\to\mathcal{G}^\flat\xrightarrow{\pi^\flat}\mathcal{G}^{Lie}\to 0. \tag{2.7.2.1}$$

To define $\mathcal{G}^\flat$, consider the exact sequence[85] of the $\mathcal{D}$-modules on $X\times X$

$$V\boxtimes V'\xrightarrow{\varepsilon} j_*j^*(V\boxtimes V')\xrightarrow{\pi}\Delta_*(V\otimes V')\to 0. \tag{2.7.2.2}$$

Here $\Delta\colon X\hookrightarrow X\times X$ is the diagonal embedding, $j\colon U := X\times X\smallsetminus\Delta(X)\hookrightarrow X\times X$ is the complementary embedding, and π is the canonical arrow. Namely, according to (2.2.2.1), one has $V\otimes V' = H^1\Delta^!(V\boxtimes V') = \operatorname{Coker}\varepsilon$; explicitly, π sends $(t_2-t_1)^{-1}v\boxtimes v'\in j_*j^*V\boxtimes V'$ to $v\otimes v'(dt)^{-1}\in\Delta_\cdot(V\otimes V')\subset\Delta_*(V\otimes V')$. Note that $\langle\ \rangle$ vanishes on $\operatorname{Ker}\varepsilon$.[86] So, pushing out the above exact sequence by $\langle\ \ \rangle$, we get an extension of $\Delta_*(V\otimes V')$ by $\Delta_*\omega_X$. This extension is supported on the diagonal; applying $\Delta^!$, we get the Tate extension $\mathcal{G}^\flat$. We denote the canonical morphism $j_*j^*(V\boxtimes V')\to\Delta_*\mathcal{G}^\flat$ by $\mu=\mu_{\mathcal{G}}$.

2.7.3. Let us define now the Lie* bracket $[\ ]_{\mathcal{G}^\flat}$ on $\mathcal{G}^\flat$.

Let L be a Lie* algebra that acts on V and V' preserving the pairing. Then the action of L on $V\otimes V'=\mathcal{G}$ lifts canonically to $\mathcal{G}^\flat$. Indeed, the action of the sheaf of Lie algebras $h(L)$ on $\mathcal{G}$ lifts canonically to $\mathcal{G}^\flat$. Namely, global sections of $h(L)$ act on V, V' preserving the $*$ pairing; hence they act on $j_*j^*(V\boxtimes V')$ and $\mathcal{G}^\flat$ by transport of structure, and everything is compatible with localization of X. One checks that the $h(L)$-action on $\mathcal{G}^\flat$ is good in the sense of 2.5.4, so we actually have an L-action.

In particular, the action of $\mathcal{G}$ on V, V' yields a canonical action of $\mathcal{G}^{Lie}$ on $\mathcal{G}^\flat$.

PROPOSITION. *There is a unique Lie* algebra structure on $\mathcal{G}^\flat$ such that the adjoint action of $\mathcal{G}^\flat$ on $\mathcal{G}^\flat$ is the above action composed with $\mathcal{G}^\flat\to\mathcal{G}^{Lie}$. The projection $\mathcal{G}^\flat\to\mathcal{G}^{Lie}$ is a morphism of Lie* algebras, so $\mathcal{G}^\flat$ is a central extension of $\mathcal{G}^{Lie}$. This extension satisfies the cyclic property* (*see* 2.7.1(iii)).

Proof. The uniqueness is clear, as well as the latter property (notice that the commutator on $\mathcal{G}^\flat$ comes from the action of $\mathcal{G}^{Lie}$ on $\mathcal{G}^\flat$ that lifts the adjoint action).

Let us show that our picture fits into the format of 2.7.1(iii); this implies the existence statement. The cyclic property follows then from the lemma in 2.7.1(iii).

Consider the ordered set of all submodules $N_\alpha\subset j_*j^*V\boxtimes V'$ such that $j_*j^*V\boxtimes V'/N_\alpha$ is supported on the diagonal. Set $\mathcal{G}^\natural_\alpha := \Delta^!(j_*j^*V\boxtimes V'/N_\alpha)$ and $\mathcal{G}^\natural := \varprojlim\mathcal{G}^\natural_\alpha$.

The pro-$\mathcal{D}_X$-module $\mathcal{G}^\natural$ carries a canonical structure of the topological $\mathcal{G}$-bimodule. This means that for every *coherent* $\mathcal{D}_X$-submodule $\mathcal{G}_\beta\subset\mathcal{G}$ we have mutually commuting $*$ pairings (left and right action) in $\cdot^l\in P_2^*(\{\mathcal{G}_\beta,\mathcal{G}^\natural\},\mathcal{G}^\natural)$ and $\cdot^r\in P_2^*(\{\mathcal{G}^\natural,\mathcal{G}_\beta\},\mathcal{G}^\natural)$ which are compatible with respect to the embeddings of the $\mathcal{G}_\beta$'s and satisfy the left and right $\mathcal{G}$-action properties.[87]

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[85] It is also exact from the left if V, V' have no $\mathcal{O}_X$-torsion.

[86] Since $\operatorname{Ker}\varepsilon=\Delta_*H^{-1}(V\overset{L}{\otimes}V')=\Delta_*\operatorname{Tor}_1^{\mathcal{O}_X}(V,V'\omega_X^{-1})$, its sections have finite supports.

[87] This makes sense since for every two coherent $\mathcal{D}_X$-submodules $\mathcal{G}_\beta$, $\mathcal{G}_{\beta'}$ of $\mathcal{G}$ their product (i.e., the image in $\mathcal{G}$ of $\mathcal{G}_\beta\boxtimes\mathcal{G}_{\beta'}$ by the product morphism) is coherent.

To define $\cdot^l$, consider the morphism $\cdot_V \boxtimes id_{V'} : \mathcal{G}\boxtimes V\boxtimes V' \to \Delta_*^{1=2}(V\boxtimes V')$. Here $\cdot_V$ is the $*$ action of $\mathcal{G}$ on V and $\Delta^{1=2}$ is the diagonal embedding $(x,y)\mapsto(x,x,y)$. Localizing, we get the morphism $\tilde{\cdot}^l : \mathcal{G}\boxtimes j_*j^*(V\boxtimes V') \to \Delta_*^{1=2} j_*j^*(V\boxtimes V')$. It satisfies the following continuity property: for every N_α, $\mathcal{G}_\beta$ as above there exists an $N_{\alpha'}$ such that $\tilde{\cdot}^l(\mathcal{G}_\beta \boxtimes N_{\alpha'}) \subset \Delta_*^{1=2}(N_\alpha)$.[88] Now $\cdot^l$ is the completion of $\tilde{\cdot}^l$ with respect to the N_α-topology.

One defines $\cdot^r$ in a similar way using the action of $\mathcal{G}$ on V' instead of V. It is clear that the $\cdot^{l,r}$ define on $\mathcal{G}^\natural$ a topological $\mathcal{G}$-bimodule structure.

The projection π from (2.7.2.2) yields the projection $\mathcal{G}^\natural \twoheadrightarrow \mathcal{G}$ which we also denote by π. This is a morphism of (topological) $\mathcal{G}$-bimodules. The $*$ pairing $\langle\ \ \rangle$ yields a $\mathcal{G}^{Lie}$-invariant morphism $\mathcal{G}_0^\natural := \operatorname{Ker}\pi \to \omega_X$ which we denote by tr.

One has[89] $\mathcal{G}^\flat = \mathcal{G}^\natural_{\rm tr}$, and the $h(\mathcal{G}^{Lie})$-module structure on $\mathcal{G}^\flat$ defined in 2.7.2 comes from the $\mathcal{G}^\flat$-module structure on $\mathcal{G}^\natural_{\rm tr}$. Therefore, by 2.7.1(i), our proposition follows from the next lemma:

LEMMA. tr *is a central morphism.*

Proof of Lemma. (a) It suffices to check our lemma in case $V = \mathcal{D}_X^n$ and $V^\circ = \omega\mathcal{D}_X^n$ is its dual. Indeed, our problem is local, so we can choose a surjective morphism $\mathcal{D}_X^n \twoheadrightarrow V$. Then $\langle\ \rangle$ yields a morphism $V^\circ \to \omega\mathcal{D}_X^n$. Our constructions are functorial with respect to the morphisms of the $(V, V^\circ, \langle\ \rangle)$'s. Looking at $(V,V^\circ) \twoheadleftarrow (\mathcal{D}_X^n, V^\circ) \to (\mathcal{D}_X^n, \omega\mathcal{D}_X^n)$, we get the desired reduction.

(b) For such V, V° our $\mathcal{G}$ is the unital associative algebra and $\mathcal{G}^\natural$ is a unital $\mathcal{G}$-module. One has $\mathcal{G}^{Lie} = \mathfrak{gl}(\mathcal{D}_X^n)$ (see 2.5.6(a)), so, by 2.7.1(iv), it remains to check the following fact:

SUBLEMMA.. *The Lie* algebra $\mathfrak{gl}(\mathcal{D}_X^n)$ is perfect.*

Proof of Sublemma. It suffices to consider the case $n=1$. One has $\mathfrak{gl}(\mathcal{D}_X) = D_{\mathcal{D}}^{Lie}$ where D^{Lie} is the sheaf $D = \mathcal{D}_X$ of differential operators considered as a Lie algebra in $\mathcal{D}iff(X)^*$; see Example (iii) in 2.5.6(b). We want to show that the adjoint action of $D^{Lie} = h(D_{\mathcal{D}}^{Lie})$ on $D_{\mathcal{D}}$ has trivial coinvariants. In fact, the action of the subalgebra $\Theta_X \subset D^{Lie}$ of vector fields has trivial coinvariants. To see this, notice that this action preserves the filtration by the degree of the differential operator, and $\operatorname{gr} D_{\mathcal{D}} = \bigoplus_{n\geq 0} \Theta_{X\mathcal{D}}^{\otimes n}$. Let us show that the coinvariants of the Θ_X-action on $\Theta_{\mathcal{D}}^{\otimes n}$ are trivial for $n \neq -1$. Since we deal with a coherent $\mathcal{D}$-module (see (i) in the lemma from 2.1.6), this amounts to the equality $[\Theta_X, \Theta_X^{\otimes n}] = \Theta_X^{\otimes n}$. If $v \in \Theta_X$, $\eta \in \omega_X$, then $[v, v^{n+1}\eta] = v^{n+1}[v,\eta] = v^{n+1}d(v,\eta)$. Such sections for $v(x)\neq 0$ generate the stalk of $\Theta_X^{\otimes n}$ at x if $n\neq -1$. □

2.7.4. If V is a vector $\mathcal{D}_X$-bundle (see 2.2.16), V° is its dual, and $\langle\ \rangle$ is the canonical pairing, then $\mathcal{G}^{Lie} = \mathfrak{gl}(V)$ (see 2.5.6(a)). Therefore we have a canonical central extension $\mathfrak{gl}(V)^\flat$ of $\mathfrak{gl}(V)$ by ω_X. Passing to homology, we get a central extension $\bar{\mathfrak{gl}}(V)^\flat$ of the sheaf of Lie algebras $\bar{\mathfrak{gl}}(V) = h(\mathfrak{gl}(V))$ (which is the sheaf of endomorphisms of V considered as a Lie algebra) by $h(\omega_X)$.

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[88]Let g_i be a finite set of generators of $\mathcal{G}_\beta$, so we have the correponding classes $\bar{g}_i \in h(\mathcal{G}_\beta)$ and endomorphisms $\bar{g}_i\tilde{\cdot}^l$ of $j_*j^*V\boxtimes V'$. Our $N_{\alpha'}$ is the intersection of preimages of N_α by these endomorphisms.

[89]We use the notation of 2.7.1(i); the material of 2.7.1 renders itself to the setting of topological modules in the obvious way.

The Tate extension satisfies the following properties:

(i) *Additivity:* Let $\mathcal{F}$ be a finite flag $V_1 \subset \cdots \subset V_n = V$ such that every $\mathrm{gr}_i V$ is a vector $\mathcal{D}_X$-bundle. Let $\mathfrak{gl}(V,\mathcal{F}) \subset \mathfrak{gl}(V)$ be the corresponding parabolic Lie* subalgebra, so we have the projections $\mathfrak{gl}(V,\mathcal{F}) \to \mathfrak{gl}(\mathrm{gr}_i V)$. Let $\mathfrak{gl}(V,\mathcal{F})^\flat$ be the restriction of the Tate extension $\mathfrak{gl}(V)^\flat$ to $\mathfrak{gl}(V,\mathcal{F})$, and let $\mathfrak{gl}(V,\mathcal{F})^{\flat'}$ be the sum of the pull-backs of the Tate extensions $\mathfrak{gl}(\mathrm{gr}_i V)^\flat$. Now there is a canonical isomorphism of extensions

$$\mathfrak{gl}(V,\mathcal{F})^\flat \xrightarrow{\sim} \mathfrak{gl}(V,\mathcal{F})^{\flat'}. \tag{2.7.4.1}$$

To see this, consider the submodule $P := \Sigma V_j \boxtimes (V/V_{j-1})^\circ \subset V \boxtimes V^\circ$; we have the obvious projections $\pi_i : P \to \mathrm{gr}_i V \boxtimes (\mathrm{gr}_i V)^\circ$. Now $\mathfrak{gl}(V,\mathcal{F}) = \Sigma V_i \otimes (V/V_{i-1})^\circ \subset V \otimes V^\circ$; i.e., $\mathfrak{gl}(V,\mathcal{F}) = \Delta^! P \subset \Delta^!(V \boxtimes V^\circ) = V \otimes V^\circ$. So the exact sequence $0 \to \Delta_*\omega_X \to \mathfrak{gl}(V,\mathcal{F})^\flat \to \mathfrak{gl}(V,\mathcal{F}) \to 0$ is the push-out of $0 \to P \to j_*j^*P \to \Delta^! P \to 0$ by $f : P \to \Delta_*\omega_X$, where f is the restriction of the canonical pairing $V \boxtimes V^\circ \to \Delta_*\omega_X$. Since f is the sum of the canonical pairings $\mathrm{gr}_i V \boxtimes (\mathrm{gr}_i V)^\circ \to \Delta_*\omega_X$ composed with π_i, we get (2.7.4.1). We leave it to the reader to check compatibility with Lie* brackets.

(ii) *Self-duality:* There is a canonical isomorphism of ω_X-extensions

$$\mathfrak{gl}(V)^\flat \xrightarrow{\sim} \mathfrak{gl}(V^\circ)^\flat \tag{2.7.4.2}$$

which lifts the standard isomorphism of Lie* algebras $\mathfrak{gl}(V) \xrightarrow{\sim} \mathfrak{gl}(V^\circ)$. It comes from the transposition of the multiples symmetry of (2.7.2.2).

2.7.5. Induced $\mathcal{D}_X$-bundles; fitting Kac-Moody and Virasoro into Tate. Consider the case of the induced $\mathcal{D}_X$-bundle, so $V = F_{\mathcal{D}} := F \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$, where F is a locally free $\mathcal{O}_X$-module of finite rank. We have $\bar{\mathfrak{gl}}(V) = \mathrm{Diff}(F,F)_X$ and $\mathfrak{gl}(V) = \mathrm{Diff}(F,F)_{\mathcal{D}}$ (see Example (iii) in 2.5.6(b)). The dual $\mathcal{D}$-module V° equals $F^\circ_{\mathcal{D}}$ where $F^\circ := F^*\omega_X$, so $j_*j^*(V \boxtimes V^\circ) = (j_*j^*F \boxtimes F^\circ)_{\mathcal{D}}$ and $\Delta_*(V \otimes V^\circ) = (j_*j^*(F \boxtimes F^\circ)/F \boxtimes F^\circ)_{\mathcal{D}}$. Applying h, we get a canonical identification

$$\mathrm{Diff}(F,F)_X \xrightarrow{\sim} (j_*j^*F \boxtimes F^\circ)/F \boxtimes F^\circ. \tag{2.7.5.1}$$

This is the usual Grothendieck isomorphism which can be described as follows. The action of $\mathrm{Diff}(F,F)_X$ on F makes $(j_*j^*F \boxtimes F^\circ)/F \boxtimes F^\circ$ a $\mathrm{Diff}(F,F)_X$-module, and (2.7.5.1) is a unique morphism of $\mathrm{Diff}(F,F)_X$-modules which maps $1 \in \mathrm{Diff}(F,F)$ to $1 \in \mathrm{End}\, F = F \boxtimes F^\circ(\Delta)/F \boxtimes F^\circ$.[90] The inverse isomorphism assigns to a "kernel" $k(x,y) \in j_*j^*F \boxtimes F^\circ$ the differential operator $f \mapsto k(f), k(f)(x) := \mathrm{Res}^{(y)}_{y=x}\langle k(x,y), f(y)\rangle$.

We see that $\Delta.\bar{\mathfrak{gl}}(V)^\flat$ is the push-forward of the extension

$$0 \to F \boxtimes F^\circ \to j_*j^*F \boxtimes F^\circ \xrightarrow{\pi} \mathrm{Diff}(F,F)_X \to 0 \tag{2.7.5.2}$$

by the map $F \boxtimes F^\circ \to \Delta.\omega_X \to h(\omega_X)$, $f(x) \boxtimes g(y) \mapsto \langle f(x), g(x)\rangle$. The left and right actions of $\bar{\mathfrak{gl}}(V) = \mathrm{Diff}(F,F)_X$ on $\Delta^{\cdot} h(j_*j^*(V \boxtimes V^\circ)) = \Delta^{\cdot} j_*j^*(F \boxtimes F^\circ)$ come from the usual left and right actions of $\mathrm{Diff}(F,F)_X$ on F and F^o, respectively; the adjoint action yields our Lie bracket on the quotient $\bar{\mathfrak{gl}}(V)^\flat$. Therefore $\bar{\mathfrak{gl}}(V)^\flat$ coincides with the Tate extension of the Lie algebra $\mathrm{Diff}(F,F)_X$ as defined in [**BS**].

[90] One can consider the right $\mathrm{Diff}(F,F)_X$-module structure defined by the right $\mathrm{Diff}(F,F)_X$-action on F° as well.

EXERCISES. Assume that $F = \mathcal{O}_X$, so $F^\circ = \omega_X$ and $\bar{\mathfrak{gl}}(F) = \mathcal{D}_X$.

(i) A local coordinate t defines a section[91] $\delta_t := \frac{dy}{y-x} \in j_* j^* \mathcal{O}_X \boxtimes \omega_X$ such that $\pi(\delta_t) = 1 \in \mathcal{D}_X$. It yields a splitting $s_t : \mathcal{D}_X \to \mathcal{D}_X^\flat$, $A \mapsto A_x \cdot \delta_t$ (here A_x denotes the differential operator A acting along the x variable). Show that in terms of this splitting the Lie bracket on $\mathcal{D}_X^\flat$ is given by the 2-cocycle

$$f(t)\partial_t^m, g(t)\partial_t^n \mapsto (-1)^{m+1} \frac{m!n!}{(m+n+1)!} f^{(m+n+1)}(t) g(t) dt. \tag{2.7.5.3}$$

(ii) Suppose that X is compact of genus g. Show that the image of $1 \in \mathcal{D}_X$ by the boundary map $H^0(X, \mathcal{D}_X) \to H^1(X, h(\omega)) = k$ for the Tate extension equals $g - 1$.[92]

Let $\mathcal{A}(F) \hookrightarrow \mathrm{Diff}(F,F)_X$ be the Lie subalgebra of differential operators of first order whose symbol lies in $\Theta_X = \Theta_X \cdot \mathrm{id}_F \subset \Theta_X \otimes \mathrm{End}(F)$. Thus $\mathcal{A}(F)$ is an extension of Θ_X by $\mathfrak{gl}(F)$; its elements are pairs $(\tau, \tilde{\tau})$ where τ is a vector field and $\tilde{\tau}$ is an action of τ on F; i.e., $\mathcal{A}(F)$ is the Lie algebroid of infinitesimal symmetries of (X, F). The pull-back of the Tate extension by $\mathcal{A}(F)_{\mathcal{D}} \hookrightarrow \mathrm{Diff}(F,F)_{\mathcal{D}} = \mathfrak{gl}(V)$ is an ω_X-extension $\mathcal{A}(F)_{\mathcal{D}}^\flat$ of $\mathcal{A}(F)_{\mathcal{D}}$; we refer to it again as the Tate extension.

Consider the two natural ω_X-extensions of the Lie* algebra $\mathfrak{gl}(F)_{\mathcal{D}}$: the Kac-Moody extension $\mathfrak{gl}(F)_{\mathcal{D}}^\kappa$ where κ is of the form $a, b \mapsto -\mathrm{tr}(ab)$ on $\mathfrak{gl}$ (see 2.5.9) and the restriction $\mathfrak{gl}(F)_{\mathcal{D}}^\flat$ of $\mathcal{A}(F)_{\mathcal{D}}^\flat$ to $\mathfrak{gl}(F)_{\mathcal{D}} \subset \mathcal{A}(F)_{\mathcal{D}}$.

PROPOSITION. *There is a canonical isomorphism of ω_X-extensions*

$$\mathfrak{gl}(F)_{\mathcal{D}}^\flat \xrightarrow{\sim} \mathfrak{gl}(F\omega_X^{-1/2})_{\mathcal{D}}^\kappa \tag{2.7.5.4}$$

which lifts the obvious identification $\mathfrak{gl}(F) \xrightarrow{\sim} \mathfrak{gl}(F\omega_X^{-1/2})$.

Proof. According to the corollary in 2.5.9, the promised isomorphism amounts to a rule that assigns to each connection ∇ on $F\omega_X^{-1/2}$ a splitting $s_\nabla : \mathfrak{gl}(F) \to \mathfrak{gl}(F)^\flat$ in a way compatible with the $\mathrm{Aut}(F)$-action and so that for every $\psi \in \mathfrak{gl}(F)\omega_X$ one has $s_{\nabla+\psi}(a) = s_\nabla(a) + \mathrm{tr}(\psi a)$. Strictly speaking, we have to define s_∇ in the presence of the parameters $\mathrm{Spec}\, R^\ell$, $R^\ell \in \mathcal{C}omu_{\mathcal{D}}(X)$ (see 2.5.9) which brings no complications except notational ones.

Let $X^{(1)} \subset X \times X$ be the first infinitesimal neighborhood of the diagonal (see 2.5.10(a)). By (2.7.5.2) one has the projection of sheaves $\mu : F \boxtimes F^\circ(\Delta)|_{X^{(1)}} \to \mathfrak{gl}(F)^\flat$. Let $1^\flat \in (\omega_X^{\otimes 1/2} \boxtimes \omega_X^{\otimes 1/2})(\Delta)|_{X^{(1)}}$ be the section which lifts $1 \in \mathcal{O}_X = (\omega_X^{\otimes 1/2} \boxtimes \omega_X^{\otimes 1/2})(\Delta)|_X$ and is anti-invariant with respect to the transposition of coordinates. Consider ∇ as an identification[93] $(p_2^* F\omega_X^{-1/2})|_{X^{(1)}} \xrightarrow{\sim} (p_1^* F\omega_X^{-1/2})|_{X^{(1)}}$. Tensoring it by the identity map for $p_2^*(F^*\omega_X^{1/2}) = p_2^*(F^\circ \omega_X^{-1/2})$, one gets an isomorphism $\mathfrak{s}_\nabla : p_2^*(F \otimes F^*)|_{X^{(1)}} \xrightarrow{\sim} (F\omega_X^{-1/2}) \boxtimes (F^\circ\omega_X^{-1/2})|_{X^{(1)}}$. We define $s_\nabla : \mathfrak{gl}(F) \to \mathfrak{gl}(F\omega_X^{\otimes 1/2})^\flat$ as $s_\nabla(a) := \mu(\mathfrak{s}_\nabla(a) \cdot 1^\flat)$. The required properties are obvious. □

[91] Here x, y are coordinates on $X \times X$ defined by t.

[92] Hint: any non-constant meromorphic function t on X provides a meromorphic lifting δ_t of 1. It remains to compute the residues of its singular parts.

[93] Here $p_i : X \times X \to X$ are the projections.

Let us pass to the Virasoro extension. Assume that F is a Θ_X-equivariant bundle such that the Θ_X-action morphism $\Theta_X \to \mathcal{A}(F)$ is a differential operator.[94] Then we can restrict $\mathcal{A}(F)^\flat_{\mathcal{D}}$ to $\Theta_{\mathcal{D}} \subset \mathcal{A}(F)_{\mathcal{D}}$. Thus every such F yields an ω_X-extension $\Theta^\flat_{\mathcal{D}}(F)$ of $\Theta_{\mathcal{D}}$ which is a Lie* subalgebra of $\mathcal{A}(F)^\flat_{\mathcal{D}}$.

Consider the case $F = \omega_X^{\otimes j}$ with the usual Lie action of Θ_X.

PROPOSITION. *There is a unique isomorphism of ω_X-extensions*

$$\Theta_{\mathcal{D}}^{(2(6j^2-6j+1))} \xrightarrow{\sim} \Theta^\flat_{\mathcal{D}}(\omega_X^{\otimes j}). \tag{2.7.5.5}$$

Proof. The uniqueness is clear since ω_X-extensions of $\Theta_{\mathcal{D}}$ are rigid; see Remark in 2.5.10(c). Thus it suffices to define our isomorphism locally in terms of a coordinate t. Let us identify $\omega_X^{\otimes j}$ with $\mathcal{O}_X$ by means of the section $(dt)^{\otimes j}$. Then $\tau = f(t)\partial_t \in \Theta_X$ acts on ω_X as the first order differential operator $f(t)\partial_t + jf'(t)$. So the section s_t from the Exercise (i) in 2.7.5 yields a splitting of $\Theta^\flat_{\mathcal{D}}(\omega_X^{\otimes j})$. As follows from (2.7.5.3) its 2-cocycle is $f(t)\partial_t, g(t)\partial_t \mapsto (j^2 - j + 6^{-1})f'''(t)g(t)dt$. As was shown in the end of 2.5.10, t also yields a splitting of the Virasoro extension of central charge 1 with the cocycle $f(t)\partial_t, g(t)\partial_t \mapsto \frac{1}{12}f'''(t)g(t)dt$. Now (2.7.5.5) is the isomorphism which identifies the splittings. □

2.7.6. The construction of 2.7.2 renders itself easily to the situation with extra parameters, i.e., to the case of $A^\ell[\mathcal{D}_X]$-modules. Namely, suppose we have $A^\ell \in \mathcal{C}omu_{\mathcal{D}}(X)$, $V, V' \in \mathcal{M}(X, A^\ell)$ and $\langle\ \rangle \in P^*_{A2}(\{V, V'\}, A)$. This datum yields the associative* matrix algebra[95] $V \underset{A}{\otimes} V' \in \mathcal{M}(X, A)$; denote by $\mathcal{G}_A$ the corresponding Lie* algebra. Assume that A is $\mathcal{O}_X$-flat and $\mathrm{Tor}_1^{A^\ell}(V, V') = 0$.[96] We will define a canonical central extension of Lie* algebras in $\mathcal{M}(X, A)$

$$0 \to A \to \mathcal{G}^\flat_A \to \mathcal{G}_A \to 0. \tag{2.7.6.1}$$

Let us construct $\mathcal{G}^\flat_A$ as a plain $A^\ell[\mathcal{D}_X]$-module. Consider the short exact sequence $0 \to V \boxtimes V' \to j_*j^*V \boxtimes V' \to \Delta_*(V \otimes V') \to 0$ of $A^\ell[\mathcal{D}_X]^{\boxtimes 2}$-modules. Its push-forward by $\langle\ \rangle : V \boxtimes V' \to \Delta_* A$ is an $A^\ell[\mathcal{D}_X]^{\boxtimes 2}$-module extension of $\Delta_*(V \otimes V')$ by $\Delta_* A$. Let $0 \to A \to \tilde{\mathcal{G}}^\flat_A \to V \otimes V' \to 0$ be the corresponding $(A^\ell)^{\otimes 2}[\mathcal{D}_X]$-module extension.

