CHAPTER 1

Axiomatic Patterns

Да поразит тя пуще грома
Ужасна, сильна аксиома.

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Харьковский Демокрит, 1816 †

1.1. Pseudo-tensor categories

The notion of pseudo-tensor category is an immediate generalization of that of operad in the same way that the notion of category generalizes that of monoid. On the other hand, the pseudo-tensor structure appears naturally on any (not necessary closed under the tensor product) subcategory of a tensor category,[1] hence the adjective "pseudo-tensor". For example, you can play with Lie algebras and their representations, but the enveloping algebras are out of reach.

Pseudo-tensor categories were considered sporadically and under various names for quite a while; see [**La**] ("multicategories") or [**Li**], [**B2**] ("multilinear categories"). Operads were introduced by P. May [**May**] in the topological context; today they are very popular: see, e.g., [**GK**], [**H**], [**Kap1**], [**KrM**], [**L**], [**J**], and [**MSS**].

Pseudo-tensor categories are defined in 1.1.1, pseudo-tensor functors and adjunction are in 1.1.5, examples are in 1.1.6, the linear setting is considered in 1.1.7, the tensor product of pseudo-tensor categories is defined in 1.1.9. A funny lemma in 1.1.10 will be used later to identify commutative chiral algebras with commutative $\mathcal{D}_X$-algebras; its explanation in 1.1.11 is due to V. Ginzburg. In 1.1.13 we explain that a pseudo-tensor structure on the category of R-modules, R an associative algebra, can be naturally defined the moment one knows the operad of endooperations of R. In 1.1.16 we fix "super" conventions.

1.1.1. Denote by $\mathcal{S}$ the category of finite non-empty sets and surjective maps. For a morphism $\pi\colon J \twoheadrightarrow I$ in $\mathcal{S}$ and $i \in I$ set $J_i := \pi^{-1}(i) \subset J$; let $\cdot \in \mathcal{S}$ be the one element set.

Definition. A *pseudo-tensor category* is a class of objects $\mathcal{M}$ together with the following datum:

† Let thee be smitten as by a thunderbolt
By a dreadful, potent axiom.
A. A. Nakhimov, "A Poet and a Mathematician",
Kharkovski Democritus, 1816

[1] In fact, every pseudo-tensor category appears in this manner in a universal way; see Remark in 1.1.6(i).

(a) For any $I \in \mathcal{S}$, an I-family of objects $L_i \in \mathcal{M}$, $i \in I$, and an object $M \in \mathcal{M}$ one has the set $P_I^{\mathcal{M}}(\{L_i\}, M) = P_I(\{L_i\}, M)$; its elements are called I-operations.

(b) For any map $\pi\colon J \twoheadrightarrow I$ in $\mathcal{S}$, families of objects $\{L_i\}_{i\in I}$, $\{K_j\}_{j\in J}$, and an object M one has the composition map

$$P_I(\{L_i\}, M) \times \prod_I P_{J_i}(\{K_j\}, L_i) \to P_J(\{K_j\}, M), \quad (\varphi, (\psi_i)) \mapsto \varphi(\psi_i). \tag{1.1.1.1}$$

Here in the notation $P_{J_i}(\{K_j\}, L_i)$ we assume that $j \in J_i$.

The following properties should hold:

(i) The composition is associative: if $H \twoheadrightarrow J$ is another surjective map, $\{F_h\}$ an H-family of objects, $\chi_j \in P_{H_j}(\{F_h\}, K_j)$, then $\varphi(\psi_i(\chi_j)) = (\varphi(\psi_i))(\chi_j) \in P_H(\{F_h\}, M)$.

(ii) For any $M \in \mathcal{M}$ there is an element $id_M \in P_\cdot(\{M\}, M)$ such that for any $\varphi \in P_I(\{L_i\}, M)$ one has $id_M(\varphi) = \varphi(id_{L_i}) = \varphi$.

We often use notation $P_{J/I}(\{K_j\}, \{L_i\}) = P_\pi(\{K_j\}, \{L_i\})$ for $\prod_I P_{J_i}(\{K_j\}, L_i)$ to write the composition map as $P_I \times P_{J/I} \to P_J$. We also write $P_n := P_{\{1,\dots,n\}}$.

We denote our pseudo-tensor category simply by $\mathcal{M}$ by abuse of notation.

1.1.2. Let $\mathcal{M}$ be a pseudo-tensor category. For $L, M \in \mathcal{M}$ set $\mathrm{Hom}(L, M) = P_1(\{L\}, M)$. Composition of morphisms comes from (b), so we have a usual category with the same objects as $\mathcal{M}$; we denote it also by $\mathcal{M}$. The composition with morphisms makes P_I a functor $(\mathcal{M}^\circ)^I \times \mathcal{M} \to \mathit{Sets}$. Therefore, we may consider the pseudo-tensor categories as usual categories equipped with extra *pseudo-tensor structure* formed by a collection of functors P_I, $|I| \geq 2$, together with composition morphisms.

Every category admits a trivial pseudo-tensor structure with $P_I = \emptyset$ for $|I| > 1$.

1.1.3. Assume that a family $\{L_i\}_{i\in I}$ has the property that the functor $M \mapsto P_I(\{L_i\}, M)$ is representable. We denote the corresponding object of $\mathcal{M}$ as $\underset{I}{\otimes} L_i$ and call it the *pseudo-tensor product* of L_i's. If the pseudo-tensor product exists for any family of objects, then we call our pseudo-tensor structure *representable*. In other words, a representable pseudo-tensor structure on a category $\mathcal{M}$ is a rule that assigns to any $I \in \mathcal{S}$ a functor $\underset{I}{\otimes}\colon \mathcal{M}^I \to \mathcal{M}$ and to any map $\pi\colon J \twoheadrightarrow I$ a natural compatibility morphism

$$\varepsilon_\pi : \underset{J}{\otimes} K_j \to \underset{I}{\otimes}(\underset{J_i}{\otimes} K_j). \tag{1.1.3.1}$$

The morphisms ε_π must behave naturally with respect to composition of the maps π, and $\underset{\cdot}{\otimes}$ is the identity functor.

If all the compatibility morphisms ε_π are isomorphisms, then we arrive at the usual notion of tensor (= symmetric monoidal) category.

1.1.4. A pseudo-tensor category $\mathcal{B}$ with single object is the same as an *operad*. Namely, such $\mathcal{B}$ amounts to a collection of sets $\mathcal{B}_n = P_n^{\mathcal{B}}$, $n \geq 1$, equipped with the action of the symmetric group Σ_n, together with appropriate "composition of operations" maps between them, and our axioms coincide here with those of an operad.

EXAMPLES. (i) The *operad* Σ *of linear orders*: $\Sigma_I = \{\text{linear orders on } I\}$; the composition is defined by the lexicographic rule. Notice that an operad over Σ, i.e., an operad $\mathcal{B}$ equipped with a morphism of operads $\mathcal{B} \to \Sigma$, is the same as "colored operad".

(ii) The *operad of projections* which assigns to I the same set I; the composition map for $J \twoheadrightarrow I$ is $I \times \prod J_i \to J$, $(i_0, (j_i)) \mapsto j_{i_0}$.

(iii) The *operad of trees* which assigns to I the set T_I of (homeomorphism classes of) trees with I ingoing and one outgoing edge; the composition is the gluing of the corresponding outgoing and ingoing edges. Notice that T_I is naturally ordered (one tree is larger than another if the latter can be obtained from the former by contracting some connected subgraphs to points); the composition preserves the order.

1.1.5. Let $\mathcal{M}$, $\mathcal{N}$ be pseudo-tensor categories. A *pseudo-tensor functor* $\tau: \mathcal{N} \to \mathcal{M}$ is a law that assigns to any object $G \in \mathcal{N}$ an object $\tau(G) \in \mathcal{M}$ and to $\{F_i\}_{i\in I}$, $G \in \mathcal{N}$, a map $\tau_I: P_I^{\mathcal{N}}(\{F_i\}, G) \to P_I^{\mathcal{M}}(\{\tau(F_i)\}, \tau(G))$ so that τ_I are compatible with composition and $\tau_\cdot(id_G) = id_{\tau(G)}$. One defines a morphism between pseudo-tensor functors in the obvious way. So, if $\mathcal{N}$ is equivalent to a small category, pseudo-tensor functors $\mathcal{N} \to \mathcal{M}$ form a category. The composition of pseudo-tensor functors is a pseudo-tensor functor in the obvious way.

Any pseudo-tensor functor τ defines an usual functor between the corresponding usual categories. We call τ a *pseudo-tensor extension* of this usual functor.

An *adjoint pair* is a pair of pseudo-tensor functors $\mathcal{M} \underset{\tau}{\overset{\nu}{\rightleftarrows}} \mathcal{N}$ together with identifications $a: P_I^{\mathcal{N}}(\{\nu(M_i)\}, G) \xrightarrow{\sim} P_I^{\mathcal{M}}(\{M_i\}, \tau(G))$ for $M_i \in \mathcal{M}$, $G \in \mathcal{N}$, compatible with composition of operations.[2] We say that ν is left adjoint to τ and τ is right adjoint to ν. The corresponding usual functors are obviously adjoint.

EXERCISE. Show that for a given pseudo-tensor functor its left (resp. right) adjoint pseudo-tensor functor is determined uniquely up to a unique isomorphism.

1.1.6. EXAMPLES. (i) A *pseudo-tensor subcategory* of a pseudo-tensor category $\mathcal{M}$ is given by a subclass of objects $\mathcal{T} \subset \mathcal{M}$ together with subsets $P_I^{\mathcal{T}}(\{L_i\}, M) \subset P_I^{\mathcal{M}}(\{L_i\}, M)$ for L_i, $M \in \mathcal{T}$ such that $P_I^{\mathcal{T}}$ are closed under the composition of operations, and $id_M \in P_\cdot^{\mathcal{T}}(\{M\}, M)$ for any $M \in \mathcal{T}$. Then $(\mathcal{T}, P^{\mathcal{T}})$ is a pseudo-tensor category, and $\mathcal{T} \hookrightarrow \mathcal{M}$ is a pseudo-tensor functor. If $P_I^{\mathcal{T}}(\{L_i\}, M) = P_I^{\mathcal{M}}(\{L_i\}, M)$ for any L_i, $M \in \mathcal{P}$, then we call $\mathcal{T}$ a *full* pseudo-tensor subcategory of $\mathcal{M}$.

Any subclass of objects of $\mathcal{M}$ may be considered as a full pseudo-tensor subcategory of $\mathcal{M}$. In particular, for any $M \in \mathcal{M}$ we have the full pseudo-tensor subcategory with single object M. We denote by $P(M) = P^{\mathcal{M}}(M)$ the corresponding operad.

REMARK. Any pseudo-tensor category $\mathcal{M}$ can be realized as a full pseudo-tensor subcategory of a *tensor* category. Here is a universal construction of such an embedding $\mathcal{M} \hookrightarrow \mathcal{M}^{\otimes}$. By definition, an object of $\mathcal{M}^{\otimes}$ is a collection $\{M_i\}_{i\in I}$ of objects of $\mathcal{M}$ labeled by some finite non-empty set I; we denote it by $\otimes M_i = \otimes_I M_i$. A morphism $\varphi : \otimes_J N_j \to \otimes_I M_i$ is a collection $(\pi, \{\varphi_i\}_{i\in I})$ where $\pi : J \twoheadrightarrow I$ and $\varphi_i \in P_{J_i}(\{N_j\}, M_i)$; we write $\varphi = \otimes\varphi_i = \otimes_\pi \varphi_i$. The composition of morphisms is defined using composition of operations. The tensor structure on $\mathcal{M}^{\otimes}$ and the embedding $\mathcal{M} \hookrightarrow \mathcal{M}^{\otimes}$ are the obvious ones.

[2] I.e., we demand that for $\varphi \in P_I^{\mathcal{N}}(\{\nu(M_i)\}, G)$, $\psi_i \in P_{J_i}^{\mathcal{M}}(\{L_j\}, M_i)$ one has $a(\varphi(\nu(\psi_i))) = a(\varphi)(\psi_i)$ and for $\chi \in P_K^{\mathcal{N}}(\{G_k\}, H)$, $\varphi_k \in P_{I_k}^{\mathcal{N}}(\{\nu(M_i)\}, G_k)$ one has $a(\chi(\varphi_k)) = \tau(\chi)(a(\varphi_k))$.

(ii) Assume that $\mathcal{M}$, $\mathcal{N}$ have representable pseudo-tensor structures. Then a pseudo-tensor functor $\mathcal{N} \to \mathcal{M}$ is the same as a usual functor $\tau: \mathcal{N} \to \mathcal{M}$ together with natural morphisms $\nu_I: \underset{I}{\otimes} \tau(N_i) \to \tau(\underset{I}{\otimes} N_i)$ for any finite families of objects $N_i \in \mathcal{N}$, such that ν_I commute with the compatibility morphisms ε_π (i.e., for any $\pi: J \twoheadrightarrow I$ and a J-family $K_j \in \mathcal{N}$ the diagram

$$\begin{array}{ccc} \underset{J}{\otimes}\tau(K_j) & \xrightarrow{\nu_J} & \tau(\underset{J}{\otimes}K_j) \\ \big\downarrow \varepsilon_\pi^{\mathcal{M}} & & \\ \underset{I}{\otimes}(\underset{J_i}{\otimes}\tau(K_j)) & & \big\downarrow \tau(\varepsilon_\pi^{\mathcal{N}}) \\ \big\downarrow \underset{I}{\otimes}\nu_{J_i} & & \\ \underset{I}{\otimes}\tau(\underset{J_i}{\otimes}K_j) & \xrightarrow{\nu_I} & \tau(\underset{I}{\otimes}(\underset{J_i}{\otimes}K_j)) \end{array} \tag{1.1.6.1}$$

commutes). In particular, any tensor functor between tensor categories is a pseudo-tensor functor.

(iii) If $\mathcal{B}$ is an operad, then we call a pseudo-tensor functor $\mathcal{B} \to \mathcal{M}$ a $\mathcal{B}$ *algebra* in $\mathcal{M}$; we denote the category of $\mathcal{B}$ algebras in $\mathcal{M}$ as $\mathcal{B}(\mathcal{M})$. A $\mathcal{B}$ algebra in $\mathcal{M}$ amounts to an object $L \in \mathcal{M}$ together with a morphism of operads $\mathcal{B} \to P^{\mathcal{M}}(L)$.

For example, a Σ algebra in $\mathcal{M}$ is just a monoid, i.e., an object L equipped with an associative product $\cdot \in P_2(\{L,L\},L) = P_2^{\mathcal{M}}(L)$. An algebra for a "trivial" 1-point operad is a commutative monoid.

(iv) For any family $\{\mathcal{M}_j\}$ of pseudo-tensor categories one defines their product $\Pi\mathcal{M}_j$ in the obvious way. A pseudo-tensor functor $\mathcal{N} \to \Pi\mathcal{M}_j$ is the same as a collection of pseudo-tensor functors $\mathcal{N} \to \mathcal{M}_j$.

(v) If $\mathcal{A}$ is a tensor category, then the product functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is a pseudo-tensor functor in the obvious way. Let $\mathcal{M}$ be a pseudo-tensor category. An *$\mathcal{A}$-action on* $\mathcal{M}$ is a pseudo-tensor functor $\otimes: \mathcal{A} \times \mathcal{M} \to \mathcal{M}$ together with an identification of two pseudo-tensor functors $\mathcal{A} \times \mathcal{A} \times \mathcal{M} \to \mathcal{M}$, $(A_1 \otimes A_2) \otimes M \xrightarrow{\sim} A_1 \otimes (A_2 \otimes M)$, that satisfies the obvious compatibilities. In concrete terms, such an action is given by an action of $\mathcal{A}$ on $\mathcal{M}$ as on a usual category together with natural maps $P_I^{\mathcal{M}}(\{M_i\},N) \to P_I^{\mathcal{M}}(\{A_i \otimes M_i\}, (\otimes A_i) \otimes N)$ for $M_i, N \in \mathcal{M}$, $A_i \in \mathcal{A}$.

(vi) We are in situation of (v), so we have a pseudo-tensor functor $\otimes : \mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$. Let P be a commutative monoid in $\mathcal{A}$. Then $\mathcal{M} \to \mathcal{A} \times \mathcal{M}$, $M \mapsto (P, M)$, is naturally a pseudo-tensor functor (see (iv) above). Thus $\mathcal{M} \to \mathcal{M}$, $M \mapsto P \otimes M$, is a pseudo-tensor functor. Similarly, if F is a commutative monoid in $\mathcal{M}$, then $\mathcal{A} \to \mathcal{M}$, $A \mapsto A \otimes F$, is naturally a pseudo-tensor functor. So if a A is a $\mathcal{B}$ algebra in $\mathcal{A}$ and M is a $\mathcal{B}$ algebra in $\mathcal{M}$, then $P \otimes M$, $A \otimes F$ are $\mathcal{B}$ algebras in $\mathcal{M}$.

1.1.7. The above definitions make sense when operations $P_I(\{L_i\},M)$ are not sets but objects of a certain fixed tensor (or even pseudo-tensor!) category $\mathcal{A}$.[3] We call such thing a *pseudo-tensor $\mathcal{A}$-category.* The common example is $\mathcal{A}$ = the tensor category of k-modules, where k is a commutative ring; this is the case of *pseudo-tensor k-categories*. Other examples (super setting, DG categories, etc.) are discussed in 1.1.16.

[3] So the composition maps are morphisms $P_I \otimes (\otimes_I P_{J_i}) \to P_J$ in $\mathcal{A}$, etc.

Explicitly, a pseudo-tensor k-category is a pseudo-tensor category $\mathcal{M}$ together with a k-module structure on the sets P_I such that the composition map is k-polylinear. For example, if $\mathcal{M}$ consists of a single object, then we have a k-operad. A pseudo-tensor k-category $\mathcal{M}$ is *additive* if it is additive as a usual k-category, and $\mathcal{M}$ is *abelian* if it is abelian as a usual k-category and the polylinear functors P_I are left exact.

A pseudo-tensor functor $F\colon \mathcal{N} \to \mathcal{M}$ between the pseudo-tensor k-categories is *k-linear* if the maps F_I are k-linear. If $\mathcal{B}$ is a k-operad, we get a notion of a $\mathcal{B}$ k-algebra in $\mathcal{M}$ (we usually call them simply $\mathcal{B}$ algebras).

For example, we have the favorite k-operads $\mathcal{A}ss$, $\mathcal{C}om$, $\mathcal{L}ie$, $\mathcal{P}ois$ which give rise, respectively, to associative, commutative and associative, Lie, and Poisson algebras. Note that $\mathcal{A}ss$, $\mathcal{C}om$ are k-envelopes of the corresponding set-theoretic operads: $\mathcal{A}ss = k\,[\Sigma]$, $\mathcal{C}om = k$. To avoid problems with skew-symmetry, we always assume that $1/2 \in k$ when speaking about Lie or Poisson algebras.

1.1.8. Remarks. (i) Let $\mathcal{B}$ be any k-operad. For $I \in \mathcal{S}$ denote by $\mathcal{FB}(I)$ the free $\mathcal{B}$ algebra in the tensor category $k\operatorname{mod}$ of k-modules with generators e_i labeled by elements $i \in I$. Then the morphism of k-modules $\mathcal{B}_I \to \mathcal{FB}(I)$, $\varphi \mapsto \varphi(\underset{I}{\otimes} e_i)$, is injective, so one may consider $\mathcal{B}_I$ as the k-submodule of $\mathcal{FB}(I)$ generated by "multiplicity free monomials of degree I".

(ii) The operad $\mathcal{P}ois$ carries a natural grading such that the product and the Poisson bracket have degrees, respectively, 0 and 1. The operad $\mathcal{A}ss$ carries a natural decreasing filtration such that $\operatorname{gr}\mathcal{A}ss = \mathcal{P}ois$.

1.1.9. Let $\mathcal{N}_j$, $j \in J$, be a finite family of pseudo-tensor k-categories. Their *tensor product* $\otimes\mathcal{N}_j$ is a pseudo-tensor category defined as follows. Its objects are collections $\{N_j\}$, $N_j \in \mathcal{N}_j$; we denote this object of $\otimes\mathcal{N}_j$ by $\otimes N_j$. One has $P_I\left(\{\otimes N_{j_i}\}, \otimes K_j\right) := \underset{kJ}{\otimes} P_I^{\mathcal{N}_j}(\{N_{j_i}\}, K_j)$; the composition maps are tensor products of those in $\mathcal{N}_j$. Note that for any pseudo-tensor k-category $\mathcal{M}$ a k-linear pseudo-tensor functor $\otimes\mathcal{N}_j \to \mathcal{M}$ is the same as a k-polylinear pseudo-tensor functor $\Pi\mathcal{N}_j \to \mathcal{M}$.

1.1.10. Let $\mathcal{M}$ be any pseudo-tensor k-category. Then $\mathcal{M} \otimes \mathcal{L}ie$ coincides with $\mathcal{M}$ as a usual k-category, and one has $P_I^{\mathcal{M}\otimes\mathcal{L}ie}(\{L_i\}, M) = P_I^{\mathcal{M}}(\{L_i\}, M) \otimes \mathcal{L}ie_I$. The following funny lemma will be of use:

Lemma. *$\mathcal{L}ie(\mathcal{M} \otimes \mathcal{L}ie) = \mathcal{C}om(\mathcal{M})$.*

Proof. We have to show that Lie algebras in $\mathcal{M} \otimes \mathcal{L}ie$ are the same as commutative algebras in $\mathcal{M}$. Let M be an object in $\mathcal{M}$. One has $P_2^{\mathcal{M}\otimes\mathcal{L}ie}(M) = P_2^{\mathcal{M}}(M) \otimes \mathcal{L}ie_2$, so a skew-symmetric binary $\mathcal{M} \otimes \mathcal{L}ie$ operation $[\ \ ]_M$ on M amounts to a symmetric binary $\mathcal{M}$ operation $\cdot_M$. Let us identify $\mathcal{L}ie_n$ with the space of Lie polynomials of variables $e_1, \ldots, e_n$, so $[\ \ ]_M = \cdot_M \otimes [e_1, e_2]$. Since $\mathcal{L}ie_3$ is generated by $[e_1,[e_2,e_3]]$, $[e_3,[e_1,e_2]]$, $[e_2,[e_3,e_1]]$ with the only relation $[e_1,[e_2,e_3]] + [e_3,[e_1,e_2]] + [e_2,[e_3,e_1]] = 0$, we see that $[\ \ ]_M$ satisfies the Jacobi relation if and only if $\cdot_M$ is associative. □

1.1.11. The material of this subsection will not be used in the sequel.

The above lemma is a particular case of a general duality statement for quadratic operads. To formulate it, we need some notions from §2 of [**GK**]. Assume for simplicity that k is a field. Let $\mathcal{A}$ be a quadratic k-operad (i.e., $\mathcal{A}$ is a quotient of the free k-operad generated by a finite-dimensional vector space with Σ_2-action $\mathcal{A}_2$ modulo relations in $\mathcal{A}_3$). Its quadratic dual $\mathcal{A}^!$ is defined in [**GK**] 2.1.9 (this is a quadratic operad with $\mathcal{A}^!_2$ equal to the vector space dual to $\mathcal{A}_2$, the Σ_2-action is twisted by *sgn*). If $\mathcal{B}$ is another quadratic operad, then we have a quadratic operad $\mathcal{A}\bullet\mathcal{B}$ defined in [**GK**] 2.2.5 (one has $(\mathcal{A}\bullet\mathcal{B})_2 = \mathcal{A}_2\otimes\mathcal{B}_2$ with the Σ_2-action twisted by *sgn*) .

Notice that for any k-operad $\mathcal{P}$ the map $\mathrm{Hom}(\mathcal{A},\mathcal{P}) \to \mathrm{Hom}(\mathcal{A}_2,\mathcal{P}_2)$ is injective. Let $\mathcal{P}^{quad}$ be a quadratic operad with $\mathcal{P}^{quad}_2 = \mathcal{P}_2$ and relations coming from $\mathcal{P}_3$. Then the obvious morphism $\mathcal{P}^{quad} \to \mathcal{P}$ yields a bijection $\mathrm{Hom}(\mathcal{A},\mathcal{P}^{quad}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{A},\mathcal{P})$.

Lemma. *For any quadratic operads $\mathcal{A},\mathcal{B}$ and a pseudo-tensor k-category $\mathcal{M}$ one has*

$$\mathcal{A}(\mathcal{M}\otimes\mathcal{B}) = (\mathcal{A}\bullet\mathcal{B}^!)(\mathcal{M}) = \mathcal{B}^!(\mathcal{M}\otimes\mathcal{A}^!). \tag{1.1.11.1}$$

Proof. Since $\bullet$ is symmetric, it suffices to define the first identification. We may assume that $\mathcal{M}$ has a single object. So we have a k-operad $\mathcal{P}$, and we want to define a canonical bijection $\mathrm{Hom}(\mathcal{A},\mathcal{P}\otimes\mathcal{B}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{A}\bullet\mathcal{B}^!,\mathcal{P})$. It comes from the obvious identification $\mathrm{Hom}(\mathcal{A}_2,(\mathcal{P}\otimes\mathcal{B})_2) = \mathcal{A}^!_2\otimes\mathcal{P}_2\otimes\mathcal{B}_2 = \mathrm{Hom}((\mathcal{A}\bullet\mathcal{B}^!)_2,\mathcal{P}_2)$. To check the relations, we may replace $\mathcal{P}$ by $\mathcal{P}^{quad}$, $\mathcal{P}\otimes\mathcal{B}$ by $(\mathcal{P}\otimes\mathcal{B})^{quad} = \mathcal{P}^{quad}\circ\mathcal{B}$ (see [**GK**] 2.2.3), and we are done by [**GK**] 2.2.6(b). $\square$

If $\mathcal{A} = \mathcal{B} = \mathcal{C}om$, then $\mathcal{A}^! = \mathcal{B}^! = \mathcal{L}ie$ (see [**GK**] 2.1.11) and the above lemma becomes the lemma from 1.1.10.

1.1.12. Remark. Suppose that our $\mathcal{M}$ is actually a tensor category with tensor product $\otimes$. Let us show that the (pseudo) tensor structure of $\mathcal{M}$ reconstructs uniquely from that of $\mathcal{M}\otimes\mathcal{L}ie$.

Note that the pseudo-tensor structure of $\mathcal{M}\otimes\mathcal{L}ie$ is representable; denote by $\otimes^{Lie}$ the corresponding tensor product. One has $\underset{I}{\otimes}^{Lie} = \mathcal{L}ie^*_I\otimes\underset{I}{\otimes}$. So one has a canonical identification $M_1\otimes M_2 = M_1\otimes^{Lie} M_2$, and the commutativity constraints differ by sign. It remains to recover the associativity constraint. For $M_1,M_2,M_3\in\mathcal{M}$ consider the composition $\underset{\{1,2,3\}}{\otimes}^{Lie} M_i \to M_1\otimes^{Lie}(M_2\otimes^{Lie}M_3) \xrightarrow{\sim} M_1\otimes(M_2\otimes M_3)$; here the first arrow is the compatibility morphism for $\otimes^{Lie}$ (see 1.1.3), and the second isomorphism comes from the ordering of our indices (e.g., $\mathcal{L}ie_{\{2,3\}} \xrightarrow{\sim} k$, $[e_2,e_3]\mapsto 1$). Let $\underset{\{1,2,3\}}{\otimes}^{Lie} M_i \to M_1\otimes(M_2\otimes M_3)\oplus M_3\otimes(M_1\otimes M_2)\oplus M_2\otimes(M_3\otimes M_1)$ be the sum of these maps for cyclic transposition of indices; denote by K its cokernel. The projection maps each of the 3 terms to K isomorphically, and the associativity constraint for $\otimes$ coincides with composition $M_1\otimes(M_2\otimes M_3) \xrightarrow{\sim} K \xrightarrow{\sim} M_3\otimes(M_1\otimes M_2) = (M_1\otimes M_2)\otimes M_3$.

1.1.13. One can use operads to define pseudo-tensor structures on some categories of modules and their dual categories. Let us explain how to do this in the k-linear setting.

Let R be an associative k-algebra; denote by Rmod, mod$R$, the category of left (right) R-modules. Let $\mathcal{B}$ be a *strict R-operad*, i.e., a k-operad endowed with an isomorphism of k-algebras $R \xrightarrow{\sim} \mathcal{B}_1$. So for any $I\in\mathcal{S}$ the k-module $\mathcal{B}_I$ is actually an $(R - R^{\otimes I})$-bimodule (here $\otimes$ is the tensor power over k), and for $J \twoheadrightarrow$

I the composition map may be written as a morphism of $(R - R^{\otimes J})$-bimodules $\mathcal{B}_I \underset{R^{\otimes I}}{\otimes} \mathcal{B}_{J/I} \to \mathcal{B}_J$.

Such $\mathcal{B}$ defines the pseudo-tensor structures $P^* = P^{*\mathcal{B}}$ and $P^! = P^{!\mathcal{B}}$ on mod$R$ and Rmod$^\circ$, respectively. Namely, for $L_i, M \in$ mod$R$ set

$$P_I^*(\{L_i\}, M) := \mathrm{Hom}_{\mathrm{mod}R^{\otimes I}}\left(\otimes L_i, M \underset{R}{\otimes} \mathcal{B}_I\right). \tag{1.1.13.1}$$

The composition of $*$ operations comes from the composition law in $\mathcal{B}$. Similarly, for C_i, $D \in R$mod set

$$P_I^!(D, \{C_i\}) := \mathrm{Hom}_{R\mathrm{mod}}\left(D, \mathcal{B}_I \underset{R^{\otimes I}}{\otimes} (\underset{I}{\otimes} C_i)\right). \tag{1.1.13.2}$$

Here we write the operations in Rmod$^\circ$ as operations on R-modules with reversed order of arguments, so that $P^!_\cdot(D, \{C\}) = \mathrm{Hom}_{R\mathrm{mod}}(D, C)$ $(= \mathrm{Hom}_{R\mathrm{mod}^\circ}(C, D))$. The composition of ! operations comes in an obvious manner from the composition law in $\mathcal{B}$.

Remarks. (i) $\mathcal{B} = P^*(R) = P^!(R)$; here we consider R as right and left R-module, respectively.

(ii) The $*$ and ! structures are abelian (see 1.1.7) iff $\mathcal{B}_I$ are flat as, respectively, R- and $R^{\otimes I}$-modules.

1.1.14. Example. Assume that R has a cocommutative Hopf algebra structure. It defines a strict R-operad $\mathcal{B}^R$. Namely, $\mathcal{B}_I^R := R^{\otimes I}$, and the composition map $\mathcal{B}_I^R \otimes \mathcal{B}_{J/I}^R \to \mathcal{B}_J$ is $(\underset{I}{\otimes} r_i) \otimes (\underset{I}{\otimes}(\underset{J_i}{\otimes} r_j^{(i)})) \mapsto \underset{I}{\otimes}(\Delta^{(J_i)}(r_i) \cdot (\underset{J_i}{\otimes} r_j^{(i)})) \in \underset{I}{\otimes} R^{\otimes J_i} = R^{\otimes J}$. Here $\Delta^{(J_i)}\colon R \to R^{\otimes J_i}$ is the J_i-multiple coproduct. Note that the ! pseudo-tensor structure coincides with the standard tensor category structure on Rmod.

1.1.15. The above notions easily sheafify. Namely, let X be a topological space (or a Grothendieck topology) and $\mathcal{M}$ a sheaf of categories on X (we assume that both morphisms and objects of $\mathcal{M}$ have local nature, i.e., satisfy the gluing property). A *pseudo-tensor structure* on $\mathcal{M}$ assigns to any open U a pseudo-tensor structure on $\mathcal{M}(U)$ and to any $j\colon V \to U$ a pseudo-tensor extension of the pull-back functor $j^*\colon \mathcal{M}(U) \to \mathcal{M}(V)$. We assume that this extension is compatible with the composition of j's and the pseudo-tensor operations have local nature (i.e., for any L_i, $M \in \mathcal{M}(U)$, the prehseaf $\underline{P}_I(\{L_i\}, M)$ on U, $\underline{P}_I(\{L_i\}, M)(V) = P_I(\{L_i|_V\}, M|_V)$, is actually a sheaf). We call such $\mathcal{M}$ a *sheaf of pseudo-tensor categories*. All the notions we discuss in Chapter 1 have straightforward sheaf versions. We will tacitly assume them for the next sections.

1.1.16. Super conventions. Sometimes we need to consider super objects in a given category.

Let $\mathcal{M}$ be a k-category. Its super ($:= \mathbb{Z}/2$-graded) objects form a k-category $\mathcal{M}^s$. In other words, $\mathcal{M}^s$ is the tensor product of $\mathcal{M}$ and the category of finite-dimensional super vector spaces. $\mathcal{M}^s$ is also a “super” category: for $M, N \in \mathcal{M}^s$ the k-module of all $\mathcal{M}$-morphisms $\mathrm{Hom}(M, N)$ has an obvious $\mathbb{Z}/2$-grading, and the morphisms in $\mathcal{M}^s$ are the even part in $\mathrm{Hom}(M, N)$. If $\mathcal{M}$ is a tensor category, then $\mathcal{M}^s$ is automatically a tensor (super) category according to the Koszul rule of signs.

A $\mathbb{Z}$-graded super object M in $\mathcal{M}$ is a super object in $\mathcal{M}$ equipped with an extra $\mathbb{Z}$-grading. For such M we set $M[1] := k[1] \otimes M$ where $k[1]$ is an odd copy of k placed in degree -1; i.e., $M[1]$ is the $\mathbb{Z}$-graded super object obtained from M by simultaneous shift of $\mathbb{Z}$- and $\mathbb{Z}/2$-gradings by 1. The shift is an automorphism of the category of $\mathbb{Z}$-graded super objects, so we have $M[a]$ for every $a \in \mathbb{Z}$. We consider M^a as a subobject of M. The category $\mathcal{M}^{s\mathbb{Z}}$ of $\mathbb{Z}$-graded super objects has an obvious "$\mathbb{Z}$-graded super structure": for $M, N \in \mathcal{M}^{s\mathbb{Z}}$ we have a $\mathbb{Z}$-graded super k-module $\mathrm{Hom}(M,N)$ with components $\mathrm{Hom}(M,N)^i = \Pi \mathrm{Hom}(M^a, N^{a+i})$. If $\mathcal{M}$ is a tensor category, then so is $\mathcal{M}^{s\mathbb{Z}}$ (the signs come from the "super" grading, so the forgetting of the $\mathbb{Z}$-grading is a tensor functor).

A DG super object, or *super complex*, in $\mathcal{M}$ is a $\mathbb{Z}$-graded super object equipped with an action of the commutative graded Lie super algebra $k[-1]$, which is the same as an odd endomorphism d of degree 1 with square 0. They form a DG super category $C\mathcal{M}^s$ in the obvious manner. This is a tensor DG super category if $\mathcal{M}$ is a tensor category.

We identify the category of $\mathbb{Z}$-graded objects in $\mathcal{M}$ with a full subcategory in $\mathcal{M}^{s\mathbb{Z}}$ of those objects for which the $\mathbb{Z}$-degree mod 2 coincides with the super degree. In the same manner the category of complexes $C\mathcal{M}$ is a full subcategory of $C\mathcal{M}^s$. If $\mathcal{M}$ is a tensor category, then these are embeddings of tensor subcategories.

If $\mathcal{M}$ is a pseudo-tensor k-category, then $\mathcal{M}^s$ is automatically a pseudo-tensor super k-category (see 1.1.7). Namely, for $L_i, M \in \mathcal{M}^s$ the k-module $P_I(\{L_i\}, M)$ carries the obvious $(\mathbb{Z}/2)^I \times \mathbb{Z}/2$-grading; hence it is a super k-module with respect to the total $\mathbb{Z}/2$-grading. The composition takes the Koszul sign rule into account. Thus for any $L_i, M \in \mathcal{M}^s$ and free super k-modules V_i, U of finite rank one has a canonical identification of super k-modules $P_I(\{V_i \otimes L_i\}, U \otimes M) = \mathrm{Hom}(\bigotimes_I V_i, U) \otimes P_I(\{L_i\}, M)$.[4] One can also view $\mathcal{M}^s$ as a plain pseudo-tensor k-category considering only even operations.

Similarly, $\mathcal{M}^{s\mathbb{Z}}$ is a pseudo-tensor $\mathbb{Z}$-graded super k-category and $C\mathcal{M}^s$ is a pseudo-tensor DG super k-category. One has $P_I(\{L_i[a_i]\}, M[b]) = P(\{L_i\}, M)[b - \Sigma a_i]$.

Notation. For $L \in C\mathcal{M}$ (or $C\mathcal{M}^s$) we set $L_\dagger := \mathcal{C}one(id_L)$, or, equivalently, $L_\dagger = k_\dagger \otimes L$. Notice that $k_\dagger$ is naturally a comutative DG k-algebra (indeed, we can write $k_\dagger = k[\eta]$, $d_\dagger = \partial_\eta$, for an odd variable η of $\mathbb{Z}$-degree -1). So, by 1.1.6(vi),[5] for a pseudo-tensor $\mathcal{M}$ the functor $L \mapsto L_\dagger$ is a DG pseudo-tensor endofunctor of $C\mathcal{M}$ or $C\mathcal{M}^s$ and the canonical morphism $L \to L_\dagger$ is a morphism of DG pseudo-tensor functors. In particular, if L is a $\mathcal{B}$ algebra, then so is $L_\dagger$ and $L \to L_\dagger$ is a morphism of $\mathcal{B}$ algebras.

1.2. Complements

This section is a continuation of 1.1. We consider inner objects of operations in 1.2.1, augmented pseudo-tensor structures in 1.2.4–1.2.7, augmented pseudo-tensor functors in 1.2.8, modules over "operadic" algebras in 1.2.11–1.2.15, the corresponding monoids (or associative algebras) in 1.2.16, and the action of an augmentation functor on algebras and modules in 1.2.17–1.2.18.

1.2.1. Inner $\mathcal{H}om$. Let $\mathcal{M}$ be a pseudo-tensor category, I a finite set, $\{L_i\}_{i \in I}$, and M objects in $\mathcal{M}$. Assume we have an object $X \in \mathcal{M}$ and an element $\varepsilon \in P_{\widetilde{I}}(\{L_i, X\}, M)$ where $\widetilde{I} := I \sqcup \cdot$. Then for any $J \in \mathcal{S}$ and a J-family of objects

[4] And this identification is compatible with composition of operations.

[5] Applied to $\mathcal{A} = C(k)$ (the category of k-complexes), $P = k_\dagger$.

$\{A_j\}$ we get a natural map

$$\kappa = \kappa_J(\varepsilon) : \ P_J(\{A_j\}, X) \to P_{J\sqcup I}(\{A_j, L_i\}, M), \tag{1.2.1.1}$$

$\kappa(\varphi) := \varepsilon(\varphi, id_{L_i})$. Here "natural" means that for any $H \twoheadrightarrow J$, an H-family of objects $\{B_h\}$ and $\chi_j \in P_{H_j}(\{B_h\}, A_j)$ one has $\kappa(\varphi(\chi_j)) = \kappa(\varphi)(\chi_j, id_{L_i})$. A pair (X, ε) with the property that all the maps κ are isomorphisms is uniquely defined (if its exists); we denote it by $(\mathcal{P}_I(\{L_i\}, M), \varepsilon_{I,\{L_i\},M})$ and call it the *inner* $\mathcal{P}$ *object*. Note that $\mathcal{P}_\phi(M) = M$, and $\mathcal{P}_1(L, M) = \mathcal{H}om(L, M)$ is the "inner Hom" object.

If well defined, inner P objects depend on their arguments in a functorial manner: for a surjection $E \twoheadrightarrow I$, $\varphi \in \mathrm{Hom}(M, N)$, and $\psi_i \in P_{E_i}(\{K_e\}, L_i)$ (as usual $E_i := \pi^{-1}(i)$) there is an obvious morphism $\mathcal{P}_I(\{L_i\}, M) \to \mathcal{P}_E(\{K_e\}, N)$ which is compatible with the composition of φ's and ψ's.

1.2.2. Here is an "inner" version of the functoriality property. Let $\pi\colon E \to I$ be *any* map of finite sets, $\{K_e\}$ an E-family of objects. Assume that the objects $\mathcal{P}_I(\{L_i\}, M)$, $\mathcal{P}_E(\{K_e\}, M)$, $\mathcal{P}_{E_i}(\{K_e\}, L_i)$ exist. Then one has a canonical "composition" element

$$c \in P_{\tilde{I}}(\{\mathcal{P}_I(\{L_i\}, M),\ \mathcal{P}_{E_i}(\{K_e\}, L_i)\},\ \mathcal{P}_E(\{K_e\}, M)) \tag{1.2.2.1}$$

defined as the image of the composition $\varepsilon_{I,\{L_i\},M}\left(id_{\mathcal{P}_I(\{L_i\},M)}, \varepsilon_{E_i,\{K_e\},L_i}\right)$ by the canonical isomorphism κ^{-1}. Note that if $E = \emptyset$, then $c = \varepsilon$.

One may check that the composition is associative in the obvious sense.

Example. Consider $L \in \mathcal{M}$ such that $\mathcal{E}nd\, L = \mathcal{H}om(L, L)$ exists. Then the composition $c \in P_2(\{\mathcal{E}nd\, L, \mathcal{E}nd\, L\}, \mathcal{E}nd\, L)$ is associative; hence it makes $\mathcal{E}nd\, L$ a monoid. In the k-linear setting $\mathcal{E}nd\, L$ is an associative k-algebra.

1.2.3. Example. Suppose we are in situation 1.1.13, so $\mathcal{M} = \mathrm{mod}R$ and the pseudo-tensor structure is given by the functors P^* defined by the operad $\mathcal{B}$. For L_i, M as above set $X := \mathrm{Hom}_{R^{\otimes I}}(\underset{I}{\otimes} L_i, M \underset{R}{\otimes} \mathcal{B}_{\tilde{I}})$. This is a right R-module in the obvious way, and we have the canonical operation $\varepsilon \in P^*_{\tilde{I}}(\{L_i, X\}, M)$. The maps $\kappa_J(\varepsilon)$ are isomorphisms for $|J| = 1$; i.e., X necessarily coincides with $\mathcal{P}_I(\{L_i\}, M)$ if the latter object exists. This happens in the following situations:

(i) Assume that the composition maps $\mathcal{B}_I \underset{R^{\otimes I}}{\otimes} \mathcal{B}_{J/I} \to \mathcal{B}_J$ are isomorphisms. Then $X = \mathcal{P}_I(\{L_i\}, M)$ if the L_i are projective modules of finite rank.

(ii) Suppose (i) holds and all $\mathcal{B}_I$ are flat R-modules. Then $X = \mathcal{P}_I(\{L_i\}, M)$ if the L_i are finitely presented R-modules.

1.2.4. Augmented pseudo-tensor structures. Denote by $\hat{\mathcal{S}}$ the category of arbitrary finite sets and maps between them, so $\mathcal{S}$ is a subcategory of $\hat{\mathcal{S}}$. If in 1.1.1 we replace $\mathcal{S}$ by $\hat{\mathcal{S}}$, then we get the definition of *augmented pseudo-tensor category*. An augmented pseudo-tensor category with single object is called an *augmented operad*.

Examples. (i) A tensor category with a unit object $\mathbf{1}$ is an augmented tensor category: set $P_I(\{L_i\}, M) = \mathrm{Hom}(\underset{I}{\otimes} L_i, M)$. We assume that $\underset{\phi}{\otimes} = \mathbf{1}$.

(ii) The linear orders operad Σ (see 1.1.4) extends in the obvious way to an augmented operad $\Sigma^\wedge$ with $\Sigma^\wedge_\emptyset = \{\text{point}\}$. We also have the "trivial" augmented one-point operad, and their k-linear envelopes $\mathcal{A}ss^\wedge = k[\Sigma^\wedge]$ and $\mathcal{C}om^\wedge = k$.

1.2.5. One may consider an augmented pseudo-tensor category as a usual pseudo-tensor category equipped with some extra structure. Let us describe this structure explicitly. We need the following definition.

Let $\mathcal{M}$ be any pseudo-tensor category. Suppose we have a functor

$$h\colon \mathcal{M} \to Sets, \tag{1.2.5.1}$$

and for any finite set I of order ≥ 2 and $i_0 \in I$ we have natural maps

$$h_{I,i_0}\colon P_I(\{L_i\}, M) \times h(L_{i_0}) \to P_{I\smallsetminus\{i_0\}}(\{L_i\}, M) \tag{1.2.5.2}$$

(here, as usual, $\{L_i\}$ is an I-family of objects, $M \in \mathcal{M}$). We call these data an *augmentation functor* if the following compatibilities (i), (ii) hold:

(i) Let J be a finite set of order ≥ 2, $\pi\colon J \twoheadrightarrow I$ a surjective map, $j_0 \in J$. Then for $\varphi \in P_I(\{L_i\}, M)$, $\psi_i \in P_{J_i}(\{K_j\}, L_i)$, $a \in h(K_{j_0})$ one has

$$h_{J,j_0}(\varphi(\psi_i), a) = \varphi(\psi_{i'}, h_{J_{i_0},j_0}(\psi_{i_0}, a))$$

(here $i_0 = \pi(j_0), i' \in I \smallsetminus \{i_0\}$) if $|J_{i_0}| \geq 2$, and

$$h_{J,j_0}(\varphi(\psi_i), a) = h_{I,i_0}(\varphi, \psi_{i_0}(a))(\psi_{i'})$$

if $|J_{i_0}| = 1$.

(ii) Assume that $|I| \geq 2$; let $i_0, i_1 \in I$ be two distinct elements. Then for $\varphi \in P_I(\{L_i\}, M)$, $a_{i_0} \in h(L_{i_0})$, $a_{i_1} \in h(L_{i_1})$, one has

$$h_{I\smallsetminus\{i_0\},i_1}(h_{I,i_0}(\varphi, a_0), a_1) = h_{I\smallsetminus\{i_1\},i_0}(h_{I,i_1}(\varphi, a_1), a_0) \in P_{I\smallsetminus\{i_0,i_1\}}(\{L_i\}, M)$$

if $|I| > 2$, and

$$h_{I,i_0}(\varphi, a_0)a_1 = h_{I,i_1}(\varphi, a_1)a_0 \in h(M)$$

if $|I| = 2$.

One defines morphisms of augmentation functors in the obvious way.

Remark. Any subfunctor of an augmentation functor is itself an augmentation functor. The functor $h(M) \equiv \emptyset$ is an augmentation functor.

Definition. We say that our augmentation functor is *non-degenerate* if all the maps $P_I(\{L_i\}, M) \to \mathcal{M}aps(h(L_{i_0}), P_{I\smallsetminus\{i_0\}}(\{L_i\}, M))$ coming from (1.2.5.2) are injective.

1.2.6. Now let $\mathcal{M}^\wedge$ be an augmented pseudo-tensor category, $\mathcal{M}$ the corresponding usual pseudo-tensor category. Then $\mathcal{M}$ carries an augmentation functor $h = P_\emptyset$; h_{I,i_0} is the composition map for the embedding $I \smallsetminus \{i_0\} \hookrightarrow I$.

Lemma. *The above construction yields an* 1-1 *correspondence between augmented pseudo-tensor categories and usual pseudo-tensor categories equipped with an augmentation functor.* □

1.2.7. Note that any augmentation functor $h\colon \mathcal{M} \to Sets$ is automatically a pseudo-tensor functor (we consider $Sets$ as a tensor category with $\otimes = \Pi$); i.e., we have natural maps

$$h_I : P_I(\{L_i\}, M) \to \mathrm{Hom}(\underset{I}{\otimes} h(L_i), h(M)). \tag{1.2.7.1}$$

To see this, we can assume, by the above lemma, that h came from an augmented pseudo-tensor structure. Then h_I is the composition map for $\emptyset \to I$. Equivalently, for $\varphi \in P_n(\{L_i\}, M)$, $a_i \in h(L_i)$ one has (here $h_n := h_{\{1,\dots,n\}}$)

$$h_n(\varphi)(a_1 \otimes \cdots \otimes a_n) = h_{\{n-1,n\},n-1}(\dots(h_{\{1,\dots,n\},1}(\varphi, a_1)\dots), a_{n-1})a_n.$$

In the situation of 1.2.1 there is a canonical map

$$\bar{h} : h(\mathcal{P}_I(\{L_i\}, M)) \to P_I(\{L_i\}, M), \quad \lambda \longmapsto h_{\tilde{I},\cdot}(\varepsilon_{I,\{L_i\},M}, \lambda). \tag{1.2.7.2}$$

For c as in 1.2.2 the $\bar{h}$ arrows intertwine $h_{\tilde{I}}(c)$ and the usual composition map for P operations.

We say that $\mathcal{P}_I(\{L_i\}, M)$ is an *inner P object in the augmented sense* if (1.2.7.2) is an isomorphism. This amounts to the fact that the canonical morphism (1.2.2.1) (for $X = \mathcal{P}_I(\{L_i\}, M)$) is an isomorphism for $J = \emptyset$ as well.

1.2.8. One defines an *augmented pseudo-tensor functor* $\tau^\wedge : \mathcal{N}^\wedge \to \mathcal{M}^\wedge$ between augmented pseudo-tensor categories in the obvious way. Such $\tau^\wedge$ amounts to a usual pseudo-tensor functor $\tau : \mathcal{N} \to \mathcal{M}$ together with a morphism of augmentation functors $h_{\mathcal{N}} \to h_{\mathcal{M}}\tau$ that are compatible in the obvious manner. For example, given an augmented operad $\mathcal{B}^\wedge$, we have the notion of a $\mathcal{B}^\wedge$ algebra in any augmented pseudo-tensor category $\mathcal{M}^\wedge$.

Examples. $\Sigma^\wedge$ algebras in $\mathcal{M}^\wedge$ are semigroups; algebras for the trivial augmented operad are commutative semigroups. In the k-linear setting $\mathcal{A}ss^\wedge$ algebras are unital associative algebras, $\mathcal{C}om^\wedge$ algebras are unital commutative algebras. If A is a unital associative algebra, then the image $1_A \in h(A)$ of $1 \in k = \mathcal{A}ss_\emptyset$ is called the *unit element* in $h(A)$.

Remark. 1_A is the unit of the associative algebra $h(A)$. So a unital associative algebra in $\mathcal{M}^\wedge$ is the same as an associative algebra in $\mathcal{M}$ such that the associative algebra $h(A)$ is unital and the left and right product with the unit element of $h(A)$ is the identity endomorphism of A.

1.2.9. Following the pattern of 1.1.13, one may use augmented operads to define augmented pseudo-tensor structures on some categories of modules and their duals.

Let R be an associative k-algebra and $\mathcal{B}^\wedge$ an augmented strict R-operad; i.e., $\mathcal{B}$ is an augmented k-operad equipped with an isomoprhism $R \xrightarrow{\sim} \mathcal{B}_1^\wedge$. Such $\mathcal{B}^\wedge$ is the same as a usual strict R-operad $\mathcal{B}$ together with a left R-module $h = \mathcal{B}_0$ and a system of $R^{\otimes I \smallsetminus \{i\}}$-linear morphisms $h_{I,i_0} : \mathcal{B}_I \underset{R}{\otimes} \mathcal{B}_0 \to \mathcal{B}_{I \smallsetminus \{i_0\}}$ (here $|I| \geq 2$, $i_0 \in I$, and $\mathcal{B}_I$ is considered as a right R-module via the i_0th action) that satisfy obvious compatibilities.

According to 1.1.13, $\mathcal{B}$ defines the $*$ and ! pseudo-tensor structures on mod$R$ and Rmod°, respectively. Now h defines the augmentation functors h^*, $h^!$ for these structures. Namely, for $M \in \mathrm{mod}R$, $D \in R\mathrm{mod}$ set

$$h^*(M) := M \underset{R}{\otimes} h, \qquad h^!(D) := \mathrm{Hom}_{R\mathrm{mod}}(D, h). \tag{1.2.9.1}$$

The structure morphisms $h^*_{I,i_0} : P_I^{*\mathcal{B}}(\{L_i\}, M) \otimes h^*(L_{i_0}) \to P^{*\mathcal{B}}_{I \smallsetminus \{i_0\}}(\{L_i\}, M)$ and $h^!_{I,i_0} : P_I^{!\mathcal{B}}(D, \{C_i\}) \otimes h^!(C_{i_0}) \to P^!_{I \smallsetminus \{i_0\}}(D, \{C_i\})$ come from the h_{I,i_0} maps. Compatibilities (i) and (ii) in 1.2.5 are clear. Notice that $\mathcal{B}^\wedge = P^{*\wedge}(R) = P^{!\wedge}(R)$.