Now set $\mathcal{G}^\flat_A = \tilde{\mathcal{G}}^\flat_A / I^\ell \tilde{\mathcal{G}}^\flat_A$ where I^ℓ is the ideal of the diagonal, $I^\ell := \mathrm{Ker}(A^\ell \otimes A^\ell \xrightarrow{\cdot} A^\ell)$. The Tor_1 vanishing assures that $\mathcal{G}^\flat_A$ is, indeed, an extension of $\mathcal{G}_A$ by A. One defines the bracket on $\mathcal{G}^\flat_A$ as in 2.7.2 and 2.7.3; again, we refer to 3.8 for a natural Clifford realization of $\mathcal{G}^\flat_A$. As in 2.7.3, the Tate extension satisfies the cyclic property (for the associative* algebra structure on the matrix algebra $\mathcal{G}_A$).

The above construction is compatible with the étale localization, so it works in the setting of algebraic $\mathcal{D}_X$-spaces. Namely, let $\mathcal{Y}$ be an algebraic $\mathcal{D}_X$-space and let V, V' be (right) $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules equipped with an $\mathcal{O}^r_{\mathcal{Y}}$-valued $*$ pairing $\langle\ \rangle$; assume that $\mathcal{Y}$ is flat over X and $\mathcal{T}or_1^{\mathcal{O}_{\mathcal{Y}}}(V, V')$ vanishes. We have a Lie* algebra $\mathcal{G}_{\mathcal{Y}} = V \otimes V' \in \mathcal{M}(\mathcal{Y})$ and its Tate (central) extension $\mathcal{G}^\flat_{\mathcal{Y}}$ by $\mathcal{O}^r_{\mathcal{Y}}$. The construction of $\mathcal{G}_{\mathcal{Y}}$, $\mathcal{G}^\flat_{\mathcal{Y}}$ is compatible with the base change (with respect to the morphisms of the $\mathcal{Y}$'s). If V, V' are equipped with an action of a Lie* algebroid $\mathcal{E}$ on $\mathcal{Y}$ such that

[94] I.e., the Θ_X-action satisfies the condition from Example (i) in 2.5.6(b).

[95] As usually, we write the tensor product of right $A[\mathcal{D}_X]$-modules as $\underset{A}{\otimes}$ instead of $\underset{A}{\otimes}^!$.

[96] We do not know if one can get rid of this technical condition.

this action preserves $\langle\ \rangle$, then $\mathcal{E}$ acts on $\mathcal{G}_{\mathcal{Y}}$ as on an associative* $\mathcal{O}_{\mathcal{Y}}$-algebra; as in 2.7.2, this action lifts canonically to an $\mathcal{E}$-action on $\mathcal{G}_{\mathcal{Y}}^{\flat}$.

An important particular case of the above situation: $\mathcal{Y}$ is an $\mathcal{O}_X$-flat algebraic $\mathcal{D}_X$-space and V, V' are mutually dual vector $\mathcal{D}_X$-bundles on $\mathcal{Y}$. Then $\mathcal{G}_{\mathcal{Y}} = \mathfrak{gl}(V)$, so we have the Tate extension $\mathfrak{gl}(V)^{\flat}$. An action of a Lie* algebroid $\mathcal{E}$ on V yields its action on V' compatible with the canonical pairing, so $\mathcal{E}$ acts on $\mathfrak{gl}(V)^{\flat}$.

2.7.7. Tate's linear algebra. Let us describe the Tate construction in its original linear algebra setting.

Let F be a k-vector space equipped with a separated complete linear[97] topology. A closed vector subspace $L \subset F$ is *compact*, resp. *cocompact*, if for any open vector subspace $U \subset F$ one has $\dim L/(L \cap U) < \infty$, resp. $\dim F/(L+U) < \infty$. A *c-lattice* is an open compact subspace; dually, a *d-lattice* is a discrete cocompact subspace.

It is easy to see that F has a c-lattice if and only if it has a d-lattice. Such an F is called a *Tate vector space*. For a Tate vector space F its dual[98] F^* is again a Tate space; the canonical map $F \to F^{**}$ is an isomorphism.

Any discrete vector space[99] is a Tate space. Any compact topological vector space[100] is a Tate space. Duality interchanges discrete and compact Tate spaces. Any Tate vector space can be represented as a direct sum of a discrete and a compact space.

Let $A : F \to G$ be a morphism of Tate vector spaces. We say that A is *compact* (resp. *discrete*) if it factors through a compact (resp. discrete) vector space. Equivalently, A is compact if the closure of $\operatorname{Im} A$ is compact; A is discrete if $\operatorname{Ker} A$ is open. The composition of a compact operator with an arbitrary one is compact; the same is true for discrete operators. A is compact if and only if $A^* : G^* \to F^*$ is discrete. We denote by $\operatorname{Hom}_c(F,G)$, $\operatorname{Hom}_d(F,G) \subset \operatorname{Hom}(F,G)$ the subspaces of compact, resp. discrete, operators, and by $\operatorname{Hom}_f(F,G)$ the intersection of these subspaces. Any operator may be represented as the sum of a compact and a discrete operator.

The above vector spaces of operators carry natural topologies. Notation: for subspaces $U \subset F$ and $V \subset G$ set $\operatorname{Hom}(F,G)_{U,V} := \{A \in \operatorname{Hom}(F,G) : A(U) \subset V\}$. Now bases of topologies on $\operatorname{Hom}(F,G)$, $\operatorname{Hom}_c(F,G)$, $\operatorname{Hom}_d(F,G)$, and $\operatorname{Hom}_f(F,G)$ are formed, respectively, by subspaces $\operatorname{Hom}(F,G)_{U,V}$, $\operatorname{Hom}(F,V)$, $\operatorname{Hom}(F/U,G)$, and $\operatorname{Hom}(F/U,V)$ where $U \subset F$ and $V \subset G$ are c-lattices. These topologies are complete and separated. The exact sequence

$$0 \to \operatorname{Hom}_f(F,G) \xrightarrow{\alpha} \operatorname{Hom}_c(F,G) \oplus \operatorname{Hom}_d(F,G) \xrightarrow{\beta} \operatorname{Hom}(F,G) \to 0, \tag{2.7.7.1}$$

where $\alpha(A) := (-A, A)$, $\beta(A,B) := A + B$, is strongly compatible with the topologies.[101]

Notice that Hom_c, Hom_d and Hom_f are two-sided ideals in the obvious sense, and all the composition maps like $\operatorname{Hom}_d(G,H) \times \operatorname{Hom}_c(F,G) \to \operatorname{Hom}_f(F,H)$, etc., are continuous. The maps $\operatorname{Hom}(F,G) \xrightarrow{\sim} \operatorname{Hom}(G^*,F^*)$, $\operatorname{Hom}(F,G)_c \xrightarrow{\sim} \operatorname{Hom}(G^*,F^*)_d$, etc., which send A to A^*, are homeomorphisms.

[97] Here "linear" means that the topology has a base formed by vector subspaces of F.

[98] By definition, F^* is the space of all continuous linear functionals $F \to k$; open subspaces in F^* are orthogonal complements to compact subspaces of F.

[99] I.e., a vector space equipped with a discrete topology.

[100] Which is the same as a profinite-dimensional vector space.

[101] I.e., α is a homeomorphism onto a closed subspace and β is continuous and open.

Exercise. Consider an (abstract) vector space $G \otimes F$. It carries four natural topologies with bases of open subspaces formed, respectively, by subspaces $V \otimes F + G \otimes U$, $V \otimes F$, $G \otimes U$, and $V \otimes U$, where $V \subset G$, $U \subset F$ are c-lattices. Now replace F by F^* and consider the obvious canonical map $G \otimes F^* \to \mathrm{Hom}_f(F, G)$. Show that it yields homeomorphisms between the completions of $G \otimes F^*$ with respect to these topologies and, respectively, $\mathrm{Hom}(F, G)$, $\mathrm{Hom}_c(F, G)$, $\mathrm{Hom}_d(F, G)$, and $\mathrm{Hom}_f(F, G)$.

So for a Tate space F we have an associative algebra $\mathrm{End}\, F$ together with two-sided ideals $\mathrm{End}_c F, \mathrm{End}_d F \subset \mathrm{End}\, F$ such that $\mathrm{End}_c F + \mathrm{End}_d F = \mathrm{End}\, F$. An operator A belongs to $\mathrm{End}_f F := \mathrm{End}_c F \cap \mathrm{End}_d F$ if and only if there are c-lattices $V \subset U$ such that $A(V) = 0$, $A(F) \subset U$. For such an A its *trace* $\mathrm{tr}\, A$ is the trace of the induced operator on U/V; it does not depend on the auxiliary choice of lattices. For every $A_c \in \mathrm{End}_c F$, $A_d \in \mathrm{End}_d F$ one has $\mathrm{tr}([A_c, A_d]) = 0$. Since $\mathrm{End}\, F = \mathrm{End}_c F + \mathrm{End}_d F$, one has $\mathrm{tr}[A, B] = 0$ for any $A \in \mathrm{End}_f F$, $B \in \mathrm{End}\, F$. The functional tr is continuous with respect to our topology on $\mathrm{End}_f F$. In terms of the above exercise, tr is simply the continuous extension of the canonical pairing $F \otimes F^* \to k$.

2.7.8. The Tate extension in the linear algebra setting. For a Tate vector space F let $\mathfrak{gl}(F)$ be $\mathrm{End}\, F$ considered as a Lie algebra. By 2.7.7 we have ideals $\mathfrak{gl}_c(F), \mathfrak{gl}_d(F) \subset \mathfrak{gl}(F)$ and $\mathrm{tr} : \mathfrak{gl}_f(F) := \mathfrak{gl}_c(F) \cap \mathfrak{gl}_d(F) \to k$ that fit into the setting of 2.7.1(ii). The corresponding central extension $\mathfrak{gl}(F)^\flat$ is called the *Tate extension*. It satisfies the cyclic property (see 2.7.1(iii)). We have canonical sections $s_c = s_c^F : \mathfrak{gl}_c(F) \to \mathfrak{gl}(F)^\flat$, $s_d = s_d^F : \mathfrak{gl}_d(F) \to \mathfrak{gl}(F)^\flat$.

According to 2.7.7 the Lie algebras $\mathfrak{gl}(F)$, $\mathfrak{gl}_c(F)$, $\mathfrak{gl}_d(F)$, and $\mathfrak{gl}_f(F)$ carry canonical topologies, and the functional tr is continuous. Our $\mathfrak{gl}(F)^\flat$ is also a topological Lie algebra: the topology is defined by the condition that the map $s_c + s_d : \mathfrak{gl}_c(F) \times \mathfrak{gl}_d(F) \to \mathfrak{gl}(F)^\flat$ is continuous and open. It is complete and separated; the projection $\mathfrak{gl}(F)^\flat \to \mathfrak{gl}(F)$ is continuous and open.

Remark. The identification $\mathfrak{gl}(F) \xrightarrow{\sim} \mathfrak{gl}(F^*)$, $A \mapsto -A^*$, lifts canonically to an identification of topological k-extensions $\mathfrak{gl}(F)^\flat \xrightarrow{\sim} \mathfrak{gl}(F^*)^\flat$ which interchanges s_c and s_d.

2.7.9. The additivity property. Let $\mathcal{F}$ be a finite flag $F_1 \subset F_2 \subset \cdots \subset F_n = F$ of closed subspaces of F.[102] Then F_i are Tate vector spaces, as well as $\mathrm{gr}_i F$. Let $\mathfrak{gl}(F, \mathcal{F}) \subset \mathfrak{gl}(F)$ be the stabilizer of our flag, so one has the projections $\pi_i : \mathfrak{gl}(F, \mathcal{F}) \to \mathfrak{gl}(\mathrm{gr}_i F)$. There are two natural k-extensions $\mathfrak{gl}(F, \mathcal{F})^\flat$ and $\mathfrak{gl}(F, \mathcal{F})^{\flat'}$ of $\mathfrak{gl}(F, \mathcal{F})$: the first one is the restriction of $\mathfrak{gl}(F)^\flat$ to $\mathfrak{gl}(F, \mathcal{F})$ and the second one is the Baer sum of the π_i-pull-backs of the extensions $\mathfrak{gl}(\mathrm{gr}_i F)^\flat$.

Lemma. *The topological central extensions $\mathfrak{gl}(F, \mathcal{F})^\flat$, $\mathfrak{gl}(F, \mathcal{F})^{\flat'}$ are canonically isomorphic.*

Proof. Consider the ideals $\mathfrak{gl}_c(F, \mathcal{F})$, $\mathfrak{gl}_d(F, \mathcal{F})$ defined as intersection of $\mathfrak{gl}(F, \mathcal{F})$ with $\mathfrak{gl}_c(F)$, $\mathfrak{gl}_d(F)$. One has $\mathfrak{gl}_c(F, \mathcal{F}) + \mathfrak{gl}_d(F, \mathcal{F}) = \mathfrak{gl}(F, \mathcal{F})$, and the images of these ideals by π_i lie in, respectively, $\mathfrak{gl}_c(\mathrm{gr}_i F)$, $\mathfrak{gl}_d(\mathrm{gr}_i F)$. Both $\mathfrak{gl}(F, \mathcal{F})^\flat$ and $\mathfrak{gl}(F, \mathcal{F})^{\flat'}$ are equipped with sections over our ideals: these are, respectively, s_c, s_d

[102] Any such flag splits; i.e., it comes from a grading.

defined as restrictions of s_c^F, s_d^F, and $s'_c := \Sigma\pi_i^*(s_c^{\mathrm{gr}_i F})$, $s'_d := \Sigma\pi_i^*(s_d^{\mathrm{gr}_i F})$. Over $\mathfrak{gl}_f(F,\mathcal{F}) := \mathfrak{gl}_c(F,\mathcal{F}) \cap \mathfrak{gl}_d(F,\mathcal{F})$ one has $s_c - s_d = s'_c - s'_d = \mathrm{tr}$. Now use the uniqueness statement of the lemma from 2.7.1(ii). □

REMARK. Suppose that every $\mathrm{gr}_i F$ is either compact or discrete. Then the corresponding sections $s_c^{\mathrm{gr}_i F}$ or $s_d^{\mathrm{gr}_i F}$ (together with the lemma) provide a splitting of $\mathfrak{gl}(F)^\flat_{\mathcal{F}}$. For example, if $\mathcal{F}$ consists of a single c-lattice $P \subset F$, then s_c^P and $s_d^{F/P}$ yield a canonical splitting s_P of $\mathfrak{gl}(F)^\flat_P$. If $P' \subset P$ is another c-lattice, then on $\mathfrak{gl}(F)_P \cap \mathfrak{gl}(F)_{P'}$ one has $s_P - s_{P'} = \mathrm{tr}_{P/P'}$.

2.7.10. The local duality. We return to the geometric setting of 2.7.2. Let $x \in X$ be a (closed) point, $j_x : U_x \hookrightarrow X$ the complement to x, $O_x \subset K_x$ the completion of the local ring at x and its field of fractions. Let V be a coherent $\mathcal{D}$-module on U_x. Set $V_{(x)} := \hat{h}_x(j_{x*}V)$ (see 2.1.13). This is a Tate vector space (see Remark (i) in 2.1.14). Applying 2.1.14 to $M = j_{x*}V$, we get a canonical isomorphism of topological vector spaces

$$V \underset{\mathcal{D}_{U_x}}{\otimes} K_x \xrightarrow{\sim} V_{(x)}. \tag{2.7.10.1}$$

Here $V \underset{\mathcal{D}_{U_x}}{\otimes} K_x := \Gamma(U_x, V) \underset{\Gamma(U_x,\mathcal{D})}{\otimes} K_x = V_\eta \underset{\mathcal{D}_\eta}{\otimes} K_x$ (where η is the generic point of X) and its topology is formed by the images of $V_\xi \underset{\mathcal{D}_X}{\otimes} O_x$, $V_\xi \in \Xi_x(j_{x*}V)$ (see Remark (i) in 2.1.13).

Suppose that V is a vector $\mathcal{D}$-bundle on U_x;[103] let V° be its dual. The $*$ pairing $j_{x*}V \boxtimes j_{x*}V^\circ \to \Delta_* j_{x*}\omega_{U_x}$ composed with the canonical projection $j_{x*}\omega_{U_x} \twoheadrightarrow i_{x*}k$ yields a canonical k-valued pairing between the Tate vector spaces $V_{(x)}$ and $V^\circ_{(x)}$.

LEMMA. *This pairing is non-degenerate, so $V_{(x)}$ and $V^\circ_{(x)}$ are mutually dual Tate vector spaces.*

Proof. It is enough to consider the example of $V = \mathcal{D}_{U_x}$. Then $V^\circ = \omega_{U_x} \otimes \mathcal{D}_{U_x}$, $V_{(x)} = K_x$, $V^\circ_{(x)} = \omega_{K_x}$, and our pairing is $f, \nu \mapsto \mathrm{Res}_x(f\nu)$. It is clearly non-degenerate. □

REMARK. The above discussion remains true for families of vector $\mathcal{D}$-bundles. Precisely, let F be a commutative algebra, V a projective right $F \otimes \mathcal{D}_{U_x}$-module of finite rank, $V^\circ := \mathcal{H}om_{F\otimes\mathcal{D}_{U_x}}(V, F \otimes (\omega_{U_x})_{\mathcal{D}})$ the dual $F \otimes \mathcal{D}_{U_x}$-module. We define $V_{(x)}$ as the completion of $h(j_{x*}V)_x$ with respect to the topology formed by all $F \otimes \mathcal{D}_X$-submodules of $j_{x*}V$ which equal V on the complement to x. Then $V \underset{F\otimes\mathcal{D}_{U_x}}{\otimes} (F \hat{\otimes} K_x) \xrightarrow{\sim} V_{(x)}$ and the canonical pairing identifies $V^\circ_{(x)}$ with $(V_{(x)})^* :=$ the module of continuous F-linear morphisms $V_{(x)} \to F$.

2.7.11. Consider now the Lie* algebra $j_{x*}\mathfrak{gl}(V)$. It acts on $j_{x*}V$ and $j_{x*}V^\circ$.

Suppose we have $V_\xi \in \Xi_x(j_{x*}V)$, $V^\circ_{\xi'} \in \Xi_x(j_{x*}V')$ such that the canonical $*$ pairing sends $V_\xi \boxtimes V^\circ_{\xi'}$ to $\Delta_*\omega_X \subset \Delta_* j_{x*}\omega_{U_x}$. Then the matrix algebra $V_\xi \otimes V^\circ_{\xi'}$ is a Lie* subalgebra of $j_{x*}\mathfrak{gl}(V)$ which equals $\mathfrak{gl}(V)$ on U_x. We call such a subalgebra *special*. Special Lie* subalgebras form a base of a topology Ξ_x^{sp} on $j_{x*}\mathfrak{gl}(V)$ at x (called the *special topology*) which is weaker than the topology Ξ_x^{Lie} (see 2.5.12).[104]

[103] Then V becomes a free $\mathcal{D}$-module after replacing X by a Zariski neighborhood of x. This follows from the fact that every ideal of $\mathcal{D}_\eta$ is principal (here η is the generic point of X).

[104] In fact, as follows from 2.5.14 and 2.7.11 the special topology is stricty weaker than Ξ_x^{Lie}.

It is clear that the action of $j_{x*}\mathfrak{g}l(V)$ on $j_{x*}V$ is continuous with respect to the Ξ_x^{sp}- and Ξ_x-topologies.[105]

Denote by $\mathfrak{g}l(V)_{(x)}$ the completion of the Lie algebra $h_x(j_{x*}\mathfrak{g}l(V))$ with respect to the Ξ_x^{sp}-topology (see 2.5.12 and Remark in 2.5.13). This is a topological Lie algebra[106] which acts continuously on the Tate vector space $V_{(x)}$.

2.7.12. Lemma. *This action yields an isomorphism of topological Lie algebras*

$$r:\ \mathfrak{g}l(V)_{(x)} \xrightarrow{\sim} \mathfrak{g}l(V_{(x)}). \tag{2.7.12.1}$$

Proof. It suffices to consider the case $V = \mathcal{D}_{U_x}$. Then $h_x(j_{x*}\mathfrak{g}l(V))$ is the vector space $\mathcal{D}_\eta$ of all meromorphic differential operators. The special topology has a base formed by the subspaces $\mathcal{D}_{m,n} := \{\partial : \partial(t^{-m}\mathcal{O}_x) \subset t^n\mathcal{O}_x\}$; here t is a parameter at x. This is exactly the topology induced from the topology on $\mathfrak{g}l(V_{(x)})$ (see 2.7.8, 2.7.7). It remains to show that $\mathcal{D}_\eta/\mathcal{D}_{m,n} \xrightarrow{\sim} \mathrm{Hom}(t^{-m}O_x, K_x/t^nO_x)$ (where Hom means continuous linear maps); this is a refreshing exercise for the reader. □

2.7.13. Now consider the central extension $j_{x*}\mathfrak{g}l(V)^\flat$ of $j_{x*}\mathfrak{g}l(V)$ by $j_{x*}\omega_{U_x}$. Notice that for any special subalgebra $\mathcal{G} = V_\xi \otimes V_{\xi'}^\circ \subset j_{x*}\mathfrak{g}l(V)$ (see 2.7.11) its Tate extension[107] $\mathcal{G}^\flat$ is a Lie* subalgebra of $j_{x*}\mathfrak{g}l(V)^\flat$. We call such a Lie* subalgebra of $j_{x*}\mathfrak{g}l(V)^\flat$ *special*. Special subalgebras form a base of a topology $\Xi_x^{sp\flat}$ on $j_{x*}\mathfrak{g}l(V)^\flat$ at x (called the *special topology*) which is weaker than the Ξ_x^{Lie}-topology (see 2.5.12).

Let $\mathfrak{g}l(V)^\flat_{(x)}$ be the completion of $h_x(j_{x*}\mathfrak{g}l(V)^\flat)$ with respect to the $\Xi_x^{sp\flat}$-topology. The adjoint action of $j_{x*}\mathfrak{g}l(V)$ on $j_{x*}\mathfrak{g}l(V)^\flat$ is continuous with respect to special topologies, so $\mathfrak{g}l(V)^\flat_{(x)}$ is a topological Lie algebra. For every special subalgebra $\mathcal{G}^\flat \subset j_{x*}\mathfrak{g}l(V)^\flat$ we have $\mathcal{G}^\flat \cap j_{x*}\omega_{U_x} = \omega_X$, so the identification $\mathrm{Res}_x : j_{x*}\omega_{U_x}/\omega_X \xrightarrow{\sim} k$ tells us that $\mathfrak{g}l(V)^\flat_{(x)}$ is a topological central extension of $\mathfrak{g}l(V)_{(x)}$ by k.

2.7.14. Proposition. *The isomorphism r of 2.7.12 lifts canonically to an isomorphism of topological central extensions*

$$r^\flat:\ \mathfrak{g}l(V)^\flat_{(x)} \xrightarrow{\sim} \mathfrak{g}l(V_{(x)})^\flat. \tag{2.7.14.1}$$

Notice that such a lifting $r^\flat$ is unique since the extension $\mathfrak{g}l(V_{(x)})^\flat$ has no automorphisms.

We give two constructions of $r^\flat$. The one right below is a version of considerations from [**BS**]. A more conceptual and simple approach uses chiral Clifford algebras; it can be found in 3.8.23.

Proof. (i) Let us compare the constructions of Tate extensions:

(a) Let η be the generic point of X. Apply the functor h to (2.7.2.2) and consider sections of the resulting sheaves over $\eta\times\eta$. We get an extension of $\bar{\mathfrak{g}}l(V)_\eta$-modules[108]

$$0 \to h(V)_\eta \otimes h(V^\circ)_\eta \to \bar{\mathfrak{g}}l(V)^\natural_\eta \to \bar{\mathfrak{g}}l(V)_\eta \to 0. \tag{2.7.14.2}$$

[105] In fact, Ξ_x^{sp} is the weakest topology on $j_{x*}\mathfrak{g}l(V)$ such that the $j_{x*}\mathfrak{g}l(V)$-action on $j_{x*}V$ equipped with Ξ_x-topology is continuous.

[106] In fact, this is a topological associative algebra.

[107] With respect to the $*$ pairing $\in P_2^*(\{V_\xi, V_{\xi'}^\circ\}, \omega_X)$.

[108] We use the notation from 2.7.4.

Consider the push-forward of this extension by the map $h(V)_\eta \otimes h(V^\circ)_\eta \xrightarrow{(\,)} h(\omega)_\eta \xrightarrow{\mathrm{Res}_x} k$ (see 2.7.2). This is a central extension of $\bar{\mathfrak{g}}l(V)_\eta$ by k; the bracket comes from the $\bar{\mathfrak{g}}l(V)_\eta$-module structure on $\bar{\mathfrak{g}}l(V)^\natural_\eta$. Our $\mathfrak{g}l(V)^\flat_{(x)}$ is the completion of this Lie algebra with respect to the topology defined by the images of vector spaces $h_{(x,x)}(j_*j^*V_\xi \boxtimes V^\circ_{\xi'})$ where $V_\xi, V^\circ_{\xi'}$ are as in 2.7.11.

(b) Look at (2.7.7.1) for $F = G = V_{(x)}$. This is an extension

$$0 \to \mathfrak{g}l_f(V_{(x)}) \xrightarrow{\alpha} \mathfrak{g}l(V_{(x)})^\natural \xrightarrow{\beta} \mathfrak{g}l(V_{(x)}) \to 0 \tag{2.7.14.3}$$

of topological $\mathfrak{g}l(V_{(x)})$-modules; here $\mathfrak{g}l(V_{(x)})^\natural := \mathfrak{g}l_c(V_{(x)}) \oplus \mathfrak{g}l_d(V_{(x)})$ and $\alpha = (+,-)$, $\beta = (+,+)$. Now $\mathfrak{g}l(V_{(x)})^\flat$ is the push-forward of this extension by tr : $\mathfrak{g}l_f(V_{(x)}) \to k$; the bracket comes from the $\mathfrak{g}l(V_{(x)})$-module structure on $\mathfrak{g}l(V_{(x)})^\natural$.

(ii) The action of $\bar{\mathfrak{g}}l(V)_\eta$ on the completion $V_{(x)}$ of $h(V)_\eta$ yields the morphism $r : \bar{\mathfrak{g}}l(V)_\eta \to \mathfrak{g}l(V_{(x)})$ of Lie algebras. We will lift it to a continuous morphism of extensions

$$r^\natural : \bar{\mathfrak{g}}l(V)^\natural_\eta \to \mathfrak{g}l(V_{(x)})^\natural \tag{2.7.14.4}$$

such that $r^\natural$ is an r-morphism of $\mathfrak{g}l(V)_\eta$-modules and its restriction to $h(V)_\eta \otimes h(V^\circ)_\eta$ equals the obvious morphism $h(V)_\eta \otimes h(V^\circ)_\eta \to V_{(x)} \otimes V^\circ_{(x)} \to \mathfrak{g}l_0(V_{(x)})$. It is clear from (a), (b) above[109] that such an $r^\natural$ yields the desired $r^\flat$.