1.2.10. Example. Assume that R has a cocommutative Hopf algebra structure with counit $\varepsilon: R \to k$. It defines an augmented strict R-operad $\mathcal{B}^{R\,\wedge}$. Namely, we already defined $\mathcal{B}^R$ in 1.1.14. Set $h = k$ (the R-module structure is defined by ε), and let h_{I,i_0} be the obvious isomorphisms $R^{\otimes I} \underset{R}{\otimes} k = R^{\otimes I \smallsetminus \{i_0\}} \otimes (R \underset{R}{\otimes} k) \xrightarrow{\sim} R^{\otimes I \smallsetminus \{i_0\}}$. We leave it to the reader to check the compatibilities. Note that k is a unit object for the standard tensor structure on Rmod, and ! augmented structure is the corresponding augmented pseudo-tensor structure (see example (i) in 1.2.4).

1.2.11. Let $\mathcal{A}$ be a pseudo-tensor category and $\mathcal{P}$ a plain category. An $\mathcal{A}$-*action on* $\mathcal{P}$ is a pseudo-tensor category $\mathcal{C}$ together with an identification of $\mathcal{A}$ with a full pseudo-tensor subcategory of $\mathcal{C}$ and $\mathcal{P}$ with a full subcategory of $\mathcal{C}$ such that

(a) As a plain category, $\mathcal{C}$ is equivalent to a disjoint union of $\mathcal{A}$ and $\mathcal{P}$.

(b) If $P_I^{\mathcal{C}}(\{C_i\}, B)$ is non-empty, then either B and every C_i are in $\mathcal{A}$, or B and exactly one of C_i are in $\mathcal{P}$.

To define such a structure, one has to present for every P, Q in $\mathcal{P}$ and $\{A_i\}_{i\in I}$ in $\mathcal{A}$, where $I \in \mathcal{S}$, a set $P_{\tilde{I}}(\{A_i, P\}, Q)$ which behaves naturally with respect to morphisms of P and Q and operations of A_i's.

The definition can be rewritten in the k-linear setting in the obvious way.

Remark. If $\mathcal{A}$ is actually a tensor category, then its action on $\mathcal{P}$ as of a tensor category can be considered as an $\mathcal{A}$-action in the above sense: just set $P_{\tilde{I}}(\{A_i, P\}, Q) := \mathrm{Hom}((\otimes A_i) \otimes P, Q)$.

If both $\mathcal{A}$ and $\mathcal{P}$ have single object, then we refer to $\mathcal{C}$ as an $\mathcal{A}$ *module operad.* The $\mathcal{A}$ and $\mathcal{P}$ objects of $\mathcal{C}$ are denoted, respectively, by $\bullet$ and $*$.

1.2.12. Examples. (i) Let $\mathcal{A}$ be any pseudo-tensor category. There is a canonical $\mathcal{A}$-action on $\mathcal{A}$, considered as a plain category. The corresponding $\mathcal{C}$ is denoted by $\mathcal{A}^m$. It is equipped with a faithful pseudo-tensor functor $\mathcal{A}^m \to \mathcal{A}$ left inverse to both structure embeddings $\mathcal{A} \subset \mathcal{A}^m$ such that $P^{\mathcal{A}^m} \xrightarrow{\sim} P^{\mathcal{A}}$ unless prohibited by condition (b). This determines $\mathcal{A}^m$ uniquely.

(ii) Consider operad Σ (see 1.1.4). The Σ module operad Σ^m contains two Σ module suboperads Σ^m_ℓ, $\Sigma^m_r \subset \Sigma^m$. Namely, $\Sigma^m_I := \Sigma_{I \sqcup \{*\}}$ = set of linear orders on $I \sqcup \{*\}$, and $\Sigma^m_{\ell I}$ (resp. Σ^m_{rI}) is the subset of orders such that $*$ is the maximal (resp. the minimal) element.

(iii) Let $\mathcal{B}_i$ be operads, and let $\mathcal{C}_i$ be $\mathcal{B}_i$ module operads, $i \in I$. Denote by $\prod_I^0 \mathcal{C}_i \subset \prod_I \mathcal{C}_i$ the full pseudo-tensor subcategory with objects $\bullet := (\bullet_i)$, $* := (*_i)$. This is a $\prod_I \mathcal{B}_i$ module operad.

1.2.13. Let $\mathcal{B}$ be an operad, $\mathcal{C}$ a $\mathcal{B}$ module operad, and $\mathcal{M}$ a pseudo-tensor category. Let $\tau\colon \mathcal{C} \to \mathcal{M}$ be a pseudo-tensor functor. Then $L := \tau(\bullet)$ is a $\mathcal{B}$ algebra in $\mathcal{M}$. We call $M := \tau(*)$ an *L-module with respect to* $\mathcal{C}$, or simply an *L-module* (if $\mathcal{C}$ is fixed). Explicitly, a structure of an L-module on an object $M \in \mathcal{M}$ is given by a system of compatible "action" maps $\mathcal{C}_I \to P_{\tilde{I}}(\{L_i, M\}, M)$.

Denote by $\mathcal{C}(\mathcal{M})$ the category of pseudo-tensor functors $\tau\colon \mathcal{C} \to \mathcal{M}$ (see 1.1.5); its objects are pairs (L, M), where L is a $\mathcal{B}$ algebra and M is an L-module. If L is a fixed $\mathcal{B}$ algebra, then we denote by $\mathcal{M}(L) = \mathcal{M}(L, \mathcal{C}) \subset \mathcal{C}(\mathcal{M})$ the subcategory of L-modules: its objects are pairs (L, M); the morphisms are identity on L.

Example. Let L be an object of a pseudo-tensor category $\mathcal{M}$ such that $\mathcal{E}nd\, L \in \mathcal{M}$ exists. Then L is an $\mathcal{E}nd\, L$-module with respect to Σ^m_ℓ.

VARIANT. More generally, suppose we have $\mathcal{B}$, $\mathcal{C}$, $\mathcal{M}$ as above, and a category $\mathcal{P}$ equipped with an $\mathcal{M}$-action $\mathcal{D}$. Denote by $\mathcal{C}(\mathcal{D})$ the category of pseudo-tensor functors $\tau : \mathcal{C} \to \mathcal{D}$ which map $\bullet$ to $\mathcal{M}$, $*$ to $\mathcal{P}$. Then $L := \tau(\bullet)$ is a $\mathcal{B}$ algebra in $\mathcal{M}$ and $P := \tau(*)$ is an *L-module in* $\mathcal{P}$ (with respect to $\mathcal{C}$, $\mathcal{D}$). For fixed L we get the category $\mathcal{P}(L) = \mathcal{P}(L, \mathcal{C}, \mathcal{D})$ of L-modules in $\mathcal{P}$; its objects are pairs $(L, P) \in \mathcal{C}(\mathcal{D})$, and the morphisms are morphisms in $\mathcal{C}(\mathcal{D})$ which are identity on L. The situation described in the beginning of this subsection corresponds to $\mathcal{P} = \mathcal{M}$, $\mathcal{D} = \mathcal{M}^m$ (see 1.2.12(i)).

1.2.14. The above definitions render to the k-linear setting in the obvious way.

LEMMA. *If $\mathcal{M}$ is an abelian pseudo-tensor k-category, then $\mathcal{M}(L, \mathcal{C})$ is an abelian k-category.* □

A similar statement for $\mathcal{P}(L, \mathcal{C}, \mathcal{D})$ is left to the reader.

1.2.15. EXAMPLES. (i) $\mathcal{B} = \mathcal{L}ie$; $\mathcal{C} = \mathcal{B}^m$. Then $\mathcal{M}(L)$ is the usual category of modules over the Lie algebra L: its object is $M \in \mathcal{M}$ equipped with a pairing $\cdot \in P_2(\{L, M\}, M)$ that satisfies the condition $\ell_1 \cdot (\ell_2 \cdot m) - \ell_2 \cdot (\ell_1 \cdot m) = [\ell_1, \ell_2] \cdot m$.

(ii) $\mathcal{B} = \mathcal{A}ss = k[\Sigma]$. Let L be an associative algebra. If $\mathcal{C} = \mathcal{B}^m = k[\Sigma^m]$, then $\mathcal{M}(L, \mathcal{C})$ is the category of L-bimodules. If $\mathcal{C} = k[\Sigma^m_\ell]$ (resp. $\mathcal{C} = k[\Sigma^m_r]$), then $\mathcal{M}(L, \mathcal{C})$ is the category of left (resp. right) L-modules.

1.2.16. Let $\mathcal{B}$ be an operad, $\mathcal{C}$ a $\mathcal{B}$ module operad, and L a $\mathcal{B}$ algebra in the tensor category *Sets* (with $\otimes = \Pi$; see 1.2.7). We have the obvious "forgetting of the L-module structure" functor o: $Sets(L, \mathcal{C}) \to Sets$. Denote by $\mathcal{C}(L)$ the monoid of endormorphisms of this functor.

Here is an explicit description of $\mathcal{C}(L)$. It is generated by elements $((\ell_i), \varphi)$, where I is a finite set, $\ell_i \in L$ is an I-family of elements of L, $\varphi \in \mathcal{C}_I$. The relations for $\mathcal{C}(L)$ come from surjective maps $\pi \colon \widetilde{J} \to \widetilde{I}$ such that $\pi(\cdot) = \cdot$ and collections $\psi_i \in \mathcal{B}_{J_i}$, $i \in I$. The relation corresponding to $\pi, (\psi_i)$ is

$$((\ell_j), \varphi_1(\varphi_2, \psi_i)) = ((\psi_i(\ell_j)_{j \in J_i}, \varphi_1) \cdot ((\ell_j)_{j \in J_\cdot}, \varphi_2)). \tag{1.2.16.1}$$

Here (ℓ_j) is a J-family of elements of L, $\varphi_1 \in \mathcal{C}_I$, $\varphi_2 \in \mathcal{C}_{J_\cdot}$, where $J_\cdot := J \cap \pi^{-1}(\cdot)$. The element $1 = (\emptyset, id.)$ is the unit in $\mathcal{C}(L)$. Denote by $\mathcal{C}(L)$mod the category of sets equipped with $\mathcal{C}(L)$-action. The functor o lifts to the functor $\tilde{o}$: $Sets_{L\mathcal{C}} \to \mathcal{C}(L)$mod which is an equivalence of categories. More generally, one may define a $\mathcal{C}$-action of L on an object of any category, and this is the same as a $\mathcal{C}(L)$-action.

The above construction has its k-linear version. Namely, for a k-operad $\mathcal{B}$, a $\mathcal{B}$ module operad $\mathcal{C}$, and a $\mathcal{B}$ algebra L in $k$mod (or in the category of sheaves of k-modules on some space) we get an associative k-algebra with unit $\mathcal{C}(L)$ (or a sheaf of k-algebras) such that the category of L-modules coincides with the one of $\mathcal{C}(L)$-modules. We call $\mathcal{C}(L)$ the *enveloping algebra* of L. In situation 1.2.15(i) it coincides with the usual enveloping algebra.

1.2.17. Let now $\mathcal{M}^\wedge = (\mathcal{M}, h)$ be an augmented pseudo-tensor category, $\mathcal{B}$, $\mathcal{C}$ as above. Let L be a $\mathcal{B}$ algebra in $\mathcal{M}$, and let $M \in \mathcal{M}(L, \mathcal{C})$ be an L-module. Then $h(L)$ is a $\mathcal{B}$ algebra in *Sets* (since h is a pseudo-tensor functor), and M (considered as an object of the usual category $\mathcal{M}$) carries a canonical $\mathcal{C}(h(L))$-action. Explicitly, a generator $((\ell_i), \varphi) \in \mathcal{C}(h(L))$ acts as $h_I(a_I(\varphi))(\ell_i) \in \operatorname{End} M$. Therefore we have a canonical "identity" functor

$$\mathcal{M}(L, \mathcal{C}) \to \mathcal{C}(h(L))\text{-modules in } \mathcal{M}. \tag{1.2.17.1}$$

1.2.18. Example. Let $\mathcal{M}$ be a pseudo-tensor k-category, and let L be a Lie algebra in $\mathcal{M}$. Let $\{M_i\}_{i\in I}$, N be a finite collection of L-modules. We say that an operation $\varphi \in P_I(\{M_i\}, N)$ is *L-compatible* if $\cdot_N(id_L, \varphi) = \sum_{i_0\in I} \varphi(\cdot_{M_{i_0}}, id_{M_i})_{i\neq i_0} \in P_{\tilde{I}}(\{L, M_i\}, N)$.[6] Let $P_{LI}(\{M_i\}, N) \subset P_I(\{M_i\}, N)$ be the k-submodule of L-compatible operations. It is easy to see that composition of L-linear operations is L-linear. Therefore P_{LI} is a pseudo-tensor structure on $\mathcal{M}(L)$. If $\mathcal{M}$ was an abelian pseudo-tensor category, then $\mathcal{M}(L)$ is also an abelian pseudo-tensor category (see 1.2.14). If h is a k-linear augmentation functor on $\mathcal{M}$, then $h(L)$ is a Lie k-algebra, and for every $M \in \mathcal{M}(L)$, $h(M)$ is an $h(L)$-module. The functor h^L of $h(L)$-invariants, $M \mapsto h^L(M) := h(M)^{h(L)}$, is an augmentation functor on $\mathcal{M}(L)$.

If $\{F_j\}_{j\in J}$, G are $h(L)$-modules in $\mathcal{M}$, then the subspace $P_J^{h(L)}(\{F_j\}, G) \subset P_J(\{F_j\}, G)$ of $h(L)$-linear operations is defined in the obvious way. Therefore $h(L)$-modules form a pseudo-tensor k-category. The "identity" functor

$$\mathcal{M}(L) \to h(L)\text{-modules in } \mathcal{M} \tag{1.2.18.1}$$

is naturally a pseudo-tensor functor.

For a k-operad $\mathcal{B}$ a $\mathcal{B}$ algebra in $\mathcal{M}(L)$ is the same as a $\mathcal{B}$ algebra A in $\mathcal{M}$ on which L acts by derivations. According to (1.2.18.1) such an A carries a canonical $h(L)$-action by derivations.

1.3. Compound tensor categories

The notion of pseudo-tensor category, as opposed to that of tensor category, is *not* self-dual. Thus "algebraic geometry" in a general pseudo-tensor k-category is very poor: one knows what are commutative algebras, but there are no coalgebras, hence no group schemes, etc. The notion of compound pseudo-tensor category restores self-duality which makes it possible to develop algebro-geometric basics on these grounds as we will see in the next section.

We do recommend the reader to consider the basic example of 2.2 below before, or while, reading this section. He may also restrict himself to compound tensor categories (see 1.3.12–1.3.15) instead of general compound pseudo-tensor categories (see 1.3.4–1.3.7), but in this case the self-duality 1.3.8(i) will be lost.

The notion of transversality for quotient sets of a given finite set is considered in 1.3.1–1.3.3. We define compound pseudo-tensor structure in 1.3.6–1.3.7, compound pseudo-tensor functors in 1.3.9, augmented version in 1.3.10. We consider duality in 1.3.11 and representable setting in 1.3.12–1.3.13. The notion of a compound category is treated in 1.3.14–1.3.16. An approach to the construction of compound tensor structures parallel to 1.1.13 is in 1.3.18–1.3.19; for an example see 1.3.20.

1.3.1. We have to fix some notation. For a finite non-empty set I denote by $Q(I)$ the set of all equivalence relations on I. We identify an element of $Q(I)$ with the corresponding quotient map $\pi_S: I \to S$ and denote it by (S, π_S) or simply S by abuse of notation; as usual for $s \in S$ we set $I_s := \pi_S^{-1}(s)$. The set $Q(I)$ is ordered: we write $S_1 \leq S_2$ iff S_1 is a quotient of S_2. It is a lattice: for any $S, T \in Q(I)$ there exist $\inf\{S, T\} \in Q(I)$ and $\sup\{S, T\} \in Q(I)$. For any $S \in Q(I)$ we have the obvious identifications $Q(I)_{\leq S} := \{S' \in Q(I): S' \leq S\} = Q(S)$, $Q(I)_{\geq S} := \{S'' \in Q(I): S'' \geq S\} = \prod_S Q(I_s)$.

[6] Here $\cdot_N \in P_2(\{L, N\}, N)$, $\cdot_{M_i} \in P_2(\{L, M_i\}, M_i)$ are the L-actions.

We say that $S, T \in Q(I)$ are *transversal* if $|S|+|T|=|I|+|\inf\{S,T\}|$; S and T are called *complementary* if they are transveral and $|\inf\{S,T\}|=1$.

1.3.2. Proposition. (i) *Let V be a non-zero vector space. Let us assign to $S \in Q(I)$ the corresponding "diagonal" subspace $V^S \subset V^I$. Then S, T are transversal iff the subspaces V^S, $V^T \subset V^I$ are (i.e., $V^S+V^T=V^I$), and S, T are complementary iff, in addition, $V^S \cap V^T = V^{\cdot} \subset V^I$.*

(ii) *Consider the graph G with vertices labeled by $S \sqcup T$ and edges labeled by I, such that an edge $i \in I$ connects $\pi_S(i)$ and $\pi_T(i)$. Then S and T are complementary (resp. transversal) iff G is a tree (resp. forest).*

Proof. (i) Note that $|S| = \dim V^S/\dim V$, and for any S, T one has $V^{\inf\{S,T\}} = V^S \cap V^T$, $S=T$ iff $V^S=V^T$. Therefore $|S|+|T| \leq |\sup\{S,T\}|+|\inf\{S,T\}|$, and the equality holds iff $V^{\sup\{S,T\}} = V^S+V^T$. Hence $|S|+|T| \leq |I|+|\inf\{S,T\}|$, and equality holds iff $V^I=V^S+V^T$.

(ii) Note that $|\inf\{S,T\}|$ coincides with the number of components of G, and compute the Euler characteristic of G. □

1.3.3. Corollary. (a) *Suppose $S, T \in Q(I)$ are transversal. Then*

(i) *for any $\alpha \in \inf\{S,T\}$ the equivalence relations induced by S, T on I_α are complementary;*

(ii) *for any $J \in Q(I)$ such that $J \geq S$ the pair $T, J \in Q(I)$ is transversal as well as $\inf\{T,J\}$, $S \in Q(J)$. The map $Q(I)_{\geq S} \to Q(T)$, $J \mapsto \inf\{T,J\}$ is injective.*

(b) *Suppose $S, P \in Q(I)$ are such that $\inf\{S,P\} = \cdot$. Then $P = \inf_{T\in\mathcal{T}}\{T\}$ where $\mathcal{T}$ is the set of all $T \in Q(I)$ such that $T \geq P$ and T, S are complementary.* □

1.3.4. Let $\mathcal{M}$ be a category. For any functors $F\colon \mathcal{M} \to Sets$, $G\colon \mathcal{M}^\circ \to Sets$ we denote by $\langle F, G\rangle$ the quotient of the disjoint union of sets $F(X)\times G(X)$, $X \in \mathcal{M}$, modulo the relations $(\varphi(f), g) = (f, {}^t\varphi(g))$, where $f \in F(X)$, $g \in G(Y)$, $\varphi \in \operatorname{Hom}(X,Y)$ (and X, Y are objects in $\mathcal{M}$). We tacitly assume that $\langle F, G\rangle$ is a well-defined set; to assure this, it suffices to know that isomorphism classes of objects in $\mathcal{M}$ form a set, or that either F or G is a representable functor (if, say, $F = \operatorname{Hom}(X_F, \cdot)$, then $\langle F, G\rangle = G(X_F)$).

The above construction generalizes to the setting of 1.1.7 in the obvious way. In particular, in the k-linear situation (so $\mathcal{M}$ is a k-category and F, G are k-linear functors with values in k-modules), the k-module $\langle F, G\rangle_k$ is the quotient of $\bigoplus_{X\in\mathcal{M}} F(X) \otimes_k G(X)$ modulo the relations $\varphi(f)\otimes g = f \otimes {}^t\varphi(g)$. We have an obvious map of sets $\langle F, G\rangle \to \langle F, G\rangle_k$ which is bijective if either F or G is representable.

1.3.5. Now assume that both categories $\mathcal{M}$ and $\mathcal{M}^\circ$ carry the pseudo-tensor structures given by functors P^*_I and $P^!_I$, respectively. It will be convenient to write the variables for $P^!_I$ in reverse order, i.e., as $P^!_I(M, \{L_i\})$ so that $P^!_{\cdot}(M, L) = \operatorname{Hom}_{\mathcal{M}}(M, L)$.

For any $I, J \in \mathcal{S}$ and I- and J-families of objects $\{L_i\}, \{M_j\}$, respectively, set

$$P^{*!}_{I,J}(\{L_i\},\{M_j\}) := \left\langle P^*_I(\{L_i\},\cdot), P^!_J(\cdot,\{M_j\})\right\rangle. \tag{1.3.5.1}$$

Our $P^{*!}_{I,J}$ is a functor; moreover, the $*$ and ! operations act on $P^{*!}$. Precisely, for any surjective maps $H \twoheadrightarrow J$, $G \twoheadrightarrow I$ and families of objects $\{N_h\}$, $\{K_g\}$ we have the composition map

$$P^!_{H/J}(\{M_j\},\{N_h\}) \times P^{*!}_{I,J}(\{L_i\},\{M_j\}) \times P^*_{G/I}(\{K_g\},\{L_i\}) \to P^{*!}_{G,H}(\{K_g\},\{N_h\}).$$

Note that $P^{*!}_{I,.} = P^*_I$, $P^{*!}_{.,J} = P^!_J$.

1.3.6. Assume that for any $I \in \mathcal{S}$ and a complementary pair $S, T \in Q(I)$ we have a natural map

$$\langle\quad\rangle^I_{S,T} : P^*_{I/S}(\{L_i\},\{M_s\}) \times P^!_{I/T}(\{N_t\},\{L_i\}) \to P^{*!}_{T,S}(\{N_t\},\{M_s\}), \tag{1.3.6.1}$$

$((\varphi_s),(\psi_t)) \mapsto \langle(\varphi_s),(\psi_t)\rangle^I_{S,T}$. Here $\{L_i\}$, $\{M_s\}$, $\{N_t\}$ are arbitrary families of objects parametrized by I, S, T, respectively, and "natural" means that $\langle\quad\rangle^I_{S,T}$ is functorial with respect to M_s and N_t variables. Note that $\langle\quad\rangle^I_{S,T}$ extends uniquely to the family of maps

$$\langle\quad\rangle^I_{S,T} : \prod_S P^{*!}_{I_s,H_s}(\{L_i\},\{B_h\}) \times \prod_T P^{*!}_{G_t,I_t}(\{A_g\},\{L_i\}) \to P^{*!}_{G,H}(\{A_g\},\{B_h\})$$

labeled by surjecitons $H \twoheadrightarrow S$, $G \twoheadrightarrow T$, in a way compatible with composition of $*$ and ! operations (acting on A_g and B_h, respectively).

1.3.7. Definition. The data $(P^*, P^!, \langle\quad\rangle)$ form a *compound pseudo-tensor structure* on $\mathcal{M}$ if the following compatibility axioms hold (here I, S, T, $\{L_i\}$, $\{M_s\}$, $\{N_t\}$ are as above):

(i) Take $H \in Q(I)$ such that $H \geq S$; set $V := \inf\{H, T\}$. Then for any H-family of objects $\{K_h\}$ the following diagram commutes:

$$\begin{array}{ccc} P^*_{H/S}(\{K_h\},\{M_s\})\times P^*_{I/H}(\{L_i\},\{K_h\})\times P^!_{I/T}(\{N_t\},\{L_i\}) & \longrightarrow & P^*_{I/S}(\{L_i\},\{M_s\})\times P^!_{I/T}(\{N_t\},\{L_i\}) \\ \Big\downarrow {\scriptstyle \prod_V \langle\ \rangle^{I_v}_{H_v,T_v}} & & \Big\downarrow {\scriptstyle \langle\ \rangle^I_{S,T}} \\ P^*_{H/S}(\{K_h\},\{M_s\}) \times \prod_V P^{*!}_{T_v,H_v}(\{N_t\},\{K_h\}) & \longrightarrow & P^{*!}_{T,S}(\{N_t\},\{M_s\}) \end{array}$$

Here the upper horizontal arrow is the product of composition maps for $I_s \twoheadrightarrow H_s$, the lower one is $\langle\ \rangle^H_{S,V}$.

(ii) Take $G \in Q(I)$ such that $G \geq T$; set $U := \inf\{S, G\}$. Then for any G-family of objects $\{R_g\}$ the following diagram commutes:

$$\begin{array}{ccc} P^*_{I/S}(\{L_i\},\{M_s\})\times P^!_{I/G}(\{R_g\},\{L_i\})\times P^!_{G/T}(\{N_t\}\{R_g\}) & \longrightarrow & P^*_{I/S}(\{L_i\},\{M_s\})\times P^!_{I/T}(\{N_t\},\{L_i\}) \\ \Big\downarrow {\scriptstyle \prod_U \langle\ \rangle^{I_u}_{S_u,G_u}} & & \Big\downarrow {\scriptstyle \langle\ \rangle^I_{S,T}} \\ \prod_U P^{*!}_{G_u,S_u}(\{R_g\},\{M_s\}) \times P^!_{G/T}(\{N_t\},\{R_g\}) & \longrightarrow & P^{*!}_{T,S}(\{N_t\},\{M_s\}) \end{array}$$

Here the upper horizontal arrow is the product of composition maps for $I_t \twoheadrightarrow G_t$, the lower one is $\langle\ \rangle^G_{U,T}$.

(iii) If $S = I$, $T = .$, $M_i = L_i$, then $\langle (id_{L_i}), \psi\rangle^I_{I,\,.} = \psi$ for any $\psi \in P^!_I(N, \{L_i\})$.

(iv) If $S = .$, $T = I$, $N_i = L_i$, then $\langle \varphi, (id_{L_i})\rangle^I_{.\,,I} = \varphi$ for any $\varphi \in P^*_I(\{L_i\}, M)$.

We call a category $\mathcal{M}$ equipped with a compound pseudo-tensor structure a *compound pseudo-tensor category*. We usually denote it by $\mathcal{M}^{*!}$, while $\mathcal{M}^*$ and $\mathcal{M}^{o!}$ denote $\mathcal{M}$ and $\mathcal{M}^o$ considered as plain pseudo-tensor categories (with the operations P^* and $P^!$, respectively).