(iii) Take $u^\natural \in \bar{\mathfrak{g}}l(V)^\natural_\eta = hj_*j^*(V \boxtimes V^\circ)_{\eta\times\eta}$; let $u \in \bar{\mathfrak{g}}l(V)_\eta$ be the image of $u^\natural$. To define $r^\natural(u^\natural) \in \mathfrak{g}l(V_{(x)})^\natural = \mathfrak{g}l_c(V_{(x)}) \oplus \mathfrak{g}l_d(V_{(x)})$ we must represent $r(u) \in \mathfrak{g}l(V_{(x)})$ as $r_c(u^\natural) + r_d(u^\natural)$ where $r_c(u^\natural) \in \mathfrak{g}l_c(V_{(x)})$, $r_d(u^\natural) \in \mathfrak{g}l_d(V_{(x)})$. Notice that $r(u) : h(V)_\eta \to h(V)_\eta$ equals $R_\Delta \circ f_{u^\natural}$ where $f_{u^\natural} : h(V)_\eta \to hj_*j^*(V \boxtimes \omega_X)_{\eta\times\eta}$ maps $s \in h(V)_\eta$ to $(id_V \boxtimes s)(u^\natural)$ (here we consider s as a morphism of $\mathcal{D}_X$-modules $V^\circ \to \omega_X$ at η) and $R_\Delta : hj_*j^*(V \boxtimes \omega_X)_{\eta\times\eta} \to h(V)_\eta$ is the residue at $\eta \times x$ along the second variable. We have $R_\Delta = -R_x + \int_\eta$ where $\int_\eta$ denotes integration over the second variable and R_x is the residue at $\eta \times x$ along the second variable. It remains to prove the following lemma, which is left to the reader:

LEMMA. (a) *The map* $\int_\eta \circ f_{u^\natural} : h(V)_\eta \to h(V)_\eta$ *extends to a compact operator* $r^\natural_c(u^\natural) : V_{(x)} \to V_{(x)}$.

(b) *The map* $-R_x \circ f_{u^\natural} : h(V)_\eta \to h(V)_\eta$ *extends to a discrete operator* $r^\natural_d(u^\natural) : V_{(x)} \to V_{(x)}$.

(c) *The map* $r^\natural : \bar{\mathfrak{g}}l(V)^\natural_\eta \to \mathfrak{g}l(V_{(x)})^\natural$ *defined by* $r^\natural := (r^\natural_c, r^\natural_d)$ *satisfies the properties promised in (ii) above.* □

2.8. Tate structures and characteristic classes

The Tate extension replaces the absent trace map for the matrix Lie* algebras. In fact, for X of arbitrary dimension n and a vector $\mathcal{D}_X$-bundle V on X (or any $\mathcal{D}_X$-scheme $\mathcal{Y}$) there is a canonical Lie* algebra cohomology class $\tau_V \in H^{n+1}(\mathfrak{g}l(V), \mathcal{O})$ which is the trace map for $n = 0$ and the Tate extension class for $n = 1$. When we are in a situation with extra parameters, i.e., we have a $\mathcal{D}_X$-scheme $\mathcal{Y}$, then one can define Chern classes $ch^{\mathcal{D}}_a(V)$ of a $\mathcal{D}_X$-bundle V on $\mathcal{Y}$ repeating the usual

[109]Use the interpretation of tr at the end of 2.7.7.

Weil construction with the trace map replaced by τ_V. Below we do this for $n = 1$, leaving the case $n > 1$ to an interested reader. It would be nice to understand the Riemann-Roch story in this setting.

We are mostly interested in the first Chern class $ch_1^{\mathcal{D}}(V)$ which is an obstruction to the existence of a *Tate structure* on V. We will see in (3.9.20.2) that a Tate structure on the tangent bundle $\Theta_{\mathcal{Y}}$ (a.k.a. the Tate structure on $\mathcal{Y}$) is exactly the datum needed to construct an algebra of chiral differential operators (cdo) on $\mathcal{Y}$.

The groupoid of Tate structures $\mathcal{T}ate(V)$ on V is defined in in 2.8.1; if non-empty, this is a torsor over the Picard groupoid $\mathcal{P}^{cl}(\Theta)$ (see 2.8.2). The rest of the section consists of two independent parts:

(a) Tate structures and $ch_1^{\mathcal{D}}$: We identify $\mathcal{T}ate(V)$ with certain down-to-earth groupoids defined in terms of connections on V in 2.8.3–2.8.9, which helps to determine the obstruction to the existence of a Tate structure. The classes $ch_a^{\mathcal{D}}$ are defined in 2.8.10. If V is induced, $V = F_{\mathcal{D}}$, then $ch_a^{\mathcal{D}}(V)$ comes from the conventional de Rham Chern class $ch_{a+1}(F\omega_X^{1/2})$ (see 2.8.13).

(b) Some constructions and concrete examples of Tate structures: We consider weakly equivariant Tate structures in 2.8.14 and identify them explicitly in the case when $\mathcal{Y}$ is a torsor in 2.8.15. The descent construction of Tate structures is treated in 2.8.16. As an application, we identify weakly equivariant Tate structures on the jet scheme of a flag space with Miura opers (see 2.8.17).

The material of 2.8.3–2.8.9 is inspired by [**GMS1**], [**GMS2**]. After the identification of Tate structures on Θ with cdo, 2.8.15 becomes essentially Theorem 4.4 of [**AG**] or [**GMS3**] 2.5. The result in 2.8.17 was conjectured by E. Frenkel and D. Gaitsgory; it is a variation on the theme of [**GMS3**] 4.10 which considered the translation equivariant setting (the original statement of [**GMS3**] 4.10 is presented as the exercise at the end of 2.8.17).

As in the previous section, we assume that $\dim X = 1$.

2.8.1. We return to the setting of 2.7.6, so $\mathcal{Y}$ is an algebraic $\mathcal{D}_X$-space flat over X, V a vector $\mathcal{D}_X$-bundle on $\mathcal{Y}$, V° the dual bundle, so we have the Tate extension $\mathfrak{gl}(V)^\flat$ of $\mathfrak{gl}(V)$. From now on we assume that $\Omega_{\mathcal{Y}} := \Omega_{\mathcal{Y}/X}$ is a vector $\mathcal{D}_{\mathcal{X}}$-bundle (which happens if $\mathcal{Y}$ is smooth), so the Lie* algebroid $\mathcal{E}(V)$ (see 2.5.16) is well defined. It acts on V, V° preserving the $*$-pairing, so $\mathcal{E}(V)$ acts on $\mathfrak{gl}(V)^\flat$ (see 2.7.6); denote this action by $ad^\flat$.

The usefulness of the following definition will become clear in 3.9.20:

Definition. A *Tate structure on* V is the following datum (i), (ii):

(i) An extension $\mathcal{E}(V)^\flat$ of the Lie* algebroid $\mathcal{E}(V)$ on $\mathcal{Y}$ by $\mathcal{O}_{\mathcal{Y}}^r$.

The corresponding Lie* $\mathcal{O}_{\mathcal{Y}}$-algebra $\mathcal{E}(V)^{\flat\diamondsuit} = \mathrm{Ker}(\tau)$ is an extension of $\mathfrak{gl}(V)$ by $\mathcal{O}_{\mathcal{Y}}^r$.

(ii) An identification of this extension with $\mathfrak{gl}(V)^\flat$ compatible with the $\mathcal{E}(V)^\flat$-actions.

Here $\mathcal{E}(V)^\flat$ acts on $\mathcal{E}(V)^{\flat\diamondsuit}$ by the adjoint action and on $\mathfrak{gl}(V)^\flat$ by $ad^\flat$.

Tate structures form a groupoid $\mathcal{T}ate(V) = \mathcal{T}ate(\mathcal{Y}, V)$. They have the étale local nature, so we have a sheaf of groupoids $\mathcal{T}ate(V)_{\mathcal{Y}_{\text{ét}}}$ on $\mathcal{Y}_{\text{ét}}$. We refer to a Tate structure on $\Theta_{\mathcal{Y}}$ as the *Tate structure on* $\mathcal{Y}$ and write $\mathcal{T}ate(\mathcal{Y}) := \mathcal{T}ate(\Theta_{\mathcal{Y}})$.

Remark. We wiil see that $\mathcal{T}ate(V)$ can be empty. We do not know if Tate structures always exist locally on $\mathcal{Y}$ (this is true for locally trivial V, which is the case if V is induced).

2.8.2. For a Lie* algebroid $\mathcal{L}$ on $\mathcal{Y}$ let[110] $\mathcal{P}^{cl}(\mathcal{L}) = \mathcal{P}^{cl}(\mathcal{Y}, \mathcal{L})$ be the groupoid of Lie* algebroid extensions $\mathcal{L}^c$ of $\mathcal{L}$ by $\mathcal{O}^r_{\mathcal{Y}}$ (we assume that the adjoint action of $\mathcal{L}^c$ on $\mathcal{O}^r_{\mathcal{Y}} \subset \mathcal{L}^c$ coincides with the structure action). This is a Picard groupoid, or rather a k-vector space in categories, in the obvious way (the sum is the Baer sum operation). The objects of $\mathcal{P}^{cl}(\mathcal{L})$ have the étale local nature, so we have a sheaf $\mathcal{P}^{cl}(\mathcal{L})_{\mathcal{Y}_{\text{ét}}}$ of Picard groupoids on $\mathcal{Y}_{\text{ét}}$.

There is a canonical action of $\mathcal{P}^{cl}(\Theta_{\mathcal{Y}})$ on $\mathcal{T}ate(V)$. Namely, an object Θ^c of $\mathcal{P}^{cl}(\Theta_{\mathcal{Y}})$ amounts, by pull-back, to an extension $\mathcal{E}(V)^c$ of $\mathcal{E}(V)$ by $\mathcal{O}^r_{\mathcal{Y}}$ trivialized over $\mathfrak{gl}(V)$. The corresponding automorphism of $\mathcal{T}ate(V)$ sends $\mathcal{E}(V)^\flat$ to its Baer sum with $\mathcal{E}(V)^c$.

Lemma. *If $\mathcal{T}ate(V)$ is non-empty, then it is a $\mathcal{P}^{cl}(\Theta_{\mathcal{Y}})$-torsor.* □

2.8.3. We are going to describe $\mathcal{T}ate(V)$ "explicitly" by means of auxiliary data of connections. Let us deal with $\mathcal{P}^{cl}(\Theta_{\mathcal{Y}})$ first.

Consider the relative de Rham complex $DR_{\mathcal{Y}/X}$. This is a complex of left $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules with terms $DR^i_{\mathcal{Y}/X} = \Omega^i := \Omega^i_{\mathcal{Y}/X}$. Passing to the right $\mathcal{D}_X$-modules (which we skip in the notation), one gets a complex of sheaves $h(DR_{\mathcal{Y}/X})$ on $\mathcal{Y}_{ét}$. Consider its 2-term subcomplex $h(DR_{\mathcal{Y}/X})_{[1,2)} := \tau_{\leq 2}\sigma_{\geq 1} h(DR_{\mathcal{Y}/X})$ with components $h(\Omega^1)$ and $h(\Omega^2)^{closed} := \mathrm{Ker}(d : h(\Omega^2) \to h(\Omega^3))$.

As in [SGA 4] Exp. XVIII 1.4, our complex defines a sheaf of Picard groupoids on $\mathcal{Y}_{ét}$. Its objects, called $h(DR_{\mathcal{Y}/X})_{[1,2)}$*-torsors*, are pairs (C, c) where C is an $h(\Omega^1)$-torsor and c is a trivialization of the corresponding induced $h(\Omega^2)^{closed}$-torsor; i.e., $c : C \to h(\Omega^2)^{closed}$ is a map such that for $\nu \in h(\Omega^1)$, $\nabla \in C$ one has $c(\nabla + \nu) = c(\nabla) + d\nu$. Denote this Picard groupoid by $h(DR_{\mathcal{Y}/X})_{[1,2)}$-*tors*.

Lemma. *There is a canonical equivalence of Picard groupoids*

$$\mathcal{P}^{cl}(\Theta_{\mathcal{Y}}) \xrightarrow{\sim} h(DR_{\mathcal{Y}/X})_{[1,2)}\text{-}tors. \tag{2.8.3.1}$$

Proof. The functor is $\Theta^c \mapsto (C, c)$ where C is the $h(\Omega^1)$-torsor of connections (every Θ^c admits a connection locally) on Θ^c and c is the curvature map. It is an equivalence due to the lemma in 1.4.17. □

2.8.4. Now let us pass to Tate structures; we follow the notation of 2.8.1 and 1.4.17.

Suppose we have a connection $\nabla : \Theta_{\mathcal{Y}} \to \mathcal{E}(V)$ on V (which is the same as a connection on $\mathcal{E}(V)$). It defines a compatible pair $(d_\nabla, c(\nabla)) \in \mathfrak{D}(\mathfrak{gl}(V))$ (see 1.4.17) where d_∇ is a d-derivation of the Lie* $\Omega^\cdot$-algebra $\Omega^\cdot \otimes \mathfrak{gl}(V)$ and $c(\nabla) \in h(\Omega^2 \otimes \mathfrak{gl}(V))$. Since $\mathcal{E}(V)$ acts on the Lie* $\mathcal{O}_{\mathcal{Y}}$-algebra $\mathfrak{gl}(V)^\flat$, our ∇ also defines a d-derivation $d^\flat_\nabla$ of the Lie* $\Omega^\cdot$-algebra $\Omega^\cdot \otimes \mathfrak{gl}(V)^\flat$ such that $d^\flat_\nabla : \mathfrak{gl}(V)^\flat \to \Omega^1 \otimes \mathfrak{gl}(V)^\flat$ corresponds to $ad^\flat_\nabla \in P^*_2(\{\Theta_{\mathcal{Y}}, \mathfrak{gl}(V)^\flat\}, \mathfrak{gl}(V)^\flat)$. One checks that $d^\flat_\nabla$ lifts d_∇, its restriction to $\Omega^\cdot \subset \Omega^\cdot \otimes \mathfrak{gl}(V)^\flat$ equals d, and $(d^\flat_\nabla)^2 = ad_{c(\nabla)}$.

Suppose we have a Tate structure $\mathcal{E}(V)^\flat$ on V and a connection $\nabla^\flat$ on the Lie* algebroid $\mathcal{E}(V)^\flat$. Such a $\nabla^\flat$ yields a connection ∇ on V (we say that $\nabla^\flat$ lifts ∇). It also yields a compatible pair $(d_{\nabla^\flat}, c(\nabla^\flat)) \in \mathfrak{D}(\mathfrak{gl}(V)^\flat)$. It is clear that $d_{\nabla^\flat}$ coincides with $d^\flat_\nabla$ defined by ∇; hence $d^\flat_\nabla(c(\nabla^\flat)) = 0$, and $c(\nabla^\flat) \in h(\Omega^2 \otimes \mathfrak{gl}(V)^\flat)$ lifts $c(\nabla)$.

[110] Here the superscript "*cl*" means "classical"; see 3.9.6, 3.9.7 for an explanation.

We will consider the above objects locally on $\mathcal{Y}_{\acute{e}t}$. Denote by $\mathfrak{T}$ the sheaf of pairs $(\mathcal{E}(V)^\flat, \nabla^\flat)$ as above.[111] Let $\mathfrak{C}$ be the sheaf of pairs $(\nabla, c(\nabla)^\flat))$ where ∇ is a connection on V and $c(\nabla)^\flat \in h(\Omega^2 \otimes \mathfrak{gl}(V)^\flat)$ is a lifting of $c(\nabla)$ such that $d^\flat_\nabla(c(\nabla)^\flat) = 0$. One has a morphism

$$c : \mathfrak{T} \to \mathfrak{C}, \quad (\mathcal{E}(V)^\flat, \nabla^\flat) \mapsto (\nabla, c(\nabla^\flat)). \tag{2.8.4.1}$$

2.8.5. LEMMA. *c is an isomorphism of sheaves.*

Proof. Use the lemma in 1.4.17. □

2.8.6. Our $\mathfrak{T}$ is naturally a groupoid: a morphism between the $(\mathcal{E}(V)^\flat, \nabla^\flat)$'s is a morphism between the corresponding $\mathcal{E}(V)^\flat$'s. Since every Tate structure admits a connection locally, the corresponding sheaf of groupoids is canonically equivalent to $\mathcal{T}ate(V)_{\mathcal{Y}_{\acute{e}t}}$.

Our $\mathfrak{C}$ also carries a groupoid structure. Namely, a morphism $(\nabla, c(\nabla)^\flat) \to (\nabla', c(\nabla')^\flat)$ is a 1-form $\nu^\flat \in h(\Omega^1 \otimes \mathfrak{gl}(V)^\flat)$ such that its image in $h(\Omega^1 \otimes \mathfrak{gl}(V))$ equals $\nabla' - \nabla$ and $c(\nabla')^\flat - c(\nabla)^\flat = d^\flat_\nabla(\nu^\flat) + \frac{1}{2}[\nu^\flat, \nu^\flat]$. The composition of morphisms is the addition of the $\nu^\flat$'s (check the relation!).

Now (2.8.4.1) lifts naturally to an equivalence of groupoids

$$c : \mathfrak{T} \xrightarrow{\sim} \mathfrak{C} \tag{2.8.6.1}$$

that sends $\phi : (\Theta^c, \nabla^\flat) \to (\Theta^{c'}, \nabla'^\flat)$ to a morphism $c(\phi) : (\nabla, c(\nabla^\flat)) \to (\nabla', c(\nabla'^\flat))$ defined by the form $\nu = c(\phi) := \phi^{-1}(\nabla'^\flat) - \nabla^\flat$ (the compatibilities follow from (1.4.17.1)).

We have proved

2.8.7. PROPOSITION. *The formula* (2.8.6.1) *yields an equivalence between* $\mathcal{T}ate(V)_{\mathcal{Y}_{\acute{e}t}}$ *and the sheaf of groupoids defined by* $\mathfrak{C}$. □

REMARK. Notice that $\mathfrak{C}$ carries a natural action of the groupoid defined by the 2-term complex $h(DR_{\mathcal{Y}/X})_{[1,2)}$. Namely, the translation by $\omega \in h(\Omega^2)^{cl}$ acts on objects of $\mathfrak{C}$ as $(\nabla, c(\nabla)^\flat) \mapsto (\nabla, c(\nabla)^\flat + \omega)$ and sends a morphism defined by a 1-form $\nu^\flat$ to a morphism defined by the same form. For $\mu \in h(\Omega^1)$ the corresponding morphism $(\nabla, c(\nabla)^\flat) \to (\nabla, c(\nabla)^\flat + d\mu)$ in $\mathfrak{C}$ is defined by $\mu 1^\flat \in h(\Omega^1 \otimes \mathfrak{gl}(V)^\flat)$. Now 2.8.7, 2.8.3 identify the corresponding sheafified action with the $\mathcal{P}^{cl}(\Theta_{\mathcal{Y}})$-action on $\mathcal{T}ate(V)$ from 2.8.2.

2.8.8. The above description of $\mathcal{T}ate(V)$ can be made more concrete by introducing auxiliary choices. First, choose an étale hypercovering $\mathcal{Y}.$ such that each $\mathcal{Y}_i$ is a disjoint union of affine schemes. Let $DR^{\geq a}_{\mathcal{Y}/X}$ be the subcomplex $\Omega^a \to \Omega^{a+1} \to \cdots$ of $DR_{\mathcal{Y}/X}$. Denote by C the complex of Čech cochains with coefficients in $h(DR^{\geq 1}_{\mathcal{Y}/X})$.

REMARK. Since $\Omega^i_{\mathcal{Y}/X}$ is a direct summand of an induced $\mathcal{D}_X$-module for $i > 0$, one has $H^a(\mathcal{Y}_b, h(\Omega^i_{\mathcal{Y}/X})) = 0$ for every $a, i > 0$. Thus C computes $R\Gamma(\mathcal{Y}, h(DR^{\geq 1}_{\mathcal{Y}/X}))$.

Below, $\partial_i : \mathcal{Y}_n \to \mathcal{Y}_{n-1}$, $i = 0, \ldots, n$, are the (étale) face morphisms.

[111] Our pairs are rigid, so we do not distinguish them from the corresponding isomorphism classes.

Now choose a triple (∇, χ, μ) where ∇ is a connection for V on $\mathcal{Y}_0$, χ is a section of $h(\Omega^2 \otimes \mathfrak{gl}(V)^\flat)$ on $\mathcal{Y}_0$ which lifts $c(\nabla)$, and μ is a section of $h(\Omega^1 \otimes \mathfrak{gl}(V)^\flat)$ on $\mathcal{Y}_1$ which lifts $\partial_0^*\nabla - \partial_1^*\nabla$. Set $\zeta := \zeta_0 + \zeta_1 + \zeta_2 \in C^3$ where

$\zeta_0 := d_\nabla(\chi) \in \Gamma(\mathcal{Y}_0, h(\Omega^3)) \subset \Gamma(\mathcal{Y}_0, h(\Omega^3 \otimes \mathfrak{gl}(V)^\flat))$,

$\zeta_1 := \partial_0^*\chi - \partial_1^*\chi - d_{\partial_0^*\nabla}\mu + \frac{1}{2}[\mu, \mu] \in \Gamma(\mathcal{Y}_1, h(\Omega^2)) \subset \Gamma(\mathcal{Y}_1, h(\Omega^2 \otimes \mathfrak{gl}(V)^\flat))$,

$\zeta_2 := -\partial_0^*\mu + \partial_1^*\mu - \partial_2^*\mu \in \Gamma(\mathcal{Y}_3, h(\Omega^1)) \subset \Gamma(\mathcal{Y}_2, h(\Omega^1 \otimes \mathfrak{gl}(V)^\flat))$.

Using the second remark in 1.4.17, one checks that ζ is a cocycle whose class $ch_1^{\mathcal{D}}(V) \in H^3(C) = H^3(\mathcal{Y}, h(DR^{\geq 1}_{\mathcal{Y}/X})) = H^4(\mathcal{Y}, DR_X(DR^{\geq 1}_{\mathcal{Y}/X}))$ does not depend on the auxiliary choices. Let C_ζ be the groupoid whose set of objects is $\{a \in C^2 : d_C(a) = -\zeta\}$ and for two objects a, a' one has $\mathrm{Hom}(a, a') := d_C^{-1}(a' - a) \subset C^1$; the composition of morphisms is the addition in C^1. If C_ζ is non-empty, i.e., if $ch_1^{\mathcal{D}}(V) = 0$, then C_ζ is a $\tau_{\leq 2}C$-torsor in the obvious way.

2.8.9. COROLLARY. (i) *$ch_1^{\mathcal{D}}(V)$ vanishes if and only if V admits a Tate structure.*

(ii) *The groupoid $\mathcal{T}ate(V)$ is canonically equivalent to C_ζ.*

(iii) *$\mathcal{P}^{cl}(\Theta_{\mathcal{Y}})$ is canonically equivalent to the Picard groupoid defined by the 2-term complex $\tau_{\leq 2}C$. The action of $\tau_{\leq 2}C$ on C_ζ corresponds via the above equivalences to the $\mathcal{P}^{cl}(\Theta_{\mathcal{Y}})$-action on $\mathcal{T}ate(V)$ from 2.8.2.*

Proof. Take any $a \in C_\zeta$, so $a = (a_0, a_1)$, $a_0 \in \Gamma(\mathcal{Y}_0, h(\Omega^2))$, $a_1 \in \Gamma(\mathcal{Y}_1, h(\Omega^1))$. Then $\mathfrak{c}_a := (\nabla, \chi + a_0) \in \mathfrak{C}(\mathcal{Y}_0)$, and $\gamma_a := \mu + a_1$ is a morphism $\partial_1^*\mathfrak{c}_a \to \partial_0^*\mathfrak{c}_a$ in $\mathfrak{C}(\mathcal{Y}_2)$ that satisfies the cocycle property. The map $a \mapsto (\mathfrak{c}_a, \gamma_a)$ is a bijection between C_ζ and the set of pairs $(\mathfrak{c}, \gamma)$ with the above property. Morphisms in C_ζ correspond to γ-compatible morphisms between the $\mathfrak{c}$'s. By 2.8.6 we have identified C_ζ with the groupoid of pairs $(\mathcal{E}(V)^\flat, \nabla)$ where $\mathcal{E}(V)^\flat$ is a Tate structure on V, ∇ is a connection on $\mathcal{E}(V)^\flat|_{\mathcal{Y}_0}$, the morphisms are morphisms between the $\mathcal{E}(V)^\flat$'s. Since any $\mathcal{E}(V)^\flat$ admits a connection on $\mathcal{Y}_0$, we are done. □

2.8.10. In fact, for a vector $\mathcal{D}_X$-bundle V one has a whole sequence of characteristic classes $ch_a^{\mathcal{D}}(V) \in H^{2a+1}(\mathcal{Y}, h(DR^{\geq a}_{\mathcal{Y}/X}))$, $a \geq 0$. One constructs them as follows.

Assume for a moment that V admits a global connection ∇, so we have $c(\nabla) \in h(\Omega^2 \otimes \mathfrak{gl}(V))$. Notice that $\mathfrak{gl}(V) = \mathcal{E}nd(V)$, hence (see 1.4.1) $\Omega^\cdot \otimes \mathfrak{gl}(V)$, is an associative* algebra. Thus $h(\Omega^\cdot \otimes \mathfrak{gl}(V))$ is an associative algebra, so we have $\frac{1}{a!}c(\nabla)^a \in h(\Omega^{2a} \otimes \mathfrak{gl}(V))$. Assume further that this section admits a global lifting $\chi^a \in \Gamma(\mathcal{Y}, h(\Omega^{2a} \otimes \mathfrak{gl}(V)^\flat))$. Then $d_\nabla(\chi^a) \in h(\Omega^{2a+1}) \subset h(\Omega^{2a+1} \otimes \mathfrak{gl}(V)^\flat)$. This form is closed, and $ch_a^{\mathcal{D}}(V)$ is its class in $H^{2a+1}(\mathcal{Y}, h(DR^{\geq a}_{\mathcal{Y}/X}))$.

One extends this definition to a general situation (when global ∇ and χ^a need not exist) using a Chern-Simons type interpolation procedure. We need some notation. Below $\Delta_n \subset \mathbb{A}^{n+1}$ is the affine hyperplane $t_0 + \cdots + t_n = 1$. For a $\mathcal{D}_X$-scheme $\mathcal{Z}$ we consider $\mathcal{Z} \times \Delta_n$ as a $\mathcal{D}_X$-scheme. One has $h(\Omega^\cdot_{\mathcal{Z}\times\Delta_n/X}) = h(\Omega^\cdot_{\mathcal{Z}/X}) \otimes \Omega^\cdot_{\Delta_n}$, so the integration over the simplex $\Sigma t_i = 1$, $0 \leq t_i \leq 1$, yields a map $\rho_n : \Gamma(\mathcal{Z} \times \Delta_n, h(\Omega^\cdot_{\mathcal{Z}\times\Delta_n/X})) \to \Gamma(\mathcal{Z}, h(\Omega^{\cdot - n}_{\mathcal{Z}}))$. It satisfies the Stokes formula $\rho_n(d\phi) = d\rho_n(\phi) + \Sigma(-1)^i \rho_{n-1}(\phi|_{\Delta^i_{n-1}})$ where $\Delta^i_{n-1} \subset \Delta_n$, $i = 0, \ldots, n$, is given by the equation $t_i = 0$.

Choose a hypercovering $\mathcal{Y}_\cdot$ as in 2.8.8 and a connection ∇ for V on $\mathcal{Y}_0$. Pulling it back by the structure étale maps $p_i : \mathcal{Y}_n \to \mathcal{Y}_0$, we get $n+1$ connections $\nabla^0_n, \ldots, \nabla^n_n$ on every $\mathcal{Y}_n$. Now the pull-back of V to $\mathcal{Y}_n \times \Delta_n$ gets a canonical connection $\nabla_n :=$

$\Sigma t_i \nabla^i_n$, so we have $c(\nabla_n)^a \in h(\Omega^{2a}_{\mathcal{Y}_n\times\Delta_n/X}\otimes\mathfrak{gl}(V))$. Choose any liftings $\chi^a_n \in \Gamma(\mathcal{Y}_n\times \Delta_n, h(\Omega^{2a}_{\mathcal{Y}_n\times\Delta_n/X}\otimes\mathfrak{gl}(V)^\flat))$ of $\frac{1}{a!}c(\nabla_n)^a$. Set $\xi^a_n := d_{\nabla_n}(\chi^a_n) \in h(\Omega^{2a+1}_{\mathcal{Y}_n\times\Delta_n/X}) \subset h(\Omega^{2a+1}_{\mathcal{Y}_n\times\Delta_n/X}\otimes\mathfrak{gl}(V)^\flat)$ and $\psi^a_n := \Sigma(-1)^i(\partial^*_i\chi^a_{n-1} - \chi^a_n|_{\Delta^i_{n-1}}) \in h(\Omega^{2a}_{\mathcal{Y}_n\times\Delta_{n-1}/X}) \subset h(\Omega^{2a}_{\mathcal{Y}_n\times\Delta_{n-1}/X}\otimes\mathfrak{gl}(V)^\flat)$.

Our ∇_n are flat in Δ_n-directions, so we can (and will) choose χ^a_n with zero image in $h(\Omega_{\mathcal{Y}_n}\otimes\Omega^{\geq a}_{\Delta_n}\otimes\mathfrak{gl}(V)^\flat)$. Then $\zeta^a_n := \rho_n(\xi^a_n) + \rho_{n-1}(\psi^a_n) \in \Gamma(\mathcal{Y}_n, \Omega^{2a+1-n})$ vanish for $n > a+1$, so $\zeta^a := \Sigma\zeta^a_n$ is a Čech cochain with coefficients in $h(DR^{\geq a}_{\mathcal{Y}/X})$. This is a cocycle due to the Stokes formula whose cohomology class $ch^{\mathcal{D}}_a(V)$ does not depend on the auxiliary choices of $\mathcal{Y}_\cdot, \nabla$, and χ^a_n.

REMARKS. (i) Using induction by n, the forms χ^a_n can be choosen so that $\chi^a_n|_{\Delta^i_{n-1}} = \partial^*_i\chi^a_{n-1}$ for every n and $i = 0,\dots,n$. Then $\zeta^a = \Sigma\rho_n(\xi^a_n)$.

(ii) One has $ch^{\mathcal{D}}_a(V^\circ) = (-1)^{a+1}ch^{\mathcal{D}}_a(V)$ (use (2.7.4.2)).

Let us show that the above class $ch^{\mathcal{D}}_1(V)$ coincides with the one from 2.8.8. Indeed, for given ∇ any datum (χ,μ) as in 2.8.8 defines $\chi_n = \chi^1_n$ as above. Namely, $\chi_0 := \chi$ and for $n \geq 1$ we set $\chi_n := p^*_0\chi_1 + d_{\nabla^0_n}\mu_n + \frac{1}{2}[\mu_n,\mu_n]$ where $\mu_n \in \Gamma(\mathcal{Y}_n\times\Delta_n, h(\Omega^1_{\mathcal{Y}_n\times\Delta_n/X}\otimes\mathfrak{gl}(V)^\flat))$ is a lifting of $\nabla_n - \nabla^0_n$, $\mu_n := t_1p^*_{01}\mu + \cdots + t_np^*_{0n}\mu$. The cocycle ζ for these χ_n's coincides with ζ for χ, μ from 2.8.8.

2.8.11. Let us compare the cohomology with coefficients in $h(DR_{\mathcal{Y}/X})$ with the absolute de Rham cohomology of $\mathcal{Y}$. Below, for a given a left $\mathcal{D}_X$-module N we denote by $DR_X(N)$ the corresponding de Rham complex placed in degrees $[0,1]$, so $H^1(DR_X(N)) = h(N)$.

Denote by $DR_{\mathcal{Y}}$ the absolute de Rham complex $\mathcal{O}_{\mathcal{Y}} \to \Omega^1_{\mathcal{Y}} \to \cdots$ of $\mathcal{Y}$. Then the complex $DR_X(DR_{\mathcal{Y}/X})$ coincides with $DR_{\mathcal{Y}}$. Indeed, the $\mathcal{D}_X$-scheme structure connection provides a decomposition $\Omega^1_{\mathcal{Y}} = \Omega^1_{\mathcal{Y}/X}\oplus\pi^*\Omega^1_X$, hence the isomorphisms $\Omega^n_{\mathcal{Y}} \xrightarrow{\sim} \oplus\Omega^i_{\mathcal{Y}/X}\otimes\pi^*\Omega^j_X$ which form the isomorphism of complexes $DR_{\mathcal{Y}/X} \xrightarrow{\sim} DR_X(DR_{\mathcal{Y}/X})$. We refer to the $DR_{\mathcal{Y}/X}$ and DR_X components $d_{\mathcal{Y}/X}$, d_X of the de Rham differential $d_{\mathcal{Y}}$ as the "vertical" and "horizontal" differentials.

Since $\Omega^i_{\mathcal{Y}/X}$ is a direct summand of an induced $\mathcal{D}_X$-module for every $i > 0$, we know that $H^0(DR_X(\Omega^i_{\mathcal{Y}/X})) = 0$ for $i > 0$ and $H^0(DR_X(\mathcal{O}_{\mathcal{Y}})) = H^0(DR_{\mathcal{Y}})$.[112]

So the projection $DR_X(DR^{\geq a}_{\mathcal{Y}/X}) \to h(DR^{\geq a}_{\mathcal{Y}/X})[-1]$ is a quasi-isomorphism for $a > 0$, and for $a = 0$ the projection $DR_{\mathcal{Y}}/H^0(DR_{\mathcal{Y}}) \to h(DR_{\mathcal{Y}/X})[-1]$ is a quasi-isomorphism. Therefore $H^m(\mathcal{Y}, DR_X(DR^{\geq a}_{\mathcal{Y}/X})) \xrightarrow{\sim} H^{m-1}(\mathcal{Y}, h(DR^{\geq a}_{\mathcal{Y}/X}))$ unless $a = m = 0$ (when the right-hand side vanishes).

2.8.12. Suppose now that our V is induced, so $V = F_{\mathcal{D}}$ for a vector bundle F on $\mathcal{Y}$. Then $ch^{\mathcal{D}}_a(V)$ can be expressed in terms of conventional characteristic classes of F. Namely, a classical Weil-Chern-Simons construction[113] provides (for any algebraic space $\mathcal{Y}$ and vector bundle F) Chern classes $ch_a(F) \in H^{2a}(\mathcal{Y}, DR^{\geq a}_{\mathcal{Y}})$ where $DR^{\geq a}_{\mathcal{Y}}$ are the subcomplexes $\Omega^a_{\mathcal{Y}} \to \Omega^{a+1}_{\mathcal{Y}} \to \cdots$ of $DR_{\mathcal{Y}} = DR^{\geq 0}_{\mathcal{Y}}$. The product of forms makes $\oplus H^{2a}(\mathcal{Y}, DR^{\geq a}_{\mathcal{Y}})$ a graded commutative ring.

[112]Proof: Suppose $\varphi \in \mathcal{O}_{\mathcal{Y}}$ is killed by d_X; i.e., $d_{\mathcal{Y}}(\varphi) \subset \Omega^1_{\mathcal{Y}/X} \subset \Omega^1_{\mathcal{Y}}$. Then $d_X(d_{\mathcal{Y}}(\varphi)) = 0$; i.e., $d_{\mathcal{Y}}(\varphi) \in H^0(DR_X(\Omega^1_{\mathcal{Y}/X}))$. Therefore $d_{\mathcal{Y}}(\varphi) = 0$ by the first equality.

[113]To be recalled in the course of the proof in 2.8.13.

REMARK. The image of $ch_a(F)$ by the map $H^{2a}(\mathcal{Y}, DR_{\mathcal{Y}}^{\geq a}) \to H^{2a}_{DR}(\mathcal{Y})$ is the conventional de Rham cohomology Chern class of F. The classes ch_a form the theory of Chern classes for the cohomology theory $\mathcal{Y} \mapsto H^{\cdot}(\mathcal{Y}, DR_{\mathcal{Y}}^{\geq *})$.

Since $DR_{\mathcal{Y}}^{\geq a+1} \subset DR_X(DR_{\mathcal{Y}/X}^{\geq a}) \subset DR_{\mathcal{Y}}$,[114] one has a canonical map

(2.8.12.1)
$$H^{2a+2}(\mathcal{Y}, DR_{\mathcal{Y}}^{\geq a+1}) \to H^{2a+2}(\mathcal{Y}, DR_X(DR_{\mathcal{Y}/X}^{\geq a})) = H^{2a+1}(\mathcal{Y}, h(DR_{\mathcal{Y}/X}^{\geq a})).$$

2.8.13. THEOREM. *$ch_a^{\mathcal{D}}(V)$ equals the image of $ch_{a+1}(F\omega_X^{-1/2}) = ch_{a+1}(F) - \frac{1}{2}ch_a(F)ch_1(\omega_X)$ by (2.8.12.1). In particular, $ch_1^{\mathcal{D}}(V)$ is the image of $ch_2(F\omega_X^{-1/2})$.*

Proof. First we prove our statement on the level of chains in the case when $F\omega_X^{-1/2}$ admits a global connection. The general situation follows then by the Chern-Simons interpolation.

(a) Suppose $F\omega_X^{-1/2}$ has a global connection ∇. Then the classes $ch_a(F\omega_X^{-1/2})$ are represented by closed forms $ch_a(\nabla) := \frac{1}{a!}\mathrm{tr}(c(\nabla)^a) \in \Gamma(\mathcal{Y}, \Omega_{\mathcal{Y}}^{2a})$ where $c(\nabla) \in \Omega_{\mathcal{Y}}^2 \otimes \mathrm{End}(F\omega^{-1/2}) = \Omega_{\mathcal{Y}}^2 \otimes \mathrm{End}(F)$ is the curvature of ∇.

The structure connection on $\mathcal{Y}$ provides a decomposition $\Omega_{\mathcal{Y}}^1 = \Omega_{\mathcal{Y}/X}^1 \oplus \pi^*\Omega_X^1$ which yields a decomposition of ∇ into a sum of vertical ∇^v and horizontal ∇^h parts. Since $\dim X = 1$, one has $\Omega_{\mathcal{Y}}^2 = \Omega_{\mathcal{Y}/X}^2 \oplus \Omega_{\mathcal{Y}/X}^1 \otimes \pi^*\Omega_X^1$. Denote by $c(\nabla^v)$ and $c(\nabla)_{1,1}$ the components of $c(\nabla)$. Then $c(\nabla^v)$ is the curvature of the relative connection ∇^v, and $c(\nabla)_{1,1} = \nabla^v_{\mathcal{C}onn}(\nabla^h)$ in terms of the notation of Remark (i) in 2.5.9.

The image of $ch_{a+1}(\nabla)$ by the map $\Omega_{\mathcal{Y}}^{2a+2} \to h(\Omega_{\mathcal{Y}/X}^{2a+1})$ (see 2.8.11) coincides with that of the form $\frac{1}{a!}\mathrm{tr}(c(\nabla^v)^a c(\nabla)_{1,1})$. We will show that $ch_a^{\mathcal{D}}(V)$ is represented by the same form.

Since $\pi^*\omega_X$ is constant along the fibers of π, our ∇^v can be considered as a vertical connection on F. One has $F_{\mathcal{D}} = V$ and $(\Omega_{\mathcal{Y}/X}^1 \otimes F)_{\mathcal{D}} = \Omega_{\mathcal{Y}/X}^1 \otimes V$; set $\nabla_V := (\nabla^v)_{\mathcal{D}} : V \to \Omega_{\mathcal{Y}/X}^1 \otimes V$.[115] Then ∇_V is a $\mathcal{D}$-module connection on V whose curvature is the image of $c(\nabla^v)$ by the map $\Omega_{\mathcal{Y}/X}^2 \otimes \mathfrak{gl}(F) \to h(\Omega^2 \otimes \mathfrak{gl}(V))$.

By (2.7.5.4)[116] the restriction of $\mathfrak{gl}(V)^\flat$ to $\mathfrak{gl}(F)_{\mathcal{D}} \subset \mathfrak{gl}(V)$ is the Kac-Moody extension for the vector bundle $F\omega_X^{-1/2}$ and the invariant form $\kappa = (\ ,\ )$, $(a,a) = -\mathrm{tr}(a^2)$. So, as in 2.5.9, the horizontal connection ∇^h provides a splitting $s_{\nabla^h} : \mathfrak{gl}(F)_{\mathcal{D}} \to \mathfrak{gl}(V)^\flat$. Set $\chi^a := s_{\nabla^h}(\frac{1}{a!}c(\nabla^v)^a) \in h(\Omega_{\mathcal{Y}/X}^{2a+1} \otimes \mathfrak{gl}(V)^\flat)$. Then, according to 2.8.10, the class $ch_a^{\mathcal{D}}(V)$ is represented by the form $d_{\nabla_V}(\chi^a) \in h(\Omega_{\mathcal{Y}/X}^{2a+1}) \subset h(\Omega_{\mathcal{Y}/X}^{2a+1} \otimes \mathfrak{gl}(V)^\flat)$.

The derivative d_{∇_V} preserves the subalgebra $\mathfrak{gl}(F)_{\mathcal{D}}^\flat \subset \mathfrak{gl}(V)^\flat$ and coincides there with the derivative $d^\kappa_{\nabla^v}$ defined by ∇^v (see Remark (i) of 2.5.9). Since $d_{\nabla^v}(c(\nabla^v)^a)$ vanishes, the formula in loc. cit. gives $d_{\nabla_V}(\chi^a) = d^\kappa_{\nabla^v} s_{\nabla^h}(\frac{1}{a!}c(\nabla^v)^a) = -(\nabla_{\mathcal{C}onn}(\nabla^h), \frac{1}{a!}c(\nabla^v)^a) = \frac{1}{a!}\mathrm{tr}(c(\nabla^v)c(\nabla)_{1,1}) = ch_{a+1}(\nabla)$; q.e.d.

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[114]Due to the fact that $\dim X = 1$.

[115]So $h(V) = F$ and $h(\Omega_{\mathcal{Y}/X}^1 \otimes V) = \Omega_{\mathcal{Y}/X}^1 \otimes F$, and ∇_V is defined by the property $h(\nabla_V) = \nabla^v$.

[116]Or, rather, its immediate generalization to the situation with extra parameters $\mathcal{Y}$; see Remark (i) in 2.5.9.

(b) Consider now the general situation, so $F\omega_X^{-1/2}$ need not have a global connection. Recall the construction of $ch_a(F\omega_X^{-1/2}) \in H^{2a}(\mathcal{Y}, DR_{\mathcal{Y}}^{\geq a})$.

Choose a hypercovering $\mathcal{Y}^{\cdot}$ as in 2.8.8 and a connection ∇ for $F\omega_X^{-1/2}$ on $\mathcal{Y}_0$. As in 2.8.10 we get connections $\nabla_n^0, \dots, \nabla_n^n$ on $\mathcal{Y}_n$ and $\nabla_n := \Sigma t_i \nabla_n^i$ on $\mathcal{Y}_n \times \Delta_n$ hence forms $ch_a(\nabla_n) := \frac{1}{a!} c(\nabla_n)^a \in \Gamma(\mathcal{Y}_n \times \Delta_n, \Omega^{2a}_{\mathcal{Y}_n\times\Delta_n})$. As in 2.8.10 we have the integration map $\rho_n : \Gamma(\mathcal{Y}_n \times \Delta_n, \Omega^{\cdot}_{\mathcal{Y}_n\times\Delta_n}) = \Gamma(\mathcal{Y}_n, \Omega^{\cdot}_{\mathcal{Y}_n}) \otimes \Gamma(\Delta_n, \Omega^{\cdot}_{\Delta_n}) \to \Gamma(\mathcal{Y}_n, \Omega^{\cdot -n}_{\mathcal{Y}_n})$. The class $ch_a(F\omega_X^{-1/2})$ is represented by a Čech cocycle $\Sigma\rho_n(ch_a(\nabla_n))$.[117]

To compare $ch_{a+1}(F\omega_X^{-1/2})$ with $ch_a^{\mathcal{D}}(V)$, one computes the latter class using connection ∇_V and forms $\chi_n^a = s_{\nabla_n^h}(\frac{1}{a!} c(\nabla_n^v)^a)$ as in (a). These forms satisfy the condition from Remark (i) in 2.8.10, so $ch_a^{\mathcal{D}}(V)$ is represented by cochain $\Sigma\rho_n d_{\nabla_V}(\chi_n^a)$. According to (a) it equals $ch_{a+1}(F\omega_X^{-1/2})$. □

2.8.14. Rigidified and weakly equivariant Tate structures. In the rest of this section we describe two general procedures for constructing Tate structures together with some examples.

(a) Let ∇ be an integrable connection on a vector $\mathcal{D}_X$-bundle V. As follows from 2.8.4, 2.8.5, there is a unique pair $(\mathcal{E}(V)^\flat_\nabla, \nabla^\flat)$ where $\mathcal{E}(V)^\flat$ is a Tate structure on V and $\nabla^\flat$ is a lifting of ∇ to an integrable connection on $\mathcal{E}(V)^\flat$. By 2.8.2, one can use $\mathcal{E}(V)^\flat_\nabla$ as the base point for an identification $\mathcal{P}^{cl}(\Theta_{\mathcal{Y}}) \xrightarrow{\sim} \mathcal{T}ate(V)$, $\Theta^c_{\mathcal{Y}} \mapsto \mathcal{E}(V)^{\flat+c}_\nabla$.

An example of this situation: Suppose that $\Theta_{\mathcal{Y}}$ is L-rigidified; i.e., we have an action τ of a Lie* algebra L on $\mathcal{Y}$ which yields an isomorphism $L_{\mathcal{O}_{\mathcal{Y}}} := L \otimes \mathcal{O}_{\mathcal{Y}} \xrightarrow{\sim} \Theta_{\mathcal{Y}}$ (see 1.4.13). Such a τ yields an integrable connection ∇_τ on $\Theta = \Theta_{\mathcal{Y}}$ defined by the property $\nabla_\tau \tau = \tau_\Theta$, where $\tau_\Theta : L \to \mathcal{E}(\Theta)$ is $\tau_\Theta(\xi)(\theta) = [\tau(\xi), \theta]$ for $\xi \in L$, $\theta \in \Theta$. Thus we have $\mathcal{E}(\Theta)^\flat_\tau := \mathcal{E}(\Theta)^\flat_{\nabla_\tau} \in \mathcal{T}ate(\Theta)$. Denote by $\mathcal{P}(L)$ the Picard groupoid of Lie* algebra ω_X-extensions of L. Any $L^c \in \mathcal{P}(L)$ yields the corresponding Lie* algebroid extension $\Theta^c_{\mathcal{Y}} := L^c_{\mathcal{O}_{\mathcal{Y}}} \in \mathcal{P}^{cl}(\Theta_{\mathcal{Y}})$ (see 1.4.13); hence the Tate structure $\mathcal{E}(\Theta)^{\flat+c}_\tau := \mathcal{E}(\Theta)^{\flat+c}_{\nabla_\tau}$. We refer to it as the L^c-*rigidified* Tate structure on $\mathcal{Y}$ (with respect to τ). One has canonical embeddings of Lie* algebras $\tau^c : L^c \to \Theta^c_{\mathcal{Y}}$, $\tau^c_\Theta : L^c \to \mathcal{E}(\Theta)^{\flat+c}_\tau$ which lift τ, τ_Θ.

(b) Let G be a smooth group $\mathcal{D}_X$-scheme affine over X acting on $\mathcal{Y}$ and let V be a G-equivariant vector $\mathcal{D}_X$-bundle. Then $\mathcal{E}(V)$ is a G-equivariant Lie* algebroid (see 2.5.17) and G acts on the Lie* algebra $\mathfrak{gl}(V)^\flat$ by transport of structure. One defines a *weakly G-equivariant* Tate structure on V demanding $\mathcal{E}(V)^\flat$ to be a weakly G-equivariant Lie* algebroid such that the maps $\mathfrak{gl}(V)^\flat \to \mathcal{E}(V)^\flat \to \mathcal{E}(V)$ are compatible with the G-actions. Weakly G-equivariant Tate structures form a torsor $\mathcal{T}ate(V)^G$ over the groupoid $\mathcal{P}^{cl}(\Theta)^G$ of weakly G-equivariant $\mathcal{O}^r_{\mathcal{Y}}$-extensions of $\Theta_{\mathcal{Y}}$.

If we have ∇ as in (a) preserved by the G-action, then $\mathcal{E}(V)^\flat_\nabla$ is weakly G-equivariant and we have the equivalence $\mathcal{P}^{cl}(\Theta_{\mathcal{Y}})^G \xrightarrow{\sim} \mathcal{T}ate(V)^G$, $\Theta^c_{\mathcal{Y}} \mapsto \mathcal{E}(V)^{\flat+c}_\nabla$.

Suppose that we have an L-rigidification τ of $\Theta_{\mathcal{Y}}$ and an action of G on the Lie* algebra L such that τ is compatible with the G-actions. Then $\Theta = \Theta_{\mathcal{Y}}$ is a G-equivariant vector $\mathcal{D}_X$-bundle and the connection ∇_τ on it is compatible with the G-action, so the above constructions apply. Let $\mathcal{P}(L)^G$ be the Picard groupoid of G-equivariant Lie* algebra ω_X-extensions L^c of L (we assume that G acts on $\omega_X \subset L^c$ trivially). For $L^c \in \mathcal{P}(L)^G$ the extension $\Theta^c_{\mathcal{Y}}$ is weakly G-equivariant in

[117]One has $\rho_n(ch_a(\nabla_n)) = 0$ for $n > a$ since ∇_n is flat along Δ_n, so our cocycle is in $DR^{\geq a}_{\mathcal{Y}}$.

the evident way, so we get a morphism of Picard groupoids and their torsors

$$\mathcal{P}(L)^G \to \mathcal{P}^{cl}(\Theta_{\mathcal{Y}})^G, \quad \mathcal{P}(L)^G \to \mathcal{T}ate(\mathcal{Y})^G, \tag{2.8.14.1}$$

$L^c \mapsto \Theta^c_{\mathcal{Y}}, \mathcal{E}(\Theta)^{\flat+c}_\tau$. The embeddings τ^c, τ^c_Θ from (a) are compatible with the G-actions.

EXERCISE. Define a strongly G-equivariant Tate structure. What do we need to make $\mathcal{E}(V)^c_\tau$ strongly G-equivariant?

2.8.15. The case of a bitorsor. Let G, G' be smooth group $\mathcal{D}_X$-schemes affine over X and let $\mathcal{Y}$ be a (G, G')-bitorsor; i.e., $\mathcal{Y}$ is a $\mathcal{D}_X$-scheme equipped with a $G \times G'$-action which makes it both G- and G'-torsor. In other words, $\mathcal{Y}$ is a G-torsor, and G' is the group $\mathcal{D}_X$-scheme of its automorphisms (= the twist of G by the G-torsor $\mathcal{Y}$ with respect to the adjoint action of G). Let us describe the groupoids $\mathcal{T}ate(\mathcal{Y})^G$, $\mathcal{T}ate(\mathcal{Y})^{G'}$, $\mathcal{T}ate(\mathcal{Y})^{G\times G'}$ of weakly equivariant Tate structures on $\mathcal{Y}$; these are torsors over the corresponding Picard groupoids $\mathcal{P}^{cl}(\Theta_{\mathcal{Y}})^G$, $\mathcal{P}^{cl}(\Theta_{\mathcal{Y}})^{G'}$, $\mathcal{P}^{cl}(\Theta_{\mathcal{Y}})^{G\times G'}$.

Let L, L' be the Lie* algebras of G, G', and $\tau : L \to \Theta_{\mathcal{Y}}$, $\tau' : L' \to \Theta_{\mathcal{Y}}$ their actions on $\mathcal{Y}$. Notice that τ is compatible with the $G\times G'$-actions where the G-action on L is the adjoint action and the G'-action is trivial; in fact, $\tau : L \xrightarrow{\sim} (\Theta_{\mathcal{Y}})^{G'}$ (the Lie* subalgebra of G'-invariants in $\Theta_{\mathcal{Y}}$). The same is true for G interchanged with G'. Since both τ and τ' are rigidifications of $\Theta_{\mathcal{Y}}$, we can use (2.8.14.1) to get the morphisms of the Picard groupoids and their torsors

$$\mathcal{P}(L) \to \mathcal{P}^{cl}(\Theta_{\mathcal{Y}})^{G'}, \quad \mathcal{P}(L) \to \mathcal{T}ate(\mathcal{Y})^{G'}, \tag{2.8.15.1}$$

$$\mathcal{P}(L') \to \mathcal{P}^{cl}(\Theta_{\mathcal{Y}})^{G}, \quad \mathcal{P}(L') \to \mathcal{T}ate(\mathcal{Y})^{G}, \tag{2.8.15.2}$$

$$\mathcal{P}(L)^G \to \mathcal{P}(\Theta_{\mathcal{Y}})^{G\times G'} \leftarrow \mathcal{P}(L')^{G'}, \quad \mathcal{P}(L)^G \to \mathcal{T}ate(\mathcal{Y})^{G\times G'} \leftarrow \mathcal{P}(L')^{G'}. \tag{2.8.15.3}$$

LEMMA. *The above functors are equivalences of groupoids.*

Proof. Consider, for example, (2.8.15.1). The inverse functor $\mathcal{P}^{cl}(\Theta_{\mathcal{Y}})^{G'} \to \mathcal{P}(L)$ assigns to $\Theta^c_{\mathcal{Y}}$ its Lie* subalgebra $(\Theta^c_{\mathcal{Y}})^{G'}$ of G'-invariants. The inverse functor $\mathcal{T}ate(\mathcal{Y})^{G'} \to \mathcal{P}(L)$ sends $\mathcal{E}(\Theta)^c$ to the pull-back by $\tau_\Theta : L \to \mathcal{E}(\Theta)^{G'}$ (see (a) in 2.8.14) of the Lie* subalgebra $(\mathcal{E}(\Theta)^c)^{G'}$ of G'-invariants in $\mathcal{E}(\Theta)^c$ (the latter is an ω_X-extension of $\mathcal{E}(\Theta)^{G'}$). The inverse functors to (2.8.15.2) and (2.8.15.3) are defined in a similar way. □

The *Tate extension* $L^\flat$ of L is the pull-back of the Tate extension $\mathfrak{gl}(L)^\flat$ by the adjoint action morphism $ad : L \to \mathfrak{gl}(L)$. The group G acts on it by transport of structure. So ad lifts to a morphism of G-equivariant ω_X-extensions $ad^\flat : L^\flat \to \mathfrak{gl}(L)^\flat$. We also have the Tate extension $L'^\flat \in \mathcal{P}(L')$ and a morphism $ad'^\flat : L'^\flat \to \mathfrak{gl}(L')^\flat$.