1.3.8. Remarks. (i) The axioms (i)–(ii) and (iii)–(iv) are mutually dual. Therefore a compound pseudo-tensor structure on $\mathcal{M}$ amounts to that on $\mathcal{M}^\circ$.

(ii) Any *tensor* structure on $\mathcal{M}$ defines a compound pseudo-tensor structure: set $P^*_I(\{L_i\}, M) = \mathrm{Hom}(\underset{I}{\otimes} L_i, M)$, $P^!_I(N, \{L_i\}) = \mathrm{Hom}(N, \underset{I}{\otimes} L_i)$, and $\langle(\varphi_s), (\psi)_t\rangle^I_{S,T} = (\underset{S}{\otimes}\varphi_s) \circ (\underset{T}{\otimes}\psi_t)$ (note that $P^{*!}_{T,S}(\{N_t\}, \{M_s\}) = \mathrm{Hom}(\underset{T}{\otimes} N_t, \underset{S}{\otimes} M_s)$).

1.3.9. A *compound pseudo-tensor functor* $\tau^{*!} : \mathcal{N}^{*!} \to \mathcal{M}^{*!}$ between the compound pseudo-tensor categories is a pair of pseudo-tensor functors $\tau^* : \mathcal{N}^* \to \mathcal{M}^*$, $\tau^! : \mathcal{N}^{o!} \to \mathcal{M}^{o!}$ that give rise to the same usual functor $\tau\colon \mathcal{N} \to \mathcal{M}$, such that all the diagrams

$$\begin{array}{ccc}
P^*_{I/S}(\{L_i\},\{M_s\}) \times P^!_{I/T}(\{N_t\},\{L_i\}) & \xrightarrow{<\ >^I_{S,T}} & P^{*!}_{T,S}(\{N_t\},\{M_s\}) \\
\Big\downarrow{\scriptstyle \tau^*\times\tau^!} & & \Big\downarrow{\scriptstyle \tau^{*!}} \\
P^*_{I/S}(\{\tau L_i\},\{\tau M_s\}) \times P^!_{I/T}(\{\tau N_t\},\{\tau L_i\}) & \xrightarrow{<\ >^I_{S,T}} & P^{*!}_{T,S}(\{\tau N_t\},\{\tau M_s\})
\end{array}$$

commute. Here the arrow $\tau^{*!}$ is defined in the obvious way using τ^* and $\tau^!$.

1.3.10. Let us explain what an augmented compound pseudo-tensor category is. Let $\mathcal{M}^{*!}$ be a compound tensor category and h an augmentation functor on $\mathcal{M}^*$. We say that h is *compatible* with the compound structure, or h is a $*$-*augmentation functor* on $\mathcal{M}^{*!}$, if it satisfies the following condition. Let $S, T \in Q(I)$ be a complementary pair and $i_0 \in I$ an element such that $\pi_T^{-1}(t_0) = \{i_0\}$, $|\pi_S^{-1}(s_0)| \geq 2$ where $s_0 = \pi_S(i_0)$, $t_0 = \pi_T(i_0)$. Set $I^0 := I \smallsetminus \{i_0\}$, $T^0 := T \smallsetminus \{t_0\}$. We may consider S and T^0 as a complementary pair in $Q(I^0)$. Our condition is commutativity of the diagram

$$\begin{array}{ccc}
P^*_{I/S}(\{L_i\},\{M_s\}) \times P^!_{I/T}(\{N_t\},\{L_i\}) \times h(N_{t_0}) & \longrightarrow & P^*_{I^0/S}(\{L_i\},M_s) \times P^!_{I^0/T^0}(N_t,\{L_i\}) \\
\Big\downarrow{\scriptstyle <\ >^I_{S,T}} & & \Big\downarrow{\scriptstyle <\ >^{I^0}_{S,T^0}} \\
P^{*!}_{T,S}(\{N_t\},\{M_s\}) \times h(N_{t_0}) & \longrightarrow & P^{*!}_{T^0,S}(\{N_t\},\{M_s\})
\end{array}$$

Here the lower horizontal map is the obvious extension of h_{T,t_0} to $P^{*!}_{T,\cdot}$; the upper one is $((\varphi_s), (\psi_t), a) \mapsto ((\varphi_s, h_{I^0_{s_0},i_0}(\varphi_{s_0}, \psi_{t_0}(a))), \psi_t)$.

One spells out the compatibility property of a !-augmentation functor in a similar (dual) way. We call a compound pseudo-tensor category endowed with both $*$- and !-augmentation functors an *augmented* compound pseudo-tensor category.

The above definitions extend to the setting of 1.1.7 in the obvious way. In particular, one has the k-linear version (replace product of sets by tensor product of k-modules, etc.).

1.3.11. Duality. One can use the augmentation functors to relate the pseudo-tensor categories $\mathcal{M}^*$ and $\mathcal{M}^{o!}$. For example, consider a !-augmentation functor $h^!$ on $\mathcal{M}^{*!}$. Assume that $h^!$ is representable by an object $\mathbf{1}$, so $h^!(M) = \mathrm{Hom}_{\mathcal{M}}(M, \mathbf{1})$, and for any $M \in \mathcal{M}$ the object $M^\circ := \mathcal{H}om_{\mathcal{M}^*}(M, \mathbf{1})$ exists (here $\mathcal{H}om_{\mathcal{M}^*}$ means inner Hom with respect to $*$ structure; see 1.2.1).

Lemma. *The functor $\mathcal{M}^\circ \to \mathcal{M}$, $M \mapsto M^\circ$, admits a canonical pseudo-tensor extension*

$$\mathcal{M}^{\circ!} \to \mathcal{M}^*. \tag{1.3.11.1}$$

Proof. We need to define the maps $P^!_I(M, \{L_i\}) \to P^*_I(\{L_i^\circ\}, M^\circ)$. Let I^1, I^2 be two copies of I. Set $J := I^1 \sqcup I^2$, $\widetilde{I} = I \sqcup \cdot$, and let $\pi_I\colon J \twoheadrightarrow I$, $\pi_{\widetilde{I}}\colon J \twoheadrightarrow \widetilde{I}$ be the projections defined by formulas $\pi_I(i^1) = \pi_I(i^2) = i$, $\pi_{\widetilde{I}}(i^1) = i$, $\pi_{\widetilde{I}}(i^2) = \cdot$; here $i \in I$, and $i^1 \in I^1$, $i^2 \in I^2$ are the copies of i. Then $I, \widetilde{I} \in Q(J)$ is a complementary pair (e.g., by 1.3.2(ii)). Consider a J-family of objects $\{A_j\}$, $A_{i^1} = L_i^\circ$, $A_{i^2} = L_i$, an I-family of objects $\{\mathbf{1}_i\}$ (copies of $\mathbf{1}$), and an $\widetilde{I}$-family of objects which are L_i for $i \in I \subset \widetilde{I}$ and the $\cdot$ one is M. The desired map is the composition

$$P^!_I(M, \{L_i\}) \to P^{*!}_{\widetilde{I},I}(\{L_i^\circ, M\}, \{\mathbf{1}_i\}) \to P^*_{\widetilde{I}}(\{L_i^\circ, M\}, \mathbf{1}) \xrightarrow{\sim} P^*_I(\{L_i^\circ\}, M^\circ).$$

Here the first arrow sends $\psi \in P^!_I(M, \{L_i\})$ to $\big\langle (\kappa_i), (\psi, id_{L_i^\circ}) \big\rangle^J_{I,\widetilde{I}}$ (where $\kappa_i \in P^*_2(\{A_{i^1}, A_{i^2}\}, \mathbf{1}_i) = P^*_2(\{L_i^\circ, L_i\}, \mathbf{1})$ is the canonical pairing), the second arrow is $\lambda \mapsto h^!_I(\lambda)(id_{\mathbf{1}}^{\otimes I})$ (for $h^!_I$ see 1.2.7; we extend it to $P^{*!}$ in the obvious manner), and the last one is the canonical isomorphism (see 1.2.1). We leave it to the reader to check that the construction is compatible with the composition of operations. $\square$

Remark. Assume in addition that we have a $*$-augmentation functor h on $\mathcal{M}^{*!}$. Then $M \to h(M^\circ)$ is an augmentation functor on $\mathcal{M}^{\circ!}$. We have a canonical morphism

$$h(M^\circ) \to h^!(M) \tag{1.3.11.3}$$

defined as restriction of the structure map $P^*_2(\{M^\circ, M\}, \mathbf{1}) \times h(M^\circ) \to \mathrm{Hom}(M, \mathbf{1})$ (see (1.2.5.2)) to $\kappa \times h(M^\circ)$ where κ is the canonical pairing. We leave it to the reader to check that (1.3.11.3) is a morphism of augmentation functors on $\mathcal{M}^{\circ!}$.

1.3.12. Assume that the ! pseudo-tensor structure is representable (see 1.1.3). Then $P^*_{I,J}(\{L_i\}, \{M_j\}) = P^*_I(\{L_i\}, \overset{!}{\underset{J}{\otimes}} M_j)$, and we may rewrite the $\langle\ \ \rangle^I_{S,T}$ data as a collection of canonical morphisms (the *compound tensor product maps*)

$$\otimes^I_{S,T}\colon P^*_{I/S}(\{L_i\}, \{M_s\}) = \prod_S P^*_{I_s}(\{L_i\}, M_s) \to P^*_T(\{\overset{!}{\underset{I_t}{\otimes}} L_i\}, \overset{!}{\underset{S}{\otimes}} M_s), \tag{1.3.12.1}$$

$\otimes^I_{S,T} := \langle \cdot, (id_{\overset{!}{\otimes}_{I_t} L_i}) \rangle^I_{S,T}$. The axioms (i), (ii) of 1.3.7 can be rewritten as commutativity of the following diagrams (where $I, S, T, \dots$ have the same meaning as in 1.3.2):

(i)
$$\begin{array}{ccc}
P^*_{H/S}(\{K_h\}, \{M_s\}) \times P^*_{I/H}(\{L_i\}, \{K_h\}) & \longrightarrow & P^*_{I/S}(\{L_i\}, \{M_s\}) \\
\big\downarrow{\scriptstyle \otimes^H_{S,V} \times \prod_V \otimes^{I_v}_{H_v,T_v}} & & \big\downarrow{\scriptstyle \otimes^I_{S,T}} \\
P^*_V(\{\overset{!}{\underset{H_v}{\otimes}} K_h\}, \overset{!}{\underset{S}{\otimes}} M_s) \times P^*_{T/V}(\{\overset{!}{\underset{I_t}{\otimes}} L_i\}, \overset{!}{\underset{H_v}{\otimes}} K_h) & \longrightarrow & P^*_T(\{\overset{!}{\underset{I_t}{\otimes}} L_i\}, \overset{!}{\underset{S}{\otimes}} M_s)
\end{array}$$

(the horizontal arrows are composition maps)

$$\text{(ii)}\qquad \begin{array}{ccc} P^*_{I/S}(\{L_i\},\{M_s\}) & \xrightarrow{\otimes^I_{S,T}} & P^*_T(\{\underset{I_t}{\otimes}{}^! L_i\}, \underset{S}{\otimes}{}^! M_s) \\ \Big\downarrow {\scriptstyle \prod_U \otimes^{I_u}_{S_u,G_u}} & & \Big\downarrow \\ P^*_{G/U}(\{\underset{I_g}{\otimes}{}^! L_i\},\{\underset{S_u}{\otimes}{}^! M_s\}) & \longrightarrow & P^*_T(\{\underset{G_t}{\otimes}{}^!(\underset{I_g}{\otimes}{}^! L_i)\}, \underset{S}{\otimes}{}^! M_s) \end{array}$$

Here the right vertical arrow is the $*$ composition map with compatibility morphisms $\underset{G_t}{\otimes}{}^!(\underset{I_g}{\otimes}{}^! L_i) \to \underset{I_t}{\otimes}{}^! L_i$; the lower horizontal arrow is the composition of $\otimes^G_{U,T}$ and the map of the P^*_T arising from $\underset{U}{\otimes}{}^!(\underset{S_u}{\otimes}{}^! M_s) \to \underset{S}{\otimes}{}^! M_s$.

1.3.13. The simplest non-trivial $\otimes^I_{S,T}$ operations are those with $|S| = 2$ (the binary compound tensor products). They look as follows. If $S = \{1,2\}$, then the projection $I \twoheadrightarrow S$ amounts to a non-trivial decomposition $I = I_1 \sqcup I_2$. The complementarity of T, S means that the only non-trivial T-equivalence class I_t has two elements $i_1 \in I_1$, $i_2 \in I_2$; i.e., T coincides with $I_1 \vee I_2 =$ the union of I_1, I_2 with i_1, i_2 identified. We call the map

$$\underset{i_1,i_2}{\otimes} = \otimes^{I_1,I_2}_{i_1,i_2} := \otimes^I_{S,T} : P^*_{I_1}(\{L_i\}, M_1) \otimes P^*_{I_2}(\{L_i\}, M_2) \\ \to P^*_{I_1 \vee I_2}(\{L_i, L_{i_1} \otimes^! L_{i_2}\}, M_1 \otimes^! M_2),$$

$\varphi_1, \varphi_2 \mapsto \varphi_1 \underset{i_1,i_2}{\otimes} \varphi_2$, the *binary tensor product at* i_1, i_2 (here the objects L_i under the $P^*_{I_1 \vee I_2}$ sign are those for $i \in I \smallsetminus \{i_1, i_2\}$).

1.3.14. Assume that $\mathcal{M}^{\circ !}$ is actually a tensor category; we call such $\mathcal{M}^{*!}$ a *compound tensor category*. One may consider $\otimes^!$ as a tensor product on $\mathcal{M}$ itself; denote this tensor category as $\mathcal{M}^!$ (so $\mathcal{M}^!$ is the tensor category dual to $\mathcal{M}^{o!}$).

If we are in the k-linear situation, then we call $\mathcal{M}^{*!}$ *abelian* if $\mathcal{M}$ is an abelian k-category, the P^*_I are left exact functors, and $\otimes^!$ is right exact.

A *compound tensor functor* $\tau^{*!}: \mathcal{N}^{*!} \to \mathcal{M}^{*!}$ between the compound tensor categories is a compound pseudo-tensor functor such that all the canonical morphisms $\nu_I: \tau(\underset{I}{\otimes}{}^! N_i) \to \underset{I}{\otimes} \tau(N_i)$ from 1.1.6(ii) are isomorphisms; i.e., $\tau^!: \mathcal{N}^! \to \mathcal{M}^!$ is a tensor functor. Such $\tau^{*!}$ amounts to a pair $(\tau^*, \tau^!)$ where $\tau^*: \mathcal{N}^* \to \mathcal{M}^*$, $\tau^!: \mathcal{N}^! \to \mathcal{M}^!$ are, respectively, the pseudo-tensor and tensor extensions of the same functor $\tau: \mathcal{N} \to \mathcal{M}$ which commute with $\otimes^I_{S,T}$ maps.

REMARK. If $\mathcal{M}$ is a compound tensor k-categrory, then the category of super objects in $\mathcal{M}$ is a compound tensor category in the obvious way, as well as the categories of graded super objects and DG super objects (see 1.1.16 for the terminology).

1.3.15. If $\mathcal{M}^{*!}$ is a compound tensor category, then the right vertical arrow in diagram (ii) in 1.3.12 is an isomorphism. Therefore all the morphisms $\otimes^I_{S,T}$ are completely determined by binary ones (use this diagram to compute $\otimes^I_{S,T}$ inductively).

The binary tensor products are functorial, commutative and associative. Let us spell out these properties explicitly. Functoriality means that the binary products are compatible with composition of $*$ operations in the obvious sense, and $\otimes_{i_1,i_2}^{I_1,I_2}$ coincides with the usual ! tensor product of morphisms if $I_1 = \{i_1\}$, $I_2 = \{i_2\}$. Commutativity means that $\otimes_{i_1,i_2}^{I_1,I_2} = \otimes_{i_2,i_1}^{I_2,I_1}$. To spell out the associativity property, assume that we have three sets I_1, I_2, I_3 and the elements $i_1 \in I_1$, $i_2^1, i_2^3 \in I_2$, $i_3 \in I_3$; denote by $I_1 \vee I_2 \vee I_3$ the disjoint union of I_j's modulo the relation $i_1 = i_2^1$, $i_2^3 = i_3$. Take $\varphi_j \in P^*_{I_j}(\{L_i\}, M_j)$. The associativity says that

$$(\varphi_1 \underset{i_1,i_2^1}{\otimes} \varphi_2) \underset{i_2^3,i_3}{\otimes} \varphi_3 = \varphi_1 \underset{i_1,i_2^1}{\otimes} (\varphi_2 \underset{i_2^3,i_3}{\otimes} \varphi_3) \in P^*_{I_1 \vee I_2 \vee I_3}(\{L_i, L_{i_1} \otimes^! L_{i_2^1}, L_{i_2^3} \otimes^! L_{i_3}\}, M_1 \otimes^! M_2 \otimes^! M_3)$$

if $i_2^1 \neq i_2^3$, and

$$(\varphi_1 \underset{i_1,i_2}{\otimes} \varphi_2) \underset{i_2,i_3}{\otimes} \varphi_3 = \varphi_1 \underset{i_1,i_2}{\otimes} (\varphi_2 \underset{i_2,i_3}{\otimes} \varphi_3) = (\varphi_1 \underset{i_1,i_3}{\otimes} \varphi_3) \underset{i_1,i_2}{\otimes} \varphi_2 \in P^*_{I_1 \vee I_2 \vee I_3}(\{L_i, L_{i_1} \otimes^! L_{i_2} \otimes^! L_{i_3}\}, M_1 \otimes^! M_2 \otimes^! M_3)$$

if $i_2^1 = i_2^3 = i_2$.

It is easy to check that for a category $\mathcal{M}$ equipped with a tensor structure $\otimes^!$ and a pseudo-tensor structure P^* the data of functorial, commutative and associative binary tensor products amount to the whole data of compound tensor products, i.e., to a compound tensor structure.

Remark. (We use terminology from 1.1.6(v).) The obvious action of the tensor category $\mathcal{M}^!$ on $\mathcal{M}$ extends canonically to an $\mathcal{M}^!$-action on $\mathcal{M}^*$ defined by the maps $P^*_I(\{M_i\}, N) \to P^*_I(\{A_i \otimes^! M_i\}, (\otimes A_i) \otimes^! N)$, $\varphi \mapsto \varphi \otimes \{id_{A_i}\}$.

1.3.16. Let $\mathcal{M}^{*!}$ be a compound tensor category such that $\mathcal{M}^!$ has unit object $\mathbf{1}$ (as a tensor category). Then $h^!(M) := \mathrm{Hom}_{\mathcal{M}}(M, \mathbf{1})$ is a !-augmentation functor on $\mathcal{M}^{o!}$. We say that $\mathbf{1}$ is a *unit object* in $\mathcal{M}^{*!}$ if $h^!$ is compatible with the compound structure (see 1.3.10). Explicitly, this means that for any I-family of objects $\{L_i\}$ and an object M the binary tensor product at $i_0 \in I$,

$$\mathrm{End}\,\mathbf{1} \times P^*_I(\{L_i\}, M) \to P^*_I(\{L_i, \mathbf{1} \otimes^! L_{i_0}\}_{i \neq i_0}, \mathbf{1} \otimes^! M) = P^*_I(\{L_i\}, M),$$

sends $id_{\mathbf{1}} \times \varphi$ to φ for any $\varphi \in P^*_I(\{L_i\}, M)$.

Recall that in any tensor category $F = \mathrm{End}\,\mathbf{1}$ is an abelian semigroup (a commutative k-algebra in the k-linear situation) that acts canonically on any object M (the action $F \to \mathrm{End}\,M$ sends $f \in F$ to $f \otimes^! id_M \in \mathrm{End}(\mathbf{1} \otimes^! M) = \mathrm{End}\,M$). The above compatibility implies that the F-action on $P^*_I(\{L_i\}, M)$ through M coincides with the action through any L_i. If we are in the k-linear situation this means that the P^*_I are canonically F-modules and $\mathcal{M}^{*!}$ is actually a compound tensor F-category.

We will call a compound tensor category $\mathcal{M}^{*!}$ which has a unit and is endowed with a $*$-augmentation functor h simply an *augmented compound tensor category*. We leave the definition of an *augmented compound tensor functor* between such categories to the reader.

1.3.17. Assume we are in the k-linear situation. We say that our unit object $\mathbf{1}$ is a *strong unit object* in $\mathcal{M}^{*!}$ if for any $I \in \mathcal{S}$ and $L_i, M \in \mathcal{M}$ one has $P^*_{\tilde{I}}(\{L_i, \mathbf{1}\}, M) = 0$.

1.3.18. A *compound operad* is the same as a compound pseudo-tensor category with single object. One may use compound operads to define compound tensor structures on some categories of modules (cf. 1.1.13). As in 1.1.13 we explain this in the k-linear setting.

Let R be an associative k-algebra, and let $\mathcal{B}^{*!}$ be a compound *strict* R-operad, i.e., a compound k-operad $\mathcal{B}^{*!}$ together with isomorphism of k-algebras $R \xrightarrow{\sim} \mathcal{B}_1$. Explicitly, such $\mathcal{B}^{*!}$ is a pair $(\mathcal{B}^*, \mathcal{B}^!)$, where $\mathcal{B}^*$ is a strict R-operad, and $\mathcal{B}^!$ is a strict R^o-operad (R^o is the opposite algebra, so $\mathcal{B}^*_I$ is an $(R - R^{\otimes I})$-bimodule and $\mathcal{B}^!_I$ is an $(R^{\otimes I} - R)$-bimodule), together with the pairings

$$\langle\ \ \rangle^I_{S,T} : (\underset{S}{\otimes} \mathcal{B}^*_{I_s}) \underset{R^{\otimes I}}{\otimes} (\underset{T}{\otimes} \mathcal{B}^!_{I_t}) \to \mathcal{B}^!_S \underset{R}{\otimes} \mathcal{B}^*_T \tag{1.3.18.1}$$

defined for any complementary pair $S, T \in Q(I)$. These maps are morphisms of $(R^{\otimes S} - R^{\otimes T})$-bimodules, and the axioms in 1.3.7 just say that the diagram (here I, S, T, H, V are as in (i) in 1.3.7)

$$\begin{array}{ccc}
(\underset{S}{\otimes} \mathcal{B}^*_{H_s}) \underset{R^{\otimes H}}{\otimes} (\underset{H}{\otimes} \mathcal{B}^*_{I_h}) \underset{R^{\otimes I}}{\otimes} (\underset{T}{\otimes} \mathcal{B}^!_{I_t}) & \longrightarrow & (\underset{S}{\otimes} \mathcal{B}^*_{I_3}) \underset{R^{\otimes I}}{\otimes} (\underset{T}{\otimes} \mathcal{B}^!_{I_t}) \\
\downarrow & & \\
(\underset{S}{\otimes} \mathcal{B}^*_{H_S}) \underset{R^{\otimes H}}{\otimes} (\underset{V}{\otimes} \mathcal{B}^!_{H_v}) \underset{R^{\otimes V}}{\otimes} (\underset{V}{\otimes} \mathcal{B}^*_{T_v}) & & \downarrow \\
\downarrow & & \\
\mathcal{B}^!_S \underset{R}{\otimes} \mathcal{B}^*_V \underset{R^{\otimes V}}{\otimes} (\underset{V}{\otimes} \mathcal{B}^*_{T_v}) & \longrightarrow & \mathcal{B}^!_S \underset{R}{\otimes} \mathcal{B}^*_T
\end{array} \tag{1.3.18.2}$$

commutes as well as the corresponding "dual" diagram, and $\langle\ \ \rangle^I_{S,T}$ coincides with $id_{\mathcal{B}^*_T}$, $id_{\mathcal{B}^!_S}$ if $|S| = 1$, resp. $|T| = 1$.

Now any such $\mathcal{B}^{*!}$ defines a compound pseudo-tensor structure on mod$R$. Namely, set (see 1.1.13)

$$\begin{aligned}
P^*_I(\{L_i\}, M) &= \mathrm{Hom}_{\mathrm{modR}^{\otimes I}}(\underset{I}{\otimes} L_i,\ M \underset{R}{\otimes} \mathcal{B}^*_I), \\
P^!_I(N, \{L_i\}) &= \mathrm{Hom}_{\mathrm{modR}}\big(N, (\underset{I}{\otimes} L_i) \underset{R^{\otimes I}}{\otimes} \mathcal{B}^!_I\big).
\end{aligned} \tag{1.3.18.3}$$

The composition of operations comes from composition laws in $\mathcal{B}^*$ and $\mathcal{B}^!$. Note that the ! pseudo-tensor structure is representable (one has $\overset{!}{\underset{I}{\otimes}} L_i = (\underset{I}{\otimes} L_i) \underset{R^{\otimes I}}{\otimes} \mathcal{B}^!_I$), so we may use the format of 1.3.12.

The compound tensor product

$$\otimes^I_{S,T} \colon \underset{S}{\otimes} P^*_{I_s}(\{L_i\}, M_s) \to P^*_T(\{\overset{!}{\underset{I_t}{\otimes}} L_i\}, \underset{S}{\otimes}^! M_s) \tag{1.3.18.4}$$

sends $\otimes\varphi_s \in \underset{S}{\otimes} \mathrm{Hom}_{\mathrm{modR}^{\otimes I_s}}(\underset{I_s}{\otimes} L_i, M_s \underset{R}{\otimes} \mathcal{B}^*_{I_s})$ to the composition

$$\begin{aligned}
\underset{T}{\otimes}\big((\underset{I_t}{\otimes} L_i) \underset{R^{\otimes I_t}}{\otimes} \mathcal{B}^!_{I_t}\big) = (\underset{I}{\otimes} L_i) \underset{R^{\otimes I}}{\otimes} (\underset{T}{\otimes} \mathcal{B}^!_{I_t}) &\xrightarrow{\otimes\varphi_s} (\underset{S}{\otimes} M_s) \underset{R^{\otimes S}}{\otimes} (\underset{S}{\otimes} \mathcal{B}^*_{I_s}) \underset{R^{\otimes I}}{\otimes} (\underset{T}{\otimes} \mathcal{B}^!_{I_t}) \\
&\xrightarrow{\langle\ \ \rangle^I_{S,T}} (\underset{S}{\otimes} M_s) \underset{R^{\otimes S}}{\otimes} \mathcal{B}^!_S \underset{R}{\otimes} \mathcal{B}^*_T
\end{aligned}$$

We leave it to the reader to check the commutativity of diagrams (i) and (ii) in 1.3.12.