For $L^c \in \mathcal{P}(L)^G$ consider the corresponding $\Theta^c_{\mathcal{Y}} \in \mathcal{P}^{cl}(\Theta_{\mathcal{Y}})^{G\times G'}$, $L'^{c'} \in \mathcal{P}(L')^{G'}$ and $\mathcal{E}(\Theta)^c \in \mathcal{T}ate(\mathcal{Y})^{G\times G'}$, $L'^{\mathcal{T}c} \in \mathcal{P}(L')^{G'}$ coming from equivalences (2.8.15.3).

PROPOSITION. *There is a canonical isomorphism of G'-equivariant extensions*

$$L'^{\mathcal{T}c} \xrightarrow{\sim} L'^{c'+\flat}. \tag{2.8.15.4}$$

Proof. Let $j' = \tau' \otimes \tau'^{\circ} : \mathfrak{gl}(L') \hookrightarrow \mathfrak{gl}(\Theta)$ be the embedding whose image is the subalgebra $\mathfrak{gl}(\Theta)^G$ of G-invariants. We also have the isomorphisms $j'^{\flat} : \mathfrak{gl}(L')^{\flat} \xrightarrow{\sim} (\mathfrak{gl}(\Theta)^{\flat})^G$, $(\tau', j') : L' \ltimes \mathfrak{gl}(L') \xrightarrow{\sim} (\mathcal{E}(\Theta)^c)^G$.

Consider the flat connection ∇_τ on Θ compatible with the $G \times G'$-action (see (a) in 2.8.14). Since the images of τ_Θ, τ'_Θ commute, one has $\tau'_\Theta = \nabla_\tau \tau' + j' ad'$.

The extension $\Theta^c_{\mathcal{Y}}$ equals the pull-back of $\mathcal{E}(\Theta)^c$ by ∇_τ. Therefore $L'^{c'}$ is the pull-back of the ω_X-extension $(\mathcal{E}(\Theta)^c)^G$ of $\mathcal{E}(\Theta)^G$ by $\nabla_\tau \tau'$. Now $L'^{\mathcal{T}c}$ is the pull-back of $(\mathcal{E}(\Theta)^c)^G$ by τ'_Θ. Since the images of j' and ∇_τ commute, the pull-back by τ'_Θ of any extension of $\mathcal{E}(\Theta)^G$ is the Baer sum of its pull-backs by $\nabla_\tau \tau'$ and $j' ad'$, and we are done □

We say that $L^c \in \mathcal{P}(L)^G$ is *strongly G-equivariant* if the L-action ad^c on L^c coming from the G-action Ad^c coincides with the adjoint action of L^c (factored through L). One checks immediately that L^c is strongly G-equivariant if and only if the images of τ^c and $\tau'^{c'}$ in $\Theta^c_{\mathcal{Y}}$ mutually commute. If this happens, then the images of τ^c_Θ and $\tau'^{\mathcal{T}c}_\Theta$ in $\mathcal{E}(\Theta)^c$ also mutually commute.

Notice that there are *two* natural equivalences of Picard groupoids $\mathcal{P}(L)^G \xrightarrow{\sim} \mathcal{P}(L')^{G'}$. The first one is $L^c \mapsto L'^{c'}$ considered above. The second equivalence $L^c \mapsto (L^c)'$ is defined as follows. Consider isomorphisms of $G \times G'$-equivariant vector $\mathcal{D}_X$-bundles $L \otimes \mathcal{O}_{\mathcal{Y}} \xrightarrow{\sim} \Theta \xleftarrow{\sim} L' \otimes \mathcal{O}_{\mathcal{Y}}$ defined by τ, τ'; let $\alpha : L \otimes \mathcal{O}_{\mathcal{Y}} \xrightarrow{\sim} L' \otimes \mathcal{O}_{\mathcal{Y}}$ be *minus* the composition. Both $L \otimes \mathcal{O}_{\mathcal{Y}}$ and $L' \otimes \mathcal{O}_{\mathcal{Y}}$ are $G \times G'$-equivariant Lie* $\mathcal{O}_{\mathcal{Y}}$-algebras in the obvious way. A usual computation shows that α is compatible with the Lie* brackets.[118] There is an obvious identification of $\mathcal{P}(L)^G$ with the Picard groupoid of the $G \times G'$-equivariant Lie* $\mathcal{O}_{\mathcal{Y}}$-algebra extensions of $L \otimes \mathcal{O}_{\mathcal{Y}}$ and the same for L replaced by L'; together with α, they produce the promised equivalence $\mathcal{P}(L)^G \xrightarrow{\sim} \mathcal{P}(L')^{G'}$, $L^c \mapsto (L^c)'$.

Lemma. *For a strongly G-equivariant L^c there is a canonical identification*

$$(L^c)' \xrightarrow{\sim} L'^{-c'}. \tag{2.8.15.5}$$

Proof. Consider isomorphisms of $G \times G'$-equivariant $\mathcal{D}_X$-vector bundles $L^c \otimes \mathcal{O}_{\mathcal{Y}} \xrightarrow{\sim} \Theta^c_{\mathcal{Y}} \xleftarrow{\sim} L'^{c'} \otimes \mathcal{O}_{\mathcal{Y}}$ defined by τ^c, $\tau'^{c'}$; let $\alpha^c : L^c \otimes \mathcal{O}_{\mathcal{Y}} \xrightarrow{\sim} L'^{c'} \otimes \mathcal{O}_{\mathcal{Y}}$ be *minus* the composition. Consider $L^c \otimes \mathcal{O}_{\mathcal{Y}}$ and $L'^{c'} \otimes \mathcal{O}_{\mathcal{Y}}$ as $G \times G'$-equivariant Lie* $\mathcal{O}_{\mathcal{Y}}$-algebras. The computation from the previous footnote with L replaced by L^c, etc., shows that α^c is compatible with the Lie* brackets (here the strong G-equivariance of L^c is used). Our identification is the restriction of α^c to the G-invariants. □

Example. The Tate extension $L^\flat$ is strongly G-equivariant, and there is a canonical identification $(L^\flat)' = L'^\flat$ (for the corresponding $G \times G'$-equivariant extension of the Lie* $\mathcal{O}_{\mathcal{Y}}$-algebra $L \otimes \mathcal{O}_{\mathcal{Y}} = L' \otimes \mathcal{O}_{\mathcal{Y}}$ is its Tate extension). Now take any $a, a' \in k$ such that $a + a' = 1$. According to the last lemma and the proposition, the weakly $G \times G'$-equivariant Tate structures that correspond to $L^{a\flat} \in \mathcal{P}(L)^G$ and $L'^{a'\flat} \in \mathcal{P}(L')^{G'}$ by (2.8.15.3) are canonically identified. Denote this Tate structure by $\mathcal{E}(\Theta)^\flat_{a,a'}$. It is equipped with canonical Lie* algebra embeddings $L^{a\flat} \hookrightarrow \mathcal{E}(\Theta)^\flat_{a,a'} \hookleftarrow L'^{a'\flat}$ whose images mutually commute.

[118] Suppose that $\tau'(\chi) = \Sigma f_i \tau(\xi_i)$, $\tau'(\psi) = \Sigma g_j \tau(\eta_j)$; we want to show that $\tau'([\chi,\psi]) = -\Sigma f_i g_j \tau([\xi_i, \eta_j])$. Indeed, $\tau'([\chi,\psi]) = [\tau'(\chi), \tau'(\psi)] = -\Sigma f_i g_j [\tau(\xi_i), \tau(\eta_j)] + \Sigma f_i [\tau(\xi_i), \tau'(\psi)] - \Sigma g_j [\tau(\eta_j), \tau'(\chi)] = -\Sigma f_i g_j \tau([\xi_i, \eta_j])$; q.e.d.

Exercises. (i) Show that the construction of the proposition is symmetric with respect to the interchange of G and G'. Precisely, show that the composition $L^{\mathcal{T}(\mathcal{T}c)} \xrightarrow{\sim} L^{\mathcal{T}(c'+\flat)} \xrightarrow{\sim} L^{(c'+\flat)'+\flat} \xrightarrow{\sim} L^{c''-\flat+\flat} \xrightarrow{\sim} L^c$ coincides with the evident identification. Here the first arrow is (2.8.15.4) transformed by $\mathcal{T}: \mathcal{P}(L') \to \mathcal{P}(L)$, the second arrow is (2.8.15.4) for G, G' interchanged, the third one comes from the identification $\flat' = -\flat$ (see Example and the last lemma), and the fourth is the evident identification $c'' = c$.

(ii) Suppose our bitorsor $\mathcal{Y}$ is trivialized; i.e., we have a horizontal X-point $e \in \mathcal{Y}(X)$. It yields an identification $G \xrightarrow{\sim} G'$, $g \mapsto g'$, where $g'e = g^{-1}e$. Show that the equivalence $\mathcal{P}(L)^G \xrightarrow{\sim} \mathcal{P}(L')^{G'}$, $L^c \mapsto (L^c)'$, comes from the identification of $G = G'$.

(iii) We are in situation (ii). Suppose that L^c is strongly G-equivariant, and let $L^{\mathcal{T}c} \in \mathcal{P}(L)^G$ be the extension that corresponds to ${L'}^{\mathcal{T}c}$ by the identification $G = G'$. By (ii) we can rewrite (2.8.15.4), (2.8.15.5) as a canonical isomorphism $L^{c+\mathcal{T}c} \xrightarrow{\sim} L^\flat$. Show that it can be rephrased as follows. It suffices to describe its composition $L^{c+\mathcal{T}c} \to \mathfrak{gl}(L)^\flat$ with $ad^\flat$. Since the images of τ^c_Θ, ${\tau'}^{\mathcal{T}c}_\Theta$ commute, we have a morphism of Lie* algebras $\tau^c_\Theta + {\tau'}^{\mathcal{T}c}_\Theta L^c \times L^{\mathcal{T}c} \to \mathcal{E}(\Theta)^c$. The Baer sum $L^{c+\mathcal{T}c}$ is the subquotient of $L^c \times L^{\mathcal{T}c}$, and the above morphism yields a morphism $\tilde{\tau}^c_\Theta : L^{c+\mathcal{T}c} \to \mathcal{E}(\Theta)^c$. Its value at $e \in \mathcal{Y}$ belongs to $\mathfrak{gl}(L)^\flat \subset \mathcal{E}(\Theta)_e$; this is our canonical morphism $L^{c+\mathcal{T}c} \to \mathfrak{gl}(L)^\flat$.

2.8.16. The descent of Tate structures. Suppose that a smooth group $\mathcal{D}_X$-scheme G affine over X acts freely on a smooth $\mathcal{D}_X$-scheme $\mathcal{Y}$; i.e., we have a morphism of smooth $\mathcal{D}_X$-schemes $\pi : \mathcal{Y} \to \mathcal{Z}$ such that $\mathcal{Y}$ is a G-torsor over $\mathcal{Z}$. Let L be the Lie* algebra of G and let $L^\flat$ be its Tate extension (see 2.8.15). Suppose we have a weakly G-equivariant Tate structure $\mathcal{E}(\Theta_\mathcal{Y})^\flat$ (see 2.8.14) and a morphism of Lie* algebras $\alpha : L^\flat \to \mathcal{E}(\Theta_\mathcal{Y})^\flat$ which sends $1^\flat$ to $1^\flat$, commutes with the action of G, and lifts the morphism $L \to \mathcal{E}(\Theta_\mathcal{Y})$ corresponding to the L-action on $\Theta_\mathcal{Y}$.

Proposition. *A pair $(\mathcal{E}(\Theta_\mathcal{Y})^\flat, \alpha)$ as above defines a Tate structure $\mathcal{E}(\Theta_\mathcal{Z})^\flat_\alpha \in \mathcal{T}ate(\mathcal{Z})$ called the descent of $\mathcal{E}(\Theta_\mathcal{Y})^\flat$ with respect to α.*

Proof. Below for a Lie* algebroid $\mathcal{L}$ on $\mathcal{Y}$ we denote by $\mathcal{L}_{/\mathcal{Z}}$ the preimage of $\Theta_{\mathcal{Y}/\mathcal{Z}} \subset \Theta_\mathcal{Y}$ by the anchor map $\tau_\mathcal{L} : \mathcal{L} \to \Theta_\mathcal{Y}$; this is a Lie* subalgebroid of $\mathcal{L}$. We will consider $\mathcal{Z}$-families of Tate structure along the fibers of π (which are $\mathcal{O}^r_\mathcal{Y}$-extensions of $\mathcal{E}(\Theta_{\mathcal{Y}/\mathcal{Z}})_{/\mathcal{Z}}$) and call them simply Tate structures on $\mathcal{Y}/\mathcal{Z}$. The material of 2.8.14 and 2.8.15 immediately generalizes to the setting of families.

Consider a filtration $B \subset A^\flat \subset \mathcal{E}(\Theta_\mathcal{Y})^\flat$ defined as follows. Let $A^\flat$ be the preimage of the Lie* subalgebroid $A \subset \mathcal{E}(\Theta_\mathcal{Y})$ preserving $\Theta_{\mathcal{Y}/\mathcal{Z}} \subset \Theta_\mathcal{Y}$. Our B is the Lie* subalgebra of all endomorphisms of $\Theta_\mathcal{Y}$ having image in $\Theta_{\mathcal{Y}/\mathcal{Z}} \subset \Theta_\mathcal{Y}$, embedded in $\mathfrak{gl}(\Theta_\mathcal{Y})^\flat$ as in 2.7.4(i).[119] Now $A^\flat$ is a Lie* subalgebroid of $\mathcal{E}(\Theta_G)^\flat$, B is an ideal in $A^\flat$, and the G-action preserves our datum. Therefore we have a weakly G-equivariant Lie* algebroid A/B on $\mathcal{Y}$ and its $\mathcal{O}^r_\mathcal{Y}$-extension $A^\flat/B$.

By construction, A/B acts on $\Theta_{\mathcal{Y}/\mathcal{Z}}$ and $\pi^*\Theta_\mathcal{Z}$, and these actions yield an isomorphism $A/B \xrightarrow{\sim} \mathcal{E}(\Theta_{\mathcal{Y}/\mathcal{Z}}) \underset{\Theta_\mathcal{Y}}{\times} \mathcal{E}(\pi^*\Theta_\mathcal{Z})$. By 2.7.4(i) the restriction of $A^\flat/B$ to $\mathfrak{gl}(\Theta_{\mathcal{Y}/\mathcal{Z}}) \times \mathfrak{gl}(\pi^*\Theta_\mathcal{Z})$ is the Baer product of the Tate extensions.

[119]Consider the filtration $\Theta_{\mathcal{Y}/\mathcal{Z}} \subset \Theta_\mathcal{Y}$; by (2.7.4.1) the Tate extension of $\mathfrak{gl}(\Theta_\mathcal{Y})$ splits canonically over B.

Now A/B contains $\mathcal{E}(\Theta_{\mathcal{Y}/\mathcal{Z}})_{/\mathcal{Z}}$ as a Lie* subalgebroid. Its preimage $\mathcal{E}(\Theta_{\mathcal{Y}/\mathcal{Z}})^\flat_{/\mathcal{Z}}$ $\subset A^\flat/B$ is a weakly G-equivariant Tate structures on $\mathcal{Y}/\mathcal{Z}$. The morphism α : $L^\flat \to \mathcal{E}(\Theta_{\mathcal{Y}})^\flat$ has an image in $A^\flat$, and, modulo B, in $\mathcal{E}(\Theta_{\mathcal{Y}/\mathcal{Z}})^\flat_{/\mathcal{Z}}$. Therefore α identifies $\mathcal{E}(\Theta_{\mathcal{Y}/\mathcal{Z}})^\flat_{/\mathcal{Z}}$ with the weakly G-equivariant $L^\flat$-rigidified Tate structure on $\mathcal{Y}/\mathcal{Z}$ (see (2.8.14.1)). Equivalently, $\mathcal{E}(\Theta_{\mathcal{Y}/\mathcal{Z}})^\flat_{/\mathcal{Z}}$ identifies canonically with the Tate structure $\mathcal{E}(\Theta)^\flat_{1,0}$ from Example in 2.8.15. So the morphism of Lie* $\mathcal{O}_{\mathcal{Z}}$-algebras $L'_{\mathcal{Z}} \hookrightarrow \pi_*\Theta_{\mathcal{Y}/\mathcal{Z}}$ (whose image is the Lie* subalgebra of G-invariant vector fields) lifts naturally to $L'_{\mathcal{Z}} \hookrightarrow \pi_*\mathcal{E}(\Theta_{\mathcal{Y}/\mathcal{Z}})^\flat_{/\mathcal{Z}}$ whose image is invariant with respect to the G-action. Denote the latter embedding by i_α.

Set $C := (\pi_* A/B)^G$, $C^\flat := (\pi_* A^\flat/B)^G$. These are Lie* algebroids on $\mathcal{Z}$, and $C^\flat$ is an $\mathcal{O}^r_{\mathcal{Z}}$-extension of C. Notice that C acts on $\Theta_{\mathcal{Z}}$; the morphism $C \to \mathcal{E}(\Theta_{\mathcal{Z}})$ is surjective, and its kernel equals $(\pi_*\mathcal{E}(\Theta_{\mathcal{Y}/\mathcal{Z}})^\flat_{/\mathcal{Z}})^G$. In particular, it contains $i(L'_{\mathcal{Z}})$. Since all our constructions were natural, $i(L'_{\mathcal{Z}})$ is an ideal in $C^\flat$.

Finally, set $\mathcal{E}(\Theta_{\mathcal{Z}})^\flat := C^\flat/i_\alpha(L'_{\mathcal{Z}})$. This is a Tate structure on $\mathcal{Z}$: the identification $\mathfrak{gl}(\Theta_{\mathcal{Z}})^\flat \xrightarrow{\sim} \mathrm{Ker}(\mathcal{E}(\Theta_{\mathcal{Z}})^\flat \to \Theta_{\mathcal{Z}})$ comes from the morphism $\pi^*(\mathfrak{gl}(\Theta_{\mathcal{Z}})^\flat) = \mathfrak{gl}(\pi^*\Theta_{\mathcal{Z}})^\flat \to A^\flat/B$. We are done. □

Denote by $\phi(L)$ the vector space of $\mathcal{D}_X$-module morphisms $\phi : L \to \omega_X$ compatible with the actions of G (the adjoint and the trivial ones). Notice that such a ϕ is automatically a morphism of Lie* algebras. One has a natural morphism of Picard groupoids

(2.8.16.1) $$\phi(L) \to \mathcal{P}^{cl}(\Theta_{\mathcal{Z}}), \quad \phi \mapsto \Theta^\phi_{\mathcal{Z}}.$$

Namely, $(\pi_*\Theta_{\mathcal{Y}})^G$ is naturally a Lie* algebroid on $\mathcal{Z}$ which is an extension of $\Theta_{\mathcal{Z}}$ by $L'_{\mathcal{Z}}$. A morphism $\phi : L \to \omega_X$ from $\phi(L)$ yields, by twist, a morphism of Lie* $\mathcal{O}_{\mathcal{Z}}$-algebras $\phi'_{\mathcal{Z}} : L'_{\mathcal{Z}} \to \mathcal{O}^r_{\mathcal{Z}}$. Our $\Theta^\phi_{\mathcal{Z}}$ is the push-out of $(\pi_*\Theta_{\mathcal{Y}})^G$ by $\phi'_{\mathcal{Z}}$. It is a Lie* $\mathcal{O}_{\mathcal{Z}}$-algebroid since $\phi_{\mathcal{Z}}$ is compatible with the $(\pi_*\Theta_{\mathcal{Y}})^G$-actions.

LEMMA. *Suppose we have $(\mathcal{E}(\Theta_{\mathcal{Y}})^\flat, \alpha)$ as in the proposition. Then for any $\phi \in \phi(L)$ one has a canonical identification*

(2.8.16.2) $$\mathcal{E}(\Theta_{\mathcal{Z}})^\flat_{\alpha+\phi} \xrightarrow{\sim} \mathcal{E}(\Theta_{\mathcal{Z}})^{\flat+\phi}_\alpha.$$

Here $\alpha + \phi$ in the left-hand side is the composition of α and the automorphism $id_{L^\flat} + \phi$ of $L^\flat$, and $\mathcal{E}(\Theta_{\mathcal{Z}})^{\flat+\phi}_\alpha$ is the Baer sum of $\mathcal{E}(\Theta_{\mathcal{Z}})^\flat_\alpha$ and $\Theta^\phi_{\mathcal{Z}}$.

Proof. We use notation from the proof of the proposition. Recall that $\mathcal{E}(\Theta_{\mathcal{Z}})^\flat_\alpha$ $= C^\flat/i_\alpha(L'_{\mathcal{Z}})$. Replacing α by $\alpha+\phi$, we do not change $C^\flat$, and $i_{\alpha+\phi} = i_\alpha - \phi'_{\mathcal{Z}}$. So (2.8.16.2) amounts to a morphism of Lie* $\mathcal{O}_{\mathcal{Z}}$-algebroids $\chi : C^\flat \to \mathcal{E}(\Theta_{\mathcal{Z}})^{\flat+\phi}_\alpha$ which lifts the identity morphism of $\mathcal{E}(\Theta_{\mathcal{Z}})$, sends $1^\flat$ to $1^\flat$, and satisfies $\chi i_\alpha = \phi'_{\mathcal{Z}}$. One constructs it as follows.

Denote the projection $C^\flat \to \mathcal{E}(\Theta_{\mathcal{Z}})^\flat_\alpha$ by η. Let ϵ be the composition of morphisms of Lie* $\mathcal{O}_{\mathcal{Z}}$-algebroids $C^\flat \to (\pi_*\Theta_{\mathcal{Y}})^G \to \Theta^\phi_{\mathcal{Z}}$ where the first arrow comes from the anchor map for $A^\flat/B$. Now our χ is the composition $C^\flat \xrightarrow{(\eta,\epsilon)}$ $\mathcal{E}(\Theta_{\mathcal{Z}})^\flat_\alpha \underset{\Theta_{\mathcal{Z}}}{\times} \Theta^\phi_{\mathcal{Z}} \to \mathcal{E}(\Theta_{\mathcal{Z}})^{\flat+\phi}_\alpha$ where the second arrow is the Baer sum projection. □

2.8.17. Tate structures on the jet scheme of a flag space. We change notation: now G is a semi-simple group. Let $N \subset B \subset G$ be a Borel subgroup and its nilradical, $H := B/N$ the Cartan group, $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras, $Q := G/B$ the flag space. Set $G_X := G \times X$, $\mathcal{J}G := \mathcal{J}G_X$ (see 2.3.2 for notation), etc. So we have smooth group $\mathcal{D}_X$-schemes $\mathcal{J}B \subset \mathcal{J}G$, and $\mathcal{J}G/\mathcal{J}B \xrightarrow{\sim} \mathcal{J}Q$. The Lie* algebra of $\mathcal{J}G$ equals $\mathfrak{g}_{\mathcal{D}}$, etc.

We will consider Tate structures and related objects on $\mathcal{J}Q$ locally with respect to X. Denote by $\mathcal{P}^{cl}(\Theta_{\mathcal{J}Q})_X$ and $\mathcal{T}ate(\mathcal{J}Q)_X$ the corresponding sheaves of groupoids on X. We also have the weakly $\mathcal{J}G$-equivariant counterparts $\mathcal{P}^{cl}(\Theta_{\mathcal{J}Q})_X^{\mathcal{J}G}$, $\mathcal{T}ate(\mathcal{J}Q)_X^{\mathcal{J}G}$.

Let $\rho \in \mathfrak{h}^*$ be the half-sum of the positive roots. Let $\omega_X^\times$ be the $\mathcal{O}_X^\times$-torsor of invertible sections of ω_X. Define the *Miura* $\mathfrak{h}^* \otimes \omega_X$*-torsor* M_X as the push-out of $\omega_X^\times$ by $\rho \otimes d\log : \mathcal{O}_X^\times \to \mathfrak{h}^* \otimes \omega_X$.

Remark. Sections of M_X are sometimes called *Miura opers* (for the Langlands dual group $G^\vee$); see, e.g., [**DS**], [**BD**], [**Fr**]. They can be interpreted as connections on an $H^\vee$-torsor $(\omega_X^\times)^\rho$ (here $H^\vee$ is the torus dual to H).

Theorem. *The groupoid $\mathcal{P}^{cl}(\Theta_{\mathcal{J}Q})_X$ is discrete, so we can consider it as a sheaf of k-vector spaces on X. There are canonical identifications of sheaves of vector spaces and their torsors*

$$\mathcal{P}^{cl}(\Theta_{\mathcal{J}Q})_X^{\mathcal{J}G} \xrightarrow{\sim} \mathfrak{h}^* \otimes \omega_X, \quad \mathcal{T}ate(\mathcal{J}Q)_X^{\mathcal{J}G} \xrightarrow{\sim} M_X. \tag{2.8.17.1}$$

Proof. (a) Consider the complex $h(DR_{\mathcal{J}Q/X})_{[1,2)}$ from 2.8.3. According to (2.8.3.1), $\mathcal{P}^{cl}(\Theta_{\mathcal{J}Q})_X$ identifies canonically with the Picard groupoid defined by the 2-term complex $\tau_{\leq 1} Rp_*(\mathcal{J}Q, h(DR_{\mathcal{J}Q/X})_{[1,2)}[1])$ where $p : \mathcal{J}Q \to X$ is the projection. To compute it, we use the canonical affine projection $\pi : \mathcal{J}Q \to Q_X$. The morphism of $\mathcal{O}_{\mathcal{J}Q}$-modules $\pi^*(\Omega_Q^1 \otimes \mathcal{O}_X) \to \Omega^1_{\mathcal{J}Q/X}$ yields an isomorphism of left $\mathcal{O}_{\mathcal{J}Q}[\mathcal{D}_X]$-modules $\mathcal{D}_X \otimes \pi^*(\Omega_Q^1 \otimes \mathcal{O}_X) \xrightarrow{\sim} \Omega^1_{\mathcal{J}Q/X}$ or right $\mathcal{O}_{\mathcal{J}Q}[\mathcal{D}_X]$-modules $(\pi^*\Omega^1_Q \otimes \omega_X)_{\mathcal{D}} \xrightarrow{\sim} \Omega^{1r}_{\mathcal{J}Q/X}$. Therefore $\pi_* h(\Omega^1_{\mathcal{J}Q/X}) = R\pi_* h(\Omega^1_{\mathcal{J}Q/X}) = (\Omega^1_Q \otimes \omega_X) \otimes \pi_*\mathcal{O}_{\mathcal{J}Q}$. The group G_X of G-valued functions on X = horizontal sections of $\mathcal{J}G$ acts on the above sheaves, and the isomorphisms are compatible with this action.

The sheaf $(\pi_*\mathcal{O}_{\mathcal{J}Q})/\mathcal{O}_{Q_X}$ admits an increasing filtration compatible with the G_X-action with successive quotients isomorphic (as G_X-modules) to direct summands of sheaves of type $(\Omega^1_Q)^{\otimes n} \otimes \omega_X^{\otimes m}$, $n > 0$. Since $\Gamma(Q, (\Omega^1_Q)^{\otimes n}) = 0$ for $n > 0$,[120] one has $p_* h(\Omega^1_{\mathcal{J}Q/X}) = 0$. In particular, $\mathcal{P}^{cl}(\Theta_{\mathcal{J}Q})_X$ is discrete; hence $\mathcal{P}^{cl}(\Theta_{\mathcal{J}Q})_X = R^2 p_* h(DR_{\mathcal{J}Q/X})_{[1,2)}$.

(b) Since $H^1(Q, (\Omega^1_Q)^{\otimes n})^G = 0$ for $n > 1$,[121] we see that the embedding $\Omega^1_Q \otimes \omega_X \hookrightarrow \pi_* h(\Omega^1_{\mathcal{J}Q})$ yields an isomorphism

$$H^1(Q, \Omega^1_Q) \otimes \omega_X \xrightarrow{\sim} (R^1 p_* h(\Omega^1_{\mathcal{J}Q/X}))^{G_X}. \tag{2.8.17.2}$$

We will see in a moment that the arrow

$$(R^2 p_* h(DR_{\mathcal{J}Q/X})_{[1,2)})^{G_X} \to R^1 p_* h(\Omega^1_{\mathcal{J}Q/X})^{G_X} \tag{2.8.17.3}$$

[120]It suffices to show that for every G-equivariant line bundle subquotient $\mathcal{L}$ of $(\Omega^1_Q)^{\otimes n}$ one has $\Gamma(Q, \mathcal{L}) = 0$, which is clear since the weight of $\mathcal{L}$ is the sum of $n > 0$ negative roots.

[121]It suffices to check that for every $\mathcal{L}$ as in the previous footnote one has $H^1(Q, \mathcal{L})^G = 0$. Indeed, if $H^1(Q, \mathcal{L})^G \neq 0$, then, by Borel-Weil-Bott, its weight is a simple negative root, and the weight of our $\mathcal{L}$ is the sum of $n > 1$ negative roots.

coming from the projection $h(DR_{\mathcal{J}Q/X})_{[1,2)}[1] \to h(\Omega^1_{\mathcal{J}Q/X})$ is an isomorphism. We define the first isomorphism in (2.8.17.1) as the composition of (2.8.17.3), the inverse to (2.8.17.2), and the standard Chern class identification $H^1(Q, \Omega^1_Q) \xrightarrow{\sim} \mathfrak{h}^*$.

One has $p_* h(\Omega^2_{\mathcal{J}Q/X}) = 0$: indeed, $\Omega^2_{\mathcal{J}Q/X}$ is a direct summand in $(\Omega^1_{\mathcal{J}Q/X})^{\otimes 2} = \mathcal{D}_X \otimes (\pi^*\Omega^1_Q \otimes \mathcal{D}_X \otimes \pi^*\Omega^1_Q)$; hence $\pi_* h(\Omega^2_{\mathcal{J}Q/X})$ is a direct summand in $((\Omega^1_Q)^{\otimes 2} \otimes \omega_X \otimes \mathcal{D}_X) \otimes \pi_* \mathcal{O}_{\mathcal{J}Q}$, and the push-forward to X of the latter sheaf vanishes for the same reason that $p_* h(\Omega^1_{\mathcal{J}Q/X})$ did. Therefore the map $R^2 p_* h(DR_{\mathcal{J}Q/X})_{[1,2)} \to R^1 p_* h(\Omega^1_{\mathcal{J}Q/X})$ is injective. Its surjectivity on G_X-invariants follows from the fact that $H^1(Q, \Omega^{1 closed}_Q) \xrightarrow{\sim} H^1(Q, \Omega^1_Q)$. To see this, notice that the arrow (2.8.17.2) lifts to $R^2 p_* h(DR_{\mathcal{J}Q/X})_{[1,2)}$ since $\Omega^{1 closed}_Q \otimes \omega_X \hookrightarrow \mathrm{Ker}(\pi_* h(\Omega^1_{\mathcal{J}Q}) \xrightarrow{d} \pi_* h(\Omega^2_{\mathcal{J}Q}))$.

(c) Now let us pass to Tate structures. Let $\mathfrak{b}^{\tilde\flat}_{\mathcal{D}} \in \mathcal{P}(\mathfrak{b}_{\mathcal{D}})$ be the restriction of the Tate extension $\mathfrak{g}^\flat_{\mathcal{D}}$ of $\mathfrak{g}_{\mathcal{D}}$ to $\mathfrak{b}_{\mathcal{D}}$. Since the adjoint action of $\mathfrak{n}$ on $\mathfrak{g}$ is nilpotent, this extension, as well as the Tate extension $\mathfrak{b}^\flat_{\mathcal{D}}$, are canonically trivialized on $\mathfrak{n}_{\mathcal{D}}$ (by 2.7.4(i)). Set $\mathfrak{b}^\mu_{\mathcal{D}} := \mathfrak{b}^{\tilde\flat/2-\flat}_{\mathcal{D}}$; let $\mathfrak{h}^\mu_{\mathcal{D}} \in \mathcal{P}(\mathfrak{h}_{\mathcal{D}})$ be its descent to $\mathfrak{h}_{\mathcal{D}}$ according to the above trivialization over $\mathfrak{n}_{\mathcal{D}}$. Denote by T_X the sheaf of trivializations of the extension $\mathfrak{h}^\mu_{\mathcal{D}}$. If $\mathfrak{h}^\mu_{\mathcal{D}}$ is locally trivial, then T_X is a $\mathcal{H}om(\mathfrak{h}_{\mathcal{D}}, \omega_X) = \mathfrak{h}^* \otimes \omega_X$-torsor.

We define the second isomorphism in (2.8.17.1) as the composition of natural maps

$$M_X \to T_X \to \mathcal{T}ate(\mathcal{J}Q)^{\mathcal{J}G}_X \tag{2.8.17.4}$$

which commute with the $\mathfrak{h}^* \otimes \omega_X$-actions (here the action on $\mathcal{T}ate(\mathcal{J}Q)^{\mathcal{J}G}_X$ comes from the first isomorphism in (2.8.17.1)). Since M_X is locally non-empty, our maps are isomorphisms of $\mathfrak{h}^* \otimes \omega_X$-torsors.

(i) *The construction of the map $M_X \to T_X$*: By the definition of M_X, it amounts to a morphism $\omega^\times_X \to T_X$, $\nu \mapsto \gamma_\nu$, such that $\gamma_{f\nu} = \rho \otimes d\log f + \gamma_\nu$ for $f \in \mathcal{O}^\times_X$. To define γ_ν, we fix a non-degenerate Ad-invariant bilinear form κ on $\mathfrak{g}$. It yields an identification $\mathfrak{g}/\mathfrak{b} \xrightarrow{\sim} \mathfrak{n}^*$; hence $(\mathfrak{g}/\mathfrak{b} \otimes \omega_X)_{\mathcal{D}} \xrightarrow{\sim} (\mathfrak{n}_{\mathcal{D}})^\circ$ (see (2.2.16.1)). Thus for any $\nu \in \omega^\times_X$ we get an isomorphism $\iota^\kappa_\nu : \mathfrak{g}_{\mathcal{D}}/\mathfrak{b}_{\mathcal{D}} \xrightarrow{\sim} (\mathfrak{n}_{\mathcal{D}})^\circ$.

The adjoint action of $\mathfrak{b}_{\mathcal{D}}$ on $\mathfrak{g}_{\mathcal{D}}$ preserves the filtration $\mathfrak{n}_{\mathcal{D}} \subset \mathfrak{b}_{\mathcal{D}} \subset \mathfrak{g}_{\mathcal{D}}$ and is trivial on the subquotient $\mathfrak{b}_{\mathcal{D}}/\mathfrak{n}_{\mathcal{D}}$. By (2.7.4.1) we have a canonical identification $a : \mathfrak{b}^{\flat+\sharp}_{\mathcal{D}} \xrightarrow{\sim} \mathfrak{b}^{\tilde\flat}_{\mathcal{D}}$ where $\mathfrak{b}^\sharp_{\mathcal{D}}$ is the pull-back of the Tate extension $\mathfrak{gl}(\mathfrak{g}_{\mathcal{D}}/\mathfrak{b}_{\mathcal{D}})^\flat$ by the adjoint action map $\mathfrak{b}_{\mathcal{D}} \to \mathfrak{gl}(\mathfrak{g}_{\mathcal{D}}/\mathfrak{b}_{\mathcal{D}})$.

The isomorphism ι^κ_ν is compatible with the $\mathfrak{b}_{\mathcal{D}}$-actions. Together with (2.7.4.2), it yields an identification $\sigma^\kappa_\nu : \mathfrak{b}^\flat_{\mathcal{D}} \xrightarrow{\sim} \mathfrak{b}^\sharp_{\mathcal{D}}$. It follows from, say, (2.7.5.4) or a direct calculation that σ^κ_ν does not depend on κ and it satisfies $\sigma^\kappa_{f\nu} = \sigma^\kappa_\nu + 2\rho \otimes d\log f$. Now consider an isomorphism $a/2(id_{\mathfrak{b}^\flat_{\mathcal{D}}} + \sigma^\kappa_\nu) : \mathfrak{b}^\flat_{\mathcal{D}} \xrightarrow{\sim} \mathfrak{b}^{\tilde\flat/2}$. It is compatible with the trivializations of our extensions over $\mathfrak{n}_{\mathcal{D}}$; hence it can be considered as a trivialization of $\mathfrak{h}^\mu_{\mathcal{D}}$, i.e., an element of T_X. This is our γ_ν.

(ii) *The construction of the map $T_X \to \mathcal{T}ate(\mathcal{J}Q)_X$, $\gamma \mapsto \mathcal{E}(\Theta_{\mathcal{J}Q})^\flat_\gamma$*: Consider $\mathcal{J}G$ as (the trivialized) $(\mathcal{J}G, \mathcal{J}G)$-bitorsor with respect to the left and right translations. Let $\mathcal{E}(\Theta_{\mathcal{J}G})^\flat$ be the weakly $\mathcal{J}G \times \mathcal{J}G$-equivariant Tate structure on $\mathcal{J}G$ denoted by $\mathcal{E}(\Theta)^\flat_{1/2,1/2}$ in Example from 2.8.15. We have the canonical embedding $\tau'^{\flat/2}_\Theta : \mathfrak{g}^{\flat/2}_{\mathcal{D}} \to \mathcal{E}(\Theta_{\mathcal{J}G})^\flat$ lifting the right translation action τ' of $\mathfrak{g}_{\mathcal{D}}$. Restricting it to $\mathfrak{b}_{\mathcal{D}}$, we get $\tau'^{\tilde\flat/2}_\Theta : \mathfrak{b}^{\tilde\flat/2}_{\mathcal{D}} \to \mathcal{E}(\Theta_{\mathcal{J}G})^\flat$.

Therefore $\gamma \in T_X$ yields a morphism $\alpha_\gamma := {\tau'}^{\tilde{\mathfrak{b}}/2}_{\Theta}\gamma : \mathfrak{b}^\flat_{\mathcal{D}} \to \mathcal{E}(\Theta_{\mathcal{J}G})^\flat$. Now the projection $\mathcal{J}G$ is a $\mathcal{J}B$-torsor over $\mathcal{J}Q = \mathcal{J}G/\mathcal{J}B$. The pair $(\mathcal{E}(\Theta_{\mathcal{J}G})^\flat, \alpha_\gamma)$ satisfies the conditions from 2.8.16. So, by the proposition from loc. cit., it yields a Tate structure on $\mathcal{J}Q$ which is weakly $\mathcal{J}G$-equivariant (since $\mathcal{E}(\Theta_{\mathcal{J}G})^\flat$ is weakly $\mathcal{J}G$-equivariant with respect to left translations). This is our $\mathcal{E}(\Theta_{\mathcal{J}Q})^\flat_\gamma \in \mathcal{T}ate(\mathcal{J}Q)^{\mathcal{J}G}_X$. According to (2.8.16.2), it behaves in a right way under $\mathfrak{h}^* \otimes \omega_X$-translations of α. □

Exercise. Suppose that X is an affine line, so it carries an action of the group Aff of the affine transformations. It lifts in the obvious way to X-schemes G_X, Q_X, hence, by transport of structure, to the corresponding jet $\mathcal{D}_X$-schemes. Show that there is a unique Tate structure on $\mathcal{J}Q$ which is (weakly) Aff-equivariant in the obvious sense, and this Tate structure is automatically weakly G_X-equivariant.[122]

2.9. The Harish-Chandra setting and the setting of c-stacks

This section is quite isolated and can be skipped by the reader.

Let Y be a scheme and let $\mathcal{L}$ be a Lie algebroid on Y which is a locally free $\mathcal{O}_Y$-module of finite rank. We will play with the quotient stack $Y/\mathcal{L}$ of Y modulo the action of (the formal groupoid of) $\mathcal{L}$ and similar objects called c-stacks (where "c" is for "crystalline"). The key fact is that the category of $\mathcal{O}$-modules on a c-stack (which are $\mathcal{L}$-modules on Y in the case of $Y/\mathcal{L}$) is naturally an augmented compound tensor category.

The particular cases we already came across are the category of $\mathcal{D}$-modules (were $\mathcal{L}$ is the tangent algebroid) and, more generally, the category $\mathcal{M}(Y)$ where Y is a $\mathcal{D}_X$-scheme (here $\mathcal{L}$ is the horizontal foliation defined by the structure connection). A c-stack covered by a point amounts to a Harish-Chandra pair $(\mathfrak{g}, K)$ (the notation for this stack is $B_{(\mathfrak{g},K)}$), and $\mathcal{O}$-modules on $B_{(\mathfrak{g},K)}$ are the same as $(\mathfrak{g}, K)$-modules.

Specialists in non-linear partial differential equations explain that it is important to consider symmetries of differential equations that mix the dependent and independent variables. In the setting of 2.3, this means that for a $\mathcal{D}_X$-scheme Y the structure that really matters is the horizontal foliation, and the projection $Y \to X$ can be forgotten. Therefore, for an algebraic geometer, non-linear differential equations *are* c-stacks. For this, see [**V**] and references therein.[123]

The c-stack functoriality is quite convenient. For example, suppose a smooth variety X carries a $(\mathfrak{g}, K)$-structure, i.e., a K-torsor whose space is equipped with a simple transitive $\mathfrak{g}$-action compatible with the K-action. Such a structure yields a morphism of c-stacks $X/\Theta_X \to B_{(\mathfrak{g},K)}$, hence a compound tensor functor (the base change) from $(\mathfrak{g}, K)$-modules to $\mathcal{D}$-modules on X. Thus one gets, say, Lie* algebras on X from Lie* algebras in the category of $(\mathfrak{g}, K)$-modules. An important $(\mathfrak{g}, K)$ corresponds to the group ind-scheme of automorphisms of the formal ball $Spec\, k[\![t_1, \dots, t_n]\!]$: according to Gelfand-Kazhdan [**GeK**], every X of dimension n carries a canonical $(\mathfrak{g}, K)$-structure (the space of formal coordinate systems), so $(\mathfrak{g}, K)$-modules give rise to "universal" $\mathcal{D}$-modules.

[122]Hint: $X = \mathbb{A}^1$ carries a unique Aff-invariant Miura oper, which yields an Aff-equivariant Tate structure. One checks that $\mathcal{P}^{cl}(\Theta_{\mathcal{J}Q})^{Aff}_X = 0$ by an argument similar to part (a) of the proof of the theorem.

[123][**V**] considers, under the name of *diffieties*, c-stacks of type $Y/\mathcal{L}$ such that $\mathcal{L}$ is a foliation. The rank of $\mathcal{L}$ is called in loc. cit. the *Dimension* of the diffiety.

In 2.9.1 we recall what Lie algebroids and modules over them are;[124] for more information about Lie algerboids in the framework of the usual differential geometry see [**M**]. The enveloping algebras of Lie algebroids are considered in 2.9.2. The corresponding Poincaré-Birkhoff-Witt theorem was proved originally by Rinehart [**Rin**] for Lie R-algebroids which are projective R-modules. For yet another proof (which makes sense also in the chiral setting) see 3.9.13 below. In 2.9.4 we show that for a Lie algebroid $\mathcal{L}$ which is a locally free $\mathcal{O}$-module of finite rank, the category of $\mathcal{L}$-modules is naturally an augmented compound tensor category (generalizing thus the construction of 2.2). In 2.9.5 we define the pull-back of a Lie algebroid by a smooth morphism; the reader can skip this section (it will be used in 4.5). In 2.9.7 we define a canonical augmented compound tensor structure on the category of Harish-Chandra modules for a Harish-Chandra pair $(\mathfrak{g}, K)$ such that $\dim \mathfrak{g}/\mathfrak{k} < \infty$. In 2.9.8 we define the notion of the $(\mathfrak{g}, K)$-structure on a smooth variety and the corresponding functor from Harish-Chandra modules to $\mathcal{D}$-modules. The Gelfand-Kazhdan structure is considered in 2.9.9 (we follow the exposition of [**W**]). In the rest of the section we discuss briefly the general setting of c-stacks (for the language of stacks see [**LMB**]). The definition of c-stack is in 2.9.10; in 2.9.11 we show that the quotient of an algebraic stack modulo the action of appropriate Lie algebroid is a c-stack. In 2.9.11 we formulate (without a proof) the statement that $\mathcal{O}$-modules on a c-stack form naturally an augmented compound tensor category which combines 2.9.4 and 2.9.7.

2.9.1. Lie algebroids and modules over them. Let R be a commutative k-algebra. A *Lie R-algebroid* is a k-module $\mathcal{L}$ equipped with R-module and Lie k-algebra structures and an action τ of the Lie algebra $\mathcal{L}$ on R (so τ is a morphism of Lie algebras $\mathcal{L} \to \mathrm{Der}_k(R)$ called the *anchor* map). One demands that for $f, g \in R$, $l, l' \in \mathcal{L}$ one has $\tau(fl)g = f(\tau(l)g)$ and $[l, fl'] = \tau(l)(f)l' + f[l, l']$.

Lie R-algebroids are local objects with respect to the Zariski or étale topology of $\operatorname{Spec} R$, so we know what Lie $\mathcal{O}_X$-algebroids = Lie algebroids on X for any scheme or algebraic space X are. Lie algebroids form a category which we denote by $\mathcal{L}ie\mathcal{A}lg_R$ or $\mathcal{L}ie\mathcal{A}lg_X$.

REMARKS. (i) $\mathrm{Der}_k(R)$ is a final object of the category of Lie R-algebroids. In the setting of a scheme or an algebraic space, Lie algebroids are assumed to be quasi-coherent $\mathcal{O}_X$-modules, so if $\Theta_X := \mathrm{Der}_k(\mathcal{O}_X)$ is $\mathcal{O}_X$-quasi-coherent (which happens if X is of finite type), then it is a final object of $\mathcal{L}ie\mathcal{A}lg_X$.

(ii) For a Lie R-algebroid $\mathcal{L}$ the kernel $\mathcal{L}^\diamondsuit$ of the anchor map is a Lie R-algebra. In the setting of a scheme or an algebraic space, $\mathcal{L}^\diamondsuit$ is a Lie $\mathcal{O}_X$-algebra which is $\mathcal{O}_X$-quasi-coherent Θ_X is.

For a Lie R-algebroid $\mathcal{L}$ denote by $\mathcal{L}^-$ a copy of $\mathcal{L}$ considered as a mere Lie k-algebra equipped with an action τ on R (i.e., we forget about the R-module structure on $\mathcal{L}$).

An $\mathcal{L}^-$*-module* is an R-module M equipped with an action of $\mathcal{L}^-$ which is compatible with the $\mathcal{L}^-$-action on R; i.e., for every $f \in R$, $m \in M$, and $l \in \mathcal{L}$ one has $l(fm) = l(f)m + f(lm)$. We say that M is a *left* (resp. *right*) $\mathcal{L}$*-module* if, in addition, one has $(fl)m = f(lm)$ (resp. $(fl)n = l(fn)$).

[124] According to K. Mackenzie, Lie algebroids enjoyed some fourteen different names for a half century of existence.

The category $\mathcal{M}(R, \mathcal{L}^-)$ of $\mathcal{L}^-$-modules is an abelian k-category; the R-tensor product makes it a tensor category. The full subcategories $\mathcal{M}^\ell(R, \mathcal{L})$, $\mathcal{M}^r(R, \mathcal{L})$ of left and right $\mathcal{L}$-modules are closed under subquotients. The tensor product of two left $\mathcal{L}$-modules is a left $\mathcal{L}$-module, and that of a left and a right $\mathcal{L}$-module is a right $\mathcal{L}$-module. Therefore $\mathcal{M}^\ell(R, \mathcal{L})$ is a tensor category which acts on $\mathcal{M}^r(R, \mathcal{L})$.

Any right $\mathcal{L}$-module M defines the (homological) *de Rham-Chevalley* complex $\mathcal{C}^r_R(\mathcal{L}, M)$. As a mere graded module, it equals $M \underset{R}{\otimes} \mathrm{Sym}_R(\mathcal{L}[1])$, and the differential is determined by the property that the evident surjective map $C(\mathcal{L}^-, M) \to \mathcal{C}^r_R(\mathcal{L}, M)$ is a morphism of complexes. Here $C(\mathcal{L}^-, M)$ is the usual Lie algebra homology complex (which equals $M \underset{k}{\otimes} \mathrm{Sym}_k(\mathcal{L}[1])$ as a mere graded module). Equivalently, the differential is characterized by the property that the evident action of the Lie algebra $\mathcal{L}^-_\dagger$ on $M \underset{R}{\otimes} \mathrm{Sym}_R(\mathcal{L}[1])$ (as on a mere graded module) makes $\mathcal{C}^r_R(\mathcal{L}, M)$ a DG $\mathcal{L}^-_\dagger$-module (see 1.1.16).

Any left $\mathcal{L}$-module P yields the (cohomological) *de Rham-Chevalley* complex $\mathcal{C}^\ell_R(\mathcal{L}, P)$. As a mere graded module, it equals $\mathrm{Hom}_R(\bigoplus \mathrm{Sym}^i_R(\mathcal{L}[1]), P)$ and the differential is defined by the property that the inclusion $\mathrm{Hom}_R(\bigoplus \mathrm{Sym}^i_R(\mathcal{L}[1]), P) \hookrightarrow \mathrm{Hom}_k(\bigoplus \mathrm{Sym}^i_k(\mathcal{L}[1]), P)$ identifies $\mathcal{C}^\ell_R(\mathcal{L}, P)$ with a subcomplex of the usual Lie algebra cochain complex $C(\mathcal{L}^-, P)$.

For left $\mathcal{L}$-modules P, P' and a right $\mathcal{L}$-module M we have the evident pairings $\mathcal{C}^\ell_R(\mathcal{L}, P) \otimes \mathcal{C}^\ell_R(\mathcal{L}, P') \to \mathcal{C}^\ell_R(\mathcal{L}, P \otimes P')$, $\mathcal{C}^\ell_R(\mathcal{L}, P) \otimes \mathcal{C}^r_R(\mathcal{L}, M) \to \mathcal{C}^r_R(\mathcal{L}, P \otimes M)$ which are morphisms of complexes. In particular, $\mathcal{C}_R(\mathcal{L}) = \mathcal{C}^\ell_R(\mathcal{L}, R)$ is a commutative DG algebra and $\mathcal{C}^\ell_R(\mathcal{L}, P)$, $\mathcal{C}^r_R(\mathcal{L}, M)$ are $\mathcal{C}_R(\mathcal{L})$-modules.

The complex $\mathcal{C}^\ell_R(P)$ carries a decreasing filtration by subcomplexes $F^\cdot$, where $\mathcal{C}^\ell_R(P)/F^n = \mathrm{Hom}_R(\mathrm{Sym}^{<n}_R(\mathcal{L}[1]), P)$. The corresponding topology is complete, and the above pairings are continuous. In particular, $\mathcal{C}^\ell_R(\mathcal{L})$ is a filtered topological commutative DG super algebra.

The above definitions render itself to the DG super setting in the obvious way.

Examples. (i) Suppose we have a Lie algebra L acting on R. Then $L_R := R \underset{k}{\otimes} L$ is naturally a Lie R-algebroid; for any Lie R-algebroid $\mathcal{L}$ a Lie R-algebroid morphism $L_R \to \mathcal{L}$ is the same as a Lie algebra morphism $L \to \mathcal{L}$ compatible with the actions on R. We refer to an isomorphism $L_R \xrightarrow{\sim} \mathcal{L}$ as an *L-rigidification* of $\mathcal{L}$. For such a rigidification the restrictions of the obvious "identity" functor $\mathcal{M}(R, \mathcal{L}^-) \to \mathcal{M}(R, L)$ (:= the category of R-modules equipped with an L-action compatible with the L-action on R) to the subcategories of left and right $\mathcal{L}$-modules are equivalences of categories:

$$\mathcal{M}^\ell(R, \mathcal{L}) \xrightarrow{\sim} \mathcal{M}(R, L) \xleftarrow{\sim} \mathcal{M}^r(R, \mathcal{L}). \tag{2.9.1.1}$$

The complexes $\mathcal{C}^r_R(\mathcal{L}, M)$, $\mathcal{C}^\ell_R(\mathcal{L}, P)$ coincide with the usual Chevalley Lie algebra homology, resp. cohomology, complexes of L with coefficients in M, P.

(ii) Suppose R is an n-Poisson algebra (see 1.4.18), so $\Omega_R[-n]$ is a Lie R-algebroid. Set $\mathcal{C}^{pois}(R) := \mathcal{C}_R(\Omega_R[-n])$. As a mere graded algebra, $\mathcal{C}^{pois}(R)$ equals $\mathrm{Hom}_R(\bigoplus \mathrm{Sym}^i_R(\Omega_R[1-n]), R)$; thus it carries a canonical continuous $(n-1)$-Poisson bracket which is the usual commutator on $\mathrm{Der}(R)$ and the action of $\mathrm{Der}(R)$ on R in appropriate degrees. This bracket is compatible with the differential and filtration $F^\cdot$, so $\mathcal{C}^{pois}_R(R)$ is naturally a topological filtered $(n-1)$-Poisson algebra (cf. 1.4.18).

Left and right $\mathcal{L}$-modules and their tensor products have the étale local nature with respect to $\operatorname{Spec} R$, so they make sense on any scheme or algebraic space X; the corresponding categories are denoted by $\mathcal{M}^\ell(X,R)$, $\mathcal{M}^r(X,\mathcal{L})$. The homological de Rham-Chevalley complexes are naturally sheaves on X. The same is true for cohomological complexes if $\mathcal{L}$ is a finitely presented $\mathcal{O}_X$-module.

EXAMPLE. If X is a smooth variety and $\mathcal{L} = \Theta_X$, then left, resp. right, $\mathcal{L}$-modules are the same as left, resp. right, $\mathcal{D}_X$-modules, and the corresponding de Rham-Chevalley complexes are the usual de Rham complexes.

2.9.2. Enveloping algebras. Let $\mathcal{L}$ be a Lie algebroid on X. The *enveloping algebra* of $\mathcal{L}$ is a sheaf $U(\mathcal{L}) = U(\mathcal{O}_X, \mathcal{L})$ of associative algebras on X equiped with a morphism of algebras $\iota_\mathcal{O} : \mathcal{O}_X \to U(\mathcal{L})$ and a morphism of Lie algebras $\iota_\mathcal{L} : \mathcal{L} \to U(\mathcal{L})$ such that $\iota_\mathcal{O}$ is a morphism of Lie algebra $\mathcal{L}$-modules ($\mathcal{L}$ acts on $U(\mathcal{L})$ by $\iota_\mathcal{L}$ composed with the adjoint action) and $\iota_\mathcal{L}$ is a morphism of $\mathcal{O}_X$-modules ($\mathcal{O}_X$ acts on $U(\mathcal{L})$ by $\iota_\mathcal{O}$ and left multiplication), and $U(\mathcal{L})$ is universal with respect to this datum. Then left, resp. right, $\mathcal{L}$-modules are the same as left (resp. right) $U(\mathcal{L})$-modules which are quasi-coherent as $\mathcal{O}_X$-modules.

$U(\mathcal{L})$ carries a natural ring filtration: $U(\mathcal{L})_0 := \iota_\mathcal{O}(\mathcal{O}_X)$, $U(\mathcal{L})_i := (\iota_\mathcal{O}(\mathcal{O}_X) + \iota_\mathcal{L}(\mathcal{L}))^i$ for $i \geq 1$; we refer to it as the standard filtration. The filtration is commutative (i.e., $\operatorname{gr} U(\mathcal{L})$ is a commutative algebra), so $\operatorname{gr} U(\mathcal{L})$ is naturally a Poisson algebra.