Note that $\otimes^!$ is a tensor product (i.e., we have a compound tensor structure) iff for any $J \twoheadrightarrow I$ the composition map $(\underset{I}{\otimes} \mathcal{B}^!_{J_i}) \underset{R^{\otimes I}}{\otimes} \mathcal{B}^!_I \to \mathcal{B}^!_J$ is an isomorphism.

1.3.19. Now suppose that $\mathcal{B}^{*!}$ is a $*$-augmented compound operad (see 1.3.10). So we have a left R-module $\mathcal{B}^*_0$ and the maps $\mathcal{B}^*_I \underset{R}{\otimes} \mathcal{B}^*_0 \to \mathcal{B}_{I\setminus\{i_0\}}, i_0 \in I$, that are compatible with $\langle\ \ \rangle$ in the sense explained in 1.3.10. According to 1.2.9, we get an augmentation functor h^* for the $*$ structure on mod$R$, $h^*(M) := M \underset{R}{\otimes} \mathcal{B}^*_0$. We leave it to the reader to check that h^* is a $*$-augmentation functor for our compound pseudo-tensor structure (see 1.3.10). Similarly, a !-augmentation for $\mathcal{B}^{*!}$ defines a !-augmentation $h^!$ on mod$R$ by the formula $h^!(M) = \mathrm{Hom}_{\mathrm{mod}R}(M, \mathcal{B}^!_0)$.

1.3.20. Example. Assume we have a cocommutative Hopf algebra structure on R. It yields a similar structure on R^o (the coproduct map for R^o coincides with that for R). By 1.1.14 we have the strict R- and R^o-operads $\mathcal{B}^* = \mathcal{B}^R$ and $\mathcal{B}^! = \mathcal{B}^{R^o}$, respectively. Let us show that $\mathcal{B}^*$ and $\mathcal{B}^!$ form a compound strict R-operad. We need to define the maps $\langle\ \ \rangle^I_{S,T}$. For $S \in Q(I)$ denote by $\triangle^I_S\colon R^{\otimes S} \to R^{\otimes I}$ the morphism of k-algebras $\underset{S}{\otimes} \triangle^{(I_s)}$ (recall that $\triangle^{(I_s)}\colon R \to R^{\otimes I_s}$ is the I_s-multiple coproduct). Let us consider $R^{\otimes I}$ as a $(R^{\otimes S} - R^{\otimes T})$-bimodule with respect to the action $(\otimes r_s)(\otimes r_i)(\otimes r_t) := \triangle^I_S(\otimes r_s) \cdot (\otimes r_i) \cdot \triangle^I_T(\otimes r_t)$. We have the obvious indentifications of $(R^{\otimes S} - R^{\otimes T})$-bimodules $(\underset{S}{\otimes} \mathcal{B}^*_{I_s}) \underset{R^{\otimes I}}{\otimes} (\underset{T}{\otimes} \mathcal{B}^!_{I_t}) = (\underset{S}{\otimes} R^{\otimes I_s}) \underset{R^{\otimes I}}{\otimes} (\underset{T}{\otimes} R^{\otimes I_t}) = (R^{\otimes I}) \underset{R^{\otimes I}}{\otimes} (R^{\otimes I}) = R^{\otimes I}$, $\mathcal{B}^!_S \underset{R}{\otimes} \mathcal{B}^*_T = R^{\otimes S} \underset{R}{\otimes} R^{\otimes T}$. Consider the morphism of $(R^{\otimes S} - R^{\otimes T})$-bimodules

$$\nu\colon R^{\otimes S} \underset{R}{\otimes} R^{\otimes T} \to R^{\otimes I},$$

$\nu\big((\otimes r_s) \otimes (\otimes r_t)\big) := (\otimes r_s) 1 (\otimes r_t) = \triangle^I_S(\otimes r_s) \cdot \triangle^I_T(\otimes r_t)$. If S, T are complementary, then ν is an isomorphism (use 1.3.2(ii)), and our $\langle\ \ \rangle^I_{S,T}$ is ν^{-1}.

We leave it to the reader to check that $\mathcal{B}^{*!} = (\mathcal{B}^*, \mathcal{B}^!, \langle\ \ \rangle)$ is a compound operad. If our Hopf algebra has counit, then the operads $\mathcal{B}^*$ and $\mathcal{B}^!$ are augmented in a natural way (see 1.2.10); it is easy to see that $(\mathcal{B}^{*\hat{}}, \mathcal{B}^{!\hat{}}, \langle\ \ \rangle)$ is an augmented compound operad.

1.4. Rudiments of compound geometry

This section treats some basic algebro-geometric constructions in compound setting. We consider matrix algebras in 1.4.2, Lie* algebras and their cohomology in 1.4.4–1.4.5, commutative$^!$ algebras in 1.4.6, reliable augmentation functors in 1.4.7, differential $*$ operations in 1.4.8, Lie* algebroids and modules over them in 1.4.11–1.4.12, Lie coalgebroids and the de Rham-Chevalley complexes in 1.4.14, the formal groupoid of a Lie coalgebroid in 1.4.15, the tangent Lie* algebroid in 1.4.16, connections in 1.4.17, coisson and odd coisson algebras in 1.4.18, hamiltonian reduction in 1.4.19, coisson modules in 1.4.20, the BRST reduction in 1.4.21–1.4.26. Finally, we explain in 1.4.27 how to encode a compound tensor structure by a single pseudo-tensor structure; the construction looks ad hoc at the moment, but in 3.2 we will see that it represents the "classical limit" of the chiral structure.

The reader is invited to attend to the continuation of the list.

For usual Lie algebroids see the textbook [**M**], written in the differential geometry setting, or a brief exposition in 2.9.1 below. For the BRST in the usual (non-compound) setting see, e.g., [**KS**]. Statement 1.4.24(iii) is a version of a theorem of Akman [**A**].

In this section $\mathcal{M}^{*!}$ is an abelian augmented compound tensor k-category (see 1.3.14 and 1.3.16); everything adapts immediately to the sheafified setting of 1.1.15. We abbreviate $\otimes^!$ to $\otimes$. There are no assumptions about exactness of the $*$-augmentation functor h.

1.4.1. $\mathcal{B}^*$ and $\mathcal{B}^!$ algebras. For $M \in \mathcal{M}$ set $\Gamma(M) := \mathrm{Hom}(\mathbf{1}, M)$. Then Γ is an augmentation functor on $\mathcal{M}^!$; in particular $\Gamma\colon \mathcal{M}^! \to k\mathrm{mod}$ is a pseudo-tensor functor (see 1.2.4 and 1.2.7).

For a k-operad $\mathcal{B}$ we denote by $\mathcal{B}(\mathcal{M}^*)$ the category of $\mathcal{B}$ algebras in $\mathcal{M}^*$; we call them simply $\mathcal{B}^*$ *algebras*. In particular we have the category $\mathcal{L}ie(\mathcal{M}^*)$ of Lie* algebras. If $\mathcal{B}^{\wedge}$ is an augmented k-operad, then $\mathcal{B}^{\wedge}$ algebras in the augmented category $\mathcal{M}^*$ are called $\mathcal{B}^{\wedge *}$ algebras. In particular, we have the category of unital associative* algebras (see 1.2.8).

Similarly, we refer to $\mathcal{B}$ algebras in $\mathcal{M}^!$ as $\mathcal{B}^!$ algebras.

As follows from Remark in 1.3.15 (and 1.1.6(v)), if E is a $\mathcal{B}^*$ algebra and R a $\mathcal{B}'^{!}$ algebra, then $E \otimes^! R$ is a $(\mathcal{B}\otimes\mathcal{B}')^*$ algebra. In particular, if R is a commutative$^!$ algebra, then $E \otimes R$ is a $\mathcal{B}^*$ algebra.

1.4.2. Matrix algebras and non-degenerate $*$ pairings. Assume we have $V, V' \in \mathcal{M}$ and $\langle\ \rangle \in P_2^*(\{V', V\}, \mathbf{1})$ a $*$ pairing. Then $E = V \otimes V'$ carries a canonical structure of associative* algebra which acts on V from the left and on V' from the right. Namely, the product operation $\cdot \in P_2^*(\{E, E\}, E)$ is the image of $id_V \otimes \langle\ \rangle \otimes id_{V'} \in \mathrm{E}nd(V) \otimes P_2^*(\{V', V\}, \mathbf{1}) \otimes \mathrm{E}nd(V')$ by the $\otimes^I_{S,T}$ map, where $I = \{1, 2, 3, 4\}$, and S, T are equivalence relations $\{2 = 3\}$ and $\{1 = 2, 3 = 4\}$, respectively. The E-action on V is the image of $id_V \otimes \langle\ \rangle$ by the binary map $\otimes^{\{1\},\{2,3\}}_{1,2}\colon \mathrm{E}nd(V) \otimes P_2^*(\{V', V\}, \mathbf{1}) \to P_2^*(\{E, V\}, V)$ (see 1.3.13 for notation). We leave the definition of the right E-action on V', as well as verification of associativity, to the reader.

One often wants to know if the E-action on V identifies E with $\mathcal{E}nd\, V$. The following condition on $\langle\ \rangle$ assures this:

Definition. A $*$ pairing $\langle\ \rangle \in P_2^*(\{V', V\}, \mathbf{1})$ is *left non-degenerate* if for every finite set I and objects A_i, $B \in \mathcal{M}$, $i \in I$, the canonical map

$$P_I^*(\{A_i\}, B \otimes V') \to P_{\tilde{I}}^*(\{A_i, V\}, B), \quad \varphi \mapsto (id_B \otimes \langle\ \rangle)(\varphi, id_V), \tag{1.4.2.1}$$

is an isomorphism.[7] We say that $\langle\ \rangle$ is *right non-degenerate* if the transposed pairing is left non-degenerate; $\langle\ \rangle$ is *non-degenerate* if it is both left and right non-degenerate.

If $\langle\ \rangle$ is left non-degenerate, then for every $B \in \mathcal{M}$ one has

$$B \otimes V' \xrightarrow{\sim} \mathcal{H}om(V, B), \quad h(B \otimes V') \xrightarrow{\sim} \mathrm{Hom}(V, B) \tag{1.4.2.2}$$

(see 1.2.1 and 1.2.2, and (1.4.2.1) for $I = \emptyset$). In particular, taking $B = \mathbf{1}$, we see that $V' = \mathcal{H}om(V, \mathbf{1})$; i.e., V' equals the $*$ dual V° to V, and $E = V \otimes V' \xrightarrow{\sim} \mathcal{E}nd\, V$,

[7] Here $id_B \otimes \langle\ \rangle \in P_2^*(\{B \otimes V', V\}, B)$.

$h(E) \xrightarrow{\sim} \operatorname{End} V$. Our E is a unital associative* algebra: the unit element[8] is $id_V \in \operatorname{End} V = h(E)$. Notice that V' is $\otimes$-flat; if h is right exact, then V is a projective object.

Exercise. The above definitions make sense if we replace $\mathcal{M}$ by the DG category of complexes $C\mathcal{M}$ (or $C\mathcal{M}^s$; see 1.1.16). Show that a complex L admits a dual L° if and only if every L^n admits a dual (then $(L^\circ)^n = (L^{-n})^\circ$, $d_{L^\circ} = -{}^t d_L$). A pairing $\langle\ \rangle \in P_2^*(\{L', L\}, \mathbf{1})$ is non-degenerate if and only if L is supported in finitely many degrees and every component $\in P_2^*(\{L'^n, L^{-n}\}, \mathbf{1})$ is non-degenerate.

1.4.3. We will often use the fact that $\otimes$ behaves naturally with respect to $*$ actions on multiples. Precisely, assume we have a map $I \to J$ of finite sets and families of objects $\{L_i\}_{i\in I}$, $\{M_j\}_{j\in J}$, $\{N_j\}_{j\in J}$. Then there is a canonical map

$$(1.4.3.1) \quad \otimes^{I\sqcup J}_{J,\tilde{I}} \colon \underset{J}{\otimes} P^*_{\tilde{I}_j}(\{L_i, M_j\}, N_j) \to P^*_{\tilde{I}}(\{L_i, \underset{J}{\otimes} M_j\}, \underset{J}{\otimes} N_j)\ , \quad \otimes\varphi_j \mapsto \underset{J}{\otimes} \varphi_j,$$

compatible with composition of operations.

Example. Assume we have a family $\{\mathcal{B}_j\}_{j\in J}$ of k-operads together with the corresponding $\mathcal{B}_j$ module operads $\mathcal{C}_j$. Let L_j be a $\mathcal{B}_j^*$ algebra, M_j an L_j-module in $\mathcal{M}^*$ with respect to $\mathcal{C}_j$. Then $\underset{J}{\otimes} M_j$ carries canonical L_j-actions that mutually commute. Namely, for $j_0 \in J$ the j_0's action map $\mathcal{C}_{j_0 I} \to P^*_{\tilde{I}}(\{L_{j_0 I}, \underset{J}{\otimes} M_j\}, \underset{J}{\otimes} M_j)$ sends $c \in \mathcal{C}_{j_0 I}$ to $c_{M_{j_0}} \otimes (\underset{j\neq j_0}{\otimes} id_{M_j})$, where $c_{M_{j_0}} \in P^*_{\tilde{I}}(\{L_{j_0 I}, M_{j_0}\}, M_{j_0})$ is the c-action on M_{j_0}.

Recall that $h(L_j)$ is a $\mathcal{B}_j$ algebra and M_j is automatically an $h(L_j)$-module (see 1.2.17). It is easy to see that the action of $h(L_{j_0})$ on $\underset{J}{\otimes} M_j$ that comes from the above L_{j_0}-action on $\underset{J}{\otimes} M_j$ coincides with the action $a \mapsto a_{M_{j_0}} \otimes (\otimes id_{M_j})_{j\neq j_0}$.

1.4.4. Lie* algebras and modules. Let L be a Lie* algebra. Denote by $\mathcal{M}(L)$ the category of L-modules in $\mathcal{M}^*$.

Lemma. *$\mathcal{M}(L)$ is naturally an abelian augmented compound tensor k-category. We denote it by $\mathcal{M}(L)^{*!}$.*

Proof. According to 1.2.18, $\mathcal{M}(L)$ carries a canonical abelian pseudo-tensor structure P^*_L. If $\{M_j\}_{j\in J}$ is a finite family of L-modules, then L acts on $\underset{J}{\otimes} M_j$ via the sum of the above J commuting actions (i.e., by Leibniz rule). This is the ! tensor product of L-modules. One easily checks that the maps $\otimes^I_{S,T}$ send L-linear operations to L-linear ones; this provides the maps $\otimes^I_{S,T}$ for $\mathcal{M}(L)$. Compatibility axioms 1.3.12 are clear.

The object $\mathbf{1}$ with trivial L-action is a unit object in $\mathcal{M}_L^{*!}$. For an L-module M the L-action on M yields the morphism of k-modules $h(M) \to \operatorname{Hom}_{\mathcal{M}}(L, M)$. Denote by $h^L(M)$ its kernel. h^L is a $*$-augmentation functor on $\mathcal{M}(L)^{*!}$. □

Remark. A morphism $L' \to L$ of Lie* algebras yields the "restriction of action" augmented compound tensor functor $\mathcal{M}(L)^{*!} \to \mathcal{M}(L')^{*!}$.

[8] See 1.2.8.

1.4.5. The Chevalley complex. The constructions below (except Remark (ii)) use only $*$ pseudo-tensor structure on $\mathcal{M}$.

For a Lie* algebra L and an L-module M we have the *Chevalley complex* $C(L, M)$ which is a complex with terms $C^n(L, M) := P^*_n(\{L, \dots, L\}, M)_{sgn}$ (the skew-coinvariants of the action of the symmetric group Σ_n on operations of n variables) and differential given by the standard formula (see below). Set $H^{\cdot}(L, M) = H^{\cdot}C(L, M)$; these are the *cohomology of L with coefficients in M.*

As in the case of usual Lie algebras, the Chevalley complex is naturally defined in the more general super setting (see 1.1.16; below we often drop the adjective "super"). Namely, let L be a super Lie* algebra, M a super L-module. Consider the graded super k-module with components $P^*_n(\{L[1], \dots, L[1]\}, M)_{\Sigma_n}$ (coinvariants of the symmetric group action on super operations of n variables). It carries a canonical odd differential δ of degree 1 coming from the morphism $P^*_n(\{L[1], \dots, L[1]\}, M)[-1] \to P^*_{n+1}(\{L[1], \dots, L[1]\}, M)$, $\varphi \mapsto \cdot_M(id_{L[1]}, \varphi) - \frac{1}{2}\varphi([\ \ ], id_{L[1]}, \dots, id_{L[1]})$ where $\cdot_M \in P_2(\{L, M\}, M)$ is the action of L on M. One has $\delta^2 = 0$, so we have a DG super k-module denoted by $C(L, M)$. Let us describe its natural symmetries.

Consider the DG Lie* algebra $L_\dagger$ (see 1.1.16 for notation). The DG Lie algebra $h(L_\dagger) = h(L)_\dagger$ acts on $C(L, M)$ in a canonical way. Namely, $h(L) \subset h(L_\dagger)$ acts via the adjoint action on copies of $L[1]$ and the L-module structure on M, and the action of its complement $h(L)[1]$ comes from the map $h(L[1]) \otimes P^*_n(\{L[1] \dots .L[1]\}, M) \to P^*_{n-1}(\{L, \dots, L\}, M)$ which is minus the sum of the canonical "convolution" maps for each of the n arguments. The above constructions are functorial in the obvious manner.

REMARKS. (i) If for every n the inner P object $\mathcal{P}^*_n(\{L, \dots, L\}, M)$ is well defined (see 1.2.1), then one has the inner Chevalley complex $\mathcal{C}(L, M)$ defined in the obvious way. It carries a canonical $*$ action of $L_\dagger$. There is a canonical morphism of DG $h(L_\dagger)$-modules $h(\mathcal{C}(L, M)) \to C(L, M)$ which is an isomorphism if our $\mathcal{P}$'s are P objects in the augmented sense (see 1.2.7).

(ii) Let us assume that the $*$ dual $L^\circ := \mathcal{H}om^*(L, \mathbf{1})$ is well defined and the canonical pairing $\langle\ \ \rangle \in P^*_2(\{L^\circ, L\}, \mathbf{1})$ is non-degenerate (see 1.4.2). Then for every $n \geq 0$ and every $N \in \mathcal{M}$ one has $N \otimes L^{\circ \otimes n} = \mathcal{P}^*_n(\{L, \dots, L\}, N)$ and $h(N \otimes L^{\circ \otimes n}) \xrightarrow{\sim} P^*_n(\{L, .., L\}, N)$ (see 1.2.1 and 1.2.7). Thus the inner Chevalley complex is well-defined (for an L-module M one has $\mathcal{C}(L, M)^n = M \otimes \mathrm{Sym}^n(L^\circ[-1])$), and we have a canonical isomorphism of complexes

$$h(\mathcal{C}(L, M)) \xrightarrow{\sim} C(L, M). \tag{1.4.5.1}$$

Consider now the super DG setting (see 1.1.16), so L is a DG Lie* super algebra, M a DG super L-module. The *Chevalley super complex* $C(L, M)$ is defined then as follows. Its components are $C^a(L, M) := \prod_n P^*_n(\{L[1], \dots, L[1]\}, M)^a$ (the operations of degree a with respect to the $\mathbb{Z}$-gradings). Its differential is the sum of two components: the Chevalley component considered above and that coming from differentials of L, M by transport of structure. As above, the DG Lie super algebra $h(L_\dagger)$ acts on $C(L, M)$.

Our $C(L, M)$ carries a decreasing filtration $F^n = C^{\geq n}(L, M)$ formed by $P^*_{\geq n}$. We equip $L_\dagger$ with a decreasing filtration $F^{-1} = L_\dagger$, $F^0 = L$, $F^1 = 0$; then the $h(L_\dagger)$-action on $C(L, M)$ is compatible with the F-filtrations.

1.4.6. Commutative$^!$ algebras and modules. We can do some commutative algebra in $\mathcal{M}^!$ as in any abelian tensor k-category. Denote by $\mathcal{C}omu(\mathcal{M}^!)$ the category of commutative and associative algebras with unit in $\mathcal{M}^!$. An object $A \in \mathcal{C}omu(\mathcal{M}^!)$ (we call it a *commutative$^!$ algebra*) is an object $A \in \mathcal{M}$ together with a commutative and associative product $\cdot = \cdot_A \colon A \otimes A \to A$ and a morphism $1 = 1_A \colon \mathbf{1} \to A$ which is a unit for $\cdot$. For a finite family $\{A_i\}$ of commutative$^!$ algebras the tensor product $\underset{I}{\otimes} A_i$ has an obvious structure of commutative$^!$ algebra; this is the coproduct of $\{A_i\}$ in $\mathcal{C}omu(\mathcal{M}^!)$. Therefore $\otimes$ defines a tensor category structure on $\mathcal{C}omu(\mathcal{M}^!)$; the object $\mathbf{1}$ endowed with an obvious product is a unit in $\mathcal{C}omu(\mathcal{M}^!)$.

For $A \in \mathcal{C}omu(\mathcal{M}^!)$ we denote by $\mathcal{M}(A)$ the category of unital A-modules in $\mathcal{M}^!$; this is an abelian k-category. The usual tensor product $\underset{A}{\otimes}$ of A-modules is right exact, so $\mathcal{M}(A)$ is an abelian tensor k-category with unit object A. The "forgetting of A-action" functor $\mathcal{M}(A) \to \mathcal{M}$ admits a left adjoint "induction" functor $\mathcal{M} \to \mathcal{M}(A)$, $M \mapsto A \otimes M$, which is a tensor functor. Below for an A-module N we denote by $\cdot_N \colon A \otimes N \to N$ the action map.

Lemma. *$\mathcal{M}(A)$ is naturally an abelian augmented compound tensor k-category. We denote it by $\mathcal{M}(A)^{*!}$.*

Proof. The tensor structure has been discussed already, so it remains to define the $*$ operations P^*_{AI}, the compound tensor product maps $\otimes^I_{S,T}$, and the $*$-augmentation functor.

Take $I \in \mathcal{S}$ and objects $\{L_i\}_{i \in I}$, M of $\mathcal{M}$. Assume that some L_{i_0}, $i_0 \in I$, and M are A-modules. We say that an operation $\varphi \in P^*_I(\{L_i\}, M)$ is *A-linear with respect to the i_0th variable* if the operations $\cdot_M(\varphi \otimes_{i_0} id_A), \varphi(id_{L_i}, \cdot_{L_{i_0}})_{i \neq i_0} \in P^*_I(\{L_i, A \otimes L_{i_0}\}_{i \neq i_0}, M)$ coincide. Here $\varphi \otimes_{i_0} id_A \in P^*_I(\{L_i, A \otimes L_{i_0}\}_{i \neq i_0}, A \otimes M)$. If all L_i are A-modules, then we denote by $P^*_{AI}(\{L_i\}, M) \subset P^*_I(\{L_i\}, M)$ the k-submodule of *A-polylinear* (i.e., A-linear with respect to each variable) operations. The composition of A-polylinear operations is again A-polylinear, so P^*_{AI} form a pseudo-tensor structure on $\mathcal{M}(A)$. The functors P^*_{AI} are left exact.

To define the structure maps $\otimes^I_{S,T} \colon \underset{S}{\otimes} P^*_{AI_s}(\{L_i\}, M_s) \to P^*_{AT}(\{\underset{AI_t}{\otimes} L_i\}, \underset{AS}{\otimes} M_s)$, consider the embedding $P^*_{AT}(\{\underset{AI_t}{\otimes} L_i\}, \underset{AS}{\otimes} M_s) \hookrightarrow P^*_T(\{\underset{I_t}{\otimes} L_i\}, \underset{AS}{\otimes} M_s)$ defined by the projections $\underset{I_t}{\otimes} L_i \to \underset{AI_t}{\otimes} L_i$. The composition $\underset{S}{\otimes} P^*_{AI}(\{L_{i_s}\}, M_s) \to \underset{S}{\otimes} P^*_{I_s}(\{L_i\}, M_s) \xrightarrow{\otimes^I_{S,T}} P^*_T(\{\underset{I_t}{\otimes} L_i\}, \underset{S}{\otimes} M_s) \to P^*_T(\{\underset{I_t}{\otimes} L_i\}, \underset{AS}{\otimes} M_s)$ takes values in the image of the embedding. Therefore we can consider it as a morphism $\underset{S}{\otimes} P^*_{AI_s}(\{L_i\}, M_s) \to P^*_T(\{\underset{AI_t}{\otimes} L_{i_t}\}, \underset{AS}{\otimes} M_s)$; this is the $\otimes^I_{S,T}$ map for $\mathcal{M}(A)^{*!}$.

Compatibility axioms 1.3.12 for $\mathcal{M}(A)^{*!}$ follow from those for $\mathcal{M}^{*!}$.

The object A, considered as an A-module, is a unit object in $\mathcal{M}(A)^{*!}$. Note that the morphism $1 \colon \mathbf{1} \to A$ yields a canonical isomorphism $End_{\mathcal{M}(A)}(A) \xrightarrow{\sim} \Gamma(A)$ of commutative k-algebras. Therefore, by 1.3.16, $\mathcal{M}(A)^{*!}$ is actually a compound tensor $\Gamma(A)$-category.

We define the $*$-augmentation functor on $\mathcal{M}^{*!}_A$ simply as the restriction of h to $\mathcal{M}(A)$ (note that h_{I,i_0} maps preserve A-polylinear operations). $\square$

REMARKS. (i) The "induction" functor $\mathcal{M} \to \mathcal{M}(A)$ extends naturally to an augmented compound tensor functor $\mathcal{M}^{*!} \to \mathcal{M}(A)^{*!}$. Namely, we already mentioned that the induction is a ! tensor functor. The map

$$P_I^*(\{F_i\}, G) \to P_{AI}^*(\{A \otimes F_i\}, A \otimes G) \tag{1.4.6.1}$$

is the composition $P_I^*(\{F_i\}, G) \to P_I^*(\{A \otimes F_i\}, A^{\otimes I} \otimes G) \to P_{AI}^*(\{A \otimes F_i\}, A \otimes G)$ where the first arrow sends φ to $\otimes_{I,I}^{I \sqcup I}(\varphi, \{id_A\}_{i \in I})$ and the second arrow is the composition with $\cdot_I \otimes id_G$ where $\cdot_I \colon A^{\otimes I} \to A$ is the I-fold product. We leave it to the reader to check the compatibility with $\otimes_{S,T}^I$ maps (see 1.3.14). The arrow $h(F) \to h(A \otimes F)$ is $h(1_A \otimes id_F)$.