REMARKS. (i) Considered as an $\mathcal{O}_X$-bimodule, $U(\mathcal{L})$ is a quasi-coherent $\mathcal{O}_{X\times X}$-module supported set-theoretically on the diagonal $X \hookrightarrow X \times X$.

(ii) The evident forgetful functors[125] $\mathcal{M}^\ell(X,\mathcal{L}) \xrightarrow{o^\ell} \mathcal{M}_\mathcal{O}(X) \xleftarrow{o^r} \mathcal{M}^r(X,\mathcal{L})$ admit left adjoints sending an $\mathcal{O}_X$-module N to, respectively, ${}_{U(\mathcal{L})}N \in \mathcal{M}^\ell(X,\mathcal{L})$ and $N_{U(\mathcal{L})} \in \mathcal{M}^r(X,\mathcal{L})$ (the *induced* left and right $\mathcal{L}$-modules). One has ${}_{U(\mathcal{L})}N = U(\mathcal{L}) \underset{\mathcal{O}_X}{\otimes} N$, $N_{U(\mathcal{L})} := N \underset{\mathcal{O}_X}{\otimes} U(\mathcal{L})$.

Notice that $\operatorname{Sym}\mathcal{L} := \operatorname{Sym}_{\mathcal{O}_X}\mathcal{L}$ carries a natural Poisson structure characterized by the property that for $\ell, \ell' \in \mathcal{L} \subset \operatorname{Sym}\mathcal{L}$ and $f \in \mathcal{O}_X \subset \operatorname{Sym}\mathcal{L}$ one has $\{\ell,\ell'\} = [\ell,\ell']$, $\{\ell, f\} = \tau(\ell)f$. The morphisms $\iota_\mathcal{O}$ and $\iota_\mathcal{L}$ yield a surjective morphism of graded Poisson algebras

$$\operatorname{Sym}\mathcal{L} \twoheadrightarrow \operatorname{gr}.U(\mathcal{L}). \tag{2.9.2.1}$$

One has the following version of the Poincaré-Birkhoff-Witt theorem:

PROPOSITION. *If $\mathcal{L}$ is $\mathcal{O}_X$-flat, then (2.9.2.1) is an isomorphism.*

Proof. Consider a graded $\mathcal{O}_X$-algebra $\mathcal{O}_X[t]$ with t of degree 1, a graded Lie $\mathcal{O}_X[t]$-algebroid $\mathcal{L}[t]$, and its subalgebroid $\mathcal{L}_t := t\mathcal{L}[t]$. It yields a graded $k[t]$-algebra $U_t := U(\mathcal{O}_X[t], \mathcal{L}_t)$; denote by $U_t^{(a)}$ its components. One has $U_t/tU_t = \operatorname{Sym}(\mathcal{L})$ and $U_t/(t-1)U_t = U(\mathcal{L})$. So it suffices to show that the $\mathcal{O}_X$-flatness of $\mathcal{L}$ implies that U_t is $k[t]$-flat. Our U_t carries the standard filtration, and we will actually prove that $(\operatorname{Sym}\mathcal{L})[t] \xrightarrow{\sim} \operatorname{gr} U_t$; i.e., $\operatorname{Sym}^a\mathcal{L} \oplus t\operatorname{Sym}^{a-1}\mathcal{L} \oplus \cdots \oplus t^a\mathcal{O}_X \xrightarrow{\sim} \operatorname{gr} U_t^a$ for every $a \geq 0$. Denote the latter statement by $(*)_a$.

Consider U_t as a right $\mathcal{L}_t$-module. Let $\mathcal{C}_t$ be the corresponding de Rham-Chevalley complex; this is a complex of graded $k[t]$-modules, and we denote by $\mathcal{C}_t^{(a)}$

[125]Here $\mathcal{M}_\mathcal{O}(X)$ denotes the category of quasi-coherent $\mathcal{O}_X$-modules.

its components with respect to the grading. Notice that

$$\mathcal{O}_X[t] \oplus U_t\mathcal{L}_t \xrightarrow{\sim} U_t; \tag{2.9.2.2}$$

i.e., $\mathcal{O}_X[t] \xrightarrow{\sim} U_t/U_t\mathcal{L}_t = H^0\mathcal{C}_t$.[126]

We will prove $(*)_a$ by induction. $(*)_0$ is evident. Suppose $(*)_i$ is known for each $i < a$. We will show in a moment that $H^{-1}\mathcal{C}_t^{(a)} = 0$. This implies $(*)_a$. Indeed, look at the truncated complex $\mathcal{C}^{(a)'} := \mathcal{C}_t^{(a)}/\mathcal{C}_t^{(a)0}$. The above vanishing of H^{-1} means that the differential in $\mathcal{C}$

$$H^{-1}\mathcal{C}^{(a)'} \xrightarrow{d} (U_t\mathcal{L}_t)^{(a)} \tag{2.9.2.3}$$

is an isomorphism. We have seen that $(U_t\mathcal{L}_t)^{(a)} \xrightarrow{\sim} U_t^{(a)}/t^a\mathcal{O}_X$. The exactness of the Koszul complexes (here the flatness of $\mathcal{L}$ is used) together with $(*)_i$ for $i < a$ imply that $H^j\mathrm{gr}\,\mathcal{C}^{(a)'} = 0$ for $j \neq -1$ and $H^{-1}\mathrm{gr}\,\mathcal{C}^{(a)'} = \mathcal{L} \oplus \cdots \oplus \mathrm{Sym}^a\mathcal{L}$. The latter group equals $H^{-1}\mathcal{C}^{(a)'}$, which gives $(*)_a$.

It remains to prove that $H^{-1}\mathcal{C}_t^{(a)} = 0$, i.e., that (2.9.2.3) is an isomorphism. Our U_t is a universal right $\mathcal{L}_t$-module, so to get an inverse to (2.9.2.3), it suffices to define on $F := U_t^{(0)} \oplus \cdots \oplus U_t^{(a-1)} \oplus (H^{-1}\mathcal{C}^{(a)'} \oplus t^a\mathcal{O}_X)$ (the latter summand has degree a) a structure of the graded right $\mathcal{L}_t$-module such that on $F/F^{(a)} = U_t/U_t^{(\geq a)}$ it is the usual $\mathcal{L}_t$-module structure, and the action of $\ell \in \mathcal{L}_t^{(1)} = \mathcal{L}$ on $m \in F^{(a-1)} = U_t^{(a-1)}$ is the class of $m \otimes \ell \in \mathcal{C}^{(a)'-1}$ in $H^{-1}\mathcal{C}^{(a)'} \subset F^{(a)}$. This formula defines a Lie algebra action of $\mathcal{L}$ on F. The multiplication by t operator $F^{(a-1)} \to F^{(a)}$ is the composition $U_t^{(a-1)} \xleftarrow{\sim} H^{-1}\mathcal{C}^{(a-1)'} \oplus t^{a-1}\mathcal{O}_X \xrightarrow{t} H^{-1}\mathcal{C}^{(a)'} \oplus t^a\mathcal{O}_X$ where the first arrow comes from isomorphism (2.9.2.3) for $a-1$ (known by induction) and (2.9.2.2). The $\mathcal{O}_X$-action on $F^{(a)}$ recovers uniquely from the Lie algebroid property. We leave it to the reader to check the compatibilities. □

2.9.3. Suppose that $\mathcal{L}$ is a locally free $\mathcal{O}_X$-module of finite rank. The adjoint action of $\mathcal{L}$ on $\mathcal{L}$ makes it an $\mathcal{L}^-$-module. One checks that the line bundle $\omega_{\mathcal{L}} := \det(\mathcal{L}^*)[rk\mathcal{L}]$ is a right $\mathcal{L}$-module. There is a canonical equivalence of categories

$$\mathcal{M}^\ell(X,\mathcal{L}) \xrightarrow{\sim} \mathcal{M}^r(X,\mathcal{L}), \qquad M \mapsto M^r := M \otimes \omega_{\mathcal{L}}. \tag{2.9.3.1}$$

We denote these categories thus identified by $\mathcal{M}(X,\mathcal{L})$. This is a tensor category since $\mathcal{M}^\ell(X,\mathcal{L})$ is.

Below we denote by $\otimes^!$ the corresponding tensor product on $\mathcal{M}^r(X,\mathcal{L})$; i.e., $\otimes^! N_i = (\otimes(N_i \otimes \omega_{\mathcal{L}}^{-1})) \otimes \omega_{\mathcal{L}}$.

Remark. Suppose that $\mathcal{L}$ is L-rigidified (see Example (i) in 2.9.1). Then the composition of equivalences (2.9.1.1) differs from (2.9.3.1) by the twist by $\det L$.

2.9.4 The compound tensor structure. Let us show that for $\mathcal{L}$ as in 2.9.3, $\mathcal{M}(X,\mathcal{L})$ is naturally an augmented compound tensor category (we already know that it is a tensor category). The construction is a generalization of the one from 2.2 (the case of $\mathcal{L} = \Theta_X$). We present it first in a formal way as in 2.2.12, explaining a more geometric picture similar to 2.2.1–2.2.7 later.

[126] Our map is evidently surjective, and it admits a left inverse which sends the class in $H^0\mathcal{C}_t$ of $u \in U_t$ to $u \cdot 1 \in \mathcal{O}_X[t]$ where $\cdot$ is the left U_t-action on $\mathcal{O}_X[t]$.

The structure has a local origin, so we can assume that X is affine. The whole compound tensor structure comes then from a compound augmented $U(\mathcal{L})$-operad $\mathcal{B}^{*!}$; see 1.3.18 and 1.3.19 (cf. 1.3.20). We define $\mathcal{B}^*_I$ as the tensor product of I copies of $U(\mathcal{L})$ considered as a *left* $\mathcal{L}$-module. This is an $(U(\mathcal{L}) - U(\mathcal{L})^{\otimes I})$-bimodule: namely, the right $U(\mathcal{L})^{\otimes I}$-action comes by transport of structure from the right $U(\mathcal{L})$-action on $U(\mathcal{L})$ and the left $U(\mathcal{L})$-module structure comes since $\mathcal{B}^*_I$ is a left $\mathcal{L}$-module. Similarly, $\mathcal{B}^!$ is the $\otimes^!$ tensor product of I copies of $U(\mathcal{L})$ considered as a *right* $U(\mathcal{L})$-module. This is a $(U(\mathcal{L})^{\otimes I} - U(\mathcal{L}))$-bimodule.

The operad structure on $\mathcal{B}^*$: for $J \twoheadrightarrow I$ the map $\mathcal{B}^*_I \otimes (\underset{I}{\otimes} \mathcal{B}^*_{J_i}) \to \mathcal{B}^*_J$ is the projection $(\underset{\mathcal{O}_X}{\otimes} U(\mathcal{L})_i) \otimes (\underset{I}{\otimes}(\underset{\mathcal{O}_X}{\otimes} U(\mathcal{L})_{j_i})) \to \underset{\mathcal{O}_X}{\otimes} (U(\mathcal{L})_i \underset{U(\mathcal{L})}{\otimes} (\underset{\mathcal{O}_X}{\otimes} U(\mathcal{L})_{j_i})) = \mathcal{B}^*_J$. The operad structure on $\mathcal{B}^!$ is defined in a similar way.

So for $M_i, N \in \mathcal{M}^r(X, \mathcal{L})$ one has $\otimes^! M_i = (\otimes M_i) \underset{U(\mathcal{L})^{\otimes I}}{\otimes} \mathcal{B}^!$ and $P^*_I(\{M_i\}, N) := \mathrm{Hom}_{U(\mathcal{L})^{\otimes I}}(\otimes M_i, N \otimes_{U(\mathcal{L})} \mathcal{B}^*_I)$.

The construction of structure morphisms $< >^I_{S,T}$: $(\underset{S}{\otimes} \mathcal{B}^*_{I_s}) \underset{U(\mathcal{L})^{\otimes I}}{\otimes} (\underset{T}{\otimes} \mathcal{B}^!_{I_t}) \to \mathcal{B}^!_S \underset{U(\mathcal{L})}{\otimes} \mathcal{B}^*_T$ repeats the one we considered in 2.2.12 for the particular case of $\mathcal{D}$-modules (i.e., $\mathcal{L} = \Theta_X$).

The axioms of the augmented compound operad are immediate.

Here is a geometric explanation of the picture. Let $\mathcal{G}$ be the formal groupoid on X that corresponds to $\mathcal{L}$ (see 1.4.15). For $I \in \mathcal{S}$ one has a formal groupoid $\mathcal{G}^I$ on X^I; it contains subgroupoids $\mathcal{G}_i := \mathcal{G} \times X^{I \smallsetminus \{i\}}$, $i \in I$. Denote the corresponding Lie algebroids by $\mathcal{L}^{(I)}$, $\mathcal{L}_i$. So $\mathcal{L}_i := \mathcal{O}_X^{I \smallsetminus \{i\}} \boxtimes \mathcal{L}$ and $\mathcal{L}^{(I)} = \prod_{i \in I} \mathcal{L}_i$.

Let $\mathcal{T}^{(I)}$ be the restriction of $\mathcal{G}^I$, considered as a mere formal $X^I \times X^I$-scheme, to $X^I \times X \hookrightarrow X^I \times X^I$, $((x_i), x) \mapsto ((x_i), (x, \dots, x))$. This is a formal $X^I \times X$-scheme equipped with an action of the groupoid $\mathcal{G}^I \times \mathcal{G}$. The action of the subgroupoid $X^I \times \mathcal{G}$ is free; denote by $\mathcal{T}^{(I)}$ the corresponding quotient. This is a formal X^I-scheme equipped with an $\mathcal{L}^{(I)}$-action; let $p^{(I)} : \mathcal{T}^{(I)} \to X^I$ be the projection and $\Delta^{(I)}_{\mathcal{L}} : X \to \mathcal{T}^{(I)}$ the lifting of the diagonal embedding that comes from the "unit" section of the groupoid. We consider $\mathcal{O}_{\mathcal{T}^{(I)}}$ as a topological $\mathcal{O}_{X^I}$-algebra; then $\mathcal{O}_X := \Delta^{(I)}_{\cdot}\mathcal{O}_X = \mathcal{O}_{\mathcal{T}^{(I)}}/\mathcal{J}$, and the topology of $\mathcal{O}_{\mathcal{T}^{(I)}}$ is the $\mathcal{J}$-adic one. Notice also that the $\mathcal{L}^{(I)}$-action identifies $\mathcal{J}/\mathcal{J}^2$ with the $\mathcal{O}_X$-module $\mathcal{L}^I/\mathcal{L}$ (the quotient modulo the image of the diagonal embedding) and $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{J}/\mathcal{J}^2) \xrightarrow{\sim} \Sigma\, \mathcal{J}^a/\mathcal{J}^{a+1}$.

Let $\mathcal{M}^r(\mathcal{T}^{(I)}, \mathcal{L}^{(I)})$ be the category of right $\mathcal{L}^{(I)}$-modules equipped with a compatible discrete $\mathcal{O}_{\mathcal{T}^{(I)}}$-action. For any of its objects the $\mathcal{O}_X$-submodule of sections killed by $\mathcal{J}$ is naturally a right $\mathcal{L}$-module; let $\Delta^{(I)!}_{\mathcal{L}} : \mathcal{M}^r(\mathcal{T}^{(I)}, \mathcal{L}^{(I)}) \to \mathcal{M}^r(X, \mathcal{L})$ be the corresponding functor. It follows easily from Kashiwara's lemma that it is an equivalence of categories; let $\Delta^{(I)}_{\mathcal{L}*} : \mathcal{M}^r(X, \mathcal{L}) \xrightarrow{\sim} \mathcal{M}^r(\mathcal{T}^{(I)}, \mathcal{L}^{(I)})$ be the inverse equivalence.

We leave it to the reader to check that for any $N \in \mathcal{M}^r(X, \mathcal{L})$ the $\mathcal{L}^{(I)}$-module $\Delta^{(I)}_{\mathcal{L}*} N$ equals $N \otimes_{U(\mathcal{L})} \mathcal{B}^*_I$. Thus $P^*_I(\{M_i\}, N) = \mathrm{Hom}_{\mathcal{M}^r(X^I, \mathcal{L}^{(I)})}(\boxtimes M_i, \Delta^{(I)}_{\mathcal{L}*} N)$. The constructions of 2.2.3, 2.2.6, 2.2.7 generalize directly to our setting with the $\Delta^{(I)}_*$ functor replaced by $\Delta^{(I)}_{\mathcal{L}*}$, and we arrive at the same augmented pseudo-tensor structure as above.

2.9.5. Smooth localization of Lie algebroids. Let $\pi : Y \to X$ is a smooth morphism of schemes, $\mathcal{L}$ any Lie algebroid on X. One defines its pull-back $\pi^\dagger(\mathcal{L})$, which is a Lie algebroid on Y, as follows. Let $\Theta_\pi \subset \Theta_Y$ be the subsheaf of vector fields preserving $\pi^{-1}\mathcal{O}_X \subset \mathcal{O}_Y$. It is a Lie $\pi^{-1}\mathcal{O}_X$-algebroid on Y (in the obvious sense) which is an extension of $\pi^{-1}\Theta_X$ by Θ_Y.[127] Set $\pi^\sharp\mathcal{L} := \pi^{-1}\mathcal{L} \underset{\pi^{-1}\Theta_X}{\times} \Theta_\pi$; this is a Lie $\pi^{-1}\mathcal{O}_X$-algebroid which is an extension of $\pi^{-1}\mathcal{L}$ by $\Theta_{Y/X}$. By construction, it acts on $\mathcal{O}_Y$, so $\mathcal{O}_Y \underset{\pi^{-1}\mathcal{O}_X}{\otimes} \pi^\sharp\mathcal{L}$ is a Lie $\mathcal{O}_Y$-algebroid. Finally, our $\pi^\dagger(\mathcal{L})$ is the push-out of $\mathcal{O}_Y \underset{\pi^{-1}\mathcal{O}_X}{\otimes} \pi^\sharp\mathcal{L}$ by the product map $\mathcal{O}_Y \underset{\pi^{-1}\mathcal{O}_X}{\otimes} \Theta_{Y/X} \to \Theta_{Y/X}$. It is a Lie $\mathcal{O}_Y$-algebroid quotient of $\mathcal{O}_Y \underset{\pi^{-1}\mathcal{O}_X}{\otimes} \pi^\sharp\mathcal{L}$. As an $\mathcal{O}_Y$-module, $\pi^\dagger\mathcal{L}$ is an extension of $\pi^*\mathcal{L}$ by $\Theta_{Y/X}$; thus $\pi^\dagger\mathcal{L}$ is $\mathcal{O}_Y$-quasi-coherent if $\mathcal{L}$ is $\mathcal{O}_X$-quasi-coherent. The projection $\mathcal{O}_Y \underset{\pi^{-1}\mathcal{O}_X}{\otimes} \pi^\sharp\mathcal{L} \to \pi^\dagger\mathcal{L}$ identifies $\pi^\sharp\mathcal{L}$ with the normalizer of $\pi^{-1}\mathcal{O}_X \subset \mathcal{O}_Y$.

If M is a left $\mathcal{L}$-module, then its $\mathcal{O}$-module pull-back π^*M is naturally a left $\pi^\dagger\mathcal{L}$-module. Namely, a section $f(\ell,\theta)$ of $\pi^\dagger\mathcal{L}$ (here $f \in \mathcal{O}_Y$, $(\ell,\theta) \in \pi^\sharp\mathcal{L}$) acts on π^*M as $f(\ell,\theta)(gm) = f\theta(g)m + fg\ell(m)$. Similarly, if N is a right $\mathcal{L}$-module, then $\pi^\circ N := \pi^*M \otimes \omega_{Y/X}$ is a right $\pi^\dagger\mathcal{L}$-module; here $\omega_{Y/X}$ is the line bundle det $\Omega_{Y/X}$ on Y. Namely, the action is $(n\nu)(f(\ell,\pi)) = (n\ell)f\nu - nLie_\theta(f\nu)$; here $\nu \in \omega_{X/Y}$, and Lie_θ is the natural ("Lie derivative") action of $\theta \in \Theta_\pi$ on $\omega_{Y/X}$.

One easily checks that the functor $\pi^\dagger$ is compatible with the composition of the π's and satisfies the descent property. The functors π^* and π° for the left and right modules also satisfy the smooth descent property. Therefore Lie algebroids and left/right modules over them are objects of a local nature with respect to smooth topology; hence they make sense on any algebraic stack.

REMARKS. (i) The functor π^* commutes with tensor products, so the category of left $\mathcal{L}$-modules is a tensor category in the stack setting.

(ii) For a Lie algebroid the property of being a locally free $\mathcal{O}$-module of finite rank is local with respect to the smooth topology. Pull-backs π^* and π° are naturally identified by (2.9.3.1), so the picture of 2.9.3 makes sense in the stack setting.

(iii) For π, $\mathcal{L}$ as above there is an evident isomorphism of Lie $\mathcal{O}_Y$-algebras $\pi^*(\mathcal{L}^\diamondsuit) \xrightarrow{\sim} \pi^\dagger(\mathcal{L})^\diamondsuit$ (we assume that $\mathcal{L}^\diamondsuit$ is $\mathcal{O}_X$-quasi-coherent; see Remark (ii) in 2.9.1). Thus a Lie algebroid $\mathcal{L}$ on an algebraic stack $\mathcal{Y}$ defines a Lie $\mathcal{O}_\mathcal{Y}$-algebra $\mathcal{L}^\diamondsuit$.

(iv) Let π be as above and let $\psi : \mathcal{L} \to \mathcal{L}'$ be a morphism of Lie algebroids on X, so we have the morphism $\psi^\dagger : \pi^\dagger(\mathcal{L}) \to \pi^\dagger(\mathcal{L}')$ of Lie algebroids on Y. There are evident isomorphisms of $\mathcal{O}_Y$-modules $\operatorname{Ker}\pi^\dagger(\psi) = \pi^*(\operatorname{Ker}\psi)$ and $\operatorname{Coker}\pi^\dagger(\psi) = \psi^*(\operatorname{Coker}\psi)$. Thus a morphism ψ of Lie algebroids on an algebraic stack $\mathcal{Y}$ yields the $\mathcal{O}$-modules $\operatorname{Ker}\psi$, $\operatorname{Coker}\psi$ on $\mathcal{Y}$.

2.9.6. Let $\mathcal{Y}$ be an algebraic stack. Let $Z \to \mathcal{Y}$ be any smooth surjective morphism where Z is a scheme, so $\mathcal{Y}$ is the quotient stack of Z modulo the action of a smooth groupoid $\mathcal{K} = Z \underset{\mathcal{Y}}{\times} Z$. Denote by $\Theta_{Z/\mathcal{Y}}$ the corresponding Lie algebroid on Z. For any smooth $\pi : Z' \to Z$ there is a natural identification $\Theta_{Z'/\mathcal{Y}} \xrightarrow{\sim} \pi^\dagger\Theta_{Z/\mathcal{Y}}$, so the $\Theta_{Z/\mathcal{Y}}$ form a Lie algebroid $\Theta_{/\mathcal{Y}}$ on $\mathcal{Y}$.

LEMMA. *$\Theta_{/\mathcal{Y}}$ is the initial object of the category of Lie algebroids on $\mathcal{Y}$.*

[127] We use the fact that π is smooth.

Proof. Let $\mathcal{L}$ be a Lie algebroid on $\mathcal{Y}$. We want to show that there is a unique morphism of Lie algebroids

$$\phi : \Theta_{/\mathcal{Y}} \to \mathcal{L}. \tag{2.9.6.1}$$

Uniqueness: For Z as above let $p_1, p_2 : \mathcal{K} \to Z$ be the structure projections, $e : Z \to \mathcal{K}$ the "unit" section of $\mathcal{K}$ over the diagonal $Z \hookrightarrow Z \times Z$. Denote by $\Theta^{(1)}_{\mathcal{K}/Z}$, $\Theta^{(2)}_{\mathcal{K}/Z}$ the tangent bundle along the fibers of p_1, p_2.

One has canonical embeddings $\Theta^{(1)}_{\mathcal{K}/Z} \hookrightarrow p_1^\dagger \Theta_{Z/\mathcal{Y}} = \Theta_{\mathcal{K}/Y}$, $\Theta^{(2)}_{\mathcal{K}/Z} \hookrightarrow p_2^\dagger \Theta_{Z/\mathcal{Y}} = \Theta_{\mathcal{K}/Y}$, and one checks easily that $\Theta^{(1)}_{\mathcal{K}/Z} \oplus \Theta^{(2)}_{\mathcal{K}/Z} \xrightarrow{\sim} \Theta_{\mathcal{K}/Y}$. Since $\phi_{\mathcal{K}} = p_1^\dagger \phi_Z$, the restriction of $\phi_{\mathcal{K}}$ to $\Theta^{(1)}_{\mathcal{K}/Z}$ coincides with the canonical embedding $\Theta^{(1)}_{\mathcal{K}/Z} \hookrightarrow p_1^\dagger \mathcal{L}_Z$. The same is true for the restriction of $\phi_{\mathcal{K}}$ to $\Theta^{(2)}_{\mathcal{K}/Z}$. Thus $\phi_{\mathcal{K}}$ is uniquely defined, hence ϕ is.

Existence: Consider a canonical embedding $\Theta^{(1)}_{\mathcal{K}/Z} \hookrightarrow p_2^\dagger \mathcal{L}_Z = \mathcal{L}_{\mathcal{K}}$ and a projection $\mathcal{L}_{\mathcal{K}} = p_1^\dagger \mathcal{L}_Z \twoheadrightarrow p_1^* \mathcal{L}_Z$. Pulling the composition back by e, we get a morphism of $\mathcal{O}_Z$-modules $\phi_Z : \Theta_{Z/\mathcal{Y}} := e^* \Theta^{(1)}_{Z/\mathcal{Y}} \to \mathcal{L}_Z$. We leave it to the reader to check that ϕ_Z comprize a morphism $\phi : \Theta_{/\mathcal{Y}} \to \mathcal{L}$ of Lie algebroids on $\mathcal{Y}$. □

EXERCISE. Suppose that $\mathcal{Y}$ is covered by a point $\cdot$, so $\mathcal{Y} = B_K$ where K is an algebraic group. Set $\mathfrak{k} := \mathrm{Lie} K$. Consider the category $\mathcal{HC}_K$ of triples $(\mathfrak{g}, \phi, \alpha)$ where $\mathfrak{g}$ is a Lie algebra, $\phi : \mathfrak{k} \to \mathfrak{g}$ is a morphism of Lie algebras, α is a K-action on $\mathfrak{g}$ (as on a Lie algebra) such that the $\mathfrak{k}$-action coming from α equals ad_ϕ. One has a functor

$$\mathcal{L}ie\mathcal{A}lg_{B_K} \to \mathcal{HC}_K, \quad \mathcal{L} \mapsto (\mathfrak{l}, \phi, \alpha), \tag{2.9.6.2}$$

where $\mathfrak{l}$ is the Lie algebra $\mathcal{L}_\cdot$, $\phi : \mathfrak{k} \to \mathfrak{l}$ is the morphism (2.9.6.1) over $\cdot$, and α arises since we have a Lie $\mathcal{O}$-algebra $\mathcal{L}^\diamondsuit$ on B_K which equals $\mathfrak{l}$ over the covering $\cdot$ (see Remark (iii) in 2.9.5). Show that (2.9.6.2) is an equivalence of categories.