More generally, any morphism of commutative$^!$ algebras $f : B \to A$ yields a compound tensor functor (base change)

$$f^* : \mathcal{M}(B)^{*!} \to \mathcal{M}(A)^{*!} \tag{1.4.6.2}$$

such that for $M \in \mathcal{M}(B)$ one has $f^*M = A \underset{B}{\otimes} M$. To define f^*, you just replace $\mathcal{M}^{*!}$ by $\mathcal{M}(B)^{*!}$ in the above considerations. The base change is compatible with the composition of f's in the obvious way.

(ii) The induction pseudo-tensor functor $\mathcal{M}^* \to \mathcal{M}(A)^*$ is left adjoint (see 1.1.5) to the obvious forgetful pseudo-tensor functor $\mathcal{M}(A)^* \to \mathcal{M}^*$. To see this, consider for $M \in \mathcal{M}(A)$ and $F_i, L_j \in \mathcal{M}$ the map $P_{I \sqcup J}^*(\{A \otimes F_i, L_j\}, M) \to P_{I \sqcup J}^*(\{F_i, L_j\}, M)$ sending an operation ψ to its composition with $1_A \otimes id_{F_i} \colon F_i \to A \otimes F_i$. This projection identifies the subspace of operations A-linear with respect to I variables with $P_{I \sqcup J}^*(\{F_i, L_j\}, M)$. The inverse map sends $\varphi \in P_{I \sqcup J}^*(\{F_i, L_j\}, M)$ to the composition of $\varphi \otimes_I (\otimes id_A) \in P_{I \sqcup J}^*(\{A \otimes F_i, L_j\}, A^{\otimes I} \otimes M)$ with the product morphism $A^{\otimes I} \otimes M \to M$. If the L_j are A-modules, then our maps preserve operations A-linear with respect to J-variables, so $P_{A\ I \sqcup J}^*(\{A \otimes F_i, L_j\}, M)$ identifies with the subspace of $P_{I \sqcup J}^*(\{F_i, L_j\}, M)$ that consists of operations A-linear with respect to the J variables.

In the setting of (1.4.6.2) the pseudo-tensor functor $f^* : \mathcal{M}(B)^* \to \mathcal{M}(A)^*$ is left adjoint to the obvious forgetful pseudo-tensor functor $\mathcal{M}(A)^* \to \mathcal{M}(B)^*$.[9]

(iii) If a $*$ pairing $\langle\ \ \rangle \in P_2^*(\{L', L\}, \mathbf{1})$ in $\mathcal{M}$ is non-degenerate (see 1.4.2), then so is the corresponding pairing $\langle\ \ \rangle_A \in P_2^*(\{A \otimes L', A \otimes L\}, A)$ in $\mathcal{M}(A)$.

1.4.7. Reliable augmentation functors. We know that h defines a pseudo-tensor functor on $\mathcal{M}^*$ with values in the tensor category of k-modules. Usually this pseudo-tensor functor is not fully faithful. Let us formulate a property of h which is an effective remedy for this nuisance.

For $M \in \mathcal{M}$ let $\tilde{h}(M) : \mathcal{C}omu(\mathcal{M}^!) \to k$mod be the functor $R \mapsto h(R \otimes M)$ (so we consider R as a variable "test" algebra). Since h is a pseudo-tensor functor on each $\mathcal{M}(R)^*$, we see that every $*$ operation $\varphi \in P_I^*(\{M_i\}, N)$ yields a morphism of functors[10] $\tilde{h}(\varphi) : \underset{I}{\otimes} \tilde{h}(M_i) \to \tilde{h}(N)$ on $\mathcal{C}omu(\mathcal{M}^!)$.

[9]To see this, replace $\mathcal{M}$ by $\mathcal{M}(B)$ in the above argument.

[10]Here the functor $\underset{I}{\otimes} \tilde{h}(M_i)$ is $R \mapsto \underset{I}{\otimes} h(R \otimes M_i)$.

Definition. We say that our augmentation functor h is *reliable* if for every non-empty family of objects $\{M_i\}$ and $N \in \mathcal{M}$ the map $\tilde{h} : P_I^*(\{M_i\}, N) \to \mathrm{Mor}(\bigotimes_I \tilde{h}(M_i), \tilde{h}(N))$ is bijective.[11]

Therefore if h is reliable, then for a k-operad $\mathcal{B}$ and an object $M \in \mathcal{M}$ a $\mathcal{B}^*$ algebra structure on M amounts to a rule that assigns to every commutative$^!$ algebra R a $\mathcal{B}$ algebra structure on $h(R \otimes M)$ such that for every morphism of commutative$^!$ algebras $R_1 \to R_2$ the corresponding morphism $h(R_1 \otimes M) \to h(R_2 \otimes M)$ is a morphism of $\mathcal{B}$ algebras.

Remarks. (i) Assume that the category Φ of all functors $F : \mathcal{C}omu(\mathcal{M}^!) \to k\mathrm{mod}$, $R \mapsto F_R$, makes sense (which happens if $\mathcal{M}$ is equivalent to a small category). Then Φ is a tensor category (one has $(\otimes F_i)_R := \otimes(F_{iR})$) and $\tilde{h} : \mathcal{M}^* \to \Phi^{\otimes}$ is a pseudo-tensor functor. Reliability of h just means that $\tilde{h}$ is a fully faithful pseudo-tensor embedding.

(ii) If we are playing with sheaves of categories, then in the above definition $k$mod should be replaced by the tensor category of sheaves of k-modules.

(iii) If h is reliable, then the corresponding augmentation functor on every $\mathcal{M}(A)^*$ is also reliable.

1.4.8. Differential $*$ operations. Many important $*$ operations between A-modules are not A-polylinear but belong to a larger class of *A-differential $*$ operations* we are going to define.

Let $\mathcal{I} \subset A \otimes A$ be the ideal of the diagonal, i.e., the kernel of the product map $A \otimes A \to A$. For an A-module L consider $A \otimes L$ as an $A \otimes A$-module and set $L^{(n)} := A \otimes L/\mathcal{I}^{n+1}(A \otimes L) = A^{(n)} \otimes_A L$. Now assume we have $M \in \mathcal{M}(A)$ and $L_i \in \mathcal{M}$. Pick some $i_0 \in I$ and identify $P_I^*(\{L_i\}, M)$ with the subspace of operations in $P_I^*(\{L_i, A \otimes L_{i_0}\}, M)$ A-linear with respect to the i_0th variable (see Remark (ii) in 1.4.6). Assume that L_{i_0} is also an A-module. The operations vanishing on $\mathcal{I}^{n+1}(A \otimes L_{i_0})$ form a subspace of $P_I^*(\{L_i\}, M)$; its elements are *differential operations of order $\leq n$ with respect to the i_0th variable.* If $n = 0$, these are the same as A-linear operations.

If all the L_i are A-modules, then the $*$ operations which are differential operations of some order with respect to each variable are called *A-polydifferential $*$ operations of finite order.* The space of such operations is $\bigcup_n P_{A\,I}^*(\{L_i^{(n)}\}, M) \subset P_{A\,I}^*(\{A \otimes L_i\}, M) = P_I^*(\{L_i\}, M)$. They are stable with respect to composition, so A-polydifferential $*$ operations form another pseudo-tensor structure on $\mathcal{M}(A)$.

1.4.9. Action of Lie* algebras on commutative$^!$ algebras. If L is a Lie* algebra, then an action of L (or an L-action by derivations) on a commutative$^!$ algebra A is an L-module structure $\tau \in P_2^*(\{L, A\}, A)$ on A (as on an object of $\mathcal{M}^*$) such that $\cdot_A : A \otimes A \to A$, $1_A : \mathbf{1} \to A$ are morphisms of L-modules.

Let τ be a fixed action of L on A. Then for $M \in \mathcal{M}(A)$ a *τ-action* (or a *τ-compatible action*) of L on M is an L-module structure on M (as on an object of $\mathcal{M}^*$) such that $\cdot_M : A \otimes M \to M$ is a morphism of L-modules. The A-modules equipped with τ-actions form an abelian category $\mathcal{M}(A, L, \tau)$.

[11] Here $\mathrm{Mor}(\bigotimes_I \tilde{h}(M_i), \tilde{h}(N))$ denotes the collection of morphisms of functors.

LEMMA. *$\mathcal{M}(A, L, \tau)$ is naturally an abelian augmented compound tensor k-category. We denote it by $\mathcal{M}(A, L, \tau)^{*!}$.*

Proof. One defines the $*$ operations in $\mathcal{M}(A, L, \tau)$ as $P^*_{A,L,I}(\{M_i\}, N) := P^*_{LI}(\{M_i\}, N) \cap P^*_{AI}(\{M_i\}, N)$ (the intersection is taken in $P^*_I(\{M_i\}, N)$). Notice that $\underset{AI}{\otimes} M_i$ is a quotient L-module of $\underset{I}{\otimes} M_i$, so we define the ! tensor product on $\mathcal{M}(A, L, \tau)$ as $\underset{A}{\otimes}$. The $\otimes^I_{S,T}$ operations all come from $\mathcal{M}(A)^{*!}$. The unit object of $\mathcal{M}(A, L, \tau)$ is A. The $*$-augmentation functor on $\mathcal{M}(A, L, \tau)^{*!}$ is $h^{\mathbf{L}}$ (see 1.4.4). □

1.4.10. The inner Chevalley complex. Let L be a Lie* algebra such that the $*$ dual $L^\circ := \mathcal{H}om^*(L, \mathbf{1})$ exists and the canonical pairing $\langle\ \rangle \in P^*_2(\{L^\circ, L\}, \mathbf{1})$ is non-degenerate (see 1.4.2). Below we give another interpretation of the inner Chevalley complexes from Remark (ii) in 1.4.5. We deal with the general super DG setting (skipping the adjectives).

Consider first the graded commutative$^!$ algebra $\mathrm{Sym}(L^\circ[-1])$. It carries a canonical action of the graded Lie* algebra $L_\dagger$ (we forget about the differentials at the moment). Namely, $L \subset L_\dagger$ acts according to the coadjoint action, and its complement $L[1] \subset L_\dagger$ acts on the generators $L^\circ[-1]$ by $-\langle\ \rangle \in P^*_2(\{L[1], L^\circ[-1]\}, \mathbf{1})$. Now the graded algebra $\mathrm{Sym}(L^\circ[-1])$ admits a unique differential δ (odd of degree 1; see 1.1.16) such that the action of $L_\dagger$ is compatible with the differentials. Explicitly, the restriction of δ to $L^\circ[-1]$ is equal to the composition $L^\circ[-2] \xrightarrow{\nu} L^\circ[-1] \otimes L^\circ[-1] \xrightarrow{-1/2\cdot} \mathrm{Sym}^2(L^\circ[-1])$ where ν is the morphism dual to the Lie* bracket.[12]

Our DG commutative$^!$ algebra carries a decreasing filtration by the DG ideals $\mathrm{Sym}^{\geq i}(L^\circ[-1])$. The topological DG algebra $\varprojlim \mathrm{Sym}(L^\circ[-1])/\mathrm{Sym}^{\geq i}(L^\circ[-1])$ is denoted by $\mathcal{C}(L)$; it carries a natural (continuous) $L_\dagger$-action. The above filtration yields a filtration $\mathcal{C}^{\geq i}(L)$ on $\mathcal{C}(L)$, and the $L_\dagger$-action is compatible with the filtrations.[13]

REMARK. Here "topological algebra" means simply the (formal) projective limit of algebras. We pass to the completion in order to assure that the functor $\mathcal{C}$ preserves quasi-isomorphisms.

Let M be any L-module. Consider M as an $L_\dagger$-module (we forget about the differential on $L_\dagger$ at the moment) via the projection $L_\dagger \to L_\dagger/L[1] = L$. Then $L_\dagger$ acts on the topological $\mathcal{C}(\mathcal{L})$-module $M\hat{\otimes}\mathcal{C}(L) := \varprojlim M \otimes (\mathrm{Sym}(L^\circ[-1])/\mathrm{Sym}^{\geq i}(L^\circ[-1]))$ via the tensor product of the our actions. As above, $M\hat{\otimes}\mathcal{C}(L))$ admits a unique differential such that the actions of $L_\dagger$ and $\mathcal{C}(L)$ are compatible with the differentials. Explicitly, on the $\mathcal{C}(L)$-generators M the differential is the sum of the differential on M and the morphism $M \to M \otimes L^\circ$ dual to the L-action. We denote this topological DG $\mathcal{C}(L)$-module equipped with the $L_\dagger$-action by $\mathcal{C}(L, M)$. It carries a natural filtration by DG submodules $\mathcal{C}^{\geq i}(L, M)$ compatible with the filtration $\mathcal{C}^{\geq i}(L)$. We leave it to the reader to check that $\mathcal{C}(L)$ coincides with the inner Chevalley complex from Remark (ii) of 1.4.5.

If L acts on a commutative$^!$ algebra A by derivations, then $L_\dagger$ acts on $A\hat{\otimes}\mathcal{C}(L)$ also by derivations, and the Chevalley differential is a derivation. Thus $\mathcal{C}(L, A)$

[12] So if $\langle\ \rangle^2 \in P^*_3(\{L^\circ \otimes L^\circ, L, L\}, \mathbf{1})$ is the tensor product of two copies of $\langle\ \rangle$, then ν is uniquely determined by property $\langle\ \rangle^2 (\nu, id_L, id_L) = \langle\ \rangle (id_{L^\circ}, [\ ]) \in P^*_3(\{L^\circ, L, L\}, \mathbf{1})$.

[13] Here $L_\dagger$ is equipped with the filtration defined at the end of 1.4.5.

is a topological DG commutative$^!$ algebra equipped with a DG $L_\dagger$-action. It is a $\mathcal{C}(L)$-deformation of $A = \mathcal{C}(L,A)/\mathcal{C}^{\geq 1}(L,A)$. If M is an A-module equipped with a compatible L-action (see 1.4.9), then $\mathcal{C}(L,M)$ is a topological DG $\mathcal{C}(L,A)$-module equipped with a compatible $L_\dagger$-action.

REMARK. The morphism $\nu : L^\circ \to L^\circ \otimes L^\circ$ is a Lie cobracket. The duality isomorphism $P_2^*(\{L,M\},M) \xrightarrow{\sim} \mathrm{Hom}(M, L^\circ \otimes M)$ identifies $*$ actions of L on M with ! coactions of L° on M. So the category of L-modules in $\mathcal{M}^*$ coincides with the category of L°-comodules in $\mathcal{M}^!$. If M is an L-module, then $\mathcal{C}(L,M)$ is the usual Chevalley complex of the Lie coalgebra L° with coefficients in L°-comodule M in the tensor category $\mathcal{M}^!$.

1.4.11. Lie* algebroids. Let A be any commutative unital algebra in $\mathcal{M}^!$.

DEFINITION. A *Lie* A-algebroid* is an A-module $\mathcal{L}$ together with a Lie* bracket $[\ \ ] \in P_2^*(\{\mathcal{L},\mathcal{L}\},\mathcal{L})$ and an action $\tau = \tau_{\mathcal{L}} \in P_2^*(\{\mathcal{L},A\},A)$ of $\mathcal{L}$ (considered as a Lie* algebra) on A by derivations. This datum should satisfy the following properties:

(i) τ is A-linear with respect to the $\mathcal{L}$-variable;

(ii) the adjoint action $ad_{\mathcal{L}}$ is a τ-action of $\mathcal{L}$ (as a Lie* algebra) on $\mathcal{L}$ (as an A-module).

Notice that the structure $*$ operations (a), (b) are A-polydifferential (see 1.4.8).

Lie* A-algebroids form a category $\mathcal{L}ie\mathcal{A}lg_A^*$. Note that $\mathcal{L}ie(\mathcal{M}(A)^*)$ coincides with the full subcategory of $\mathcal{L}ie\mathcal{A}lg_A^*$ that consists of $\mathcal{L}$ with $\tau = 0$.

1.4.12. For $\mathcal{L} \in \mathcal{L}ie\mathcal{A}lg_A^*$ an $\mathcal{L}$-*module* M is an A-module M equipped with a τ-action $\tau_M \in P_2^*(\{\mathcal{L},M\},M)$ of $\mathcal{L}$ (considered as a plain Lie* algebra) such that τ_M is A-linear with respect to $\mathcal{L}$-variable. $\mathcal{L}$-modules form a full subcategory[14] $\mathcal{M}(A,\mathcal{L}) \subset \mathcal{M}(A,\mathcal{L},\tau)$ closed under subquotients and $\underset{A}{\otimes}$; clearly $A \in \mathcal{M}(A,\mathcal{L})$. Define the $*$ operations in $\mathcal{M}(A,\mathcal{L})$ as $P_{A,\mathcal{L}}^*$; i.e., let us consider $\mathcal{M}(A,\mathcal{L})$ as a full augmented compound tensor subcategory $\mathcal{M}(A,\mathcal{L})^{*!}$ in $\mathcal{M}(A,\mathcal{L},\tau)^{*!}$.

LEMMA. *$\mathcal{M}(A,\mathcal{L})^{*!}$ is an abelian augmented compound tensor k-category.* □

REMARK. If our compound tensor structure is a mere tensor one (see 1.3.8(ii)), then Lie* algebroids are the same as Lie algebroids. The above $\mathcal{L}$-modules are the same as *left* $\mathcal{L}$-modules for $\mathcal{L}$ considered as a usual Lie algebroid (see 2.9.1). There seems to be no adequate compound analog for the notion of *right* module over a usual Lie algebroid.

1.4.13. Let L be any Lie* algebra, τ an action of L on A.

LEMMA. (i) *The induced A-module $L_A := A \otimes L$ has a unique structure of a Lie* A-algebroid such that the obvious morphism $1_A \otimes id_L : L \to L_A$ is a morphism of* Lie* *algebras compatible with their actions on A. The functor $L \mapsto L_A$ from the category of Lie* algebras acting on A to that of Lie* A-algebroids is left adjoint to the "forgetting of the A-action" functor.*

(ii) *The functor $\mathcal{M}(A,L_A)^{*!} \to \mathcal{M}(A,L,\tau)^{*!}$ assigning to an L_A-module M the same M considered as an L-module is an equivalence of augmented compound tensor categories.* □

[14] In the notation $\mathcal{M}(A,\mathcal{L},\tau)$ we consider $\mathcal{L}$ as a plain Lie* algebra acting on A.

For a Lie* A-algebroid $\mathcal{L}$ we refer to an isomorphism of Lie* algebroids $L_A \xrightarrow{\sim} \mathcal{L}$ as (L,τ)-*rigidification* of $\mathcal{L}$.

1.4.14. The de Rham-Chevalley complex. As in 1.4.5, 1.4.10, we consider the super setting.

(i) Let $\mathcal{L}$ be a Lie* A-algebroid. For $M \in \mathcal{M}(A,\mathcal{L})$ the A-polylinear operations form a subcomplex $C_A(\mathcal{L},M) \subset C(\mathcal{L},M)$ closed under the $h(\mathcal{L}_\dagger)$-action. This is the *de Rham–Chevalley* complex of $\mathcal{L}$ with coefficients in M. Set $C_A(\mathcal{L}) := C_A(\mathcal{L},A)$.

Below we discuss an "inner" version of this complex.

(ii) The next definitions make sense in any unital abelian tensor k-category $\mathcal{M}^!$ (we do not use the compound structure at the moment).

Let A be a commutative unital algebra, $\mathcal{L}^\circ$ an A-module. A *Lie A-coalgebroid* structure on $\mathcal{L}^\circ$ is an odd differential $d = d_{\mathcal{L}^\circ}$ of degree 1 and square 0 on the algebra $\mathrm{Sym}_A(\mathcal{L}^\circ[-1])$. We denote this commutative DG algebra by $\mathcal{C}_A(\mathcal{L}^\circ)$; this is the *de Rham-Chevalley* DG algebra of $\mathcal{L}^\circ$.

Let $\mathcal{L}^\circ$ be a Lie A-coalgebroid. For an A-module M a $\mathcal{L}^\circ$-*action*, or $\mathcal{L}^\circ$-*module* structure, on M is a $d_{\mathcal{L}^\circ}$-action on $\mathrm{Sym}_A(\mathcal{L}^\circ[-1]) \underset{A}{\otimes} M$, i.e., a differential of $\mathrm{Sym}_A(\mathcal{L}^\circ[-1]) \underset{A}{\otimes} M$ that makes it a DG $\mathcal{C}_A(\mathcal{L}^\circ)$-module. Such a differential is uniquely determined by its restriction $M \to \mathcal{L}^\circ \underset{A}{\otimes} M$. We denote this DG $\mathcal{C}_A(\mathcal{L}^\circ)$-module by $\mathcal{C}_A(\mathcal{L}^\circ, M)$.

The category of $\mathcal{L}^\circ$-modules is denoted by $\mathcal{M}(A,\mathcal{L}^\circ)$. This is a tensor category: if $\mathcal{L}^\circ$ acts on (finitely many) A-modules M_α, then it acts naturally on $\otimes M_\alpha = \underset{A}{\otimes} M_\alpha$ so that $\mathcal{C}_A(\mathcal{L}^\circ, \otimes M) = \underset{\mathcal{C}_A(\mathcal{L}^\circ)}{\otimes} \mathcal{C}_A(\mathcal{L}^\circ, M_\alpha)$.

(iii) Let us return to the compound tensor category setting.

Suppose that an A-module $\mathcal{L}$ admits a dual $\mathcal{L}^\circ$ and the canonical $*$ pairing $\langle\ \rangle \in P^*_{A2}(\{\mathcal{L}^\circ,\mathcal{L}\},A)$ is non-degenerate (in $\mathcal{M}(A)^{*!}$; see 1.4.2).

Lemma. *There is a natural bijective correspondence between Lie* A-algebroid structures on $\mathcal{L}$ and Lie A-coalgebroid structures on $\mathcal{L}^\circ$.*

Sketch of a proof. Suppose we have a Lie* A-algebroid structure on $\mathcal{L}$. There is a unique action $ad^\circ_{\mathcal{L}}$ of the Lie* algebra $\mathcal{L}$ on $\mathcal{L}^\circ$ (called the *coadjoint* action) such that $\langle\ \rangle \in P^*_2(\{\mathcal{L}^\circ,\mathcal{L}\},A)$ is $\mathcal{L}$-invariant (here $\mathcal{L}$ acts on $\mathcal{L}$ by the adjoint action, on A by τ). Indeed, the operation $\tau(id_{\mathcal{L}}, \langle\ \rangle) - \langle [\ ], id_{\mathcal{L}^\circ}\rangle \in P^*_3(\{\mathcal{L},\mathcal{L},\mathcal{L}^\circ\},A)$ is A-linear with respect to the second argument $\mathcal{L}$, so, since $\langle\ \rangle$ is non-degenerate (use Remark (ii) from 1.4.6) there is a unique $ad^\circ_{\mathcal{L}} \in P^*_2(\{\mathcal{L},\mathcal{L}^\circ\},\mathcal{L}^\circ)$ such that the above operation equals $\langle\ \rangle(id_{\mathcal{L}}, ad^\circ_{\mathcal{L}})$. By the universality property of $\langle\ \rangle$, our $ad^\circ_{\mathcal{L}}$ is a τ-action.

Consider the graded commutative$^!$ algebra $\mathrm{Sym}_A(\mathcal{L}^\circ[-1])$. The Lie* algebra $\mathcal{L}$ acts on it (via τ and $ad^\circ_{\mathcal{L}}$). This action extends to an action of the graded Lie* algebra $\mathcal{L}_\dagger$ (see 1.1.16); we forget about the differential on $\mathcal{L}_\dagger$ at the moment. Namely, $\mathcal{L}[1] \subset \mathcal{L}_\dagger$ kills A and acts on $\mathcal{L}^\circ[-1]$ by $\langle\ \rangle$.

We leave it to the reader to show that $\mathcal{L}^\circ$ carries a unique Lie A-coalgebroid structure such that the $\mathcal{L}_\dagger$-action on $\mathrm{Sym}_A(\mathcal{L}^\circ[-1])$ is compatible with the differentials and to check that this map is, indeed, bijective. □

For a Lie* A-algebroid $\mathcal{L}$ as above we call $\mathcal{L}^\circ$ the *dual Lie coalgebroid* and $\mathcal{C}_A(\mathcal{L}) = \mathcal{C}_A(\mathcal{L},A) := \mathcal{C}_A(\mathcal{L}^\circ)$ the *de Rham-Chevalley* DG algebra of $\mathcal{L}$.

Remark. This DG algebra can be seen as the algebra of functions on the quotient space of $\operatorname{Spec} A$ modulo the action of $\mathcal{L}$.

Lemma. *For an A-module M there is a natural bijection between $\mathcal{L}$- and $\mathcal{L}^\circ$-module structures on M. This bijection is compatible with tensor products, so we get a natural equivalence of tensor categories*

$$\mathcal{M}(A,\mathcal{L}) \xrightarrow{\sim} \mathcal{M}(A,\mathcal{L}^\circ). \tag{1.4.14.1}$$

Sketch of a proof. The subspace of operations $\phi \in P_2^*(\{\mathcal{L},M\},M)$ which are A-linear with respect to the first variable identifies naturally with $\operatorname{Hom}(M,\mathcal{L}^\circ \underset{A}{\otimes} M)$ (see 1.4.2). We leave it to the reader to check that ϕ is an $\mathcal{L}$-module structure on M if and only if the morphism $M \to \mathcal{L}^\circ \underset{A}{\otimes} M$ is an $\mathcal{L}^\circ$-action on M. □

For $M \in \mathcal{M}(A,\mathcal{L})$ we usually denote the DG $\mathcal{C}_A(\mathcal{L})$-module $\mathcal{C}_A(\mathcal{L}^\circ,M)$ by $\mathcal{C}_A(\mathcal{L},M)$; this is the *inner de Rham-Chevalley* complex of $\mathcal{L}$ with coefficients in M. One has $h(\mathcal{C}_A(\mathcal{L},M)) \xrightarrow{\sim} C_A(\mathcal{L},M)$ (cf. (1.4.5.1)).

Remarks. (i) For $M \in \mathcal{M}(A,\mathcal{L})$ consider the graded $\operatorname{Sym}_A(\mathcal{L}^\circ[-1])$-module $M \underset{A}{\otimes} \operatorname{Sym}_A(\mathcal{L}^\circ[-1])$. It carries a natural action of the graded Lie* algebra $\mathcal{L}_\dagger$: namely, $\mathcal{L} \subset \mathcal{L}_\dagger$ acts according to the tensor product of the $\mathcal{L}$-actions, and $\mathcal{L}[1] \subset \mathcal{L}_\dagger$ acts via the above action on $\operatorname{Sym}_A(\mathcal{L}^\circ[-1])$. Now the differential in $\mathcal{C}_A(\mathcal{L},M)$ is characterized by the property that the $\mathcal{L}_\dagger$-action is compatible with the differentials.

(ii) In the situation of 1.4.13 the above constructions amount to that of 1.4.10: one has $\mathcal{C}_A(L_A) = \mathcal{C}(L,A)$ and $\mathcal{C}_A(L_A,M) = \mathcal{C}(L,M)$.

1.4.15. Lie coalgebroids and formal groupoids. In this subsection the compound tensor structure is not used, so $\mathcal{M}^!$ can be any unital abelian k-category ($\otimes$ is right exact, and k is of characteristic 0); for simplicity, we assume that $\mathcal{M}^!$ has many flat objects. We have the category of commutative unital algebras in $\mathcal{M}^!$ and its dual category of affine schemes in $\mathcal{M}^!$. Many notions of affine algebraic geometry generalize in a straightforward manner to such a setting. In particular, we know what are the affine formal schemes in $\mathcal{M}^!$.

Let A be a commutative unital algebra in $\mathcal{M}^!$, $Y := \operatorname{Spec} A$ the corresponding affine scheme. Let $\mathcal{G} = \operatorname{Spf} E$ be a formal groupoid on Y. Thus we have the structure projection $\mathcal{G} \to Y \times Y$ and the "identity morphism" embedding $i : Y \hookrightarrow \mathcal{G}$; let $I \subset E$ be the ideal of i. We say that $\mathcal{G}$ is *flat* if E is an I-adic topological algebra, I/I^2 is a flat A-module, and $\operatorname{Sym}_A(I/I^2) \xrightarrow{\sim} \sum I^n/I^{n+1}$. We say that a Lie A-coalgebroid P is *flat* if P is flat as an A-module.

Proposition. *The category of flat formal groupoids on* $\operatorname{Spec} A$ *is naturally equivalent to the category of flat Lie A-coalgebroids.*

Sketch of a proof. (a) Let P be a flat Lie A-coalgebroid, $C := \mathcal{C}_A(P)$ its de Rham-Chevalley DG algebra. Let us construct the corresponding formal groupoid $\mathcal{G} = \operatorname{Spf} E$ on Y. Idea of the construction: $\operatorname{Spec} C$ is a DG version of the orbit space of $\mathcal{G}$; thus $\mathcal{G}$ is fiber square of Y over $\operatorname{Spec} C$ with its evident groupoid structure. Of course, the fiber square should be taken in the DG sense and appropriately completed.

We consider C as a filtered topological pro-algebra: $C = \varprojlim C_n$ where $C_n := C/F^{n+1}$ and $F^n := C^{\geq n}$ is the "stupid" filtration, so $\operatorname{gr}_F C = \operatorname{Sym}_A(P[-1])$. Set

$T_n := A \overset{L}{\underset{C_n}{\otimes}} A$; one computes it using a C_n-flat resolution of A. One has the projections $\ldots \to C_2 \to C_1 \to C_0 = A$ and $\ldots \to T_2 \to T_1 \to T_0 = A$. Set $E_n := H^0 T_n$; these are commutative algebras. The natural morphisms $A = A \underset{C_n}{\otimes} C_n \to T_n \leftarrow C_n \underset{C_n}{\otimes} A = A$ show that T_n and E_n are $A \otimes A$-algebras.

LEMMA. *One has $H^{>0}T_n = 0$ for every n. The projections $\ldots \to E_2 \to E_1 \to E_0 = A$ are surjective. Set $I_n := \mathrm{Ker}(E_n \to A)$; then $\mathrm{Ker}(E_n \to E_m) = I_n^{m+1}$ for $m < n$, and $\sum I_n^a / I_n^{a+1} = \mathrm{Sym}_A^{\le n} P$ for $n \ge 1$.*

Proof of Lemma. C_n is filtered by the stupid filtration, and we can compute T_n using a filtered resolution of A. One gets a filtration on T_n such that $\mathrm{gr}_F T_n = A \overset{L}{\underset{\mathrm{gr}_F C_n}{\otimes}} A$. Then $H^{>0}\mathrm{gr}_F T_n = 0 = H^{<0}\mathrm{gr}_F^{\le n} T_n$ and $H^0 \mathrm{gr}_F^{\cdot} T_n = \mathrm{Sym}_A^{\le n} P$ by a usual Koszul resolution computation, which implies the lemma. $\square$

Set $E := \varprojlim E_n$. By the lemma, this is an I-adic topological $A \otimes A$-algebra, where $I := \varprojlim I_n$, and $\sum I^n / I^{n+1} = \mathrm{Sym}_A P$.

It remains to define on $\mathcal{G} := \mathrm{Spf}\, E$ the structure of a formal groupoid on Y. To do this, we consider for every $a \ge 0$ a formal Y^{a+1}-scheme $\mathcal{G}_a = \mathrm{Spf}\, E^{(a)}$ defined as the appropriately completed DG fiber product of $a+1$ copies of Y over $\mathrm{Spec}\, C$ (i.e., the DG pull-back of the $(\mathrm{Spec}\, C)^{a+1}$-scheme Y^{a+1} by the diagonal embedding $\mathrm{Spec}\, C \hookrightarrow (\mathrm{Spec}\, C)^{a+1}$). Thus $\mathcal{G}_0 = Y$, $\mathcal{G}_1 = \mathcal{G}$. Now the $\mathcal{G}_\cdot$ form a formal simplicial scheme in the obvious way, and each standard projection $\mathcal{G}_a \to \mathcal{G} \underset{Y}{\times} \ldots \underset{Y}{\times} \mathcal{G}$ (a times) is an isomorphism. Therefore $\mathcal{G}$ is a groupoid on Y (so that $\mathcal{G}_\cdot$ is the "homotopy quotient" of Y modulo $\mathcal{G}$, i.e., the "classifying space" $B_{\mathcal{G}}$ of $\mathcal{G}$). The details are left to the reader.

REMARK. The formal groupoid $\mathcal{G}$ carries a canonical coaction of the Lie $A \otimes A$-coalgebroid $P_1 := (P \otimes A) \times (A \otimes P)$. Namely, for $n \ge 0$ let C_n^F be a copy of C_n considered as a filtered C_n-algebra (with filtration $F^\cdot$). Set $\Phi_n := C_n^F \overset{L}{\underset{C_n}{\otimes}} C_n^F$. This is a filtered DG C_n-algebra, and $\mathrm{gr}^\cdot \Phi_n = \mathrm{gr}^\cdot C_n^F \overset{L}{\underset{C_n}{\otimes}} \mathrm{gr}^\cdot C_n^F = (\mathrm{Sym}_A^{\le n}(P[-1]))^{\otimes 2} \underset{A \otimes A}{\otimes} T_n$. Therefore $\underset{a}{\oplus} H^a \mathrm{gr}^a \Phi_n = (\mathrm{Sym}_A^{\le n}(P[-1]))^{\otimes 2} \underset{A \otimes A}{\otimes} E_n$. Being the first term of the spectral sequence for the filtration on Φ_n, this graded algebra carries a natural differential. Passing to the projective limit by n, we see that $(\mathrm{Sym}^\cdot_{A \otimes A}(P_1[-1])) \underset{A \otimes A}{\otimes} E$ carries a natural differential, which is the promised P_1-coaction on $\mathcal{G}$.

This coaction is compatible with the groupoid structure on $\mathcal{G}$ in an evident way. In fact, each Y^{a+1}-scheme $\mathcal{G}_a$ carries a natural action of the Lie coalgebroid $P_a :=$ the product of copies of P acting along each of the coordinates, and the structure projections are compatible with these actions.

(b) Suppose we have a flat formal groupoid $\mathcal{G} = \mathrm{Spf}\, E$. The corresponding Lie A-coalgebroid is $P := I/I^2$. We will identify $\mathrm{Sym}_A(P[-1])$ with the de Rham algebra of invariant forms on $\mathcal{G}$; the Lie coalgebroid structure on P is given then by the de Rham differential.

Let $DR^{(1)}_{\mathcal{G}/Y}$ be the de Rham complex of $\mathcal{G}$ relative to the first structure projection $\mathcal{G} \to Y$. Similarly, let $DR^{(1)}_{\mathcal{G}_2/\mathcal{G}}$ be the de Rham complex of $\mathcal{G}_2 := \mathcal{G} \underset{Y}{\times} \mathcal{G}$

relative to the first projection $\mathcal{G} \underset{Y}{\times} \mathcal{G} \to \mathcal{G}$. The second projection and the product map $\mathcal{G} \underset{Y}{\times} \mathcal{G} \to \mathcal{G}$ yield the pull-back morphisms of DG algebras $DR^{(1)}_{\mathcal{G}/Y} \to DR^{(1)}_{\mathcal{G}_2/\mathcal{G}}$. Their equalizer is the DG algebra $DR^{(\ell)}_{\mathcal{G}/Y}$ of *left invariant forms* on $\mathcal{G}$.

Forgetting about the differential, we have an evident morphism of graded A-algebras $DR^{(\ell)}_{\mathcal{G}/Y} \to i^* DR^{(1)}_{\mathcal{G}/Y} = \mathrm{Sym}_A(P[-1])$; one checks immediately that this is an isomorphism. We have defined the promised Lie A-coalgebroid structure on P.

Remark. We can also consider the DG algebra $DR^{(r)}_{\mathcal{G}/Y}$ of *right* invariant differential forms (which lies in the relative de Rham complex for the second structure projection $\mathcal{G} \to Y$). There is a canonical identification $DR^{(\ell)}_{\mathcal{G}/Y} \xrightarrow{\sim} DR^{(r)}_{\mathcal{G}/Y}$ coming from the involution $g \mapsto g^{-1}$ of $\mathcal{G}$.

(c) It remains to show that the functors defined in (a), (b) are mutually inverse:

Suppose we have a flat Lie coalgebroid P; let $\mathcal{G} = \mathrm{Spf}\, E$ be the corresponding formal groupoid as in (a), and P' its Lie coalgebroid as in (b). To define a canonical isomorphism $P \xrightarrow{\sim} P'$, consider the canonical P_1-coaction on $\mathcal{G}$ from Remark in (a). Its $A \otimes P$-component is a differential on the graded algebra $E \underset{A}{\otimes} \mathrm{Sym}(P[-1])$; denote this formal DG algebra by $DR^{(1)}_{\mathcal{G},P}$. One checks that the canonical morphism of DG algebras $DR^{(1)}_{\mathcal{G}/Y} \to DR^{(1)}_{\mathcal{G},P}$ is an isomorphism. Its inverse identifies $\mathrm{Sym}(P[-1])$ with $DR^{(\ell)}_{\mathcal{G}/Y}$. This identification is compatible with the differential; this is the promised isomorphism of Lie coalgebroids $P \xrightarrow{\sim} P'$.

Now suppose we have a flat formal groupoid $\mathcal{G} = \mathrm{Spf}\, E$. Let P be the corresponding Lie A-coalgebroid, C its de Rham-Chevalley DG algebra as in (b), and $\mathcal{G}' = \mathrm{Spf}\, E'$ the formal groupoid defined by P as in (a). Let us define a canonical isomorphism $\mathcal{G} \xrightarrow{\sim} \mathcal{G}'$.

The "identity morphism" embedding $Y \hookrightarrow \mathcal{G}$ yields a morphism of algebras $DR^{(1)}_{\mathcal{G}/Y} \twoheadrightarrow A$. Its kernel is a DG ideal; denote by F^n the filtration on $DR^{(1)}_{\mathcal{G}/Y}$ by powers of this ideal. By construction, we have an embedding of DG algebras $C \hookrightarrow DR^{(1)}_{\mathcal{G}/Y}$ compatible with the $F^{\cdot}$ filtrations. Notice that each projection $DR^{(1)}_{\mathcal{G}/Y}/F^{n+1} \twoheadrightarrow DR^{(1)}_{\mathcal{G}/Y}/F^1 = A$ is a quasi-isomorphism. It yields a morphism of $A \otimes A$-algebras $E'_n := H^0(A \overset{L}{\underset{C_n}{\otimes}} A) \to (DR^{(1)}_{\mathcal{G}/Y}/F^{n+1}) \underset{C_n}{\otimes} A = E/I^{n+1} =: E_n$. One checks that this is an isomorphism; passing to the limit, we get an isomorphism of formal $Y \times Y$-schemes $\mathcal{G} \xrightarrow{\sim} \mathcal{G}'$. We leave it to the reader to check that it is compatible with the groupoid composition law. □

Remark. Here is another (equivalent) way to define the Lie coalgebroid of a formal groupoid $\mathcal{G}$. Consider the formal simplicial scheme $\mathcal{G}_{\cdot} = B_{\mathcal{G}}$, $\mathcal{G}_a = \mathcal{G} \underset{Y}{\times} \dots \underset{Y}{\times} \mathcal{G}$ (a times) as in (a). Let $\mathrm{Spec}\, Q^{\cdot} \subset \mathcal{G}_{\cdot}$ be the maximal simplicial subscheme such that $Q^1 = E/I^2$. Thus $Q^{\cdot}$ is a cosimplicial algebra; our C is the corresponding normalized complex. This is a DG algebra in the usual way; one has $C^0 = A$, $C^1 = P$. We leave it to the reader to check that C is strictly commutative, equals $\mathrm{Sym}_A(P[-1])$ as a mere graded algebra, and the corresponding Lie coalgebroid structure on P coincides with one from (b).

EXERCISE. Let P be an A-flat A-coalgebroid and $\mathcal{G}$ the corresponding formal groupoid. Show that for any A-module a P-action on it amounts to a $\mathcal{G}$-action. Thus $\mathcal{M}(A,P)$ coincides with the category of $\mathcal{G}$-equivariant A-modules.

Returning to the compound tensor category setting, consider a Lie* A-algebroid $\mathcal{L}$ as in 1.4.14(iii). Its dual Lie A-coalgebroid $\mathcal{L}^\circ$ is A-flat (see 1.4.2). We get the corresponding formal groupoid $\mathcal{G} = \mathcal{G}_{\mathcal{L}}$. By Exercise and (1.4.14.1), $\mathcal{L}$-modules are the same as $\mathcal{G}_{\mathcal{L}}$-equivariant A-modules.

1.4.16. The tangent algebroid. For $M \in \mathcal{M}(A)$ we have the k-module of derivations (with respect to the ! tensor structure) $\mathrm{Der}(A,M) \subset \mathrm{Hom}_{\mathcal{M}}(A,M)$. The functor $\mathrm{Der}(A,\cdot)$ is representable: there is a universal derivation $d\colon A \to \Omega_A$.

More generally, for any finite collection of objects $\{N_i\}_{i\in I}$ of $\mathcal{M}$ we have a k-submodule $\mathrm{Der}(A,M)(\{N_i\})_I \subset P^*_{\tilde{I}}(\{N_i, A\}, M)$ of operations φ that are derivations with respect to A, i.e., satisfy $\varphi(id_{N_i}, \cdot_A) = \cdot_M(\varphi \otimes_1 id_A) + \cdot_M(\varphi \otimes_2 id_A) \in P^*_{\tilde{I}}(\{N_i, A\otimes A\}, M)$. Here $\varphi \otimes_1 id_A$, $\varphi \otimes_2 id_A \in P^*_{\tilde{I}}(\{N_i, A\otimes A\}, M)$ are the operations defined in 1.4.3; the subscripts 1 and 2 show which copy of A in $A\otimes A$ we insert in φ (the transposition of factors symmetry of $A\otimes A$ interchanges them). Note that for any N_i-valued $*$ operations χ_i the composition $\varphi(\chi_i, id_A)$ is again a derivation with respect to A, and in case $I = \varnothing$ we have our old $\mathrm{Der}(A,M)$.

By Remark (ii) in 1.4.6 (and left exactness of P^*), the map $P^*_{\tilde{I}}(\{N_i, \Omega_A\}, M) \to P^*_{\tilde{I}}(\{N_i, A\}, M)$, $\psi \mapsto \psi(id_{N_i}, d)$, identifies the subspace of A-linear with respect to Ω_A operations with $\mathrm{Der}(A,M)(\{N_i\})_I$. If the N_i are also A-modules, then this map identifies $P^*_{A\tilde{I}}(\{N_i, \Omega_A\}, M)$ with the subspace of derivations A-linear with respect to N_i.

Suppose that the object $\mathcal{D}er(A,M) := \mathcal{H}om_{\mathcal{M}(A)^*}(\Omega_A, M) \in \mathcal{M}(A)$ exists (see 1.3.12). By the above and Remark (ii) in 1.4.6 for any finite collection $\{G_i\}$, $i\in I$, one has $P^*_I\big(\{G_i\}, \mathcal{D}er(A,M)\big) = \mathrm{Der}(A,M)(\{G_i\})_I \subset P^*_{\tilde{I}}(\{G_i, A\}, M)$.

Let us show that $\Theta_A := \mathcal{D}er(A,A)$ (if it exists) is a Lie* A-algebroid in a natural way; this is the *tangent algebroid* of A. Namely, the action $\tau \in P^*_2(\{\Theta_A, A\}, A)$ is the universal derivation (that corresponds to id_{Θ_A} under the above identification), and $[\ ] \in P^*_2(\{\Theta_A, \Theta_A\}, \Theta_A)$ is uniquely defined by the condition that τ is a Lie* action of Θ_A on A. Compatibilities (i) and (ii) in 1.4.11 are clear.

REMARKS. (i) Θ_A is a universal Lie* A-algebroid: for any $L \in \mathcal{L}ie\mathcal{A}lg^*_A$ there is a unique morphism of Lie* A-algebroids $\tau : L \to \Theta_A$ called the *anchor* morphism. Notice that $L^\diamondsuit := \mathrm{Ker}\,\tau$ is the maximal Lie* A-subalgebra of L.

(ii) If the canonical $*$ pairing $\in P^*_A(\{\Theta_A, \Omega_A\}, A)$ is non-degenerate, then the de Rham-Chevalley DG algebra $\mathcal{C}_A(\Theta_A)$ equals the de Rham complex DR_A, $DR^i = \Omega^i_A := \Lambda^i \Omega_A$.

(iii) Let M be an A-module. One may look for a universal Lie* A-algebroid $\mathcal{E}(M)$ acting on M. If $\mathcal{E}(M)$ exists, then $\mathcal{E}(M)^\diamondsuit \subset \mathcal{E}(M)$ is the Lie* A-algebra that corresponds to the associative* A-algebra $\mathcal{E}nd\,M$; we denote it by $\mathfrak{gl}(M)$.

1.4.17. Connections. The notion of connection on an A-module M is clear (it uses only the tensor category structure); we know what the curvature of a connection is (this is a morphism $M \to \Omega^2_A \otimes M$), etc. Let us explain what the connections in the context of Lie* algebroids are.

Let $\mathcal{L}$ be a Lie* A-algebroid as in 1.4.14(ii), so we have the de Rham-Chevalley DG commutative$^!$ algebra $\mathcal{C} = \mathcal{C}_A(\mathcal{L})$. Let P be a Lie* A-algebra, and $\mathcal{E}$ a P-extension of $\mathcal{L}$. So $\mathcal{E}$ is an A-algebroid, and we have a short exact sequence $0 \to P \xrightarrow{i} \mathcal{E} \xrightarrow{\pi} \mathcal{L} \to 0$ where i and π are morphisms of Lie* A-algebroids.

An $\mathcal{L}$-*connection* on $\mathcal{E}$ is an A-linear section $\nabla : \mathcal{L} \to \mathcal{E}$ of π. Its *curvature* is $c(\nabla) := [\nabla, \nabla] - \nabla([\ ]) \in P_2^*(\{\mathcal{L}, \mathcal{L}\}, P) \subset P_2^*(\{\mathcal{L}, \mathcal{L}\}, \mathcal{E})$. One checks that $c(\nabla)$ is skew-symmetric and A-bilinear. Thus, by (1.4.2.1), one can write $c(\nabla) \in h(\mathcal{C}^2 \otimes P)$.

Consider the Lie* $\mathcal{C}^{\cdot}$-algebra $\mathcal{C} \otimes P$. Then ∇ defines a $d_{\mathcal{C}}$-derivation d_∇ of the $\mathcal{C}$-module $\mathcal{C} \otimes P$ such that $d_\nabla : P \to \mathcal{C}^1 \otimes P$ corresponds to $[\nabla, id_P] \in P_2^*(\{\mathcal{L}, P\}, P)$ via (1.4.2.1).

If non-empty, the set of $\mathcal{L}$-connections on $\mathcal{E}$ is a torsor for $\mathrm{Hom}(\mathcal{L}, P) = h(\mathcal{C}^1 \otimes P)$. For $\nu \in h(\mathcal{C}^1 \otimes P)$ one has

$$c(\nabla + \nu) = c(\nabla) + d_\nabla(\nu) + \frac{1}{2}[\nu, \nu], \quad d_{\nabla+\nu} = d_\nabla + ad_\nu. \tag{1.4.17.1}$$

REMARK. We call Θ_A-connections simply connections (here π is the canonical projection). If M is an A-module such that $\mathcal{E}(M)$ is well defined (see Remark (iii) in 1.4.16), then a connection on $\mathcal{E}(M)$ is the same as a connection on M; the identification $\mathrm{Hom}(M, \Omega_A^2 \otimes M) = h(\Omega_A^2 \otimes \mathfrak{g}l(M))$ identifies the corresponding curvatures.

The set of Lie* algebroids equipped with an $\mathcal{L}$-connection can be described as follows. Namely, for a given Lie* A-algebra P consider the following two sets:

(i) the set[15] $\mathfrak{F}(P)$ of pairs $(\mathcal{E}, \nabla)$ where $\mathcal{E}$ is a P-extension of $\mathcal{L}$, ∇ a connection on $\mathcal{E}$;

(ii) the set $\mathfrak{D}(P)$ of pairs $(\mathfrak{d}, \mathfrak{c})$ where $\mathfrak{d}$ is a $d_{\mathcal{C}}$-derivation of the Lie* $\mathcal{C}$-algebra $\mathcal{C} \otimes P$,[16] and $\mathfrak{c} \in h(\mathcal{C}^2 \otimes P)$ is an element such that $\mathfrak{d}(\mathfrak{c}) = 0$ and $\mathfrak{d}^2 = ad_{\mathfrak{c}}$. We call such a pair $(\mathfrak{d}, \mathfrak{c})$ *compatible.*

The group $h(\mathcal{C}^1 \otimes P)$ acts on both sets: for $\nu \in h(\mathcal{C}^1 \otimes P)$ the ν-translations are $(\mathcal{E}, \nabla) \mapsto (\mathcal{E}, \nabla + \nu)$, $(\mathfrak{d}, \mathfrak{c}) \mapsto (\mathfrak{d} + ad_\nu, \mathfrak{c} + \mathfrak{d}(\nu) + \frac{1}{2}[\nu, \nu])$.

LEMMA. *For any $(\mathcal{E}, \nabla) \in \mathfrak{F}(P)$ the pair $(d_\nabla, c(\nabla))$ is compatible. The map $\mathfrak{F}(P) \to \mathfrak{D}(P)$, $(\mathcal{E}, \nabla) \mapsto (d_\nabla, c(\nabla))$, is bijective and compatible with the $h(\mathcal{C}^1 \otimes P)$-actions.* □

REMARK. Consider a larger set $\mathfrak{D}'(P)$ of pairs $(\mathfrak{d}, \mathfrak{c})$ which satisfy only the second compatibility $\mathfrak{d}^2 = ad_{\mathfrak{c}}$ from (ii). The $h(\mathcal{C}^1 \otimes P)$-action extends to $\mathfrak{D}'(P)$ by the same formula. The function $(\mathfrak{d}, \mathfrak{c}) \mapsto \mathfrak{d}(\mathfrak{c})$ is constant on the $h(\mathcal{C}^1 \otimes P)$-orbits.

1.4.18. Coisson and odd coisson algebras. A *compound Poisson*, or simply *coisson*, bracket on $A \in \mathcal{C}omu(\mathcal{M}^!)$ is a Lie* bracket $\{\ \} \in P_2^*(\{A, A\}, A)$ such that the adjoint action $\tau_{\{\ \}}$ is an action of A (considered as a Lie* algebra) on A (as on a commutative$^!$ algebra). In other words, $\{\ \}$ satisfies the Leibniz rule with respect to either of the variables. Such pair $(A, \{\ \})$ is called a *coisson* algebra.

By 1.4.13 (i) a coisson bracket $\{\ \}$ yields a Lie* A-algebroid structure on $A \otimes A$. One checks easily that the kernel of the projection $A \otimes A \to \Omega_A$, $a \otimes b \mapsto adb$, is an ideal in $A \otimes A$, so Ω_A inherits the Lie* A-algebroid structure. In other words, Ω_A carries a unique Lie* A-algebroid structure such that $d\colon A \to \Omega_A$ is a morphism of Lie* algebras and $\tau_{\Omega_A}(d, id_A) = \{\ \}$.

[15] Pairs $(\mathcal{E}, \nabla)$ are rigid, so we make no distinction between $(\mathcal{E}, \nabla)$ and its isomorphism class.

[16] I.e., $\mathfrak{d}$ is a $d_{\mathcal{C}}$-derivation for the $\mathcal{C}$-module structure and a derivation for the Lie* bracket.

Conversely, a structure of a Lie* A-algebroid on Ω_A yields a binary $*$ operation $\{\ \} := \tau_{\Omega_A}(d, id_A)$ on A. If $\{\ \}$ is skew-symmetric and $d\{\ \} = [d,d] \in P_2^*(\{A,A\}, \Omega_A)$, then $\{\ \}$ is a coisson bracket. Therefore coisson brackets on A are the same as Lie* A-algebroid structures on Ω_A that satisfy the above conditions.

The category of coisson algebras $\mathcal{C}ois(\mathcal{M}^{*!})$ is a tensor category. Namely, for a finite family $(A_i, \{\ \}_i)$, $i \in I$, of coisson algebras their tensor product $(A, \{\ \})$ is $A = \otimes A_i$, $\{\ \} = \sum\limits_{i\in I} \{\ \}_i^\sim$ where for $j \in I$ the bracket $\{\ \}_j^\sim \in P_2^*(\{A,A\}, A)$ is the $\underset{I\smallsetminus\{j\}}{\otimes} A_i$-linear extension of $\{\ \}_j$ (see Remark (ii) in 1.4.6). The algebra $\mathbf{1}$ with zero coisson bracket is a unit object for this tensor structure.