2.9.7. The setting of Harish-Chandra modules. The compound pseudotensor structure from 2.9.4 is a particular case of a canonical compound structure on the category of $\mathcal{O}$-modules on a c-stack. We will sketch the general picture in 2.9.10–2.9.12 below. Prior to this, let us consider another particular case of the c-stack setting which is especially relevant for applications.

There are two equivalent ways to describe the notion of a Harish-Chandra pair:

(i) We consider a reasonable ind-affine group formal scheme G. Here the formal scheme is the same as an ind-scheme such that G_{red} is a scheme; "reasonable" (see 2.4.8) amounts to the property that for any subscheme $G_{red} \subset P \subset G$ the ideal of G_{red} in $\mathcal{O}_P$ is finitely generated.

(ii) We consider a collection $(\mathfrak{g}, K) = (\mathfrak{g}, K, \phi, \alpha)$ where K is an affine group scheme, $\mathfrak{g}$ a Lie algebra which is a Tate vector space, $\phi : \mathfrak{k} := \mathrm{Lie}\, K \hookrightarrow \mathfrak{g}$ a continuous embedding of Lie algebras with an open image,[128] and α a K-action on $\mathfrak{g}$ (as on a Lie algebra). We demand that the $\mathfrak{k}$-action on $\mathfrak{g}$ coming from α is equal to ad_ϕ.

Notice that for any G as in (i) the pair $(\mathfrak{g}, G_{red})$ with obvious ϕ and α fits (ii). One can show that this functor from the category of G's as in (i) to that of $(\mathfrak{g}, K)$'s as in (ii) is an equivalence of categories.

[128] Notice that $\mathfrak{k}$ is naturally a profinite-dimensional Lie algebra.

For G as in (i) a G-*module* is a vector space V equipped with a G-action.[129] In the setting of (ii), a $(\mathfrak{g}, K)$-*module* is a (discrete) vector space equipped with a K-action and a $\mathfrak{g}$-action which are compatible in the obvious manner. We denote the corresponding categories by $\mathcal{M}(G)$ and $\mathcal{M}(\mathfrak{g}, K)$; these are tensor categories in the obvious way. One checks that G-modules are the same as $(\mathfrak{g}, G_{red})$-modules.

Proposition. *If* $\dim \mathfrak{g}/\mathfrak{k} < \infty$, *then* $\mathcal{M}(\mathfrak{g}, K)$ *is naturally an abelian augmented compound tensor category.*

Sketch of a proof. We already have the tensor product. Let us construct the $*$ operations. Set $n := \dim \mathfrak{g}/\mathfrak{k}$.

The G-module picture is convenient here. Write $G = \operatorname{Spf} F$ where F is a topological algebra. For $I \in \mathcal{S}$ set $F^{(I)} := (F^{\hat{\otimes} I})^G$, so $\operatorname{Spf} F^{(I)} = G^I/G$, where we consider the diagonal right translation action of G on G^I. Notice that G^I/G is a formally smooth formal scheme whose reduced scheme K^I/K has codimension $(|I|-1)n$. Set $E^{(I)} := \varinjlim \operatorname{Ext}^{(|I|-1)n}_{F^{(I)}}(F^{(I)}/\mathcal{J}, F^{(I)})$ where $\mathcal{J}$ runs the set of all open ideals in $F^{(I)}$. This is a discrete $F^{(I)}$-module equivariant with respect to the left translation action of G^I on G^I/G.

For a G-module V let $\hat{\Delta}_* N := (\mathcal{O}_{G^I} \hat{\otimes} N)^G$ be the topological vector space of N-valued functions on G^I equivariant with respect to the diagonal right G-translations on G^I. This is a topological $F^{(I)}$-module equivariant with respect to the G^I-action, and $\hat{\Delta}_* N = \varprojlim \hat{\Delta}_* N/\mathcal{J}\hat{\Delta}_* N$. Set $\Delta_*^{(I)} N := E^{(I)} \underset{F^{(I)}}{\otimes} \hat{\Delta}_* N = \varinjlim \operatorname{Ext}^{(|I|-1)n}_{F^{(I)}}(F^{(I)}/\mathcal{J}, \hat{\Delta}_*^{(I)} N)$; this is a discrete G^I-equivariant $F^{(I)}$-module. The functor $\Delta_*^{(I)} : \mathcal{M}(G) \to \mathcal{M}^{\delta}_{G^I}(G^I/G) :=$ the category of discrete G^I-equivariant $F^{(I)}$-modules is actually an equivalence of categories; the inverse functor $P \mapsto \operatorname{Tor}_{(|I|-1)n}(k_e, P)$ where k_e is the skyscraper sheaf at the distinguished point $e \in G^I/G$ fixed by the action of $G \subset G^I$.

Now, for a collection $\{M_i\}$, $i \in I$, of G-modules we set

$$P_I^*(\{M_i\}, N) := \operatorname{Hom}_{G^I}(\otimes M_i, \Delta_*^{(I)} N) \otimes \lambda_I^{\otimes n} \tag{2.9.7.1}$$

where λ_I is as in 2.2.2. Equivalently, a $*$ operation is a morphism of G^I-equivariant $F^{(I)}$-modules $(\otimes M_i) \otimes F^{(I)} \to \Delta_*^{(I)} N \otimes \lambda_I^{\otimes n}$. The composition of $*$ operations comes, as in the case of $\mathcal{D}$-modules, from natural identifications $\Delta_*^{(J)} N = \Delta_*^{(J/I)} \Delta_*^{(I)} N$ (here $J \twoheadrightarrow I$ is a surjection) where $\Delta_*^{(J/I)} : \mathcal{M}(G^I) \to \mathcal{M}(G^J)$ is a functor defined in the same manner as above (using the space G^J/G^I).

The compound tensor product maps $\otimes^I_{S,T}$ (see 1.3.12) are defined as in the $\mathcal{D}$-module case using restrictions to the diagonal $G^T/G \subset G^I/G$.

The augmentation functor is $h(N) := \operatorname{Ext}^n(k, N)$ where k is the trivial G-module, and Ext is computed in $\mathcal{M}(G)$. The definition of the compatibility morphisms is left to the reader. □

2.9.8. $(\mathfrak{g}, K)$**-structures.** One uses the above picture as follows (this is an example of c-functoriality; see 2.9.12).

Let X be a smooth variety, $(\mathfrak{g}, K)$ a Harish-Chandra pair, G the corresponding group formal scheme. A $(\mathfrak{g}, K)$-*structure* on X is a scheme Y equipped with a

[129] Which is the linear action of the group valued functor $R \mapsto G(R)$ on the functor $R \mapsto R \otimes V$ and R is a test commutative algebra.

projection $\pi : Y \to X$ and a $(\mathfrak{g}, K)$-action (which is the same as a G-action) such that

(i) K acts along the fibers of π and makes Y a K-torsor over X.

(ii) The action of $\mathfrak{g}$ is formally simply transitive.

Notice that (ii) can be replaced by

(ii)′ The morphism $(\mathfrak{g}/\mathfrak{k}) \otimes \mathcal{O}_Y \to \pi^*\Theta_X$ defined by the $\mathfrak{g}$-action and $d\pi$, is an isomorphism.

Notice that for any left $\mathcal{D}_X$-module its pull-back to Y is a G-equivariant $\mathcal{O}_Y$-module, and this functor is an equivalence of categories between $\mathcal{M}^\ell(X)$ and the category of G-equivariant (quasi-coherent) $\mathcal{O}$-modules on Y. Any G-module V yields a G-equivariant $\mathcal{O}_Y$-module $V \otimes \mathcal{O}_Y$, so we have defined an exact faithful tensor functor $\mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}^\ell(X)$, $V \mapsto V^{(Y)}$.

Here is a more explicit definition of $V^{(I)}$. Our Y defines a G-torsor $\hat{Y}$ on X equipped with a flat connection. Namely, let $\hat{X} \subset X \times X$ be the formal completion at the diagonal. Then $\tilde{Y}$ is the pull-back of Y by the first projection $\hat{X} \to X$ considered as an X-scheme via the second projection $\hat{X} \to X$; the G-action on $\hat{Y}$ comes from the G-action on Y and the connection is the derivation along the second variable in $\hat{X}$. Now $V^{(Y)}$ is simply the $\hat{Y}$-twist of V. One has $Y \subset \hat{Y}$; i.e., as a mere G-torsor (we forget about the connection) our $\hat{Y}$ is the G-torsor induced from the K-torsor Y. Therefore as a mere $\mathcal{O}_X$-module $V^{(Y)}$ is the twist of V by the K-torsor Y.

Now the above functor extends to an augmented compound tensor functor

$$\mathcal{M}(\mathfrak{g}, K)^{*!} \to \mathcal{M}(X)^{*!}. \tag{2.9.8.1}$$

Our functor is evidently a tensor functor. Let us explain the how it transforms the $*$ operations; the compatibility with compound tensor products is defined in a similar way. For $I \in \mathcal{S}$ we have the $(\mathfrak{g}, K)^I$-structure Y^I on X^I. Let $\hat{X}^{(I)}$ be the formal completion of X^I at the diagonal and $\hat{Y}^{(I)}$ the pull-back of Y^I to $\hat{X}^I$. One has the diagonal embedding $Y \hookrightarrow \hat{Y}^{(I)}$, and the G^I-action on $Y^{(I)}$ yields an isomorphism $(G^I \times Y)/G \xrightarrow{\sim} Y^{(I)}$. Thus we have an affine projection $\mathfrak{p} : Y^{(I)} \to G^I/G$. Therefore for a G-module V we have a G^I-equivariant "discrete" $\mathcal{O}$-module $\mathfrak{p}^*\Delta_*^{(I)}V$ on the formal scheme $\hat{Y}^{(I)}$. Its K^I-invariant sections form a discrete $\mathcal{O}$-module on $\hat{X}^{(I)}$ which is naturally a left $\mathcal{D}$-module, i.e., a left $\mathcal{D}$-module Q on X^I supported at the diagonal. One checks immediately that $Q = (\Delta_*^{(I)}(V^{(Y)})^r)^\ell$. Now it is clear that a $*$ operation $\in P_I^*(\{M_i\}, V)$ in $\mathcal{M}(G)$ yields a morphism of left $\mathcal{D}_{X^I}$-modules $\boxtimes M_i^{(Y)} \to Q \otimes \lambda^{\otimes n}$, i.e., a $*$ operation $\in P_I^*(\{M_i^{(Y)r}\}, V^{(Y)r})$, and we are done.

Remark. Instead of a single X equipped with a $(\mathfrak{g}, K)$-structure, we can consider smooth families X/S of varieties equipped with a fiberwise $(\mathfrak{g}, K)$-structure, so (2.9.10.2) becomes a functor with values in the corresponding fiberwise $\mathcal{D}$-modules. Suppose we have a morphism of smooth families $f : X'/S' \to X/S$ which is fiberwise étale. Then for any $(\mathfrak{g}, K)$-structure Y on X/S its pull-back $Y' := Y \underset{X}{\times} X'$ is naturally a $(\mathfrak{g}, K)$-structure on X', and for $V \in \mathcal{M}(\mathfrak{g}, K)$ one has $V^{(Y')} = f^*V^{(Y)}$.

Exercise. Let $G^\sim$ be a group scheme which contains K and its formal completion at K is identified with G (so $\mathfrak{g}$ is the Lie algebra of $G^\sim$). Then $X := K \diagdown G^\sim$ is equipped with an evident $(\mathfrak{g}, K)$-structure given by the projection $Y = G^\sim \to X$.

Show that for each $(\mathfrak{g}, K)$-module V the corresponding $\mathcal{D}$-module $V^{(Y)}$ on X is naturally weakly $G^{\sim}$-equivariant[130] with respect to the right $G^{\sim}$-action on X, and the functor $V \mapsto V^{(Y)}$ establishes an equivalence between $\mathcal{M}(\mathfrak{g}, K)$ and the augmented compound pseudo-tensor category of weakly $G^{\sim}$-equivariant $\mathcal{D}$-modules on X.

For example, for $G^{\sim} = \mathbb{G}_a^n$, $K = 1$, we get an equivalence between the augmented compound pseudo-tensor category of $k[\partial_1, \ldots, \partial_n]$-modules and that of the $\mathcal{D}$-modules on $\mathbb{A}^n$ weakly equivariant with respect to translations.

2.9.9. The Gelfand-Kazhdan structure. Set $O_n := k[\![t_1, \ldots, t_n]\!]$. One has a group formal scheme $G :=\operatorname{Aut} O_n$: for a commutative algebra R an R-point of G is a continuous automorphism g of $R\hat{\otimes} O_n = R[\![t_1, \ldots, t_n]\!]$ over R. Then $\mathfrak{g} := \operatorname{Lie} G = \operatorname{Der}_k O_n$, and an R-point of G lies in $K := G_{red}$ if and only if g preserves the ideal generated by $t_1, \ldots, t_n$.

According to [**GeK**], each smooth variety X of dimension n admits a *canonical* $(\mathfrak{g}, K)$-structure Y_{GK} called the *Gelfand-Kazhdan structure*. Namely, Y_{GK} is the space of all formal coordinate systems on X; i.e., an R-point of Y_{GK} is an $R\hat{\otimes} O_n$-point ξ of X such that $d\xi : \Theta_{R\hat{\otimes} O_n/R} \xrightarrow{\sim} \xi^* \Theta_X$. The group G acts on Y_{GK} by transport of structure; the axioms of the $(\mathfrak{g}, K)$-structure are immediate.

As in Remark in 2.9.8, we can consider families of smooth n-dimensional varieties, and the Gelfand-Kazhdan structure is functorial with respect to fiberwise étale morphisms of families.

A *universal left $\mathcal{D}$-module* (in dimension n) is a rule M which assigns to every smooth family X/S of fiberwise dimension n an $\mathcal{O}_X$-quasi-coherent left $\mathcal{D}_{X/S}$-module $M_{X/S}$ on X and to every fiberwise étale morphisms of families $f : X'/S' \to X/S$ an identification $M_{X'/S'} \xrightarrow{\sim} f^* M_{X/S}$; the obvious compatibility with composition of the f's should hold. Universal $\mathcal{D}$-modules form an abelian augmented compound pseudo-tensor category $\mathcal{M}_n^{*!}$. By 2.9.8, the Gelfand-Kazhdan structure yields an augmented compound pseudo-tensor functor

$$\mathcal{M}(\operatorname{Aut} O_n)^{*!} = \mathcal{M}(\mathfrak{g}, K)^{*!} \to \mathcal{M}_n^{*!}. \tag{2.9.9.1}$$

PROPOSITION. *This is an equivalence of augmented compound pseudo-tensor categories.*

Proof. Left to the reader. □

EXERCISE. Define the category of universal $\mathcal{O}$-modules in dimension n and show that it is canonically equivalent, as a tensor category, to the category of K-modules where K is as above.

One can use $(\mathfrak{g}, K)$-modules to deal with "non-universal" $\mathcal{D}$-modules as well. Namely, let X be an *affine* smooth variety of dimension n. So Y_{GK} is also affine. Then $A := \Gamma(Y_{GK}, \mathcal{O}_{Y_{GK}})$ carries a $(\mathfrak{g}, K)$-action; i.e., it is a commutative$^!$ algebra in $(\mathfrak{g}, K)$mod. As in 1.4.6, we have the augmented compound pseudo-tensor category $\mathcal{M}(A)^{*!}$ of A-modules equipped with a compatible $(\mathfrak{g}, K)$-action. For a left $\mathcal{D}$-module M on X, set $\Gamma_{GK}(M) := \Gamma(Y_{GK}, \pi^* M)$. This is an object of $\mathcal{M}(A)$ in the

[130] Recall (see, e.g., [**BB**] or [**Kas1**] where the term "quasi-equivariant" is used) that a weakly $G^{\sim}$-equivariant $\mathcal{D}$-module on X is a $\mathcal{D}$-module M together with a lifting of the $G^{\sim}$-action on X to M considered as an $\mathcal{O}_X$-module so that for every $g \in G$ the action of the corresponding translation on M is compatible with the action of $\mathcal{D}_X$.

evident way; since Y_{GK} is affine, we get an equivalence of categories

$$\Gamma_{GK} : \mathcal{M}^{\ell}(X) \xrightarrow{\sim} \mathcal{M}(A). \tag{2.9.9.2}$$

EXERCISES. (i) The composition of (2.9.9.2) with the forgetful functor $\mathcal{M}(A) \to \mathcal{M}(\mathfrak{g}, K)$ is right adjoint to the functor $\mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}(X)$, $V \mapsto V^{(Y_{GK})}$.

(ii) Show that (2.9.9.2) extends naturally to an augmented compound pseudo-tensor equivalence.

2.9.10. C-stacks. The rest of the section is a brief sketch of a general setting which includes the settings of 2.9.4 and 2.9.7.

We need a dictionary.

(a) We play with sheaves and (1-)stacks on the category of affine k-schemes equipped with the fpqc topology, calling them simply *spaces* and *stacks*. Schemes, formal schemes, etc., are identified with the corresponding spaces, and spaces with stacks having trivial "inner symmetries".

A morphism of stacks $f : \mathcal{P} \to \mathcal{Q}$ is said to be *space* morphism if for every morphism $S \to \mathcal{Q}$ where S is a space, the fibered product $\mathcal{P} \underset{\mathcal{Q}}{\times} S$ is a space. It suffices to check this for affine schemes S. If $\mathcal{P}$ is a space, then f is a space morphism.

(b) Suppose one has a class of morphisms of spaces whose target is a scheme which is stable by the base change. We say that a space morphism $f : \mathcal{P} \to \mathcal{Q}$ belongs to our class if for every morphism $S \to \mathcal{Q}$ where S is a scheme, the morphism $\mathcal{P} \underset{\mathcal{Q}}{\times} S \to S$ is there.

E.g., we have *schematic* morphisms coming from the class of morphisms whose source is a scheme, flat schematic morphisms, quasi-compact schematic morphisms, *fpqc coverings* := surjective flat quasi-compact schematic morphisms, etc.

Suppose, in addition, that our class is stable under composition with flat quasi-compact schematic morphisms from the right.[131] We say that f as above belongs to our class *fpqc locally* if for every morphism $S \to \mathcal{Q}$ where S is a scheme, there exists an fpqc covering $K \to \mathcal{P} \underset{\mathcal{Q}}{\times} S$ such that the composition $K \to \mathcal{P} \underset{\mathcal{Q}}{\times} S \to S$ is in the class.

(c) A formal scheme F is said to be *smooth in formal directions of rank n* if there exists an open ideal $\mathcal{J} \subset \mathcal{O}_F$ such that the topology of $\mathcal{O}_F$ is $\mathcal{J}$-adic, $\mathcal{J}/\mathcal{J}^2$ is a locally free $\mathcal{O}_F/\mathcal{J}$-module of finite rank n, and $\mathrm{Sym}_{\mathcal{O}_F/\mathcal{J}}(\mathcal{J}/\mathcal{J}^2) \xrightarrow{\sim} \Sigma \mathcal{J}^a/\mathcal{J}^{a+1}$. One checks that n does not depend on the choice of $\mathcal{J}$.

Consider the class of morphisms $F \to Z$ where Z is a scheme, F a formal scheme smooth in formal directions of rank n, and $\mathcal{J}$ as above can be chosen so that $\mathrm{Spec}\, \mathcal{O}_F/\mathcal{J}$ is flat and quasi-compact over Z. It satisfies both conditions of (b). We say that a space morphism f of stacks is a *c-morphism of c-rank n* if it belongs fpqc locally to our class. If, in addition, f is surjective, then it is called a *c-covering of c-rank n*.

DEFINITION. A *c-stack of c-rank n* is a stack which admits a c-covering of c-rank n by a scheme. A morphism $\mathcal{P} \to \mathcal{Q}$ of c-stacks is called *c-morphism* if for any c-covering $S \to \mathcal{Q}$ where S is a scheme, there exists a surjective flat quasi-compact schematic morphism $T \to \mathcal{P} \underset{\mathcal{Q}}{\times} S$ where T is a scheme.

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[131] I.e., for $Y \xrightarrow{g} Z \xrightarrow{f} T$ such that g is a flat quasi-compact schematic morphism and f in the class, the composition fg is in the class.

Thus c-stacks are the same as quotient stacks $S/\mathcal{G}$ where S is a scheme and $\mathcal{G}$ a groupoid on S such that either of the projections $\mathcal{G} \to S$ is a c-covering.

REMARKS. (i) In the definition of c-morphism it suffices to consider a single c-covering $S \to \mathcal{Q}$.

(ii) c-morphisms exist only between c-stacks of the same c-rank.

EXAMPLES. (o) Algebraic stacks are c-stacks of c-rank 0.

(i) In the situation of 2.9.3 the quotient $Y/\mathcal{G}_{\mathcal{L}}$ of Y modulo the action of the formal groupoid $\mathcal{G}_{\mathcal{L}}$ defined by $\mathcal{L}$ is a c-stack of c-rank equal to the rank of $\mathcal{L}$ as an $\mathcal{O}_Y$-module.

(ii) For G as in 2.9.7 with $\dim \mathfrak{g}/\mathfrak{k} < \infty$ the classifying stack B_G is a c-stack of c-rank equal to $\dim \mathfrak{g}/\mathfrak{k}$.

Here is a common generalization of Examples (i) and (ii) (for $\dim G < \infty$):

2.9.11. Let $\mathcal{Y}$ be an algebraic stack and $\mathcal{L}$ a Lie algebroid on $\mathcal{Y}$. Suppose that $\operatorname{Ker} \phi = 0$ and $\operatorname{Coker} \phi$ is a locally free $\mathcal{O}_{\mathcal{Y}}$-module of rank n; here ϕ is the canonical morphism (2.9.6.1).

PROPOSITION. *Such a $(\mathcal{Y}, \mathcal{L})$ yields a c-stack $\mathcal{Q} = \mathcal{Y}/\mathcal{L}$ of c-rank n equipped with a morphism $\mathcal{Y} \to \mathcal{Q}$.*

Proof. Let $Z \to \mathcal{Y}$ a smooth covering where Z is a scheme, so $\mathcal{Y}$ is the quotient of Z modulo the action of the smooth groupoid $\mathcal{K} := Z \underset{\mathcal{Y}}{\times} Z$. We have the Lie algebroid $\mathcal{L}_Z$ on Z; let $\mathcal{G}^{\hat{}}$ be the corresponding formal groupoid. By our condition, $\mathcal{G}^{\hat{}}$ contains the formal completion[132] $\mathcal{K}^{\hat{}}$ of $\mathcal{K}$ (which is the formal groupoid of $\Theta_{Z/\mathcal{Y}}$).

We will embed $\mathcal{K}$ into a larger formal groupoid $\mathcal{G}$ on Y whose formal completion equals $\mathcal{G}^{\hat{}}$, and we will define Q as the quotient stack $Z/\mathcal{G}$.

Let $\mathcal{G}^\ell$ be the quotient of $\mathcal{G}^{\hat{}} \underset{Z}{\times} \mathcal{K}$ modulo the relation $(gr, k) = (g, rk)$ where $g \in \mathcal{G}^{\hat{}}$, $r \in \mathcal{K}^{\hat{}}$, $k \in \mathcal{K}$, and the two products are well defined. Our condition assures that $\mathcal{G}^\ell$ is a formal scheme over $Z \times Z$. Denote by $\mathcal{G}^r$ the quotient of $\mathcal{K} \underset{Z}{\times} \mathcal{G}^{\hat{}}$ modulo a similar relation. We write the points of $\mathcal{G}^\ell$, $\mathcal{G}^r$ as, respectively, $g \cdot k$, $k' \cdot g'$.

There is a canonical isomorphism $\nu : \mathcal{G}^\ell \xrightarrow{\sim} \mathcal{G}^r$ of formal $Z \times Z$-schemes defined as follows.

Consider the Lie algebroid $\mathcal{L}_{\mathcal{K}}$ on $\mathcal{K}$; let $\mathcal{H}$ be its formal groupoid. Since $\mathcal{L}_{\mathcal{K}} = p_1^\dagger \mathcal{L}_Z = p_2^\dagger \mathcal{L}_Z$, $\mathcal{H}$ coincides with the formal completion of the pull-back of $\mathcal{G}^{\hat{}}$ either by p_1 or by p_2. Therefore a point of $\mathcal{H}$ over a pair of infinitely close points $(k, k') \in \mathcal{K} \times \mathcal{K}$ can be written as either a point $g^\ell \in \mathcal{G}^{\hat{}}$ over $(p_1(k), p_1(k'))$ or $g^r \in \mathcal{G}^{\hat{}}$ over $(p_1(k), p_1(k'))$. Now ν identifies $g^\ell \cdot k$ with $k' \cdot g^r$. One easily checks that ν is well defined.

Our $\mathcal{G}$ equals $\mathcal{G}^\ell$ or $\mathcal{G}^r$ identified by ν. The groupoid structure on $\mathcal{G}$ is defined using ν. The details (such as the independence of the construction of the choice of Z) are left to the reader. □

2.9.12. Recall that for a stack $\mathcal{Q}$ an $\mathcal{O}$-*module on* $\mathcal{Q}$ is a rule N that assigns to any $f \in \mathcal{Q}(S)$, S a scheme, a quasi-coherent $\mathcal{O}_S$-module f^*N, to any arrow $f \to f'$ in $\mathcal{Q}(S)$ an isomorphism $f^*N \xrightarrow{\sim} f'^*N$, and to any $g : S' \to S$ an isomorphism $(fg)^*N \xrightarrow{\sim} g^*f^*N$, so that the evident compatibilities are satisfied. The category of $\mathcal{O}$-modules on $\mathcal{Q}$ is denoted by $\mathcal{M}_{\mathcal{O}}(\mathcal{Q})$. This is a tensor category.

[132] I.e., the formal completion at the "identity", which is a formal groupoid.

For example, for $\mathcal{Q} = \mathcal{Y}/\mathcal{L}$ as in 2.9.11, one has $\mathcal{M}_{\mathcal{O}}(\mathcal{Q}) = \mathcal{M}^{\ell}(\mathcal{Y}, \mathcal{L})$ (see 2.9.5). In the situation of Example (ii) of 2.9.10, this is the category of $(\mathfrak{g}, K)$-modules. One has the following generalization of 2.9.4:

PROPOSITION. *If* $\mathcal{Q}$ *is a c-stack, then* $\mathcal{M}(\mathcal{Q})$ *is naturally an abelian augmented compound pseudo-tensor category. The pull-back functor for a c-morphism of c-stacks is naturally an augmented pseudo-tensor functor.* □

The particular cases corresponding to Examples (i) and (ii) from 2.9.10 were considered in 2.9.4 and 2.9.7. The construction in the general situation is essentially a combination of those from 2.9.4 and 2.9.7, and we omit it.

REMARK. Functoriality (2.9.8.1) is a particular case of functoriality with respect to c-morphisms of c-stacks. Namely, for our smooth variety X consider the c-stack X^{-} which is the quotient of X modulo the formal groupoid $\mathcal{G}$ generated by Θ_X; i.e., $\mathcal{G}$ = the formal completion of $X \times X$ at the diagonal. The category $\mathcal{M}_{\mathcal{O}}(X^{-})$ is the category of left $\mathcal{D}$-modules $\mathcal{M}^{\ell}(X)$. Now for a $(\mathfrak{g}, K)$-structure Y the projection $G \backslash Y \to X^{-}$ defined by π is an equivalence of c-stacks, and (2.9.8.1) is the composition of the pull-back functors for the c-morphisms $X^{-} \stackrel{\sim}{\leftarrow} G \backslash Y \to B_G$.