Remark. The evident morphisms $A_i \to A$ are morphisms of coisson algebras which mutually coisson commute in the obvious sense. This datum is universal: for any coisson algebra B a morphism of coisson algebras $A \to B$ amounts to a collection (φ_i) of mutually coisson commuting morphisms $\varphi_i : A_i \to B$.

Examples. (i) If L is a Lie* algebra, then the symmetric$^!$ algebra $\operatorname{Sym} L$ carries a unique coisson bracket $\{\ \}$ such that $L = \operatorname{Sym}^1 L \hookrightarrow \operatorname{Sym} L$ is a morphism of Lie* algebras; this is the Kostant-Kirillov bracket. The functor $\mathcal{L}ie(\mathcal{M}^*) \to \mathcal{C}ois(\mathcal{M}^*), L \mapsto (\operatorname{Sym} L, \{\ \})$, is left adjoint to the "forgetting of commutative$^!$ algebra structure" functor $\mathcal{C}ois(\mathcal{M}^{*!}) \to \mathcal{L}ie(\mathcal{M}^*)$.

(ii) More generally, if A is a commutative$^!$ algebra and $\mathcal{L}$ is a Lie* A-algebroid, then the symmetric$^!$ algebra $\operatorname{Sym}\mathcal{L} := \operatorname{Sym}_A\mathcal{L}$ of the A-module $\mathcal{L}$ carries a unique coisson bracket $\{\ \}$ such that $\{\operatorname{Sym}^p\mathcal{L}, \operatorname{Sym}^q\mathcal{L}\} \subset \operatorname{Sym}^{p+q-1}\mathcal{L}$, and $\{\ \}$ coincides with $[\ ]$ on $\operatorname{Sym}^1 \times \operatorname{Sym}^1$ and with $\tau_{\mathcal{L}}$ on $\operatorname{Sym}^1 \times \operatorname{Sym}^0$. For example, $\operatorname{Sym}\Theta_A$ is the algebra of functions on the cotangent bundle to $\operatorname{Spec} A$ and $\{\ \}$ is a canonical coisson structure on it.

(iii) The above constructions often come in a twisted version. Let L be a Lie* algebra and $L^\flat$ a central extension of L by $\mathbf{1}$. The *twisted symmetric algebra* $\operatorname{Sym}^\flat L$ is the quotient of $\operatorname{Sym}(L^\flat)$ modulo the ideal generated by the image of the difference of embeddings $\mathbf{1} = \operatorname{Sym}^0(L^\flat) \hookrightarrow \operatorname{Sym}(L^\flat)$, $\mathbf{1} \subset L^\flat = \operatorname{Sym}^1(L^\flat) \hookrightarrow \operatorname{Sym}(L^\flat)$. This quotient algebra inherits the coisson bracket from $\operatorname{Sym}(L^\flat)$. Similarly, in the setting of (ii), if $\mathcal{L}^\flat$ is a central extension of a Lie* algebroid $\mathcal{L}$ by A, then the twisted symmetric algebra $\operatorname{Sym}_A^\flat\mathcal{L}$ inherits a coisson bracket from $\operatorname{Sym}_A(\mathcal{L}^\flat)$.

The above definition, of course, makes sense in the super, or DG super, setting (see 1.1.16). For a commutative super, or DG super, algebra A we can also consider *n-coisson* brackets on A, which are Lie* brackets $\{\ \} \in P_2^*(\{A[-n], A[-n]\}, A[-n])$ on $A[-n]$ such that the adjoint action of $A[-n]$ on A satisfies the Leibniz rule. One calls $(A, \{\ \})$ an *n-coisson* algebra. For $n = 0$ this is the usual coisson algebra. As above, for n-coisson A the A-module $\Omega_A[-n]$ is naturally a Lie* algebroid.

Remarks. (i) For the plain super setting, it is only the parity of n that matters, so we speak of *odd* and *even* (= usual) coisson algebras.

(ii) In (the particular case of) the mere tensor (non-compound) category setting, we talk of *n-Poisson* and *odd* or *even* Poisson algebras.

Example. For any Lie* A-algebroid $\mathcal{L}$ the symmetric$^!$ algebra $\operatorname{Sym}_A(\mathcal{L}[n])$ is naturally an n-coisson algebra whose bracket is characterized by the property that it coincides with $[\ ]$ on $\operatorname{Sym}^1 \times \operatorname{Sym}^1$ and with τ_L on $\operatorname{Sym}^1 \times \operatorname{Sym}^0$. In particular, if Θ_A is well defined, then $\operatorname{Sym}_A(\Theta_A[1])$ is an odd coisson algebra; its odd coisson bracket is called the *Schouten-Nijenhuis* bracket. More generally, $\mathrm{Sym}_A(\Theta_A[n])$ is an n-coisson algebra.

EXERCISE. Any $\nu \in \mathrm{Hom}_A(\mathcal{L}, A)$ yields a degree 1 odd derivation d_ν of the algebra $\mathrm{Sym}_A(\mathcal{L}[1])$ which equals ν on Sym^1. Show that it is compatible with the odd coisson bracket if and only if ν, considered as an element of $C_A(\mathcal{L})^1$ (see 1.4.14(i)), is closed. In particular, suppose that Θ_A is well defined and the canonical pairing $\in P^*_{A2}(\{\Theta_A, \Omega_A\}, A)$ is non-degenerate. Then $\nu \in h(\Omega^1_A)$ is closed if and only if d_ν is a derivation of the Schouten-Nijenhuis bracket.

Let A be an n-coisson algebra. Suppose Θ_A is well defined and the canonical pairing $\in P^*_{A2}(\{\Omega_A, \Theta_A\}, A)$ is non-degenerate. Consider the de Rham-Chevalley algebra $\mathcal{C}_A(\Omega_A[-n])$ of the Lie* algebroid $\Omega_A[-n]$ (see 1.4.14). As a mere graded algebra, it equals $\mathrm{Sym}_A(\Theta_A[n-1])$. Since Θ_A is a Lie* algebroid, the latter is an $n-1$-coisson algebra. One checks that the de Rham-Chevalley differential is compatible with the coisson bracket, so $\mathcal{C}_A(\Omega_A)$ is a DG $n-1$-coisson algebra. In particular, for an odd coisson A, $\mathcal{C}^{cois}(A)$ is a coisson algebra.

The $n-1$-coisson bracket is compatible with the stupid filtration $\mathcal{C}_A(\Omega_A)^{\geq a}$, so the completion $\mathcal{C}^{cois}(A) := \varprojlim \mathcal{C}_A(\Omega_A[-n])/\mathcal{C}_A(\Omega_A[-n])^{\geq a}$ is naturally a topological $(n-1)$-coisson algebra. The spectrum of $\mathcal{C}^{cois}(A)$ is "the space of symplectic leaves" of $\mathrm{Spec}A$.

1.4.19. Hamiltonian reduction. The usual operations on Poisson algebras have obvious coisson versions. For example, if $(A, \{\ \})$ is a coisson algebra and $I \subset A$ is an ideal such that $\{I, I\} \subset I$, then I acts on A/I as a Lie* algebra. Set $B := (A/I)^I$ (:= the maximal I-invariant subobject of A/I; we tacitly assume that it exists). Then B is a ! subalgebra of A/I, and B inherits a coisson bracket $\{\ \}$ such that for $b_1, b_2 \in B$ and $\tilde{b}_i \in A$, $\tilde{b}_i \mathrm{mod}\, I = b_i$, one has $\{b_1, b_2\} = \{\tilde{b}_1, \tilde{b}_2\}\mathrm{mod}\, I$. In particular, for $(A, \{\ \}) \in \mathcal{C}ois(\mathcal{M}^{*!})$, $L \in \mathcal{L}ie(\mathcal{M}^*)$, and a morphism of Lie* algebras $\alpha\colon L \to A$, the commutative$^!$ algebra $B\colon = (A/\alpha(L)A)^L$ carries a canonical coisson bracket $\{\ \}$ (here $\alpha(L)A$ is the ideal generated by $\alpha(L) \subset A$, so L acts on A via α and $\{\ \}$, $A/\alpha(L)A$ carries the quotient action, and $(\ )^L$ is the subobject of L-invariants). We call $(B, \{\ \})$ the *hamiltonian reduction* of $(A, \{\ \})$.

1.4.20. Coisson modules. Let $(A, \{\ \})$ be a coisson algebra. A *coisson A-module*, or *$(A, \{\ \})$-module*, is an A-module M (here A is considered as a plain commutative$^!$ algebra) together with a $\tau_{\{\ \}}$-action of A on M (here A is considered as a Lie* algebra) $\tau^{(A)}_M \in P^*_2(\{A, M\}, M)$ which is a derivation with respect to the A-variable. The coisson modules form a category $\mathcal{M}(A, \{\ \})$.

In fact $\mathcal{M}_{(A,\{\ \})}$ coincides with the category $\mathcal{M}(A, \Omega_A)$ where Ω_A is the Lie* A-algebroid from 1.4.18. The equivalence $\mathcal{M}(A, \Omega_A) \xrightarrow{\sim} \mathcal{M}(A, \{\ \})$ assigns to an Ω_A-module M the same M with $\tau^{(A)}_M$ coming from the morphism of Lie* algebras $d : A \to \Omega_A$. Therefore $\mathcal{M}(A, \{\ \})$ carries a canonical augmented compound tensor structure; we denote it $\mathcal{M}(A, \{\ \})^{*!}$.

1.4.21. BRST reduction (coisson version). This is a refined version of the hamiltonian reduction; both are compared in 1.4.26. Below we tacitly deal with DG super objects, i.e., with the category $C\mathcal{M}^s$, following the conventions of 1.1.16. For $A \in C\mathcal{M}^s$, $h(A)$ is a complex of super k-modules. We assume that $k \supset \mathbb{Q}$.

Suppose we have $L, L' \in C\mathcal{M}^s$ and a $*$ pairing $\langle\ \rangle \in P^*_2(\{L', L\}, \mathbf{1})$. Consider $L[1] \oplus L'[-1]$ as a commutative Lie* algebra. Let $(L[1] \oplus L'[-1])^\flat = L[1] \oplus$

$L'[-1] \oplus \mathbf{1}$ be its central extension by $\mathbf{1}$ with commutator $\in P_2^*(\{L'[-1], L[1]\}, \mathbf{1}) = P_2^*(\{L', L\}, \mathbf{1})$ equal to $\langle\ \rangle$. Set $\mathcal{C}l_c := \mathrm{Sym}^\flat(L'[-1] \oplus L[1])$ (see Example (iii) in 1.4.18).[17] This is the *Clifford* coisson algebra. It carries an extra $\mathbb{Z}$-grading $\mathcal{C}l_c^{(\cdot)}$ determined by the conditions $L[1] \subset \mathcal{C}l^{(-1)}$, $L'[-1] \subset \mathcal{C}l^{(1)}$. One has:

(i) There are obvious embeddings $\mathrm{Sym}(L'[-1]), \mathrm{Sym}(L[1]) \hookrightarrow \mathcal{C}l_c$; the product morphism $\mathrm{Sym}(L'[-1] \oplus L[1]) = \mathrm{Sym}(L'[-1]) \otimes \mathrm{Sym}(L[1]) \to \mathcal{C}l_c$ is an isomorphism of commutative$^!$ algebras.

(ii) The coisson bracket vanishes on both $L'[-1]$ and $L[1]$; the $\{\ \}$ pairing between $L'[-1]$ and $L[1]$ is equal to $\langle\ \rangle$.

(iii) The subobject $L \otimes L' = L[1] \otimes L'[-1] \subset \mathcal{C}l_c^{(0)}$ is a Lie* subalgebra. The coisson bracket on it coincides with the bracket that comes from the associative* algebra structure as defined in 1.4.2. This subalgebra normalizes $L[1], L'[-1] \subset \mathcal{C}l_c^{(\pm 1)}$, and the corresponding adjoint action coincides with the action from 1.4.2.

We will need a simple lemma. For $A \in C\mathcal{M}^s$ consider the coisson $\mathcal{C}l_c$-module $A \otimes \mathcal{C}l_c$. Let $F_c \subset h(A \otimes \mathcal{C}l_c)$ be the centralizer of $L[1] \subset \mathcal{C}l_c$. In other words, F_c is the kernel of the morphism $\psi : h(A \otimes \mathcal{C}l_c) \to \mathrm{Hom}(L[1], A \otimes \mathcal{C}l_c)$ that comes from the adjoint action of $L[1]$ on $\mathcal{C}l_c$. It is clear that $F_c \supset h(A \otimes \mathrm{Sym}(L[1]))$.

1.4.22. Lemma. *If the $*$ pairing $\langle\ \rangle$ is non-degenerate (see 1.4.2), then* $F_c = h(A \otimes \mathrm{Sym}(L[1]))$.

Proof. We have $h(A \otimes \mathcal{C}l_c^{(a)}) = \oplus\, h(A \otimes \mathrm{Sym}^n(L'[-1]) \otimes \mathrm{Sym}^{n-a}(L[1]))$. Take any $\nu \in h(A \otimes \mathcal{C}l_c^{(a)})$; let n be the largest number such that the nth component of ν is non-zero; denote it by $\bar{\nu} \in h(A \otimes \mathrm{Sym}^n(L'[-1]) \otimes \mathrm{Sym}^{n-a}(L[1]))$. The morphism $\chi := \psi(\nu)$ sends $L[1]$ to the degree $a-1$ part of $A \otimes \mathrm{Sym}^{\leq n-1}(L'[-1]) \otimes \mathrm{Sym}(L[1])$; let $\bar{\chi}$ be its composition with the projection to $A \otimes \mathrm{Sym}^{n-1}(L'[-1]) \otimes \mathrm{Sym}^{n-a}(L[1])$. Thus (see 1.4.2) $\bar{\chi} \in \mathrm{Hom}(L[1], A \otimes \mathrm{Sym}^{n-1}(L'[-1]) \otimes \mathrm{Sym}^{n-a}(L[1])) = h(A \otimes (L'[-1] \otimes \mathrm{Sym}^{n-1}(L'[-1])) \otimes \mathrm{Sym}^{n-a}(L[1]))$. According to the Leibnitz formula the image of $\bar{\chi}$ by the product map $L'[-1] \otimes \mathrm{Sym}^{n-1}(L'[-1]) \to \mathrm{Sym}^n(L'[-1])$ equals $n\bar{\nu}$. If ν lies in F_c, i.e., $\chi = 0$, then $n\bar{\nu} = 0$, hence $n = 0$. □

1.4.23. From now on L is a Lie* algebra. We assume that the $*$ dual $L^\circ := \mathcal{H}om(L, \mathbf{1})$ exists and the canonical pairing $\langle\ \rangle \in P_2^*(\{L^\circ, L\}, \mathbf{1})$ is non-degenerate (see 1.4.2). Then the adjoint action of L on L, hence on $L[1]$, yields a morphism of Lie* algebras $\beta : L \to L \otimes L^\circ = L[1] \otimes L^\circ[-1] \subset \mathcal{C}l_c^{(0)}$. We can consider β as an element of $h(\mathcal{C}l_c^{(0)} \otimes L^\circ)$. Denote by $\tilde{\beta}_c \in h(\mathcal{C}l_c[1])$ its image by the product map $\mathcal{C}l_c \otimes L^\circ[-1] \to \mathcal{C}l_c$.

Let A be a coisson algebra, and let $\alpha : L \to A$ be a morphism of Lie* algebras. Consider the graded coisson algebra $A \otimes \mathcal{C}l_c^{\cdot}$. We have a morphism of mere graded Lie* algebras

$$\ell : L_\dagger \to A \otimes \mathcal{C}l_c. \qquad (1.4.23.1)$$

Here $L_\dagger$ is a contractible DG Lie* algebra defined in 1.1.16 (we forget about its differential for the moment), and the components of ℓ are $\ell^{(0)} = \alpha + \beta : L \to A \otimes \mathcal{C}l^{(0)}$ and $\ell^{(-1)} :=$ the composition $L[1] \hookrightarrow \mathcal{C}l^{(-1)} \to A \otimes \mathcal{C}l$.

[17] The subscript "c" is for "coisson" or "classical."

Let us rewrite α as $\tilde{\alpha} \in h(A \otimes L^\circ)^{even} \subset h(A \otimes \mathcal{C}l_c[1])$. Let $\tilde{\beta} \in h(A \otimes \mathcal{C}l_c[1])$ be the image of $\tilde{\beta}_c$ by the morphism of algebras $\mathcal{C}l_c \to A \otimes \mathcal{C}l_c$. Set

$$\mathfrak{d}_c := \tilde{\alpha} + \frac{1}{2}\tilde{\beta} \in h(A \otimes \mathcal{C}l_c[1]) = \mathrm{Hom}(k[-1], h(A \otimes \mathcal{C}l_c); \tag{1.4.23.2}$$

this is the *(classical) BRST charge*. Its adjoint action is an odd derivation d of degree 1 of the graded coisson algebra $A \otimes \mathcal{C}l_c$ called the *BRST differential*.

1.4.24. PROPOSITION. (i) *One has* $\{\mathfrak{d}_c, \mathfrak{d}_c\} = 0$, *so* $d^2 = 0$.
We denote the DG coisson algebra $(A \otimes \mathcal{C}l_c, d)$ *by* $\mathcal{C}_{\mathrm{BRST}}(L, A)_c$.

(ii) ℓ *commutes with the differentials, so we have a morphism of DG Lie** *algebras*

$$\ell : L_\dagger \to \mathcal{C}_{\mathrm{BRST}}\,(L, A)_c. \tag{1.4.24.1}$$

We refer to this as the BRST property of $\mathfrak{d}$ *or* d. *It determines* $\mathfrak{d}$ *and* d *uniquely:*

(iii) $\mathfrak{d}_c$ *is a unique even element of degree zero in* $h(A \otimes \mathcal{C}l_c[1])$ *such that* $[\mathfrak{d}_c, \ell^{(-1)}]$ *equals* $\ell^{(0)}$, *and* d *is a unique odd derivation of* $A \otimes \mathcal{C}l$ *of degree 1 such that* $d\ell^{(-1)} = \ell^{(0)}$.

Proof. (i), (ii) are straightforward computations. (iii) follows from the next lemma. Let Der_c be the graded Lie algebra of derivations of the coisson algebra $A \otimes \mathcal{C}l_c$.

1.4.25. LEMMA. *The maps* $h(A \otimes \mathcal{C}l_c^{(a)}) \to Der_c^a \to \mathrm{Hom}(L[1], A \otimes \mathcal{C}l_c^{(a-1)})$, *where the first arrow is the adjoint action and the second one is* $\delta \mapsto \delta\ell^{(-1)}$, *are injective for* $a > 0$.

Proof. (a) Let δ be a coisson derivation of $A \otimes \mathcal{C}l_c$ of degree $a > 0$ such that $\delta\ell^{(-1)} = 0$; i.e., δ kills $L[1] \subset \mathcal{C}l_c$. Then it sends $L^\circ[-1]$ and A to the centralizer of $L[1]$ in, respectively, $A \otimes \mathcal{C}l_c^{(a+1)}$ and $A \otimes \mathcal{C}l_c^{(a)}$, which is zero by 1.4.22. Thus $\delta = 0$.

(b) The kernel of the first arrow is contained in the centralizer of $L[1]$; by 1.4.22 it vanishes if $a > 0$. $\square$

Considered as an object of an appropriate homotopy category of coisson algebras, $\mathcal{C}_{\mathrm{BRST}}(L, A)_c$ is called the *BRST reduction of* A with respect to α. Its cohomology is a graded coisson algebra $H^{\cdot}_{\mathrm{BRST}}(L, A)_c$. The reduction is *regular* if $H^{\neq 0}_{\mathrm{BRST}}(L, A)_c = 0$. Then the BRST reduction is a plain coisson algebra $H^0_{\mathrm{BRST}}(L, A)_c$.

1.4.26. Let us compare the BRST and hamiltonian reductions (see 1.4.19). Set $\bar{A} := A/\alpha(L)A$. This is a commutative$^!$ algebra; the L-action on A (via α and the coisson bracket) yields an L-action on $\bar{A}$. Let I be the DG ideal of $\mathcal{C}_{\mathrm{BRST}}(L, A)_c$ generated by the image of $\ell^{(-1)} : L[1] \to A \otimes \mathcal{C}l_c^{(-1)}$. The morphism of graded algebras $A \otimes \mathrm{Sym}(L^\circ[-1]) \to A \otimes \mathcal{C}l_c/I$ is surjective; by 1.4.24 its kernel is the ideal generated by $\alpha(L)$. Thus $\mathcal{C}_{\mathrm{BRST}}(L, A)_c/I = \bar{A} \otimes \mathrm{Sym}(L^\circ[-1])$. One checks immediately that the BRST differential yields on $\bar{A} \otimes \mathrm{Sym}(L^\circ[-1])$ the usual Chevalley differential (see 1.4.10), so we have a canonical surjective morphism of DG commutative$^!$ algebras

$$\mathcal{C}_{\mathrm{BRST}}(L, A)_c \twoheadrightarrow \mathcal{C}(L, \bar{A}). \tag{1.4.26.1}$$

Passing to zero cohomology, we get a canonical morphism of commutative algebras

$$H^0_{\mathrm{BRST}}(L,A)_c \to \bar{A}^L. \tag{1.4.26.2}$$

It is easy to see that it is compatible with the coisson brackets.

1.4.27. The compound tensor structure on $\mathcal{M}$ can be encoded in a single pseudo-tensor structure $\mathcal{M}^c$. For the meaning of this construction see 3.2.

So let us bring the sundry polylinear operations of a compound tensor category under one roof. For an I-family $L_i \in \mathcal{M}$, $M \in \mathcal{M}$, and $S \in Q(I)$ (see 1.3.1) set

$$P^c_I(\{L_i\}, M)_S := P^*_S(\{\underset{i\in I_s}{\otimes} L_i\}, M) \otimes (\underset{s\in S}{\otimes} \mathcal{L}ie_{I_s}). \tag{1.4.27.1}$$

The k-module of *c operations* is

$$P^c_I(\{L_i\}, M) := \underset{S\in Q(I)}{\oplus} P^c_I(\{L_i\}, M)_S. \tag{1.4.27.2}$$

The c operations compose in a natural manner. Namely, assume we have $J \twoheadrightarrow I$ and a J-family of objects $K_j \in \mathcal{M}$. Take $\varphi \in P^c_I(\{L_i\}, M)_S$, $\psi_i \in P^c_{J_i}(\{K_j\}, M)_{T_i}$ (where $T_i \in Q(J_i)$). For $R \in Q(J)$ the R-component $\varphi(\psi_i)_R \in P^c_J(\{K_j\}, M)_R$ vanishes unless $R \le T$,[18] I and R are transversal in $Q(T)$, and $\inf\{I, R\} = S$ (see 1.3.1). If these conditions are satisfied, then $\varphi(\psi_i)_R = \varphi(\gamma_s)$ where for $s \in S$ one has[19] $\gamma_s := \otimes^{T_s}_{I_s,R_s} \psi_i \in P^*_{R_s}(\{\underset{j\in J_r}{\otimes} K_j\}, \underset{i\in I_s}{\otimes} L_i) \otimes (\underset{t\in T_s}{\otimes} \mathcal{L}ie_{J_t})$ and the composition $\varphi(\gamma_s)$ is the tensor product of compositions of $*$ and Lie operations.[20] We leave it to the reader to check its associativity.

We have defined an abelian (see 1.1.7) pseudo-tensor structure $\mathcal{M}^c$ on $\mathcal{M}$.

Set $P^c_I(\{L_i\}, M)^n := \underset{S\in Q(I,|I|-n)}{\oplus} P^c_I(\{L_i\}, M)_S$ where $Q(I,m) \subset Q(I)$ consists of quotients $I \twoheadrightarrow S$, $|S| = m$. This is a natural $\mathbb{Z}$-grading on P^c compatible with the composition of c operations.

One has $P^c_I(\{L_i\}, M)^{|I|} = \mathrm{Hom}(\underset{I}{\otimes} L_i, M) \otimes \mathcal{L}ie_I$, $P^c_I(\{L_i\}, M)^0 = P^*_I(\{L_i\}, M)$. The corresponding embedding and projection define the maps $P^! \otimes \mathcal{L}ie \to P^c \to P^*$ which are compatible with the composition of operations, so we have the pseudo-tensor functors

$$\mathcal{M}^! \otimes \mathcal{L}ie \to \mathcal{M}^c \to \mathcal{M}^* \tag{1.4.27.3}$$

that extend the identity functor on $\mathcal{M}$.

REMARK. By 1.1.12 the compound pseudo-tensor structure $\mathcal{M}^{*!}$ is uniquely determined by the pseudo-tensor structure $\mathcal{M}^c$ together with the $\mathbb{Z}$-grading on P^c.

1.4.28. LEMMA. *Lie algebras in $\mathcal{M}^c$ are the same as coisson algebras without unit in $\mathcal{M}^{*!}$.*

Proof. A binary c operation $\mu \in P^c_2(M)$ is the same as a pair $(\cdot, \{\ \ \})$ where[21] $\cdot \in \mathrm{Hom}(M \otimes M, M)$ and $\{\ \} \in P^*_2(M)$. The skew-symmetry of μ amounts to the

[18] Here $T := \sqcup T_i \in Q(J)$.

[19] See 1.3.12 for notation.

[20] The composition of Lie operations is the tensor product of maps $\mathcal{L}ie_{I_s} \otimes (\underset{t\in T_r}{\otimes} \mathcal{L}ie_{J_t}) \to \mathcal{L}ie_{J_r}$ labeled by $r \in R_s$. We use the fact that $T_r \xrightarrow{\sim} I_s$ (the transversality condition).

[21] We use the identification $k \xrightarrow{\sim} \mathcal{L}ie_2$, $1 \mapsto [\ ]$.

symmetry of $\cdot$ and the skew-symmetry of $\{\ \ \}$. The Jacobi identity for μ amounts to three identities (corresponding to the graded components):

– the Jacobi identity for $\cdot \otimes [\ ]$ which, by 1.1.10, is the same as the associativity of $\cdot$,

– the Jacobi identity for $\{\ \}$,

– the identity which says that $\{\ \}$ acts by derivations of $\cdot$. $\square$